The distribution of contract durations across firms: a unified framework for understanding and comparing dynamic wage and price setting models.*

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April 3, 2006

Abstract

This paper shows how any steady state distribution of ages and related hazard rates can be represented as a distribution across firms of completed contract lengths. The distribution is consistent with a Generalised Taylor Economy or a Generalised Calvo model with duration dependent reset probabilities. Equivalent distributions have different degrees of forward lookingness and imply different behaviour in response to monetary shocks. We also interpret data on the proportions of firms changing price in a period, and the resultant range of average contract lengths.

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JEL: E50.

Keywords: Contract length, steady state, hazard rate, Calvo, Taylor.

*This research began when I was a visitor to the ECB in January 2002. I would like to thank Vitor Gaspar, Luigi Siciliani, Peter N Smith, John Treble for helpful clues in my quest. I would like to thank Engin Kara for the simulations in section 6. I have only myself to blame for any remaining faults.

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1 Introduction

Dynamic pricing and wage-setting models have become central to macroeconomic modelling in the new neoclassical synthesis approach. It has become apparent that different models of pricing have different implications for matters such as the persistence of output, prices and inflation to monetary shocks. In this paper I show that there is a unified approach which can be used to understand and compare the distribution of durations across firms (DAF) implied by models of price and wage-setting, and also data on pricing. We start from the idea of modelling the class of all steady state distributions of durations across a given population (in this case, the firms or unions that set prices or wages). In steady state there are three equivalent ways of interpreting the distribution of durations: first there is the cross-sectional distribution of *ages*: how long has the price or wage contract lasted until now? This is like the population census. Second, we can look at the distribution in terms of *survival probabilities*: from the cross-section of ages, what is the probability of progressing from one age to the next one. Lastly, we can look at the cross-section of contracts in steady-state across firms (the DAF) and ask what is the distribution of completed contract lengths (lifetimes) of this cross-section. This corresponds to the average contract length across firms. The main innovation of the paper is to develop a transparent framework that allows us to move between these concepts. The first two concepts (distribution of ages and hazard rates) are of course very well understood in statistics, being basic tools in demography, evolutionary biology and elsewhere. The third concept, the distribution of completed durations across the population of firms is a more novel concept, but it is what we need if we are to answer questions such as what is the average length of contracts across firms and to apply these concepts to understand and compare different models of pricing.

Each of these three ways of looking at the class of all steady-state distributions has a natural application to modelling price and wage setting. In the Generalised Taylor Economy (GTE) introduced in Kara and Dixon (2005), we model the distribution of completed contract lengths (lifetimes). There are many sectors, each with sector specific contract lengths. Hence any steady state distribution of contract lifetimes can be modelled as uniform[1] GTE. The simple Taylor economy where all contract lengths are the same

[1] A Uniform GTE is one where in each sector, the cohorts are of equal size and one cohort moves each period. Thus, if the contract length is T, then cohort size is T^{-1}. A non-uniform GTE with different cohort sizes would be inconsistent with a steady state.

is a special case of the GTE. In the case of the Calvo approach, we have a reset probability which may be constant (as in the classical Calvo model) or duration dependent (Wolman 1999). We show that the Calvo model with duration dependent reset probabilities (denoted as the *Generalised Calvo model* GC) is coextensive with the set of all steady state distributions: each possible steady state age distribution has exactly one GC and one GTE which corresponds to it. Hence, using this framework, we are able to compare the different models of pricing *for a given distribution of durations across firms*. This enables us to isolate the precise effect of the pricing model as opposed to the difference in the distribution of contract lengths. As Dixon and Kara (2006a) showed, existing comparisons of simple Taylor and simple Calvo models of pricing have failed to even ensure that the mean contract lengths are the same, let alone the overall distribution of contract lengths across firms (see for example Kiley 2002). The reason for this confusion has been that researchers have taken the established statistical models from demography, evolutionary biology and the study of unemployment spells which in this context yield the distribution of lifetimes across the population of *contracts*[2] rather than firms. In order to understand pricing you need the distribution of contract lengths across firms.

