Imperfect Demand Expectations and Endogenous Business Cycles

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Abstract

Optimal growth models aim at explaining long run trends of growth under the strong assumption of full efficiency in the allocation of resources. As a result, the steady state time paths of the main economic aggregates reflect constant, exogenous or endogenous, growth. To introduce business cycles in this optimality structure one has to consider some source of inefficiency. By assuming that firms adopt a simple non optimal rule to predict future demand, we let investment decisions to depart from the ones that would guarantee the total efficiency outcome. The new investment hypothesis is considered under three growth setups (the simple one equation Solow model of capital accumulation, the Ramsey model with consumption utility maximization, and a two sector endogenous growth setup); for each one of the models, we find that endogenous business cycles of various orders (regular and irregular) are observable.

Keywords: Endogenous business cycles, Nonlinear dynamics, Growth models, Bifurcation analysis.

JEL classification: C61, E32, O41

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1. Introduction

Growth models are specially designed to characterize long term trends of growth. Both the neoclassical framework [i.e., the Solow (1956) and the Ramsey (1928) – Cass (1965) – Koopmans (1965) models] and the endogenous growth setup [i.e., the interpretation of the growth process pioneered by Romer (1986), Lucas (1988), Rebelo (1991) and Aghion and Howitt (1992)] characterize accumulation processes which culminate in a steady state result where the main economic aggregates (output, capital stock, investment and consumption) grow at constant rates, which can be respectively determined by exogenous or endogenous factors.

Because growth is not a linear process it is important to inquire why business cycle fluctuations are not captured under standard growth setups. The obvious answer is that optimal growth models are built over a perfect competition market structure, where no place is left for inefficiencies. A market clearing Walrasian setup allows answering only to the following question: which is the trend of evolution of macroeconomic aggregates when all imperfections are ruled out? Several attempts to explain business cycles have been suggested in the economic literature. The most important debate is the one between the strand of literature that considers that the utility maximization scenario is not suitable to explain fluctuations that occur in a context of market failures and the theory that has attempted to reconcile the basic intertemporal growth framework with cycles caused by real factors.

The first approach, originally presented by Phelps (1970) and Lucas (1972), can be thought as a Keynesian theory of fluctuations. In this view, markets fail for various reasons. First, there are coordination problems, which emerge from the strategic nature of the relations among economic agents; in this case, low levels of aggregate investment can explain why each individual firm does not invest. Therefore, expansions would be periods of generalized confidence while recessions would be associated with low levels of economic activity triggered by a wave of pessimism throughout the economy. Second, in a Keynesian perspective, fluctuations may be understood as well as the result of incomplete nominal adjustment or price stickiness. Prices and wages do not adjust instantaneously and simultaneously and, hence, nominal sluggishness provokes real shocks on demand. As a whole, the Keynesian view sets aside the competitive optimization framework in order to emphasize that cycles are essentially the result of nominal disturbances over a far from perfect economic system.

The second approach, which relies on the work set forth by Kydland and Prescott (1982), Long and Plosser (1983), King, Plosser and Rebelo (1988), and Christiano and Eichenbaum (1992), among others, corresponds to the Real Business Cycles (RBC) theory. As Rebelo (2005) highlights, the literature on RBC allows for several important thoughts about cycles; besides the relevance and appealing in terms of empirically testable results, this theory has the merit of allowing to unify business cycles analysis and growth theory, since fluctuations are approached under the dynamic general equilibrium models that characterize the study of economic growth since the work of Solow. In the words of Rebelo (2005, page 2), 'business cycles can be studied using dynamic general equilibrium models. These models feature atomistic agents who operate in competitive markets and form rational expectations about the future'.

To allow for cycles under a market efficiency setup, RBC models consider real shocks, like technology shocks or sudden changes in fiscal policy. These disturbances introduce a stochastic component on growth models, and it is this non deterministic element that produces fluctuations around the output trend. The revolutionary idea behind RBC theory is that it is not necessarily demand that mainly determines cycles, but instead it is probably a supply side shock regarding total factor productivity (TFP). Furthermore, fluctuations occur under perfectly competitive markets and thus recessions and expansions are simply the response to real economy events. The whole RBC mechanism works through the labor market: TFP growth induces a change in the behaviour of profit and utility maximizing agents; firms will want to hire additional work; the higher demand for work rises wages and introduces a change in households choices; these will be available to trade leisure by work hours in order to gain increased income that can be used in consumption. Under an analytical viewpoint, RBC models introduce over the simple Ramsey model an intertemporal choice between labor and leisure, with leisure an argument of the utility function.

The RBC theory bases its explanation of cycles on the occurrence of exogenous disturbances. Under this perspective, one can establish a critical argument that resembles the one that motivated the rise of endogenous growth models. Neoclassical growth theory offered an exogenous interpretation of long run growth: exogenous technology shocks were the engine of sustained growth. Similarly, in RBC models the engine of fluctuations is also the exogenous productivity disturbances (or other kind of shocks). Thus, although RBC models continue to be a fundamental benchmark in the understanding of business cycles, as documented in, e.g., King and Rebelo (1999) and

Jones, Manuelli and Siu (2000), it is important to look beyond this ‘exogenous business cycles’ theory.

This paper searches for a model that explains business cycles within an optimal growth framework, as in RBC models, but that can simultaneously be presented entirely as a deterministic setup. This requires introducing some kind of Keynesian feature in the model, that is, market failures are the ingredient that is necessary to obtain endogenous business cycles in an intertemporal growth scenario.

There is an important strand of literature that presents an endogenous interpretation of cycles. Studies on ‘endogenous business cycles’ are motivated by the work on nonlinear macroeconomics first developed in the 1980s by various authors including Medio (1979), Stutzer (1980), Benhabib and Day (1981), Day (1982) and Grandmont (1985). Authoritative surveys about this literature can be found in Baumol and Benhabib (1989), Boldrin and Woodford (1990), Chiarella (1992) and Bullard and Butler (1993). It is with the work of Christiano and Harrison (1999) that RBC theory and the endogenous cycles interpretation are crossed. These authors use an optimal growth model with endogenous labor – leisure decisions just as in the RBC theory, but stochastic shocks are replaced by a production externality. This externality allows for increasing marginal returns in production which generate a complex dynamic behaviour that for some combinations of parameter values is characterized by long term periodic and a-periodic motion: cycles are an endogenous phenomenon and no external event is necessary to justify them.

Other authors have reemphasized the importance of this approach; this is the case of Wen (1998), Schmitt-Grohé (2000), Guo and Lansing (2002), Goenka and Poulsen (2004) and Coury and Wen (2005). This last work highlights the little empirical relevance of the externalities model, given the unreasonably high level of externalities that is necessary to produce nonlinear motion. This literature is also criticized, following Reichlin (1997), in the grounds of the extreme complexity of the obtained growth paths – multiple equilibria and chaotic motion frequently arise as the outcome of the referred analytical structures, raising doubts about the underlying hypothesis regarding agents expectations and also raising the question of how such outcomes can be subject to empirical scrutiny.

Nevertheless, the referred approach to business cycles is able to maintain the structure of the prototype growth model and simultaneously generate endogenous cycles through a kind of market imperfection, which is precisely where one wants to aim in the present work.

Other studies remark the relevance of expectations in the determination of cycles; this is other concern of ours. Maintaining the analysis close to the RBC benchmark, Cochrane (1994), Danthine, Donaldson and Johnson (1998), Beaudry and Portier (2004), Lorenzoni (2005) and Jaimovich and Rebelo (2006), develop models where the central issue consists in highlighting the role of expectations about the future as a fundamental source of cycles. For instance, periods of expansion may be the result of optimistic expectations about TFP growth, which can be triggered by news announcing, for instance, a technological revolution (e.g., as a result of the introduction of the internet); if the expectations become an overvaluation of what in fact ends up by occurring this may produce a downfall in investment and as a consequence a period of recession may arise. The notion that news about the future or some kind of change in agents' expectations can be an important source of fluctuations is a matter that is under discussion in macroeconomics since the work of the most prominent economists of the early twentieth century, like Pigou.

Another work that calls the attention for the relevance of expectations in the determination of cycles is Dosi, Fagiolo and Roventini (2005). The argument of this group of authors is that agents do not act on a fully rational way; instead, firms tend to employ routinized behavioural investment rules that are less costly than the rules underlying a profit maximizing behaviour. Firms are risk averse, or prudent, and they cannot as well fully anticipate future levels of demand. Therefore, choices concerning investment decisions are always based on non optimal estimates of future demand. Lack of full knowledge and prudence are the main features that do not allow for a complete coincidence between real world demand expectations and demand expectations obeying to the benchmark rational expectations optimality setup.

Following the previous discussion of growth and cycles, next sections develop a group of models with the following characteristics:

i) They are all simple optimal growth models;

ii) No stochastic features are introduced;

iii) Endogenous cycles are generated through firms' investment decisions;

iv) Investment decisions are disturbed by a market failure. Firms are unable to predict future levels of demand that are exactly identical to the demand level corresponding to rational expectations;

v) Firms are generally prudent, and thus investment lies below the optimal level, but periods of overconfidence are not excluded from the analysis;

vi) Three scenarios are considered: the one equation Solow model of capital accumulation, the consumption utility maximization Ramsey model and an endogenous growth setup with two sectors which produce, with different technologies, physical and human capital;

vii) The analytical structures will consider solely investment and consumption decisions and, thus, the labor market (the labor-leisure trade-off) is excluded from the analysis.

In the proposed framework, the main idea is that the level of investment chosen by firms does not coincide in every moment with the level of investment that is compatible with the optimal setup; in some time periods investment is below its potential level, reflecting the risk averse nature of firms' decisions; in other periods, overconfidence may lead to investment above the optimum.

The rule that we will adopt concerning expected demand growth will lead to a piecewise difference equation that resembles a logistic / tent map. This type of map is known to produce a great variety of nonlinear dynamic results ranging from low and high periodicity cycles to a-periodicity / chaos. Therefore, by introducing a source of inefficiency (the absence of coincidence between effective and potential investment) we generate endogenous fluctuations under the structure of the standard intertemporal growth setup.

The dynamic analysis will proceed in two steps: first, we study the existence of local bifurcations in the steady state vicinity. Although it gives important guidance about the dynamic properties of the problem, the local analysis tends to be misleading when nonlinearities are present. Hence, on a second step, global dynamics are discussed, considering particular examples with specific parameter values. The main result is that cycles of various orders (regular and irregular) are observable, and in this way the developed framework intends to be a contribution to the literature on endogenous business cycles.

