Business Cycles with Endogenous Mark-ups

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Abstract

Endogenous mark-ups have been a matter of interest in macroeconomics, especially from the middle 1990's onwards. However, the complexity of this class of models, does not allow general qualitative conclusions in most cases, and there is plenty of room for investigation, especially in the reasons driving the emergence of multiple equilibria and non-saddle-point dynamics. In this article we extend a simple dynamic general equilibrium model to include the possibility of strategic interaction between producers in each industry, and entry affects the level of macroeconomic efficiency through an endogenous mark-up. We demonstrate multiple equilibria is a likely outcome even in an exogenous labour-supply framework. A pair of equilibria exists (a stable and an unstable one) and they are connected through

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a heteroclinic orbit. When we allow labour supply to vary, a third equilibrium may emerge if the government is present in the economy, and local indeterminacy may exist.

KEY WORDS: Endogenous mark-ups; Multiple equilibria; Local dynamics; Global dynamics.
JEL CLASSIFICATION: C6; D4; D5; E3; L1.

1 Introduction

Endogenous mark-ups have been a matter of interest in macroeconomics, especially from the middle 1990's onwards. Despite the fact we can find older references to endogenous mark-ups in macroeconomics, especially in Dunlop (1938) critique to Keynes' counter-cyclical real wage due to demand shocks, the generalised interest was established with the seminal works of Rotemberg and Woodford (1991) and Rotemberg and Woodford (1995).

More recently, Goodfriend and King (1997) and Clarida et al. (1999) brought a special type of endogenous mark-up - due to nominal rigidity - to the centre of the policy analysis in what is now called the New Neoclassical/Keynesian Sinthesis.

However, sticky prices are not the only source of endogenous mark-ups, and they may not be the most important one. Furthermore, the interaction between sources of mark-up variation and with other real rigidities may play an important role in explaining the business-cycle phenomena. For a survey of the literature refer to Rotemberg and Woodford (1999).

The dynamic features of several types of endogenous-mark-up models have been studied in Galí (1994a), Galí (1994b) Galí (1995) and Rotemberg and Woodford (1995), amongst others. However, the complexity of this class of models, does not allow general qualitative conclusions in most cases, and there is plenty of room for investigation, especially in the reasons driving the emergence of multiple equilibria and non-saddle-point dynamics. Global-dynamics behaviour is also an open field for research.

One type of endogenous-mark-up model has produced several works, but usually rely on a discrete-time overlapping-generations framework: the variable-entry Cournotian class of models. Examples of this type of framework can be found in Chatterjee et al. (1993), D'Aspremont et al. (1995), or Kaas and Madden (2005).

On the other hand, the empirical literature has shown there is strong evidence of a mildly counter-cyclical mark-up - see, *inter alia*, Martins et al. (1996). This pattern is consistent with a Cournotian model of the mark-up with frequent demand shocks and relatively rare supply shocks. Additionally, the business creation/destruction pattern observed in reality is also consistent with this type of models.

Thus, applying this type of models to replicate the business-cycle features of real economies seem promising, but it has produced very few works, e.g. Portier (1995) for the French economy and Jaimovich (2004) for the U.S. However, the technology used (log-linearisation, calibration, etc.) assumes the equilibrium exists, it is unique, and it is saddle-point stable, as in the Real Business Cycles literature.

In this article we extend a simple dynamic general equilibrium model to include the possibility of strategic interaction between producers in each industry, and entry affects the level of macroeconomic efficiency through an endogenous mark-up. We demonstrate multiple equilibria is a likely outcome even in an exogenous labour-supply framework. A pair of equilibria exists (a stable and an unstable one) and they may be linked through a heteroclinic trajectory. When we allow labour supply to vary, a third equilibrium may emerge if the government is present in the economy, and local indeterminacy may exist.

In section 2 we extend the Ramsey model to allow for an endogenous mark-up. The existence of equilibria, and their local and global stability features are studied. In section 3 we allow for an elastic labour supply. Section 4 concludes.

2 A Ramsey model with endogenous mark-ups

2.1 Households

We assume there is a single infinitely living household, total population is constant and has been normalised to unity. Thus, quantity variables mays be interpreted as *per capita* values. Exogenous population growth does not change the main message of the model.

The household is assumed to maximise an intertemporal utility function in the absence of uncertainty:

$$\max_{C(t)} U = \int_0^{\infty} e^{-\rho.t}.u\left[C\left(t\right)\right].dt \tag{1}$$

where $\rho > 0$ represents the rate of time preference and C stands for consumption. For sake of simplicity the felicity function is logarithmic:

$$u\left[C\left(t\right)\right] = \ln\left[C\left(t\right)\right] \tag{2}$$

Notice we assume a unit elasticity of intertemporal substitution in consumption, however using a different constant-elasticity function does not change the main results.

The household sells human and physical capital services (L) to firms obtaining labour and non-labour income in exchange. The final good can be used either for consumption or for capital accumulation. The price of the final good P is normalised to unity, i.e. the final good is used as *numéraire*. Therefore, the instantaneous budget constraint is given by

$$\dot{K}\left(t\right) = w\left(t\right).L\left(t\right) + R\left(t\right).K\left(t\right) + \Pi\left(t\right) - C\left(t\right) - T\left(t\right) - \delta.K\left(t\right) \tag{3}$$

where K represents the capital stock, w is the wage rate, R stands for the rental price of capital, Π represents real pure profits, T is a lump-sum tax levied on households, and $0 < \delta < 1$ stands for a constant depreciation rate.

Optimal consumption and labour supply paths can be obtained maximising a current-value Hamiltonian, and the first-order conditions can be expressed by the following behavioural equations:

$$\frac{\dot{C}\left(t\right)}{C\left(t\right)} = r\left(t\right) - \rho \tag{4}$$

$$L\left(t\right) = \overline{L} \tag{5}$$

$$\lim_{t\to\infty} e^{-\rho.t}.\frac{K\left(t\right)}{C\left(t\right)} = 0 \tag{6}$$

where the real interest rate as $r = R - \delta$. Notice that the labour supply is always equal to the maximum amount of labour available to the household, as it does not generate disutility.

2.2 Government

We assume government can choose its level of real consumption. Since Ricardian equivalence holds here, we can ignore government borrowing without lost of generality. Therefore, government follows a balanced-budget rule over time:

$$T(t) = G(t) \tag{7}$$

2.3 The final-good sector

The final good, Y, is produced in a competitive retail sector using a CES technology that transforms a continuum of intermediate goods, with mass equal to unity, into a final homogeneous good. The technology exhibits constant returns to specialisation:

$$Y(t) = \left[\int_0^1 y_j(t)^{\frac{\sigma-1}{\sigma}} .dj\right]^{\frac{\sigma}{\sigma-1}} \tag{8}$$

where $\sigma > 0$ represents the elasticity of substitution between inputs and y_j stands for intermediate consumption of variety $j \in [0,1]$.

The maximisation problem can be solved in two steps: (i) determining demand functions for each input that minimises total cost for a given level of final output; (ii) determining the optimal level of output for the representative firm. The first step gives us the following intratemporal demand function for each input:

$$y_j(t) = \left[\frac{p_j(t)}{P(t)}\right]^{-\sigma} .Y(t) \tag{9}$$

where p_j stands for the price of good j and P is the appropriate cost-of-producing index form this firm given by

$$P(t) = \left[\int_0^1 p_j(t)^{1-\sigma} .dj\right]^{\frac{1}{1-\sigma}} \tag{10}$$

The cost function can be written as $P(t).Y(t)$. Therefore, the second step in the maximisation program equals the price of the final good to its marginal cost:

$$1 = P(t). \tag{11}$$

2.4 The intermediate goods sector

Industry j (J) is composed of $m_j \geq 1$ producers[1], each facing the following technology:

$$y_i(t) = \max\{F[K_i(t), A(t).L_i(t)] - 1.\Phi, 0\} \tag{12}$$

where y_i represents the output of firm i, $A(t) > 0$ stands for labour efficiency, K_i and L_i represent its capital and labour inputs, $F(.)$ is homogeneous of degree one (HoDO), and $\Phi > 0$ induces increasing returns to scale.

Using the terminology in D'Aspremont et al. (1997), we assume Cournotian Monopolistic Competition (CMC), i.e. firms compete over quantities within the same industry, and they compete over prices across industries. Therefore, each firm faces the following demand for its variety:

$$y_i(t) + \sum_{k \neq i \in J} y_k(t) = \left[\frac{p_j(t)}{P(t)}\right]^{-\sigma}.D(t) \tag{13}$$

where $D = C + I + G$ represents total demand for the final good in the economy, and I stands for gross investment defined as

$$I(t) = \dot{K}(t) + \delta.K(t) \tag{14}$$

The representative firm maximises its real profits given by

$$\begin{aligned}
\max_{L_i(t), K_i(t)} \Pi_i(t) &= \frac{p_j(t)}{P(t)}.y_i(t) - \frac{w(t)}{P(t)}.L_i(t) - \frac{R(t)}{P(t)}.K_i(t) \qquad (15) \\
p_j(t) &= \left[\frac{y_j(t)}{D(t)}\right]^{-\sigma}.P(t) \\
y_i(t) + \Phi &= F[K_i(t), A(t).L_i(t)] \\
& y_j(t) = y_i(t) + \sum_{k \neq i \in J} y_k(t)
\end{aligned}$$

[1] Of course this number is an integer. However, we will treat it as a real number, for simplicity. We can think about it as the average number of firms in each industry.

Notice this is a static problem, as the firm does not accumulate capital. The first-order conditions are given by

$$[1-\mu_i(t)].F_{L,i}(t) = \frac{w(t)}{p_j(t)} \tag{16}$$

$$[1-\mu_i(t)].F_{K,i}(t) = \frac{R(t)}{p_j(t)} \tag{17}$$

where $\mu_i = (p_j - MC_i)/p_j \in (0,1)$ is the *Lerner* index for firm i, $MC_i = w/F_{L,i} = R/F_{K,i}$ represents the marginal cost of production, and

$$\begin{aligned} F_{L,i}(t) &= \frac{\partial F}{\partial L_i}[K_i(t), L_i(t), A(t)] \\ F_{K,i}(t) &= \frac{\partial F}{\partial K_i}[K_i(t), A(t).L_i(t)] \end{aligned}$$

stand for marginal products.

2.5 From micro to macro

Let us now assume an intra-industrial symmetric exists for all industries. In this case, we have $\mu_i = \mu_j = 1/(\sigma.m_j)$. Notice that for an equilibrium to exist we must have $\sigma.m_j > 1$. Considering $F(\cdot)$ is HoDO, its partial derivatives are homogeneous of degree zero. Thus, we can rewrite equations (16) and (17) to represent the entire industry[2]:

$$(1-\mu_j).F_L(K_j, L_j, A) = \frac{w}{p_j} \tag{18}$$

$$(1-\mu_j).F_K(K_j, A.L_j) = \frac{R}{p_j} \tag{19}$$

where $L_j = \sum_{i\in J} L_i = m_j.L_i$ and $K_j = \sum_{i\in J} K_i = m_j.K_i$ represent labour and capital demand in industry j. Furthermore, we can use *Euler*'s theorem and the above-mentioned equations to obtain total profits in the industry:

$$\Pi_j = \frac{p_j}{P}.[\mu_j.F(K_j, A.L_j) - m_j.\Phi] \tag{20}$$

[2] For simplicity with drop the time indices from this point onwards.

If all industries are identical, i.e. if there is an inter-industrial symmetric equilibrium, we have $m_j = m$ (consequently $\mu_j = \mu$), and $p_j = P$.

If the final output market is in equilibrium we have $Y(t) = D(t)$. Furthermore, market demand for inputs can be written as

$$(1-\mu).F_L(K, L, A) = \frac{w}{P} \tag{21}$$

$$(1-\mu).F_K(K, A.L) = \frac{R}{P} \tag{22}$$

where $L = \int_0^1 L_j.dj = L_j$ and $K = \int_0^1 K_j.dj = K_j$ represent total labour and capital demand. the market-clearing condition for the labour market is given by $L = \overline{L}$. Therefore, using the properties of $F(\cdot)$ we can derive an aggregate production function for final output given by

$$Y = F\left(K, A.\overline{L}\right) - m.\Phi \tag{23}$$

2.6 Entry

Profit income is obtained by aggregating industries' profits, i.e. $\Pi = \int_0^1 \Pi_j.dj = \Pi_j = Y - w.L - R.K$. Considering the equilibrium factor prices and the aggregate production function, total profits can be expressed as

$$\begin{aligned}\Pi &= \{F(K, A.L) - [F_K(K, A.L).K + F_{A.L}(K, A.L).A.L]\} + \\ &\quad +\mu.[F_K(K, A.L).K + F_{A.L}(K, A.L).A.L] - m.\Phi\end{aligned}$$

where $F_{A.L} = F_L/A$. Since $F(\cdot)$ is HoDO, we can use *Euler*'s theorem to simplify the previous expression. The theorem implies that $F = F_K.K + F_{A.L}.(A.L)$. Thus, total profits are given by

$$\Pi = \mu.F(K, A.L) - m.\Phi \tag{24}$$

Assuming instantaneous free entry, the number of firms in each industry adjusts in order to keep pure profits equal to zero. Taking into account that $m = 1/(\sigma.\mu)$, we obtain the rule governing the endogenous mark-up:

$$\mu = \sqrt{\frac{\Phi}{\sigma.F\left(K, A.\overline{L}\right)}} \tag{25}$$

Since μ cannot be larger than 1, there is a minimal level of capital $\underline{K}$ such that $F(\underline{K}, A.\overline{L}) = \frac{\Phi}{\sigma}$. Therefore, we have the feasibility condition given by $K > \underline{K}$.

If we substitute this result in the aggregate production function, we obtain a reduced-form aggregate production function that depends on inputs and an efficiency index that is decreasing with the mark-up

$$Y = (1-\mu).F\left(K, A.\overline{L}\right). \tag{26}$$

Had we considered the number of firms per industry was fixed (e.g. $m(t) = 1$ as is the monopolistic competition case) and the mass of industries was given by $n(t)$, free entry would mean that $n = F\left(K, A.\overline{L}\right)/(\sigma.\Phi)$. Thus, equation (26) would be given by $Y = (1-1/\sigma).F\left(K, A.\overline{L}\right)$. It is easy to see, that free-entry monopolistic competition model without externalities is formally equivalent to a Walrasian Ramsey model with a less efficient production function. In that case, all the main properties of the Ramsey model are kept: unique equilibrium, saddle-point stability, and the two fundamental theorems of welfare economics would hold.

From now on, we assume that $F(.)$ is a Cobb-Douglas production function given by

$$Y = K_i(t)^{\alpha}.\left[A(t).L_i(t)\right]^{1-\alpha}$$

with $0 < \alpha < 1$. One can easily notice the results in the previous subsections do not depend on the Cobb-Douglas technology, but hold for all linear homogeneous production functions.

2.7 General equilibrium

Let us now study the existence and uniqueness of general equilibrium in this economy.