The remainder of the paper is organized as follows. Section 2 discusses the expected demand rule. Section 3 considers three different growth models and in each one of them we introduce investment decisions that depart from the optimum as a consequence of non perfect demand expectations by firms. In this section, a local dynamics analysis is developed. Section 4 analyzes global dynamics by means of numerical examples that are graphically illustrated. Finally, section 5 concludes. Proofs of propositions are presented in appendix.

2. Demand Expectations

An optimal demand level is defined here as the level of demand corresponding to market clearing (i.e., to the absence of inefficiencies in the allocation of resources). Such level is the one underlying the decisions agents undertake when considering optimal growth modelling structures. In practice, aggregate demand can differ from the optimal benchmark value; hence, we start the analysis by considering a variable d_t which represents the ratio between effectively observed and optimal levels of demand. Demand is often below the reference level (when agents predict an economic slowdown or given some kind of inertia), and it can also be above optimal values (for periods of generalized optimistic behaviour); thus, while d_t is commonly taken as equal to 1 in most growth models, it is often lower or higher than 1.

A central assumption underlying the analysis that follows is that firms make today's investment decisions based on expectations about future demand. Therefore, it is necessary to define a rule translating how firms predict the evolution of the ratio d_t. Consider $E_t\gamma_d$ as the expected growth rate of d_t from t to t+1. If firms adopt always a behaviour compatible with the optimal growth setup, this implies $E_t\gamma_d=(1/d_t)-1$, such that $E_td_{t+1}=1$, that is, independently of the value of d_t, the next period expected demand will coincide with the optimal demand level. Figure 1 sketches this boundary; as we shall see, the nature of business cycles implies that values above and below this line are observable, meaning that cycles are synonymous of deviations from optimal expectations regarding future demand. In the figure, we present a lower bound, $E_t\gamma_d\geq-1$, that guarantees non negative values for E_td_{t+1}.

*** Figure 1 here ***

To generate cycles in standard growth models, we consider instead of the optimal rule of expectations formation [$E_t\gamma_d=(1/d_t)-1$], an approximated rule, which is composed by a piecewise function containing two straight lines. The first, defined in $0\leq d_t \leq\phi_0$ with ϕ_0 some positive value below unity, is a linear approximation of the optimal curve around $d_t=\phi_0$, that is, $E_t\gamma_d = 2/\phi_0 - 1 - (1/\phi_0^2)\cdot d_t$. The second, defined for $d_t\geq\phi_0$, is a linear equation passing through points (d_t, $E_t\gamma_d$)=(ϕ_0;1/ϕ_0-1) and (d_t, $E_t\gamma_d$)=(1;ϕ_1), with $0<\phi_1<1$.

Figure 2 gives an example of a possible function obeying the imposed conditions. The proposed approximation to the optimality scenario serves the suggested purposes: it changes the optimal rule in order to generate endogenous cycles (as we will regard below), and it allows for the possibility of demand expectations growth below and above the benchmark values.

*** Figure 2 here ***

The right-hand equation in figure 2 is analytically given by $E_t\gamma_d = \frac{1-\phi_0+\phi_0^2\phi_1}{\phi_0\cdot(1-\phi_0)} - \frac{1-\phi_0+\phi_0\phi_1}{\phi_0\cdot(1-\phi_0)}\cdot d_t$.

The central assumption in this paper is that firms do not necessarily adopt an optimal rule in terms of expected demand growth. Instead, they take a simpler approximated rule, that we can present in terms of a relation between d_t and d_{t+1} (we consider perfect foresight, and thus the operator of expectations is hereafter neglected),

$$d_{t+1} = \begin{cases} \frac{2}{\phi_0}\cdot d_t \cdot \left(1-\frac{1}{2\phi_0}\cdot d_t\right), & 0 \le d_t \le \phi_0 \\ \frac{1}{\phi_0\cdot(1-\phi_0)}\cdot d_t \cdot \left[1-\phi_0^2\cdot(1-\phi_1)-(1-\phi_0\cdot(1-\phi_1))\cdot d_t\right], & d_t \ge \phi_0 \\ d_0 \text{ given} \end{cases} \tag{1}$$

Figure 3 represents equation (1) for specific values of ϕ_0 and ϕ_1. The figure displays a hump-shaped function similar to the logistic equation. This equation is known to lead, for some parameter values, to periodic and a-periodic long term time series. Therefore, it is through equation (1) that we will propose a framework of endogenous cycles that can be associated to any one of the most influential growth setups (section 3 considers neoclassical growth by taking the Solow and the Ramsey models, and endogenous growth by assuming a two sector model with physical goods and human capital generated by different technologies).

*** Figure 3 here ***

Having defined the rule through which firms form expectations about future demand, it is now necessary to associate this mechanism to firms' investment decisions. We consider per capita investment variables: j_t represents potential investment and i_t translates the amount of effective investment. The relation between the two is given by considering demand expectations, that is, $i_t = f(E_{t-1}d_t)\cdot j_t$, with f'>0. In order to simplify the analysis in the next section, we consider an explicit function $f(E_{t-1}d_t) = d_t^{\theta}$, θ>0. If d_t=1, then i_t=j_t, that is, for an optimal level of demand, the investment level corresponds to the potential level; because in some time moments d_t<1, there are periods of underinvestment, while in other moments d_t>1, meaning overinvestment. Considering a non optimal investment variable that is determined by the proposed demand expectations rule, endogenous fluctuations are generated, and these will propagate to the main macroeconomic aggregates as we introduce the rule in standard intertemporal growth models.

Before analysing growth models, let us briefly study the dynamic properties of system 1. In a first moment consider a local analysis in the vicinity of the steady state. The demand ratio steady state is a point $\overline{d} = d_t = d_{t+1}$. Under this definition, besides the trivial point $\overline{d} = 0$, two equilibrium points are found: $\overline{d} = \phi_0 \cdot (2-\phi_0)$ for $d_t \leq \phi_0$, and $\overline{d} = 1$ for $d_t \geq \phi_0$. However, since $\phi_0 \cdot (2-\phi_0) \geq \phi_0$, we can concentrate on point $\overline{d} = 1$; hence, the first equation of (1) may be neglected in the analysis of the steady state.

Stability analysis of system (1) allows finding the result in proposition 1.

Proposition 1. *The demand expectations rule (1) has a unique steady state point* $\overline{d} = 1$. *This is a stable equilibrium point for* $\phi_1 < \dfrac{\phi_0 \cdot (3-2\phi_0)-1}{\phi_0 \cdot (2-\phi_0)}$*; it is unstable for* $\phi_1 > \dfrac{\phi_0 \cdot (3-2\phi_0)-1}{\phi_0 \cdot (2-\phi_0)}$*; and a bifurcation is observable under condition* $\phi_1 = \dfrac{\phi_0 \cdot (3-2\phi_0)-1}{\phi_0 \cdot (2-\phi_0)}$.

Proof: appendix 1.

A rigorous inquire about the nature of the bifurcation in proposition 1 is hard to undertake, leading to a large quantity of repetitive calculation. As an illustration, we state a simple particular result,

Proposition 2. *For ϕ_0=0.5, the bifurcation referred in proposition 1 is a flip bifurcation.*

Proof: appendix 2.

Figure 4 plots the regions of stability (*S*) and instability (*U*), given the space of parameters. The line dividing the two areas corresponds to the bifurcation condition.

*** Figure 4 here ***

Note that the stability result of proposition 1, which is illustrated in figure 4, is a result attained through a local analysis. The 'logistic' shape of our equation (1) allows to suspect that a global analysis will reveal another qualitative behaviour, namely periodic and a-periodic long term time series of d_t will be found for different values of parameters ϕ_0 and ϕ_1.

A diagram similar to the one in figure 4 can be displayed, now considering a global analysis instead of the local analysis previously undertaken.[1] Taking some initial value for d_t in the interval (0,1), figure 5 presents the long term qualitative nature of the d_t time series for different pairs (ϕ_0,ϕ_1).

*** Figure 5 here ***

The stability area in figure 5 (where a fixed point is found) is the same as in figure 4. The difference respect to the area that under a local analysis is identified as unstable, but that global analysis reveals to be an area where multiple long term qualitative outcomes are possible: period 2, 4, 8, … cycles are identified and regions of high periodicity (higher than 35 period cycles) or a-periodicity are also present. We draw in figures 6 and 7 the series of Lyapunov characteristic exponents (LCEs) for selected values of ϕ_0 and ϕ_1, in order to identify the regions with chaos or, more rigorously, the regions where exponential divergence of nearby orbits is evidenced.

[1] The various figures relating global analysis are drawn using IDMC software (interactive Dynamical Model Calculator). This is a free software program available at www.dss.uniud.it/nonlinear, and copyright of Marji Lines and Alfredo Medio.

*** Figures 6 and 7 here ***

Figures 6 and 7 can be analyzed alongside with figure 5: chaos emerges for values of parameters to which we cannot identify a low order periodicity (chaos is associated with positive values of the Lyapunov characteristic exponent).

The chaotic nature of system 1 can be understood as well by looking at bifurcation diagrams [figure 8 plots a bifurcation diagram (ϕ_0,d_t) for ϕ_1=0.5; figure 9 plots the bifurcation diagram (ϕ_1,d_t) for ϕ_0=0.75. These are the same values chosen to draw the LCEs, and the two pairs of diagrams can be jointly analyzed].

*** Figures 8 and 9 here ***

Finally, a long term time series of the demand function is presented in figure 10, revealing the pattern of evolution that is followed when ϕ_0 and ϕ_1 are such that the system falls in the chaotic zone.

*** Figure 10 here ***

In the previous paragraphs we have characterized the dynamics underlying difference equation (1). Recall that this function describes how firms perceive the evolution of demand. Remind also that demand expectations are the central influence over firms' investment decisions and, thus, demand expectations characterized by endogenous fluctuations will imply an investment process that is also governed by a cyclical / chaotic behaviour, which then spreads to the whole main economic aggregates. The following section debates the role of this process when associated to optimal growth frameworks.

3. Growth Models with Non Optimal Investment Levels

In this section we introduce the non optimal investment decisions of firms, derived from an incorrect evaluation of demand evolution, in growth models. We consider three growth models with a common feature: they will be all represented by two equations systems, where one of the equations is the demand rule (1) and the other characterizes the process of capital accumulation. We begin with the Solow growth model.