Definition 1 *General equilibrium: it is a flow of consumption, capital stock, and mark-up such that: (i) households and firms optimise; (ii) all markets clear.*

Using the equations previously derived, we can represent the general equilibrium using a system on $[C(t), K(t)]$ such that

$$\dot{C} = \left[(1-\mu).F_K\left(K, A.\overline{L}\right) - (\rho+\delta)\right].C, \quad t \in \mathbb{R}_+ \tag{27}$$

$$\dot{K} = (1-\mu) . F\left(K, A.\overline{L}\right) - C - G - \delta.K, \quad t \in \mathbb{R}_+ \tag{28}$$

$$\lim_{t\to\infty} e^{-\rho.t}.\frac{K}{C} = 0 \tag{29}$$

$$K(0) = K_0, given \tag{30}$$

The steady state is thus defined by the values $(C^*, K^*, \mu^*) \in \mathbb{R}^2_+ \times (0,1)$, which may be determined by equations (27) to (30) when $\dot{C} = \dot{K} = 0$.

Assumption 2 *Assume that* $0 \leq G \leq G_0(K)$ *with*

$$G_0(K) \leq (1-\mu) . F\left(K, A.\overline{L}\right) - \delta.K \tag{31}$$

This assumption means government consumption cannot be so high that it takes the net output. Equivalently, household consumption should always be non-negative.

Proposition 3 *Let assumption 2 hold. Then there is at least one steady state if and only if the following condition holds*

$$\eta \leq \eta_0 = A.\overline{L}.B(\beta) . \varrho^{-\frac{2}{\beta}} \tag{32}$$

where

$$\begin{aligned} \eta &\equiv \frac{\Phi}{\sigma} > 0, \varrho \equiv \rho + \delta > 0, \beta \equiv 2.\frac{1-\alpha}{\alpha} > 0 \\ B(\beta) &\equiv \left[\frac{2}{(2+\beta) . (1+\beta)^{1+\beta}}\right]^{\frac{2}{\beta}} .\beta^2 > 0 \end{aligned}$$

Proof. It is straightforward to see that aggregate consumption is obtained by the aggregate budget constraint, i.e. $C^* = C(\mu, K, .)$ is determined uniquely by equation (28) with $\dot{K} = 0$. Thus, the first condition is easily identified as the necessary condition for $C^* \geq 0$, i.e. government consumption cannot be larger than net output given by the solutions to the system.

Now, if we substitute the $F_K(.)$ function in (27) (with $\dot{C} = 0$) and use (25) to obtain the capital stock as a decreasing function of the mark-up, we obtain a steady-state equilibrium function stating that $r^* = \varrho$, for $C^* > 0$:

$$f_1(\mu) \equiv Q_1.(1-\mu).\mu^{\beta} - 1 = 0 \tag{33}$$

where

$$Q_1 = \frac{\alpha}{\varrho}.\left(\frac{A.\overline{L}}{\eta}\right)^{\frac{\beta}{2}} > 0$$

This function has no closed-form solution, but it can easily be studied, as we know that $f_1(0) = f_1(1) = -1$ and there is a unique stationarity point given by

$$f_1'(\overline{\mu}) = 0 \Leftrightarrow \overline{\mu} = \frac{2.(1-\alpha)}{2.(1-\alpha)+\alpha} \in (0,1)$$

Notice this value to the steady-state mark-up level corresponds to a maximum of $f_1(.)$, as $f_1(\overline{\mu}) > -1$. However, we cannot guarantee that $f_1(\overline{\mu}) > 0$, therefore equilibria may not exist for specific values of the parameters. The position of $f_1(.)$ is governed by the value of Q_1 that depends negatively on η, ϱ and $\left(A.\overline{L}\right)$.[3] Furthermore, $\overline{\mu}$ depends negatively on α and it goes to zero (one) when the capital share in income tends to one (zero). Using figure 1 and varying the value of η, we notice an equilibrium exists if there is at least one solution for equation (33):

Thus, if the value of Q_1, is not large enough a solution exists. Furthermore, we know that if $f_1(\overline{\mu}) = 0$ there is only one equilibrium in this model (a bifurcation). Thus the combination of values that guarantee the existence of at least one equilibrium is given by

$$\frac{\alpha}{\varrho}.\left(\frac{A.\overline{L}}{\eta}\right)^{\frac{\beta}{2}}(1-\overline{\mu}).\overline{\mu}^{\beta} - 1 \geq 0$$

and solving it in order to η the condition obtained is $\eta \leq \eta_0$. ■

One interpretation for the existence condition $\eta \leq \eta_0$ is that the level of "standardised" barriers to entry (the fixed cost divided by the elasticity of substitution) cannot be too large or else, there would not be an incentive for any firm to stay in these industries.

[3] The sign of the derivative with respect to α cannot be determined unambiguously.

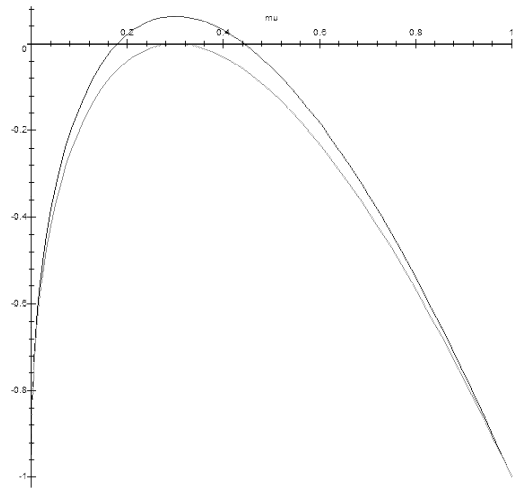

Figure 1: Equilibrium in the model with exogenous labour supply

Proposition 4 *Multiple equilibria: a single equilibrium exists for $\eta = \eta_0$ and exactly a pair of equilibria exists for $\eta < \eta_0$.*

Proof. The equilibrium function $f_1(.)$ (continuous and twice-differentiable) is strictly increasing in the range $\mu \in (0, \overline{\mu})$ and it is strictly decreasing in the range $\mu \in (\overline{\mu}, 1)$ as it can be expressed in the following form

$$f_1'(\mu) = Q_1.\mu^{1+\beta}.(1+\beta).(\overline{\mu} - \mu)$$

The existence of the single solution for $\eta = \eta_0$ was demonstrated in the previous proposition. With a single-peaked function, for $\eta < \eta_0$ there is a maximum of two fixed points for function $f_1(.)$, one to the left of $\overline{\mu}$ - $\mu^*_{(1)} \in (0, \overline{\mu})$ - and one to its right $\mu^*_{(2)} \in (\overline{\mu}, 1)$. ■

The choice of η as an important parameter was not a matter of chance. Notice it is closely linked to the degree of imperfect competition as it increases with the fixed cost and it decreases with the elasticity of demand directed to each variety. Now, if we fix the values of α, A and $\overline{L}$, standard parameters in the Walrasian Ramsey model, $\eta = \eta_0$ gives us a separation curve in the (η, ϱ) space, represented in figure 2:

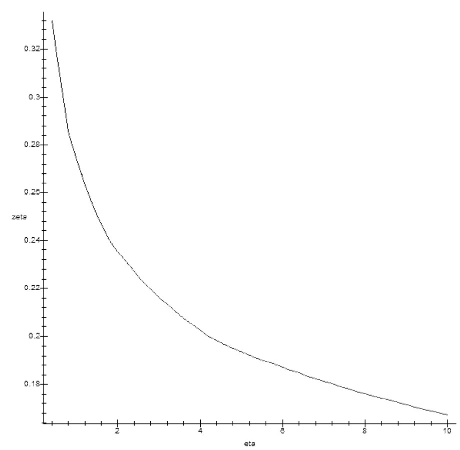

Figure 2: Existence and multiplicity of equilibria in the Cournotian Ramsey model

Along the schedule depicted there a unique equilibrium in the model. In the NE area no equilibrium exists in the model, and in the SW area two equilibria exist, provided the government consumption condition holds.