3.1 The Solow Model

In the Solow growth setup, an exogenous saving rate, $0<s<1$, is considered. Recalling that j_t and i_t represent potential and effective investment respectively (in per capita values), and defining y_t as per capita output, we can write $j_t=sy_t$ or $i_t=sd_t^{\theta}y_t$. We consider a Cobb-Douglas production function $y_t = Ak_t^{\alpha}$, with $A>0$ a technology index and $0<\alpha<1$ the output-capital elasticity; the process of capital accumulation is described by equation $k_{t+1} - k_t = i_t - \delta k_t$, with $\delta>0$ the capital depreciation rate.

Putting together the previous information, we get to the capital accumulation equation in the Solow model, with non optimal investment,

$$k_{t+1} = sAd_t^{\theta} k_t^{\alpha} + (1-\delta)\cdot k_t, \quad k_0 \text{ given} \tag{2}$$

The dynamics of the original Solow model [equation (2) when $d_t=1$] is well known – in the presence of diminishing marginal returns, the steady state is characterized by a zero growth result, that can be changed only with some external event like technological progress. Adding d_t, and the corresponding evolution process in (1), we introduce endogenous cycles. Note that if the steady state is eventually reached, then $\bar{d} = 1$ and thus the equilibrium level of per capita capital is the one in the original model: $\bar{k} = (sA/\delta)^{1/(1-\alpha)}$.

The local properties of the model are mainly the ones discussed for equation (1). To this equation we have added a capital accumulation constraint, that under the assumption of decreasing marginal returns involves a stable steady state. The linearization of system (1) – (2) in the steady state vicinity implies the matricial system

$$\begin{bmatrix} k_{t+1} - \bar{k} \\ d_{t+1} - 1 \end{bmatrix} = \begin{bmatrix} 1-(1-\alpha)\cdot\delta & \theta\delta\bar{k} \\ 0 & -\dfrac{1-(1-\phi_1)\cdot\phi_0\cdot(2-\phi_0)}{\phi_0\cdot(1-\phi_0)} \end{bmatrix} \cdot \begin{bmatrix} k_t - \bar{k} \\ d_t - 1 \end{bmatrix} \tag{3}$$

Let J_1 be the Jacobian matrix in 3. The local properties of the system are stated in proposition 3.

Proposition 3. *The Solow model with non optimal demand expectations / investment decisions can be locally characterized by saddle-path stability for* $\phi_1 > \frac{\phi_0 \cdot (3-2\phi_0)-1}{\phi_0 \cdot (2-\phi_0)}$. *A stable equilibrium is found for* $\phi_1 < \frac{\phi_0 \cdot (3-2\phi_0)-1}{\phi_0 \cdot (2-\phi_0)}$.

Proof: appendix 3.

The main new result obtained with the introduction of a Solow physical capital constraint over our initial system is that the space of parameters identified in figure 4 as unstable is, in the two-dimensional version of the model, an area of saddle-path stability. The region of stability continues to be exactly the same. Once again, keep in mind that this is a steady state vicinity result and, in fact, the true qualitative dynamic results are the ones evidenced in figure 5. In section 4 we study some numerical examples in order to illustrate the presence of endogenous fluctuations in this model.

3.2 The Ramsey Model

In this sub-section we introduce optimal consumption decisions by a representative agent engaged in maximizing utility. Consider a consumption utility function, $U(c_t)$, where c_t is per capita consumption. We need to work with an explicit function U, hence, we consider $U(c_t)=\ln(c_t)$, a function that fulfils the main requirement of positive and diminishing marginal utility. Taking a discount factor $\beta\in(½,1)$, the intertemporal problem that the agent solves in t=0 is $Max\sum_{t=0}^{+\infty}\beta^t \cdot \ln(c_t)$.

The constraint over this problem is a capital accumulation equation that is similar to the one in the Solow model. Taking the same production function and the same process of capital accumulation and replacing the exogenous process of investment by the demand equation $y_t=j_t+c_t$, the wanted equation is

$$k_{t+1} = d_t^{\theta} \cdot (Ak_t^{\alpha} - c_t) + (1-\delta) \cdot k_t \tag{4}$$

The dynamics of the Ramsey model is also widely known from the literature. In the original version (d_t=1) steady state vicinity analysis leads to a saddle-path equilibrium, where we identify a unique one-dimensional stable relation between endogenous variables k_t and c_t. Therefore, if one wants to study the dynamics of the model with endogenous cycles, we can do it only over the stable arm. Thus, this has to be found.

Proposition 4. *In the Ramsey model with non optimal investment decisions, the capital accumulation difference equation*

$$k_{t+1} = Ad_t^{\theta}k_t^{\alpha} + \left[1 - \frac{1-\beta\lambda_{21}}{\beta}d_t^{\theta} - \delta\right]\cdot k_t + \left[\frac{1-\beta\lambda_{21}}{\beta}\bar{k} - \bar{c}\right]\cdot d_t^{\theta} \qquad (5)$$

describes the economy's transitional dynamics, when the convergence to the steady state is guaranteed by the fact that the stable trajectory $c_t - \bar{c} = \frac{1-\beta\lambda_{21}}{\beta}\cdot(k_t - \bar{k})$ *is followed. The value* λ_{21} *is the eigenvalue inside the unit circle of the Jacobian's linearized version of the model and* $\bar{k}$ *and* $\bar{c}$ *are steady state values of the model's variables.*

Proof: appendix 4.

Equation (5) synthesizes the dynamics of the Ramsey model in the particular case when the saddle-path is followed (because c_t is a control variable, the representative agent can choose an initial level of consumption over the stable trajectory, what implies that this trajectory will be followed until the equilibrium point is accomplished). The equation gives the dynamic behaviour of only one of the variables, k_t; however, to gain access to the dynamics of c_t, we just have to look to the relation between k_t and c_t given by the stable trajectory.

As in the Solow model, the Ramsey model with the possibility of endogenous fluctuations is now a two equation system, (5) – (1). Once more, we linearize the system around the steady state point,

$$\begin{bmatrix} k_{t+1} - \bar{k} \\ d_{t+1} - 1 \end{bmatrix} = \begin{bmatrix} \lambda_{21} & \theta\delta\bar{k} \\ 0 & -\frac{1-(1-\phi_1)\cdot\phi_0\cdot(2-\phi_0)}{\phi_0\cdot(1-\phi_0)} \end{bmatrix} \cdot \begin{bmatrix} k_t - \bar{k} \\ d_t - 1 \end{bmatrix} \qquad (6)$$

The dynamic results relating to system (6) are very similar to the ones characterizing (3). Because λ_{21} lies inside the unit circle, the same result as in proposition 3 is derived, that is, a bifurcation line separates a region of stability from a region of local saddle-path stability, that we know to be the area where endogenous fluctuations can be identified. Again, we leave concrete global dynamic results to section 4.

3.3 Endogenous Growth

The main departure of endogenous growth models relatively to the neoclassical setup is that in opposition to the Solow and Ramsey frameworks, variables k_t and c_t do not converge in any circumstance to long term constant values. Instead, these variables will exhibit constant growth rates that are, in the simpler form of the model, the same for the several economic aggregates involved in the analysis. Thus, to accomplish an equation where, as in (2) or (5), a stable dynamic process is revealed, we will have to consider ratios of variables with a same long term rate of growth, rather than the original per capita variables. As we shall see, some simplifications are needed in order to get to that one-dimensional stable difference equation.

The endogenous growth scenario that is proposed is the Uzawa (1964) – Lucas (1988) model with physical and human capital. Consider h_t as representing per capita human capital and let u_t be the share of human capital applied to produce physical goods. One main assumption is that physical and human capital are produced with different technologies. The final goods production function is $y_t = Ak_t^{\alpha} \cdot (u_t h_t)^{1-\alpha}$; under this specification, physical capital is used entirely to generate additional physical goods. Note, as well, that both inputs exhibit decreasing marginal returns ($0<1-\alpha<1$ is the output – human capital elasticity). The production function regarding the generation of human capital is linear: $z_t = B \cdot (1-u_t) \cdot h_t,\ B > 0$. In both production functions, constant returns to scale prevail.

Considering the same utility maximization structure as in the Ramsey model, recovering the non optimal process of investment of previous models and assuming a same depreciation rate for both forms of capital, the Uzawa–Lucas endogenous growth model with non perfect demand expectations will be given by (7),

$$
\begin{aligned}
&Max\sum_{t=0}^{+\infty}\beta^t\cdot\ln(c_t)\\
\text{subject to}:\ &k_{t+1}=d_t^{\theta}\cdot(y_t-c_t)+(1-\delta)\cdot k_t\\
&h_{t+1}=z_t+(1-\delta)\cdot h_t\\
&k_0,h_0\ \text{given.}
\end{aligned}
\tag{7}
$$

For the Uzawa-Lucas model, we state proposition 5.

Proposition 5. *Let* $\omega_t \equiv k_t / h_t$ *and* $\psi_t \equiv c_t / k_t$. *Assuming that the initial values of* ψ_t *and* u_t *are already the steady state results,* $\psi_0 = \overline{\psi}$ *and* $u_0 = \overline{u}$, *then the Uzawa-Lucas endogenous growth model can be expressed through a two equations system that includes (1) and the following difference equation for the dynamics of the ratio between types of capital,*

$$
\omega_{t+1}=\frac{\omega_t}{(1+B-\delta)\cdot\beta}\cdot
\left\{Ad_t^{\theta}\cdot\left[\frac{(1-\beta)\cdot(1+B-\delta)}{B\cdot\omega_t}\right]^{1-\alpha}-\left[\left(\frac{1}{\alpha}-\beta\right)\cdot B+(1-\beta)\cdot(1-\delta)\right]\cdot d_t^{\theta}+(1-\delta)\right\}
\tag{8}
$$

Proof: appendix 5.

The study of local dynamic properties of the system (8)-(1) does not differ significantly from the analysis undertaken for Solow and Ramsey models. The main distinction is, as referred, that the constant long term expected result respects now not to the capital stock but to a ratio of capital stocks.

Following the same procedure, we linearize (8)-(1) in the steady state vicinity,

$$
\begin{bmatrix}\omega_{t+1}-\overline{\omega}\\ d_{t+1}-1\end{bmatrix}=
\begin{bmatrix}-\frac{1-\alpha}{\alpha}\cdot\frac{B}{(1+B-\delta)\cdot\beta} & -\frac{(1-\delta)\cdot\overline{\omega}}{(1+B-\delta)\cdot\beta}\\ 0 & -\frac{1-(1-\phi_1)\cdot\phi_0\cdot(2-\phi_0)}{\phi_0\cdot(1-\phi_0)}\end{bmatrix}\cdot
\begin{bmatrix}\omega_t-\overline{\omega}\\ d_t-1\end{bmatrix}
\tag{9}
$$

While in the neoclassical growth models we have regarded that the eigenvalue derived from the capital equation was always inside the unit circle, this does not happen now. If $B > \dfrac{\alpha \cdot (1-\delta) \cdot \beta}{1-\alpha \cdot (1+\beta)}$, instability will prevail. Otherwise, our known local qualitative behaviour applies: the regions of saddle-path stability and stable node stability, and the bifurcation line, in the (ϕ_0, ϕ_1) space are precisely the same as in the Solow and Ramsey models.

4. Global Dynamics

In this section, we illustrate graphically the global dynamics of each one of the growth models previously described. As benchmark parameters values we consider the following vector: [ϕ_0 ϕ_1 s A θ α δ β B]=[0.75 0.5 0.25 1 0.5 0.25 0.05 0.96 0.1]. Note that some of these parameters exist only in one or two of the models.

Let us begin by the Solow model. We have regarded in section 2, through the construction of bifurcation diagrams and computation of Lyapunov exponents, that for the chosen values of parameters (ϕ_0=0.75 and ϕ_1=0.5), the demand equation implies chaotic motion; this chaotic motion will spread, under the defined investment rule, to the accumulation of capital. Therefore, the capital stock will exhibit endogenous fluctuations under a long term perspective. This can be confirmed by looking at figure 11 (which is drawn for the set of chosen values of parameters).

*** Figure 11 here ***

The previous information can be complemented with bifurcation diagrams. Figure 12 draws the bifurcation diagram for (k_t, ϕ_0) when ϕ_1=0.5, and figure 13 draws the bifurcation diagram for (k_t, ϕ_1) when ϕ_0=0.75 (the other parameters have the values given in the presented vector).

*** Figures 12 and 13 here ***

Relatively to the Solow model, we can also present the dynamic behaviour of the other per capita variables besides capital, namely consumption, output and investment (both potential and effective). All the variables will display, for the given parameter values, a chaotic behaviour. Just as an illustration regard figures 14 and 15; these present attractors, relating in the first case the long run relationship between variables k_t and c_t, and in the second case, y_t and i_t.

*** Figures 14 and 15 here ***

In what concerns the Ramsey model, for the same parameters set we encounter a same kind of qualitative behaviour for the endogenous variables, that is, endogenous business cycles are evidenced. To compare results, one represents the same set of figures as in the Solow model: the time series concerning the equilibrium behaviour of variable k_t, two bifurcation diagrams relating to the same variable and two attracting sets involving various variables from the model.

*** Figures 16 to 20 here ***

The differences that we find when comparing the two groups of figures relate to the fact that consumption is determined in a different way: this is an exogenous variable in the Solow framework and a result of optimal decisions in the Ramsey model. Nevertheless, the main result continues to hold: endogenous business cycles occur as a result of inefficiencies regarding firms' expectations about future demand.

Finally, the endogenous growth model considers constant long run values for the consumption – physical capital ratio, for the shares of human capital used in each productive sector and for the positive long run rate of growth of per capita economic aggregates. Thus, the only variable that exhibits endogenous fluctuations is ω_t, that is, the physical capital – human capital ratio. For this variable, figures 21 to 23 display the long term time trajectory and two bifurcation diagrams. As we regard, the referred ratio can assume multiple long run values according to the different parameterizations of the demand equation, and for most of them endogenous cycles are evidenced. Nevertheless, the growth rate of the various per capita aggregates is constant and, for the chosen parameter values, equal to: $\gamma = \beta \cdot (B - \delta) - (1 - \beta) = 0.008$.

*** Figures 21 to 23 here ***

5. Final Remarks

The paper takes the simple assumption that firms make mistakes and seldom adopt basic non optimal rules when predicting future demand. Thus, investment decisions depart from the ones leading to the levels of investment that underlie the structure of the several most influential intertemporal growth problems. The considered growth setups may be seen as describing long term trends of growth, which are sketched over a competitive market structure where any kind of imperfection is ruled out. The new assumption may be understood as a market inefficiency that induces the presence of endogenous cycles.

We have studied the dynamic properties of the demand expectations rule. There is a bifurcation line that locally separates a region of stability from a region of instability; the local analysis is important to perceive where the change in the topological properties of the model occurs but it does not tell the full story. Through numerical examples, one has understood that the local instability area is indeed a region where cycles of various periodicities can be found. For some parameter values these cycles are completely a-periodic, that is, chaos exists (i.e., time series display sensitive dependence on initial conditions).

On a second stage, the non optimal investment rule was associated to a neoclassical growth setup (Solow and Ramsey models) and to an endogenous growth framework. As a result, the growth models maintain their basic features (i.e., neoclassical growth models continue to generate a long term growth rate that on average is zero, while the endogenous growth model implies a positive long run growth rate) but endogenous fluctuations arise: in the Solow / Ramsey models, physical capital and consumption time paths exhibit cyclical motion; the same is true for the physical capital – human capital ratio in the two-sector endogenous growth framework.

Appendices

Appendix 1 – Proof of proposition 1

Let $G(d_t,\phi_0,\phi_1)=\dfrac{1}{\phi_0\cdot(1-\phi_0)}\cdot d_t\cdot\left[1-\phi_0^2\cdot(1-\phi_1)-(1-\phi_0\cdot(1-\phi_1))\cdot d_t\right]$. Function G is simply the second equation in (1). Let also $\dfrac{\partial G(1;\phi_0;\phi_1)}{\partial d_t}$ represent the derivative of function G in the vicinity of $\overline{d}=1$.

Stability requires $\frac{\partial G(1;\phi_0;\phi_1)}{\partial d_t}$ to be inside the unit circle (conversely, instability means that the value of this derivative has to be outside the unit circle). Computing the derivative we get $\frac{\partial G(1;\phi_0;\phi_1)}{\partial d_t}=-\frac{1-(1-\phi_1)\cdot(2-\phi_0)\cdot\phi_0}{\phi_0\cdot(1-\phi_0)}$. This value is negative; thus, it makes sense to evaluate in which conditions it is above or below -1. The condition for stability is then $1-\frac{1-(1-\phi_1)\cdot(2-\phi_0)\cdot\phi_0}{\phi_0\cdot(1-\phi_0)}>0$. This condition is satisfied for the value of ϕ_1 obeying the inequality displayed in the proposition. The instability and bifurcation results follow accordingly ■

Appendix 2 – Proof of proposition 2

Consider $G^2(d_t,\phi_0,\phi_1)$ as the second iterate of function G, that is,

$$G^2(d_t,\phi_0,\phi_1)=\left[\frac{1}{\phi_0\cdot(1-\phi_0)}\right]^2\cdot d_t\cdot\left[1-\phi_0^2\cdot(1-\phi_1)-(1-\phi_0\cdot(1-\phi_1))\cdot d_t\right]\cdot$$
$$\left\{1-\phi_0^2\cdot(1-\phi_1)-\frac{1-\phi_0\cdot(1-\phi_1)}{\phi_0\cdot(1-\phi_0)}\cdot d_t\cdot\left[1-\phi_0^2\cdot(1-\phi_1)-(1-\phi_0\cdot(1-\phi_1))\cdot d_t\right]\right\}$$

According to Medio and Lines (2001), page 157, the following are necessary and sufficient conditions for the bifurcation referred in proposition 1 to be a flip bifurcation,

i) $\frac{\partial G(1;\phi_0^{bif};\phi_1^{bif})}{\partial d_t}=-1$;

ii) $\frac{\partial^2 G^2(1;\phi_0^{bif};\phi_1^{bif})}{\partial d_t^2}=0$ and $\frac{\partial^3 G^2(1;\phi_0^{bif};\phi_1^{bif})}{\partial d_t^3}\neq 0$;

iii) $\frac{\partial G^2(1;\phi_0^{bif};\phi_1^{bif})}{\partial \phi_0}=0$ and $\frac{\partial^2 G^2(1;\phi_0^{bif};\phi_1^{bif})}{\partial \phi_0 \partial d_t}\neq 0$ for any $0\leq\phi_1\leq 1$;

iv) $\frac{\partial G^2(1;\phi_0^{bif};\phi_1^{bif})}{\partial \phi_1}=0$ and $\frac{\partial^2 G^2(1;\phi_0^{bif};\phi_1^{bif})}{\partial \phi_1 \partial d_t}\neq 0$ for any $0\leq\phi_0\leq 1$;

Note that ϕ_0^{bif} and ϕ_1^{bif} are the values of the parameters for which the bifurcation occurs.

Since we are assuming that $\phi_0 = 0.5$ only the first three conditions need to be investigated. The first condition is satisfied for $\phi_1^{bif} = 0$, according to the result in proposition 1. The other conditions can be computed in a straightforward way. For $\phi_0 = 0.5$, the presented second iterate simplifies to

$$G^2(d_t, \phi_1) = (3+\phi_1)^2 \cdot d_t - 2\cdot(1+\phi_1)\cdot(3+\phi_1)\cdot(4+\phi_1)\cdot d_t^{\,2} \\ + 8\cdot(1+\phi_1)^2\cdot(3+\phi_1)\cdot d_t^{\,3} - 8\cdot(1+\phi_1)^3\cdot d_t^{\,4}$$

The first, second and third derivatives of G^2 in order to d_t are:

$$\frac{\partial G^2(d_t,\phi_1)}{\partial d_t} = (3+\phi_1)^2 - 4\cdot(1+\phi_1)\cdot(3+\phi_1)\cdot(4+\phi_1)\cdot d_t \\ + 24\cdot(1+\phi_1)^2\cdot(3+\phi_1)\cdot d_t^{\,2} - 32\cdot(1+\phi_1)^3\cdot d_t^{\,3}$$

$$\frac{\partial^2 G^2(d_t,\phi_1)}{\partial d_t^{\,2}} = -4\cdot(1+\phi_1)\cdot(3+\phi_1)\cdot(4+\phi_1) + 48\cdot(1+\phi_1)^2\cdot(3+\phi_1)\cdot d_t \\ - 96\cdot(1+\phi_1)^3\cdot d_t^{\,2}$$

$$\frac{\partial^3 G^2(d_t,\phi_1)}{\partial d_t^{\,3}} = 48\cdot(1+\phi_1)^2\cdot(3+\phi_1) - 192\cdot(1+\phi_1)^3\cdot d_t$$

With the previous derivatives we prove condition *ii*. In particular, we regard that: $\frac{\partial^2 G^2(1,0)}{\partial d_t^{\,2}} = 0$ and $\frac{\partial^3 G^2(1,0)}{\partial d_t^{\,3}} = -48$.