Proposition 5 *The difference* $\mu^*_{(2)} - \mu^*_{(1)} \geq 0$ *decreases with the value of* η*, for* $\eta < \eta(0)$*.*

Proof. The effect of η on an equilibrium value for the mark-up can be assessed using the parametric derivative $\partial\mu^*/\partial\eta$. Let us now divide function $f_1(.)$ into two monotonic branches with the same functional form:

$$f_1(\mu,\eta) = \begin{cases} f_{1_{(1)}}(\mu,\eta) \Leftarrow \mu^* \in (0,\overline{\mu}) \\ f_{1_{(2)}}(\mu,\eta) \Leftarrow \mu^* \in [\overline{\mu},1) \end{cases}$$

we can unambiguously express each of the two equilibria using $f_{1_{(s)}}\left(\mu^*_{(s)},\eta\right) = 0$, for $s = 1,2$. Thus, the above-mentioned derivative can be obtained using the implicit-function theorem:

$$\frac{\partial\mu^*_{(s)}}{\partial\eta} = -\frac{\frac{\partial f_{1_{(s)}}}{\partial\eta}}{\frac{\partial f_{1_{(s)}}}{\partial\mu^*_{(s)}}}\left(\mu^*_{(s)},\eta\right) = -\frac{1}{2}.\frac{\overline{\mu}.\left(1-\mu^*_{(s)}\right).\mu^*_{(s)}}{\eta.\left(\mu^*_{(s)}-\overline{\mu}\right)}$$

Since we know$\mu^*_{(1)} < \mu$ and $\mu^*_{(2)} > \mu$, we know that $\partial\mu^*_{(1)}/\partial\eta > 0$ and $\partial\mu^*_{(2)}/\partial\eta < 0$. Therefore, $\partial\left[\mu^*_{(2)} - \mu^*_{(1)}\right]/\partial\eta < 0$ as long as $\eta < \eta_0$. ■

It is clearly depicted in figure 1, as the $f_1(.)$ function moves up when we decrease the value of η, i.e. when the fixed cost decreases or the elasticity of demand increases. When η approaches zero, we end up with $\mu^*_{(1)}$ very close to zero, the Walrasian case, and $\mu^*_{(2)}$ very close to one, a degenerate monopoly in each industry. The latter cannot be ruled out for a very large elasticity of demand, $\sigma \to \infty$, as long as $\Phi > 0$ (even if it is very small), because a Walrasian equilibrium cannot subsist with increasing returns to scale.

2.8 Local dynamics

To study the dynamics of the system, we log-linearise it about a steady-state equilibrium. The log-linearised system can express it as

$$\begin{pmatrix} \dot{\widehat{C}} \\ \dot{\widehat{K}} \end{pmatrix} = \mathbf{J}_1 . \begin{pmatrix} \widehat{C} \\ \widehat{K} \end{pmatrix} \tag{34}$$

where $\widehat{H} = dH/H^*$ represents the proportional deviation of variable H from its steady-state value and $\dot{\widehat{H}} = \dot{H}/H^*$. The Jacobian matrix evaluated at a steady-state equilibrium, $\mathbf{J}_1$, is given by

$$\mathbf{J}_1 = \begin{pmatrix} 0 & \frac{\varrho.(2-\alpha)}{2.(1-\mu^*)}.\left(\mu^* - \overline{\mu}\right) \\ -\frac{\varrho.s^*_C}{\alpha} & \rho + \frac{\varrho.\mu^*}{2.(1-\mu^*)} \end{pmatrix}$$

where $s^*_C = C^*/Y^*$ is the steady-state consumption share in domestic expenditure.

Proposition 6 *Stability and bifurcations: (a) if the equilibrium is unique it is a fold bifurcation. (b) If there is a pair of equilibria, then $\mu^*_{(1)}$ is saddle-point stable and $\mu^*_{(2)}$ is totally unstable (a source).*

Proof. It is easy to see the trace is always positive and given by

$$Tr\left(\mathbf{J}_1\right) = \rho + \frac{\varrho.\mu^*}{2.\left(1-\mu^*\right)} > \rho > 0$$

and the determinant

$$\det(\mathbf{J}_1) = \frac{\varrho^2.s_C^*.(1+\beta)}{2.(1-\mu^*)}.(\mu^* - \overline{\mu}) < Tr(\mathbf{J}_1)$$

Thus, the determinant is positive for $\mu^* > \overline{\mu}$, and it is negative otherwise. Therefore, for the real part of the eigenvalues evaluated at the steady-state equilibria we have two positive values for the equilibrium with a high mark-up level ($\mu^*_{(2)} > \overline{\mu}$), and one positive and one negative value for smaller equilibrium value of the mark-up ($\mu^*_{(1)} < \overline{\mu}$).

Thus, we may have 1 or 2 equilibria, if the existence condition holds, one of these is saddle-point stable (we will call it a 'saddle') and the other one is totally unstable (we will call it a 'source'). Consequently, for $\mu^* = \overline{\mu}$ we have a fold bifurcation. ■

Proposition 7 *Real eigenvalues: both eigenvalues are real, irrespective to the steady state they are associated with.*

Proof. The discriminant of the characteristic polynomial given by $\det(\mathbf{J}_1 - \lambda.\mathbf{I}) = 0$ is given by $[Tr(\mathbf{J}_1)/2]^2 - \det(\mathbf{J}_1)$. It is always positive for the 'saddle,' as the determinant is negative in that case. Thus, no complex eigenvalues appear for the low mark-up equilibrium.

For the source equilibrium let us consider the alternative expression for the trace and the determinant:

$$\det(\mathbf{J}_1) = s_C^*.\frac{\varrho}{\alpha}.x(\mu^*), \quad \text{where } x(\mu^*) = \frac{dR}{d\mu}(\mu^*).K(\mu^*) = \frac{\varrho.(2-\alpha)}{2.(1-\mu^*)}.(\mu^* - \overline{\mu});$$

$$Tr(\mathbf{J}_1) = \frac{x(\mu^*)}{\alpha} + \left(\frac{\varrho}{\alpha} - \delta\right), \quad \text{where } \frac{\varrho}{\alpha} - \delta = \frac{C^* + G^*}{K^*} > 0.$$

It is easy to notice this is a second-degree polynomial in x. Since $x(.)$ is continuous and differentiable in $\mu^* \in (0,1)$, and we know the discriminant shows at least one positive value (for the low-mark-up equilibrium), we need to analyse if there is any value of x that turns the discriminant equal to zero. The solution for the quadratic equation is given by

$$x_0 = \alpha.\left\{-\frac{C^* + G^*}{K^*} + 2.\alpha.\frac{C^*}{K^*} \pm 2.\sqrt{-\alpha.\frac{C^*}{K^*}.\frac{(1-\alpha).C^* + G^*}{K^*}}\right\}.$$

Since the expression within the square root is always negative, there is no real value of x that turns the discriminant equal to zero. Therefore, the discriminant is always positive, i.e. no complex eigenvalues can exist in this model. ■

2.9 Global dynamics

Though the local dynamics give only information on the local manifolds associated to the two equilibrium fixed points, it can be proved that the stable manifold associated to $X_B^* \equiv (K_B^*, C_B^*)$ ($W^s(X_B^*)$) and the unstable manifold associated with $X_A^* \equiv (K_A^*, C_A^*)$ ($W^u(X_A^*)$) coincide, for values of the capital stock such that $K_A^* \leq K \leq K_B^*$ and for values of consumption such that $C_A^* \leq C \leq C_B^*$. That is, we have a heteroclinic orbit, joining the two equilibria $\Gamma_{AB}(X)$ (see Figure 3). Though the trajectories for consumption and the stock of capital belonging to the heteroclinic orbit, are non-stationary equilibria, heteroclinic orbits are robust for changes in the parameters of the economy which do not cross bifurcation values. Mathematically, heteroclinic orbits belong to the non-wandering set, but are structurally unstable (see Guckenheimer and Holmes (1990)). This means that the though they do not have the nature of a local bifurcation because they still occur for large variations of the parameters, for large increases in $\rho + \delta$ or decreases in η both the two equilibria and the heteroclinic orbit which connects them will disappear because of the presence of a fold bifurcation.