To confirm condition *iii*, we need the following derivatives,

$$\frac{\partial G^2(d_t,\phi_1)}{\partial \phi_1} = \\ 2\cdot(3+\phi_1)\cdot d_t - 2\cdot\left[(3+\phi_1)\cdot(4+\phi_1) + (1+\phi_1)\cdot(4+\phi_1) + (1+\phi_1)\cdot(3+\phi_1)\right]\cdot d_t^{\,2} \\ + 8\cdot\left[2\cdot(1+\phi_1)\cdot(3+\phi_1) + (1+\phi_1)^2\right]\cdot d_t^{\,3} - 24\cdot(1+\phi_1)^2\cdot d_t^{\,4}$$

$$\frac{\partial^2 G^2(d_t,\phi_1)}{\partial\phi_1\partial d_t}=$$
$$2\cdot(3+\phi_1)-4\cdot[(3+\phi_1)\cdot(4+\phi_1)+(1+\phi_1)\cdot(4+\phi_1)+(1+\phi_1)\cdot(3+\phi_1)]$$
$$+24\cdot\left[2\cdot(1+\phi_1)\cdot(3+\phi_1)+(1+\phi_1)^2\right]\cdot d_t^{\,2}-96\cdot(1+\phi_1)^2\cdot d_t^{\,3}$$

Considering the equilibrium value of d_t and the value of parameter ϕ_0 for which the bifurcation occurs, one observes that condition *iii* holds: $\frac{\partial G^2(1,0)}{\partial\phi_1}=0$ and $\frac{\partial^2 G^2(1,0)}{\partial\phi_1\partial d_t}=2$.

The obtained results indicate that for $\phi_0=0.5$ and $\phi_1=0$ the dynamic demand rule displays a flip bifurcation in the vicinity of the steady state■

Appendix 3 – Proof of proposition 3

Conditions for stability are:

$$1+Tr(J_1)+Det(J_1)>0$$
$$1-Tr(J_1)+Det(J_1)>0$$
$$1-Det(J_1)>0$$

Since $1-(1-\alpha)\cdot\delta\in(0,1)$, the second and third conditions are immediately verified. The first is satisfied for the condition given in the proposition, that is, when $-\frac{1-(1-\phi_1)\cdot\phi_0\cdot(2-\phi_0)}{\phi_0\cdot(1-\phi_0)}>-1$. If this condition does not hold, saddle-path stability prevails, given the other two conditions. Note that this result can be obtained directly from the straightforward computation of the eigenvalues of J_1: $\lambda_{11}=1-(1-\alpha)\cdot\delta$ is always inside the unit circle; $\lambda_{12}=-\frac{1-(1-\phi_1)\cdot\phi_0\cdot(2-\phi_0)}{\phi_0\cdot(1-\phi_0)}$ is inside the unit circle under the condition in the proposition. The bifurcation at $\phi_1=\frac{\phi_0\cdot(3-2\phi_0)-1}{\phi_0\cdot(2-\phi_0)}$ continues to occur■

Appendix 4 – Proof of proposition 4

The proof of this proposition is just a matter of solving the Ramsey model. Consider the Hamiltonian function $H(k_t, p_t, c_t) = \ln(c_t) + \beta p_{t+1} \cdot \left[d_t^{\theta} \cdot \left(A k_t^{\alpha} - c_t \right) - \delta k_t \right]$ with p_t the shadow-price of k_t. Optimality conditions are:

$$H_c = 0 \Rightarrow \beta p_{t+1} = \frac{1}{d_t^{\theta} c_t};$$

$$\beta p_{t+1} - p_t = \left[\delta - \alpha A d_t^{\theta} k_t^{-(1-\alpha)} \right] \cdot \beta p_{t+1}$$

$$\lim_{t \to +\infty} k_t \beta^t p_t = 0 \text{ (transversality condition)}$$

From the first order conditions, one withdraws a dynamic equation concerning the time evolution of the consumption variable,

$$c_{t+1} = \beta \cdot (d_t / d_{t+1})^{\theta} \cdot c_t \cdot \left\{ 1 - \delta + \alpha A d_{t+1}^{\theta} \cdot \left[d_t^{\theta} \cdot \left(A k_t^{\alpha} - c_t \right) + (1-\delta) \cdot k_t \right]^{-(1-\alpha)} \right\}$$

This last equation alongside with the capital accumulation constraint (4), constitute our Ramsey system with endogenous cycles [which are introduced through (1)]. This system is now linearized in the neighbourhood of the steady state point $(\bar{k}, \bar{c})$; remind, again, that $\bar{d} = 1$,

$$\begin{bmatrix} k_{t+1} - \bar{k} \\ c_{t+1} - \bar{c} \end{bmatrix} = \begin{bmatrix} 1/\beta & -1 \\ -\sigma & 1+\sigma \end{bmatrix} \cdot \begin{bmatrix} k_t - \bar{k} \\ c_t - \bar{c} \end{bmatrix}$$

with $\sigma \equiv \frac{1-\alpha}{\alpha} \cdot \beta \cdot \left(\frac{1-\beta}{\beta} + \delta \right) \cdot \left(\frac{1-\beta}{\beta} + \delta \cdot (1-\alpha) \right) > 0$. We denote the presented Jacobian matrix by J_2, and we verify the presence of saddle-path stability, given that $1 - Tr(J_2) + Det(J_2) = \frac{\sigma \cdot (1-2\beta)}{\beta} < 0$ and $1 + Tr(J_2) + Det(J_2) = 2 + \frac{2+\sigma}{\beta} > 0$.

Hence, an eigenvalue of J_2, λ_{21}, is inside the unit circle, while the other, λ_{22}, is a positive value larger than one. For eigenvalue λ_{21}, we compute the eigenvector $P_{21} = \begin{bmatrix} 1 & \frac{1-\beta\lambda_{21}}{\beta} \end{bmatrix}$, where the second element of the vector is the slope of the stable trajectory, as presented in the proposition. The stable trajectory is positively sloped, meaning that if c_0 is such that the saddle-path is followed, the qualitative behaviour in the adjustment process towards the steady state is the same for both variables (the consumption rises with an increase in the capital stock and falls with a decline in this stock). Replacing c_t in (4) by the stable arm expression, we get the capital accumulation equation (5), which in the absence of the dynamics of d_t is a difference equation with stability properties similar to the ones in the simple Solow equation■

Appendix 5 – Proof of proposition 5

Let p_t and q_t be shadow-prices of the physical capital and human capital variables, respectively. These prices should satisfy the transversality conditions $\lim_{t\to+\infty} k_t\beta^t p_t = \lim_{t\to+\infty} h_t\beta^t q_t = 0$. These prices also allow to determine first-order conditions of (7),

$$H_c = 0 \Rightarrow \beta p_{t+1} = \frac{1}{d_t^{\,\theta} c_t};$$

$$H_u = 0 \Rightarrow \frac{q_{t+1}}{p_{t+1}} = \frac{A}{B}\cdot d_t^{\theta}\cdot\left(\frac{k_t}{u_t h_t}\right)^{\alpha};$$

$$\beta p_{t+1} - p_t = \left[\delta - \alpha A d_t^{\,\theta}\left(\frac{u_t h_t}{k_t}\right)^{1-\alpha}\right]\cdot\beta p_{t+1};$$

$$\beta q_{t+1} - q_t = \left[\delta - (1-u_t)\cdot B\right]\cdot\beta q_{t+1} - (1-\alpha)\cdot\beta A d_t^{\theta} u_t^{1-\alpha}\cdot\left(\frac{k_t}{h_t}\right)^{\alpha}\cdot p_{t+1}.$$

Two important relations, which exclude the presence of shadow-prices, can be withdrawn from the optimality conditions, namely,

$$\frac{c_{t+1}}{c_t} = \beta\cdot\left(\frac{d_t}{d_{t+1}}\right)^{\theta}\cdot\left[1-\delta+\alpha d_{t+1}^{\theta}\cdot\left(\frac{u_{t+1}h_{t+1}}{k_{t+1}}\right)^{1-\alpha}\right];$$

$$\left(\frac{k_{t+1}}{u_{t+1}h_{t+1}}\right)^{\alpha} = \frac{c_{t+1}}{c_t}\cdot\frac{1}{(1+B-\delta)\cdot\beta}\cdot\left(\frac{k_t}{u_t h_t}\right)^{\alpha}$$

From the two constraints in problem (7) and the two above equations, we get all the necessary information concerning the model's dynamics. To analyze the steady state, we take the usual assumption that variables k_t, h_t and c_t grow at a same equilibrium rate in the long run and that $\bar{u}$ is a constant in the interval (0,1). The equation concerning human capital accumulation reveals that the equilibrium rate is $\gamma = B \cdot (1-\bar{u}) - \delta$. To make this result explicit, we have to find $\bar{u}$.

The dynamics of the capital ratio, ω_t, are given by the following difference equation,

$$\frac{\omega_{t+1}}{\omega_t} = \frac{k_{t+1}}{k_t} \cdot \frac{h_t}{h_{t+1}} = \frac{Ad_t^{\theta} \cdot \left(\frac{u_t}{\omega_t}\right)^{1-\alpha} - d_t^{\theta}\psi_t + 1 - \delta}{B \cdot (1-u_t) + 1 - \delta}$$

Observe that the motion of the capital ratio can be characterized by the consideration of variables with constant long run values (namely, ψ_t, u_t and ω_t itself).

We can also present an equation for the time movement of ψ_t following a same procedure,

$$\frac{\psi_{t+1}}{\psi_t} = \frac{c_{t+1}}{c_t} \cdot \frac{k_t}{k_{t+1}} = \frac{(1+B-\delta) \cdot \beta \cdot \left(\frac{u_t}{\omega_t} \cdot \frac{\omega_{t+1}}{u_{t+1}}\right)^{\alpha}}{Ad_t^{\theta} \cdot \left(\frac{u_t}{\omega_t}\right)^{1-\alpha} - d_t^{\theta}\psi_t + 1 - \delta}$$

The next expression comes from combining the previous two:

$$\left(\frac{u_{t+1}}{u_t}\right)^{\alpha} \cdot \left(\frac{\omega_{t+1}}{\omega_t}\right)^{1-\alpha} \cdot \frac{\psi_{t+1}}{\psi_t} = \frac{(1+B-\delta) \cdot \beta}{B \cdot (1-u_t) + 1 - \delta}$$

Since ψ_t, u_t and ω_t do not grow in the steady state, this last condition allows for a straightforward computation of the steady state value of u_t: $\bar{u} = \frac{(1-\beta) \cdot (1+B-\delta)}{B}$. In the presence of the equilibrium human capital share, we write the expression that translates the constant long term growth rate of the various per capita variables (consumption, physical capital, human capital and output), $\gamma = \beta \cdot (B-\delta) - (1-\beta)$. This result is the known steady state positive growth rate of the Uzawa-Lucas growth model.

The other steady state results are obtained in a straightforward way,

$$\overline{\psi} = \left(\frac{1}{\alpha} - \beta\right) \cdot B + (1-\beta)\cdot(1-\delta)\,;\ \overline{\omega} = \left(\alpha \cdot \frac{A}{B}\right)^{1/(1-\alpha)} \cdot \overline{u}\ .$$

The equation in the proposition, (8), is just the difference equation found in this appendix for the physical – human capital ratio with variables ψ_t and u_t replaced by the correspondent steady state values■

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FIGURES

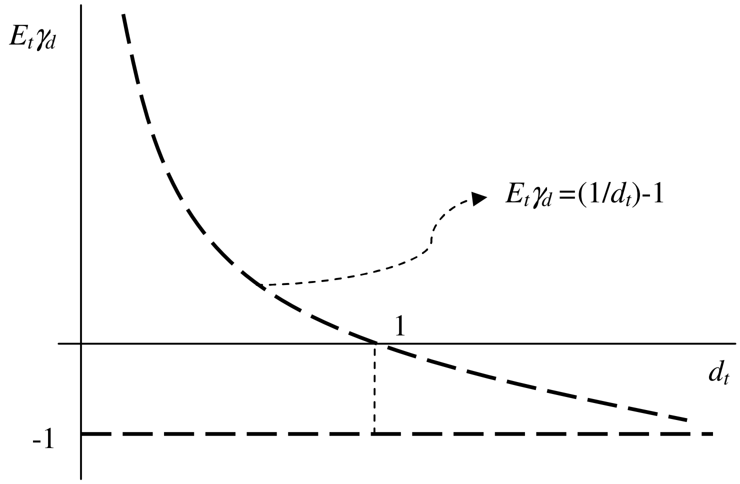

Figure 1 – The optimal growth of demand expectations.

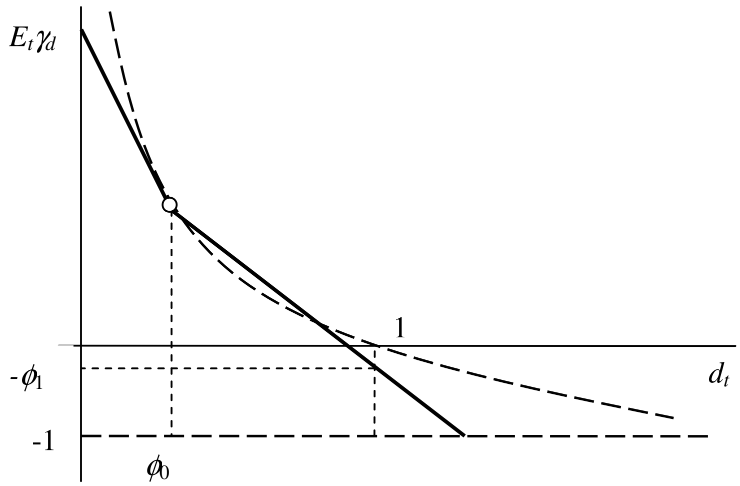

Figure 2 – A non optimal demand expectations growth rule.

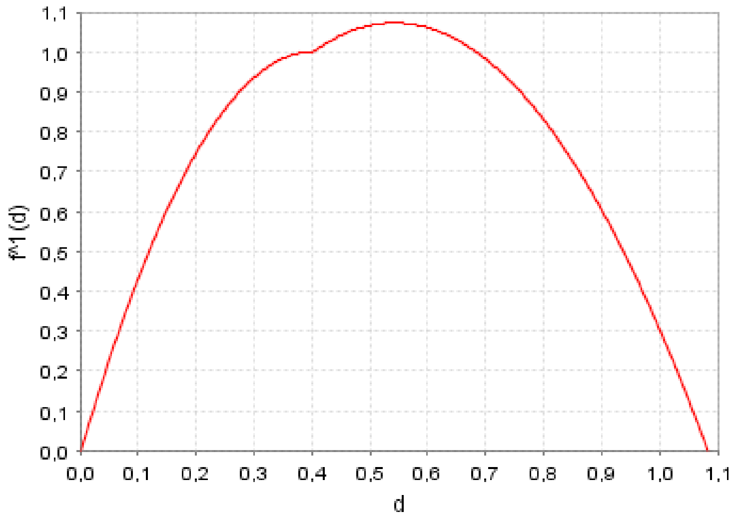

Figure 3 – Evolution of demand expectations (ϕ_0=0.4; ϕ_1=0.7).

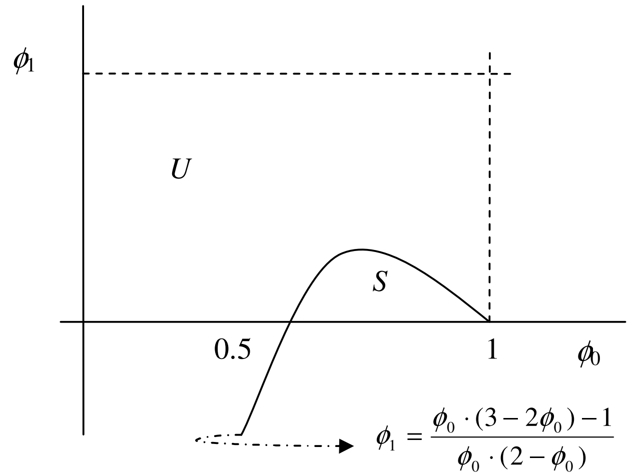

Figure 4 – Areas of stability / instability in the parameters space.

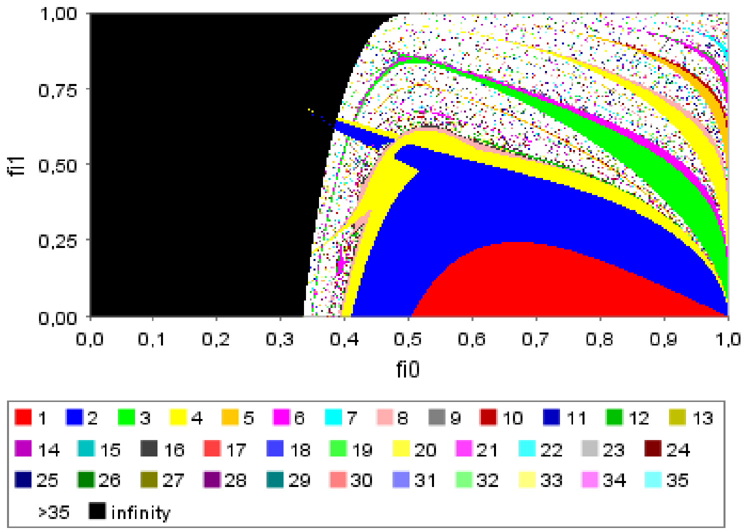

Figure 5 – Global dynamics in the parameters space.

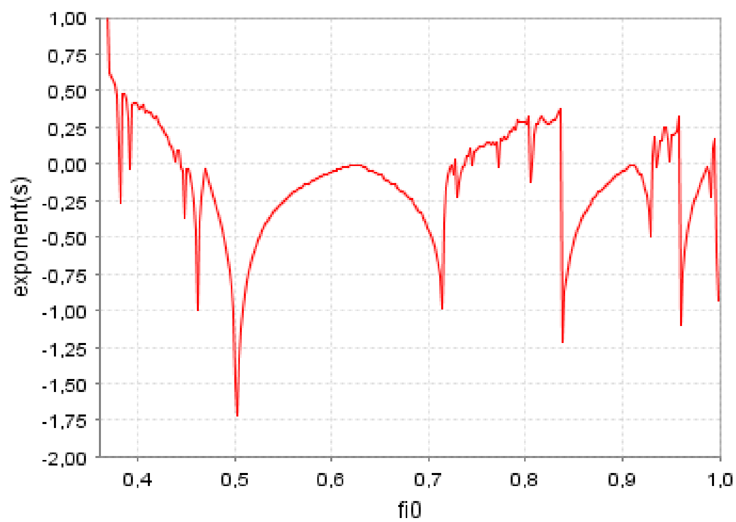

Figure 6 – LCEs for ϕ_1=0.5.

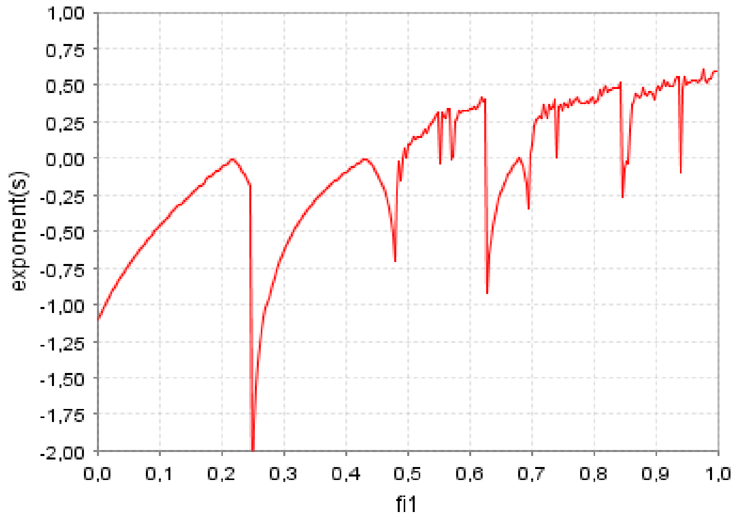

Figure 7 – LCEs for ϕ_0=0.75.