We can give a simple geometric proof for the existence of an heteroclinic orbit. Consider a square triangle whose vertices are A B and D and with sides given by $K = K_A^*$ and $C \in [C_A^*, C_B^*]$, $C = C_B^*$ and $K \in [K_A^*, K_B^*]$ and the third joins the two equilibria along the line $C = \frac{K}{\alpha}(\rho + (1-\alpha)\delta) - G$. As, except for the vertices, the combinations of (C, K) belonging to the sides of the triangle are not equilibrium points of the two differential equations (27)-(28), the solution is changing locally. If we consider the local slopes of the vector field defined by the two differential equations, it is easy to see that it points outward in all the three sides of the triangle. In addition, the local stable manifolds, whose slopes are given by the eigenvectors associated to the eigenvalues with smaller magnitudes (λ_s), evaluated locally at the two equilibrium points, bisect the two vertices of the triangle associated to the two equilibrium points. Therefore, there should be a separatrix connecting the two equilibrium points and passing through the interior of the triangle.

Heteroclinic orbits are a rare event in economics. Therefore we need to

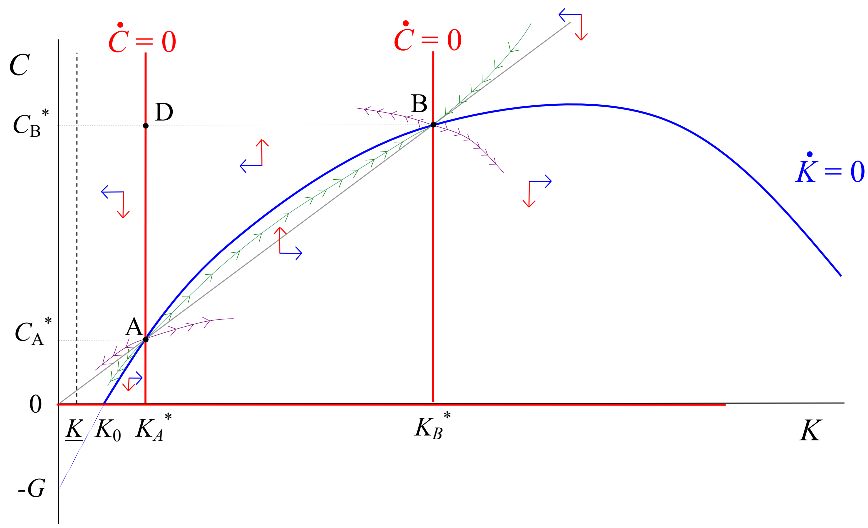

Figure 3: The phase diagram for the Cournotian Ramsey model

add some explanation. The basic reason for their occurrence is related to the fact that the stable manifold associated with equilibrium point K_B^* needs to be bounded at the left, because the capital stock has to have a minimal dimension from the feasibility conditions associated to the existence of an equilibrium. The only way for this to hold is if the left bound is a stationary equilibrium, K_A^*in our case. This stationary equilibrium is both a minimal equilibrium dimension for the economy, similar to a sunk cost, but is also a kind of a poverty trap. If a parameter of the economy changes so the the equilibrium moves left then the economy will move along the heteroclinic towards the higher equilibrium point. This can be produced by reductions of $\rho + \delta$ or by increases in η. Along the (global) transition consumption the ratio $\frac{C}{K}$ increases first and decreases afterwards and the mark-up decreases.

Notice equilibrium B corresponds to the small mark-up and equilibrium A to the large one, since we observe a decreasing relationship between the

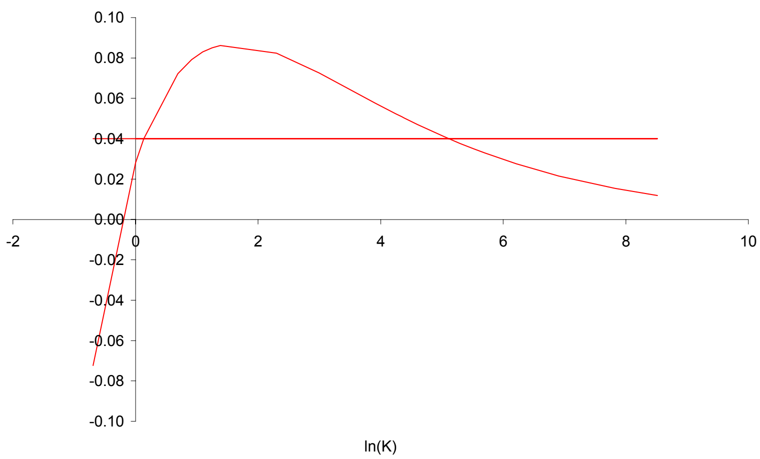

Figure 4: Real interest rate and the equilibrium in the exogenous labour model

mark-up and the capital stock in the long run given by

$$K^* = \left[\frac{\alpha. \left(1 - \mu^*\right). \left(A.\overline{L}\right)^{1-\alpha}}{\eta} \right]^{\frac{1}{1-\alpha}}$$

Figure 4 , which uses a graphical analysis as in Galí (1994b), depicts the equilibrium using the steady-state condition $\left(1 - \mu^*\right).F_k^*$.

For large values of the capital stock the mark-up is low and decreasing marginal returns force $\left(1 - \mu^*\right).F_k^*$ down. However, for small values of K^* the Inada condition is not enough to offset the effect of a very large mark-up. Therefore, we observe an increasing branch for $\left(1 - \mu^*\right).F_k^*$ at the left end of the spectrum.

3 The endogenous labour case

In this section we extend the Cournotian Ramsey model in order to allow for a labour supply with non-zero wage-elasticity. Here, this deterministic continuous-time framework makes an important step towards greater comparability with the Real Business Cycle (RBC) class of (Walrasian) models.

3.1 The leisure-consumption decision

Let us modify the household problem in order to include disutility of labour. Instead of equation (2), we will use the following additively separable isoelastic felicity function in this section:

$$u\left[C\left(t\right),L\left(t\right)\right]=\ln\left[C\left(t\right)\right]-\frac{\xi}{1+\tau}.\left[L\left(t\right)\right]^{1+\tau} \tag{35}$$

where $\xi > 0$ and $\tau \geq 0$. In this case, the elasticity of intertemporal substitution in labour supply is given by $1/\tau$. Thus, the labour supply does not correspond to (5) any longer, bur is given by the following equation instead:

$$L=\left(\frac{w}{\xi.C}\right)^{\frac{1}{\tau}} \tag{36}$$

3.2 Equilibrium in the labour market

Using the new labour supply in (5) and the labour demand in (21) we can express the equilibrium employment as

$$L=\left[\frac{\left(1-\mu\right).\left(1-\alpha\right).A^{1-\alpha}.K^{\alpha}}{\xi.C}\right]^{\frac{1}{\tau+\alpha}}$$

Thus, the general equilibrium given bu definition 1 is changed to include the previous static equation.

3.3 General equilibrium

Definition 8 *General equilibrium: it is a flow of consumption, capital stock, employment, and mark-up such that: (i) households and firms optimise; (ii) all markets clear.*

Using the equations previously derived, we can represent the general equilibrium using a system on $[C(t), K(t)]$ such that

$$\dot{C} = [(1-\mu).F_K(K, A.L) - \varrho].C, \quad t \in \mathbb{R}_+ \tag{37}$$

$$\dot{K} = (1-\mu).F(K, A.L) - C - G - \delta.K, \quad t \in \mathbb{R}_+ \tag{38}$$

$$L = \left[\frac{(1-\mu).(1-\alpha).A^{1-\alpha}.K^{\alpha}}{\xi.C}\right]^{\frac{1}{\tau+\alpha}}, \quad t \in \mathbb{R}_+ \tag{39}$$

$$\lim_{t\to\infty} e^{-\rho.t}.\frac{K}{C} = 0 \tag{40}$$

$$K(0) = K_0, \quad \text{given} \tag{41}$$

The steady state is thus defined by the values $(C^*, K^*, L^*, \mu^*) \in \mathbb{R}_+^2 \times (0, \overline{L}] \times (0,1)$, which may be determined by equations (37) to (41) when $\dot{C} = \dot{K} = 0$.

Using the same strategy to generate a representation of the general equilibrium, we obtain the following equilibrium function:

$$\begin{aligned} f_2(\mu) &\equiv q(\mu) - S_0.z(\mu) = 0 \\ q(\mu) &\equiv S_1.(1-\mu) - G.\mu^2 \\ z(\mu) &\equiv (1-\mu)^{1+\gamma}.\mu^{\beta.\gamma} \end{aligned} \tag{42}$$

where

$$\begin{aligned} \gamma &= 2.\frac{1+\tau}{\beta} > 0 \\ S_0 &= \left(\frac{\alpha}{\varrho}\right)^{\gamma}.\frac{(1-\alpha).A^{1+\tau}}{\xi.\eta^{\tau}} > 0 \\ S_1 &= \left(1 - \delta.\frac{\alpha}{\varrho}\right).\eta > 0 \end{aligned}$$

Notice $\delta.\alpha/\varrho < 1$ as we know that the steady-state capital-output ratio equals α/ϱ. Thus, if an equilibrium exists, $C^* + G > 0$, which implies $S_1 > 0$.

Proposition 9 *Equilibrium without government: if there is no government in this economy, i.e. if $G = 0$, the model exhibits the same features as the fixed-labour-supply model studied in the previous section.*

Proof. With $G = 0$ the quadratic term in $q(.)$ disappears. There fore, the equilibrium function becomes

$$f_2|_{G=0}(\mu) \equiv S_1.(1-\mu).\left\{1 - \left[Q_2.(1-\mu).\mu^{\vartheta}\right]^{\frac{1}{1+\gamma}}\right\} = 0$$

where $Q_2 = (S_0/S_1)^{1/(1+\gamma)} > 0$ and $\vartheta = \beta.\gamma/(1+\gamma) > 0$. Since $\mu = 1$ is not an equilibrium, the expression in square brackets defines the steady-state mark-up. We can easily notice this expression is very similar to $f_1(\cdot)$, and the conclusions obtained for the case with exogenous labour supply can easily be transferred to this special case, where ϑ has a similar role to β and Q_2 has a similar role to Q_1. ■

Notice, in this case, the value of the mark-up that maximises this function depends on both α and τ, and the (unique) existence condition stating a range for η depends on all the parameter values.

However, in general we have $G > 0$, so the function $f_2(.)$ has to be studied in order to evaluate the number of equilibria and the dynamics of the system.

Proposition 10 *Existence of equilibrium with government: if the government acts in this economy, i.e. if $G > 0$, the model exhibits at least one equilibrium, provided G is less than net output.*

Proof. First of all, the values of this function for the extreme values of the mark-up level are given by

$$\begin{aligned} f_2(0) &= S_1 > 0 \\ f_2(1) &= -G < 0 \end{aligned}$$

Considering the equilibrium function is continuous and its first derivative of the function can be written as

$$\begin{aligned} f_2'(\mu) &= -S_1 - 2.G.\mu - S_0.z'(\mu) \\ z'(\mu) &= z(\mu).\frac{\beta.\gamma.(1-\mu) - (1+\gamma).\mu}{(1-\mu).\mu} \end{aligned}$$

the monotonicity of the function in the vicinity of these extreme values is given by:

$$\begin{aligned} f_2'(0) &= -S_1 < 0 \\ f_2'(1) &= -S_1 - 2.G < 0 \end{aligned}$$

i.e. the equilibrium function is decreasing on the right-hand side of $\mu = 0$ and it is also decreasing on the left-hand side of $\mu = 1$.

Thus, by continuity, there is at least one fixed point for $f_2(.)$, considering $0 < G \leq (1 - \mu^*) F(K^*, A.L^*) - \delta.K^*$. ■

However, this result does not rule out the possibility of multiple equilibria to exist. To do that, we need to study the monotonicity of functions $q(.)$ and $z(.)$ when $\mu \in (0, 1)$.

Proposition 11 *Multiple equilibria with government: if the government acts in this economy, i.e. if $G > 0$, the model exhibits a maximum number of three equilibria, provided G is less than net output.*

Proof. Function $z(\mu)$ is very similar, in its structure, to $f_1(\mu)$. It is easy to notice that $z(0) = 0$ and $z(1) = 0$. Also, there is a unique stationarity point to this function given by

$$z'(\mu) = 0 \Leftrightarrow \mu = \bar{\bar{\mu}} \equiv \frac{\beta.\gamma}{1 + \gamma + \beta.\gamma} \in (0, 1)$$

where we can see the value of $\bar{\bar{\mu}}$ depends solely on the values of α and τ

$$\begin{aligned} \frac{\partial \bar{\bar{\mu}}}{\partial \alpha} &= -2.\frac{(1+\tau)^2}{(3 + 2.\tau - 2.\alpha - \alpha.\tau)^2} < 0 \\ \frac{\partial \bar{\bar{\mu}}}{\partial \tau} &= 2.\frac{(1-\alpha)^2}{(3 + 2.\tau - 2.\alpha - \alpha.\tau)^2} > 0 \end{aligned}$$

Since the value of $z\left(\bar{\bar{\mu}}\right)$ is positive, this point corresponds to a maximum.[4]

The second-degree polynomial $q(\mu)$ exhibits the following values for the extreme values of μ: $q(0) = S_1 > 0$ and $q(1) = -G < 0$. Furthermore, there

[4] Is is easy to show the second derivative of $z\left(\bar{\bar{\mu}}\right)$ is negative.

is a unique solution to $q(\mu) = 0$ given by $\mu = \mu_0 \equiv \left[\sqrt{(S_1)^2 + 4.S_1.G} - S_1\right] / (2.G) \in (0,1)$.

If $\overline{\overline{\mu}} \geq \mu_0$, then function $f_2(\cdot)$ is decreasing at least up to $\mu = \mu_0$, where its value is given by $f_2(\mu_0) = -S_0.z(\mu_0) < 0$. Thus, there is a solution for $f_2(\mu) = 0$ in the interval $\mu \in (0, \mu_0)$. In the range $\mu \in (\mu_0, 1)$ the function starts to be increasing, but its maximum value would be $f_2(1) = -G < 0$, thus there is no solution here.