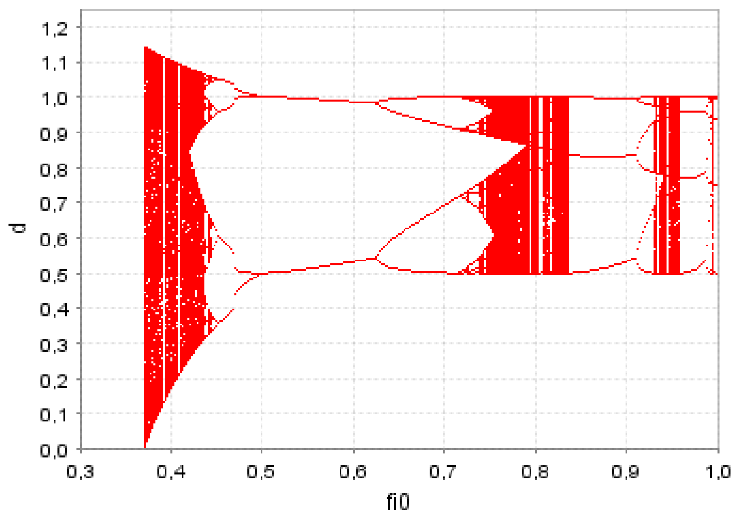

Figure 8 – Bifurcation diagram (ϕ_1=0.5).

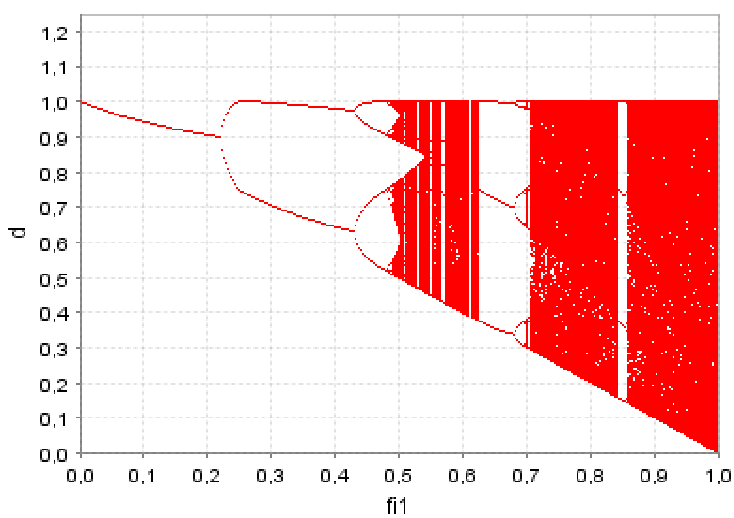

Figure 9 – Bifurcation diagram (ϕ_0=0.75).

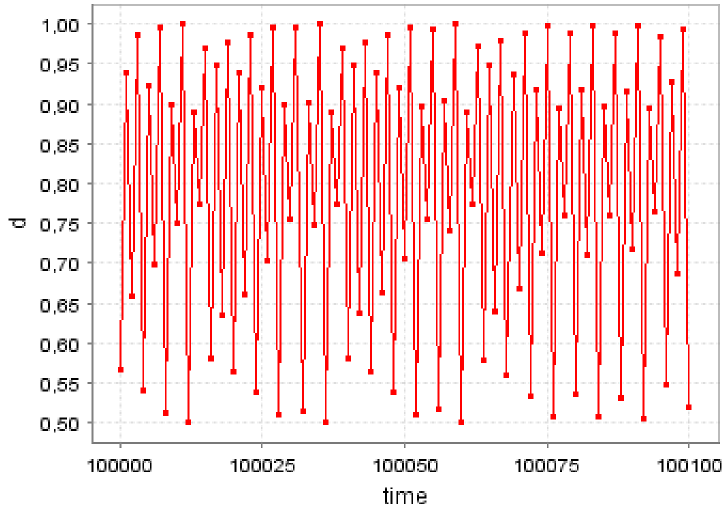

Figure 10 – d_t time series (ϕ_0=0.75; ϕ_1=0.5; transients=100,000).

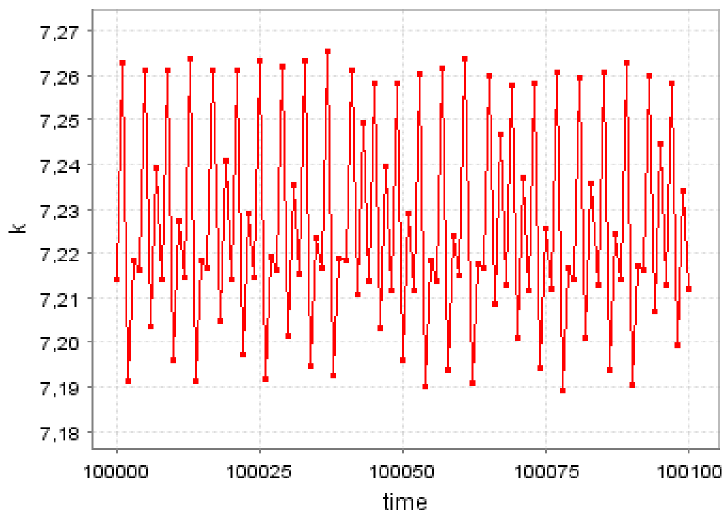

Figure 11 – Solow model: k_t time series (transients=100,000).

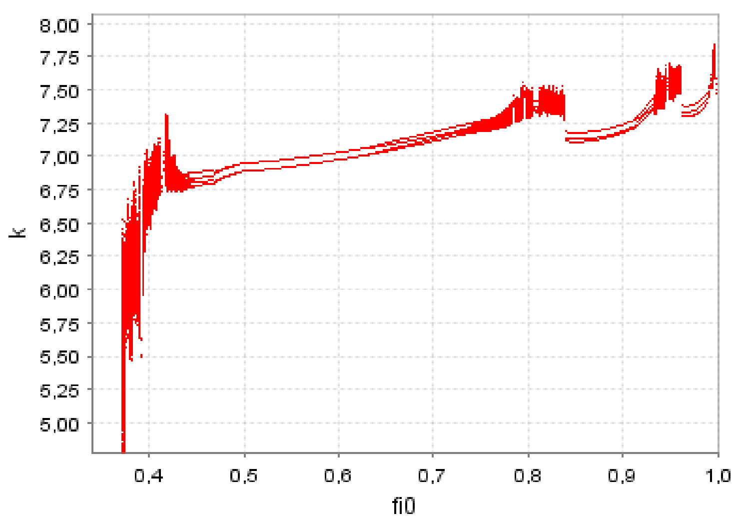

Figure 12 – Solow model: (k_t,ϕ_0) bifurcation diagram.

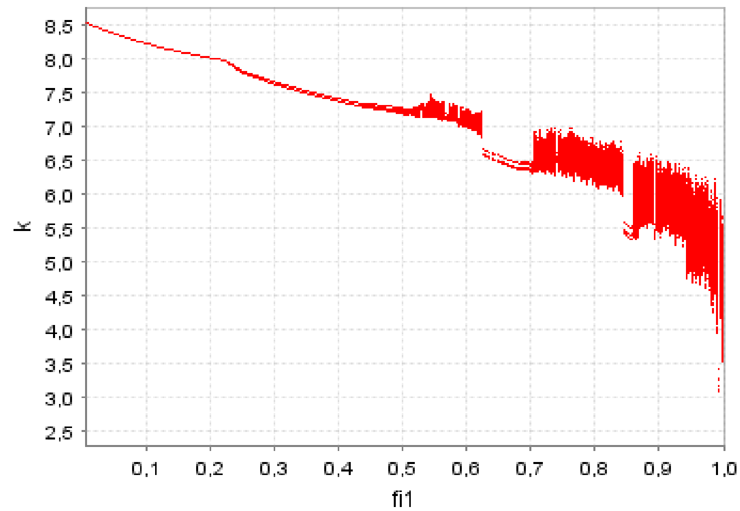

Figure 13 – Solow model: (k_t,ϕ_1) bifurcation diagram.

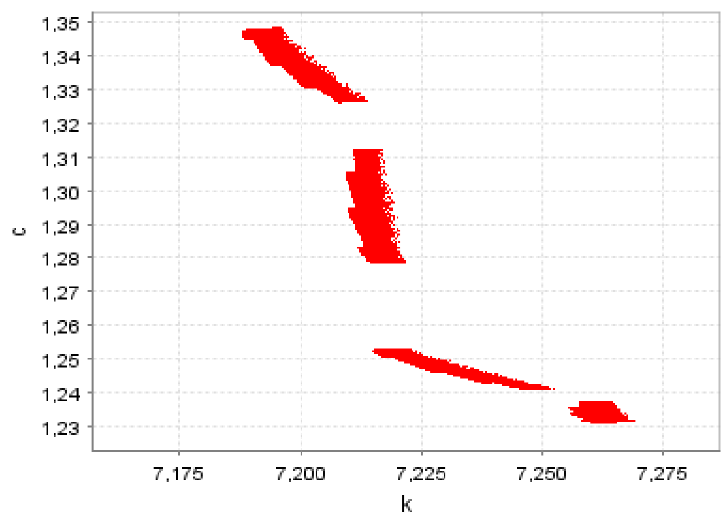

Figure 14 – Solow model: (k_t, c_t) attractor (transients=1,000).

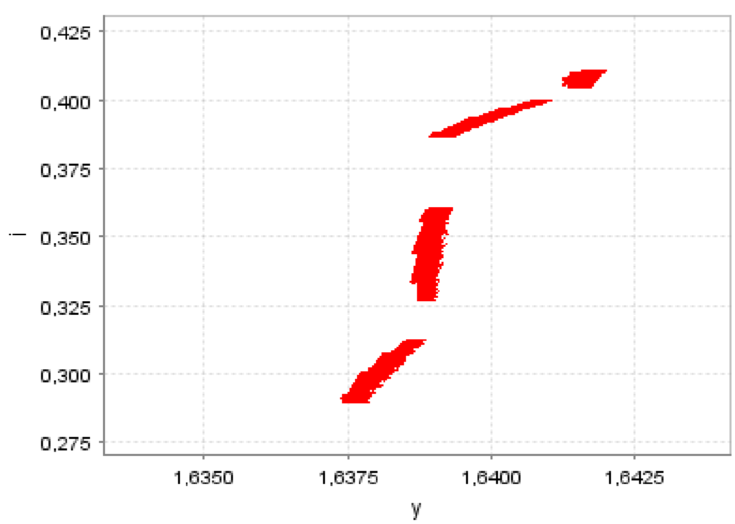

Figure 15 – Solow model: (y_t, i_t) attractor (transients=1,000).