If $\overline{\overline{\mu}} < \mu_0$, then function $f_2(\cdot)$ may exhibit more than one equilibrium. In order to obtain a better picture of this case, we need to study the first derivative of function $f_2(\cdot)$, and it is given by $f_2'(\mu) = -S_1 - 2.G.\mu - S_0.z'(\mu)$. The second derivative of $z(\cdot)$ is given by

$$\begin{aligned}
z''(\mu) &= (1-\mu)^{\gamma-1}.\mu^{\beta.\gamma-2}.\gamma.\left(a_z.\mu^2 - b_z.\mu + c_z\right) \\
a_z &= (1+\beta).[1+\gamma.(1+\beta)] > 0 \\
b_z &= 2.\beta.\gamma.(1+\beta) > 0 \\
c_z &= \beta.(\beta.\gamma - 1) > 0
\end{aligned}$$

We know that $z''(\mu) = 0$ for $\mu = 0$, and also for $\mu = \left[b_z \pm \sqrt{(b_z)^2 - 4.a_z.c_z}\right] / (2.a_z)$. Considering that $(b_z)^2 - 4.a_z.c_z = 4.\beta.(1+\beta+\gamma+\beta.\gamma) > 0$, that $b_z - 2.a_z = -2.(1+\beta+\gamma+\beta.\gamma) < 0$, and that $(b_z)^2 - 4.a_z.c_z < 2.a_z - b_z$, we can conclude that $\mu_A = \left[b_z - \sqrt{(b_z)^2 - 4.a_z.c_z}\right] / (2.a_z) \in (0,1)$ and additionally $\mu_B = \left[b_z + \sqrt{(b_z)^2 - 4.a_z.c_z}\right] / (2.a_z) \in (0,1)$. Therefore, there are two real solutions for $z''(\mu) = 0$ in the range $\mu \in (0,1)$. Furthermore, the other extreme value for the mark-up level gives rise to

$$z''(1) = \begin{cases} 0 \Longleftarrow \gamma > 1 \\ a_z - b_z + c_z > 0 \Longleftarrow \gamma = 1 \\ +\infty \Longleftarrow \gamma < 1 \end{cases}$$

Therefore, we can conclude the following for the function $z'(\mu)$:

$$\begin{aligned} z'(0) &= 0 \\ z'(\mu) &> 0 \Longleftarrow \mu \in (0, \overline{\overline{\mu}}) \\ z'\left(\overline{\overline{\mu}}\right) &= 0 \\ z'(\mu) &< 0 \Longleftarrow \mu \in (\overline{\overline{\mu}}, 1) \\ z'(1) &= 0 \end{aligned}$$

Notice that $\overline{\overline{\mu}} = b_z/(2.a_z)$. Thus, we can conclude this function has a maximum for $\mu = \mu_A$ and it exhibits a minimum for $\mu = \mu_B$.

Consequently, if $\overline{\overline{\mu}} < \mu_0$, we may find a maximum of two solutions for $q'(\mu) = S_0.z'(\mu)$ in the range $\mu \in (\overline{\overline{\mu}}, 1)$, where the decreasing $q'(\mu)$ function may intercept the U-shaped $S_0.z'(\mu)$ function none, once or twice. If there is no solution for $f_2'(\mu) = 0$, and this derivative is always negative, i.e. there is only one equilibrium. If there is a unique solution for $f_2'(\mu) = 0$, it is a minimum for $f_2(\cdot)$, and this derivative is always negative, i.e. there is also only one equilibrium. If there is a pair of solutions for $f_2'(\mu) = 0$, the first corresponds to a minimum and the second to a maximum for $f_2(\cdot)$, and this derivative is positive in the interval between solutions, i.e. there may be one, two or three equilibria. The steady-state equilibrium function $f_2(\mu)$ may exhibit the following graphical representation: see figure 5. ■

A simple economic intuition for the existence of these three equilibria can be put forward by using figure 6 which represents $(1-\mu^*).F_K^*$ as a function of the capital stock. For very small or very high values of the capital stock (very high or very low mark-up levels), the Inada conditions dominate the model and the marginal productivity of capital shows very high (close to infinity) or very low (close to zero) values. Therefore, one could always find an equilibrium where the function is equal to ϱ, as it happens in the fixed-mark-up model. However, for intermediate values of the capital stock (and mark-up), the efficiency externality may be strong enough to offset the decreasing marginal returns and two extra equilibrium may arise: one when the $(1-\mu^*).F_K^*$ function crosses ϱ from below and another one when the Inada conditions make it cross again from above.

However, government consumption plays a key role activating the decreasing branch on the left end of the spectrum, as this effect is not active when $G = 0$. Government expenditure is important in this case acting throuh taxes' income effect in labour supply and influencing the opti-

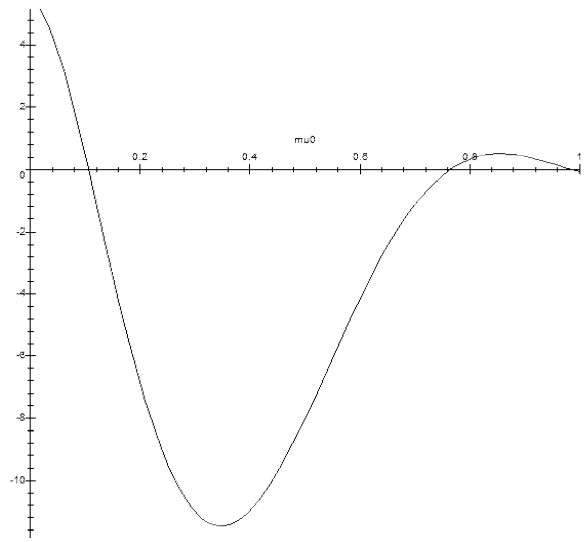

Figure 5: Equilibrium in the model with endogenous labour supply

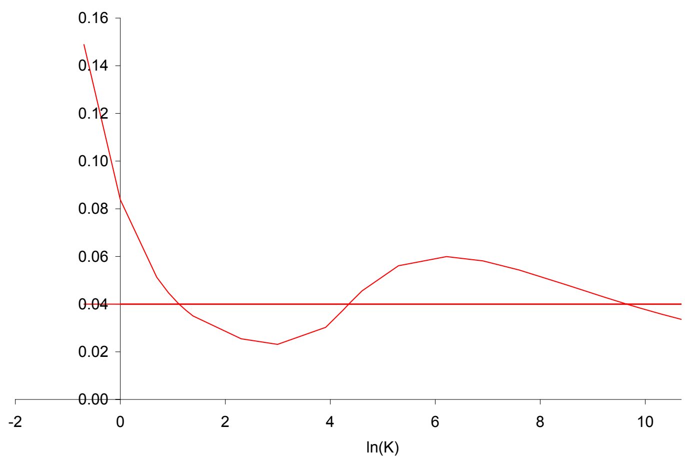

Figure 6: Real interest rate and the equilibrium in the endogenous labour model

mal labour-capital mix via mark-up: in the steady-state we have $K^*/L^* = (\varrho/\alpha)^{-1/(1-\alpha)}.A.(1-\mu^*)^{1/(1-\alpha)}$.