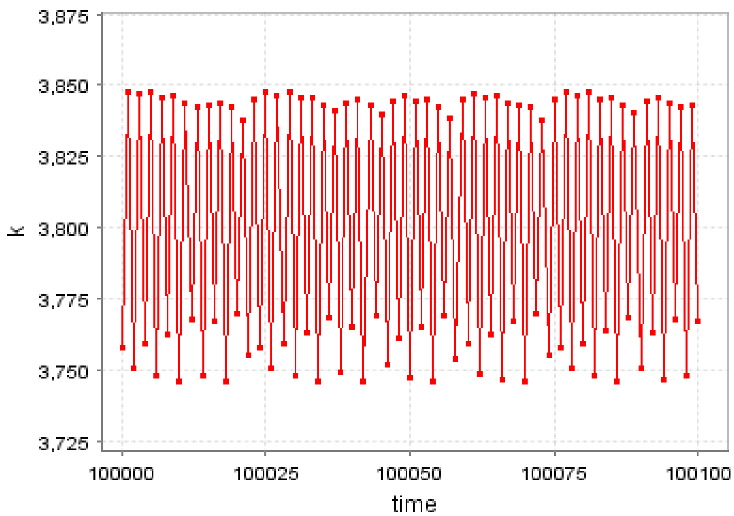

Figure 16 – Ramsey model: k_t time series (transients=100,000).

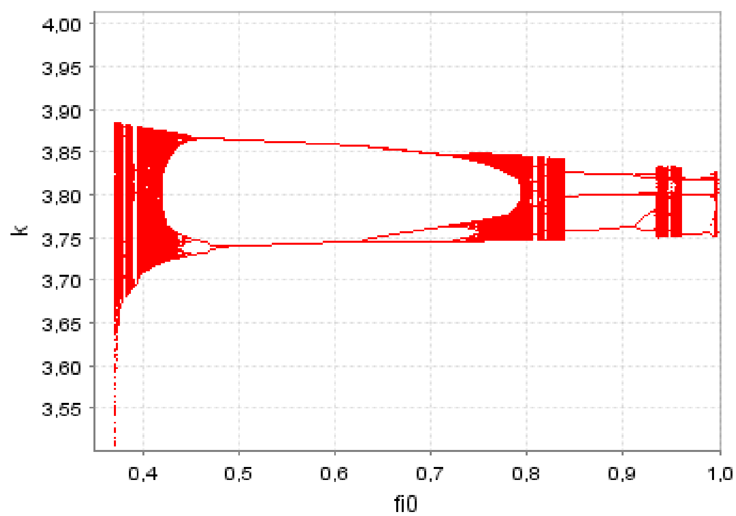

Figure 17 – Ramsey model: (k_t,ϕ_0) bifurcation diagram.

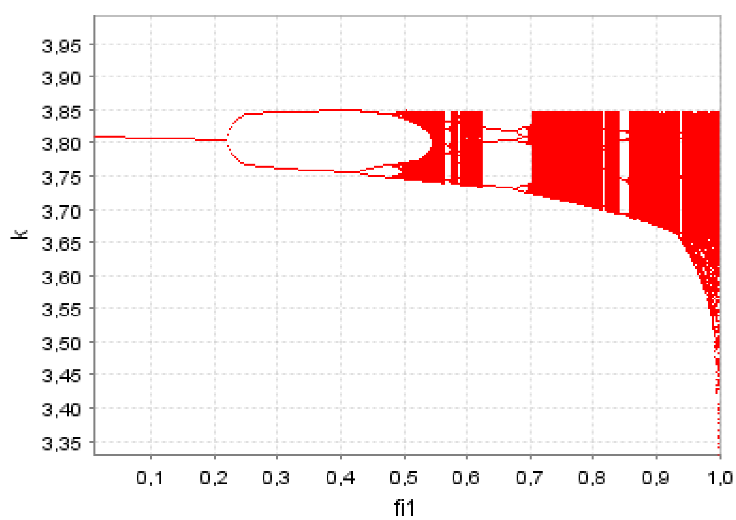

Figure 18 – Ramsey model: (k_t,ϕ_1) bifurcation diagram.

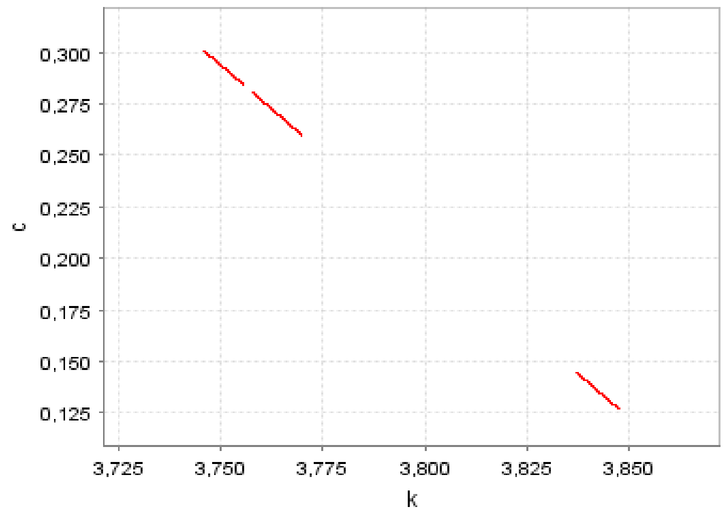

Figure 19 – Ramsey model: (k_t, c_t) attractor (transients=1,000).

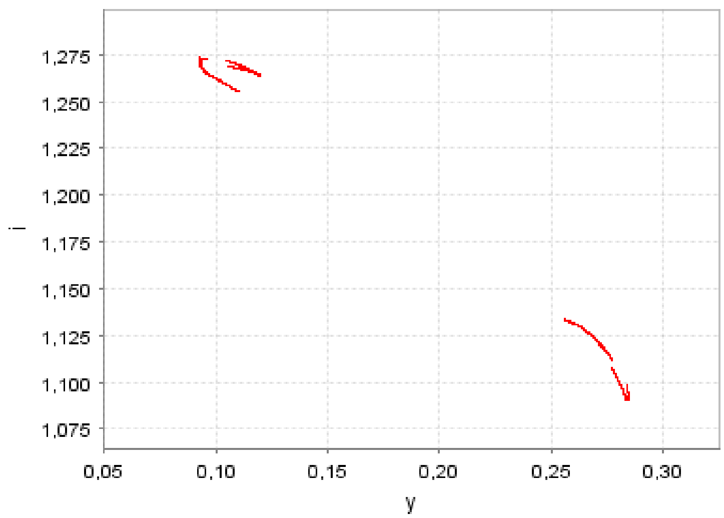

Figure 20 – Ramsey model: (y_t, i_t) attractor (transients=1,000).

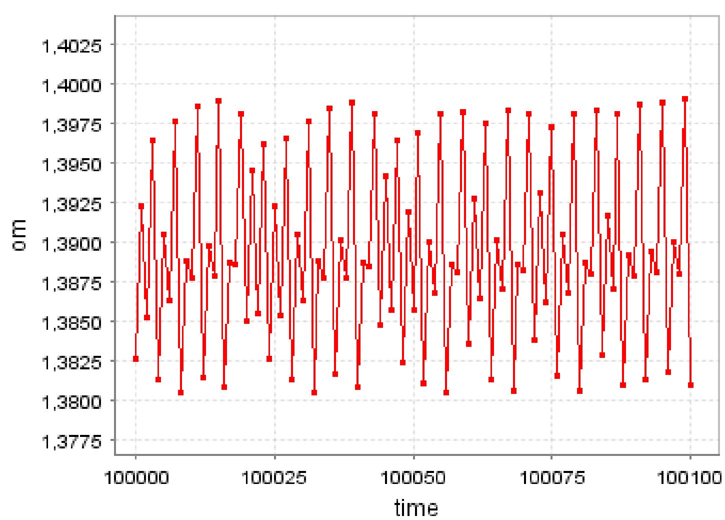

Figure 21 – Endogenous growth model: ω_t time series (transients=100,000).

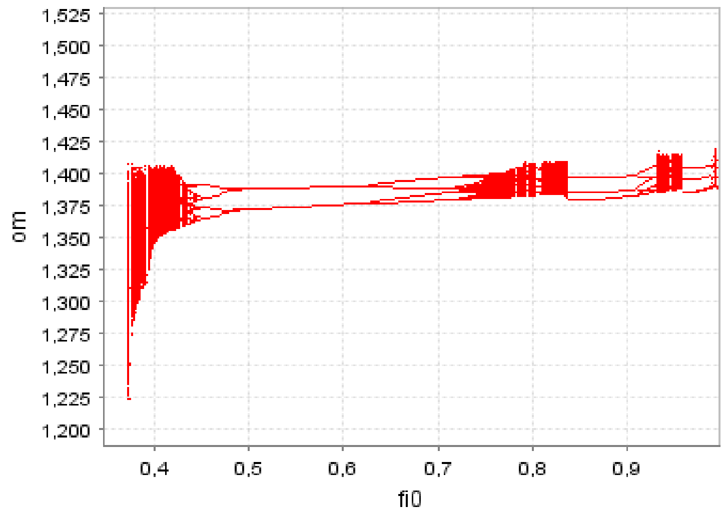

Figure 22 – Endogenous growth model: (ω,ϕ_0) bifurcation diagram.

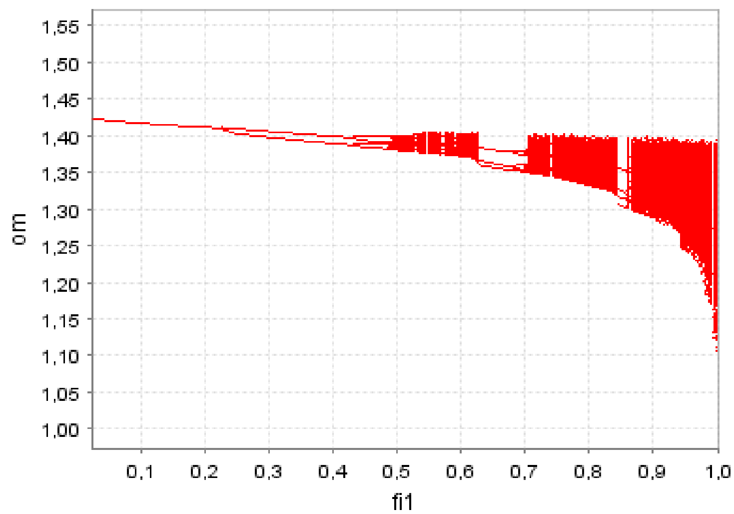

Figure 23 –Endogenous growth model: (ω,ϕ_1) bifurcation diagram.