3.4 Local dynamics

Again, we log-linearise it about a steady-state equilibrium:

$$\begin{pmatrix} \dot{\widehat{C}} \\ \dot{\widehat{K}} \end{pmatrix} = \mathbf{J}_2. \begin{pmatrix} \widehat{C} \\ \widehat{K} \end{pmatrix} \tag{43}$$

where the new Jacobian matrix $\mathbf{J}_1$ is given by

$$\mathbf{J}_2 = \begin{pmatrix} -\frac{\varrho.(2-\mu^*).(1-\alpha)}{(1+2.\tau+\alpha).\left(\mu^{\#}-\mu^*\right)} & \frac{\varrho.\{[1+\tau.(2-\alpha)].\mu^*-2.\tau.(1-\alpha)\}}{(1+2.\tau+\alpha).\left(\mu^{\#}-\mu^*\right)} \\ -\frac{\varrho.\left\{2.s_C^*.(\alpha+\tau)+(1-\alpha).\left[2-\left(1+s_C^*\right).\mu^*\right]\right\}}{\alpha.(1+2.\tau+\alpha).\left(\mu^{\#}-\mu^*\right)} & \rho+\frac{\varrho.[2.(1-\alpha)+(\tau+\alpha).\mu^*]}{(1+2.\tau+\alpha).\left(\mu^{\#}-\mu^*\right)} \end{pmatrix}$$

where $\mu^{\#} \equiv 2.(\tau+\alpha)/[2.(\tau+\alpha)+1-\alpha] \in (0,1)$. We can observe the trace is given by

$$Tr(\mathbf{J}_2) = \rho + \frac{\varrho.\mu^*.(1+\tau)}{(1+2.\tau+\alpha).(\mu^{\#}-\mu^*)}$$

One can notice there is no solution for $Tr(\mathbf{J}_2) = 0$, as it is an increasing function of μ^*, but this function is not defined for $\mu^* = \mu^{\#}$. However, we know the trace is positive in the interval $\mu^* \in (0, \mu^{\#})$ and is is always negative in the interval $\mu^* \in (\mu^{\#}, 1)$. The value for the trace with $\mu^* = 0$ is equal to ρ and its value for $\mu^* = 1$ is $-[(\alpha+\tau).\rho + (1+\tau).\delta]/(1-\alpha) < 0$. Its graphical representation is given by

The determinant is given by

$$\begin{aligned} \det(\mathbf{J}_2) &= -c + \Delta \\ c &= \frac{\varrho.(1-\alpha)}{\alpha.(\tau+\alpha)}.[\rho + (1-\alpha).\delta + s_C^*.\tau.\varrho] > 0 \\ \Delta &= \frac{Tr(\mathbf{J}_2) - \rho}{\alpha.(\tau+\alpha)}.x \\ x &= -(1-\alpha).[\rho + (1-\alpha).\delta] + s_C^*.\alpha.\varrho.(1+\tau) \end{aligned}$$

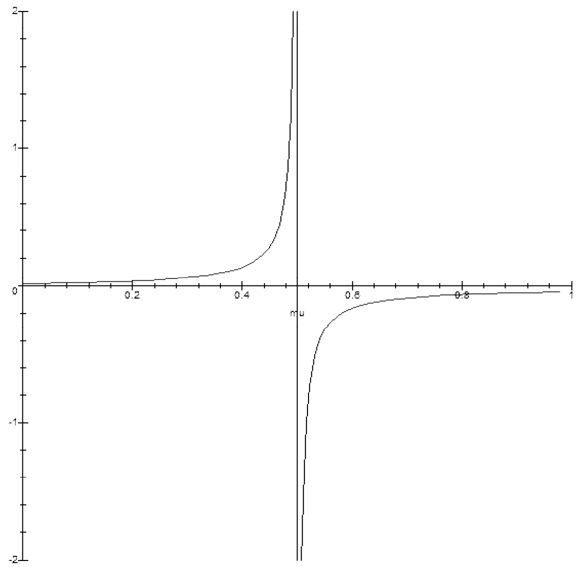

Figure 7: The mark-up and the trace of the Jacobian in the endogenous labour model

Therefore, the sign of $\det(\mathbf{J}_2)$ is given by

$$sign\left[\det\left(\mathbf{J}_2\right)\right]=\begin{cases} sign\left(x\right)\Leftarrow\mu^*\in\left(0,\mu^{\#}\right) \\ -sign\left(x\right)\Leftarrow\mu^*\in\left(\mu^{\#},1\right)\end{cases}$$

The value of x depends on μ^*, as s_C^* is a function of the steady-state mark-up. Thus, for $s_C^*>\varpi\equiv\left(1-\alpha\right).\left[\rho+\left(1-\alpha\right).\delta\right]/\left[\alpha.\varrho.\left(1+\tau\right)\right]$ we have $x>0$ and $x<0$ otherwise. One can notice that for small (large) values of α or τ, or for large (small) values of ϱ (or ρ and δ separately), ϖ approximates unity (zero). Thus, it would be more difficult to observe a positive (negative) value of x, despite the steady-state mark-up level that determines s_C^*. Since we know that

$$s_C^*\left(\mu^*\right)=S_0.\left(1-\mu^*\right)^{\gamma+1}.\left(\mu^*\right)^{\beta.\gamma}\in\left(0,1\right)$$

it is easy to notice that

$$s_C^{*'}\left(\mu^*\right)=s_C^*.\gamma.\left(1+\beta\right).\frac{\overline{\mu}-\mu^*}{\mu^*.\left(1-\mu^*\right)}$$

i.e. the function is increasing for $\mu^*\in\left(0,\overline{\mu}\right)$ and it is decreasing for

$\mu^* \in (\overline{\mu}, 1)$. Notice, however, the equilibrium conditions imply that, for a given parameter set, μ^* is such that $s_C^* \in (0,1)$).

So, what do we know about the determinant?

1) It is a function of the trace.

2) It is not defined for $\mu^* = \mu^{\#}$

3) $\lim\limits_{\mu^* \to 0} \det(\mathbf{J}_2) = -\frac{\varrho.(1-\alpha).[\rho+(1-\alpha).\delta]}{\alpha.(\tau+\alpha)} < 0$.

4) $\lim\limits_{\mu^* \to 1} \det(\mathbf{J}_2) = \frac{\varrho.[\rho+(1-\alpha).\delta]}{\alpha} > 0$.

Since we cannot obtain unambiguous signs to both the trace and the determinant, the discriminant follows the same way. Therefore, it is possible to observe complex eigenvalues for some parameter sets.

We then used numerical simulations to assess the possibility of obtaining something different in its nature from both the fixed-mark-up model and the exeogenous labour supply case. The following table presents three parameter sets that present distinct features:

Table I: Numerical values for the parameters

Set	α	δ	ξ	ρ	σ	τ	Φ	A	G
I	1/3	0.025	28.17	0.015	2	1	0.029	1	0.088
II	3/4	0.025	2.41	0.015	2	0.001	20.582	1	0.087
III	1/10	0.025	2.41	0.015	2	0.008	0.049	1	0.087

Their dynamic features are resumed in the following table:

Table II: Equilibrium mark-ups and eigenvalues

Set	μ^*	λ_1	λ_2
I	0.167	-0.049	0.070
	0.105	-0.011	0.029
II	0.764	0.016	0.187
	0.985	-0.153	-0.008
III	0.211	$-0.206 - 1.024.i$	$-0.206 + 1.024.i$

Set I generates a single saddle-point non-oscillatory stable equilibrium. Set II generates a trio of non-oscillatory equilibria: one is saddle-point stable, one is a 'source,' and one is a 'sink.' Finally, set III generates a single equilibrium with complex eigenvalues and local indeterminacy.

4 Conclusions

In this article we developed a dynamic general equilibrium model with Cournotian Monopolistic Competition where free entry induces an endogenous desired mark-up.

In the case where labour supply is innelastic (the Cournotian Ramsey model), multiple equilibria is a likely outcome. The equilibrium associated with the high mark-up is unstable and can exhibit complex roots. The low-mar-up equilibrium is Pareto preferred to the previous one and is saddle-point stable, as in a fixed mark-up model, including the competitive Ramsey model. The two equilibrium may be linked through a heteroclinic trajectory.

When labour supply induces disutility to the households, government consumption makes a big difference. In the zero-government-purchases case, the outcomes are qualitatively identical to the exogenous-labour model. In the positive-government-purchases model a third equilibrium may exist. Here, local indeterminacy is a possible outcome either for one of three equilibria or for a unique one, and this result does not depend on a overlapping-generations structure or in assuming the elasticity of substitution between varieties is smaller than unity. Complex global dynamic structures, including hysteresis may arise in this case.

This set of qualitative results imply that empirical applications of desired endogenous mark-up models have to be carefully studied before implemented.

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