It is widely recongised that there is a variety of pricing or wage-setting bahviour in most economies. This raise the question of aggregation: if we seek to represent the economy with a particular model, is the model itself consistent with this heterogeneity? This paper shows that both the GTE and GC are closed under aggregation: if we take two economies represented by a GTE, the resultant economy will also be a GTE. Likewise the GC. More importantly, we show that this is not the case for either the simple Taylor or Calvo models. If there is heterogeneity in the economy, then it cannot consistently be represented as a simple Taylor or Calvo process (except possibly as a dubious approximation). However, another generalisation of the Calvo idea, the Multiple Calvo economy MC is closed under aggregation. In the MC economy, there are many sectors, each with a sector specific reset probability.

Given that we have a particular distribution of durations, what difference does the pricing model make? Following the analysis of Dixon and Kara (2005), the concept of Forward Lookingness (FL) is employed: how far on average do agents look forward (what is the weighted mean number of periods

[2] We could use term "price spell" for a contract.

price setters look forward when the set their price?). We find that in the GTE model, firms on average are more myopic than in the GC model for a give distribution of durations. This leads to observable differences in impulse response functions in response to monetary shocks.

We also apply this approach to the Bils-Klenow data set (Bils and Klenow 2004). From the sectoral data for the proportion of changes in prices per month we are able to construct the average length of contracts under the hypothesis that there is a calvo process in each sector, and also find that the shortest possible mean duration is achieved by the assumption that there is the simplest GTE consistent with the observed proportion, which consists of 1 or 2 contract durations that yield the observed proportion of prices changing. The longest possible mean is proportional to the longest possible contract length. This paper provides a simple and transparent discrete time framework for understanding nominal price rigidity in dynamic macromodels, and also indicates how empirical evidence from price data can be applied in a consistent and relevant manner.

In section 2 we review the well known facts about the steady state distribution of ages and hazard rates. We then introduce the new concept of the distribution of durations across firms and show how all three concepts are related by simple formulae which are spreadsheet friendly. In section 3, we link the concepts to different models of pricing. In section 4 we analyse the different pricing models in terms of forward lookingness and compare the mean reset prices. In section 5, we implement these ideas using the Bils-Klenow data set.

2 Steady State Distributions of Durations across Firms.

We will consider the steady-state demographics of contracts in terms of their durations. The *lifetime* of a contract is how long it lasts from its start to its finish, a *completed* duration. The *age* of a contract at time t is how long it has been in force since it started. The age is a duration which may or may not be completed. We will first review the well known representation of steady state durations by the related concepts of the *age distribution* and *hazard rates* (see for example, Kiefer 1988).

There is a continuum agents (we will call them firms here) f which set wages or prices represented by the unit interval $f \in [0,1]$. In steady state we can take a cross-section at time t and measure the age distribution[3] of ages: α_j^s is the proportion of firms which have contracts age j, $\{\alpha_j^s\}_{j=1}^F$ where F is the oldest age in steady state[4]. In steady state, the distribution of ages is monotonic: you cannot have more older people than younger, since to become old you must first be young. Hence the set of all possible steady state age distributions is given by:

$$\Delta_M^{F-1} = \left\{ \boldsymbol{\alpha}^s \in \Delta^{F-1} : \alpha_j^s \geq 0, \alpha_j^s \geq \alpha_{j+1}^s \right\}$$

An alternative way of looking at the steady state distribution of durations is in terms of the hazard rate. The hazard rate at a particular age is the proportion of contracts at age i which do not last any longer (contracts which end at age i, people who die at age i). Hence the hazard rate is defined in terms of the age distribution: given the distribution of ages in steady-state $\boldsymbol{\alpha}^s \in \Delta_M^{F-1}$, the corresponding vector of hazard rates[5] $\boldsymbol{\omega} \in [0,1]^{F-1}$ is given by:

$$\omega_i = \frac{\alpha_i^s - \alpha_{i+1}^s}{\alpha_i^s}; i = 1 ... (F-1) \tag{1}$$

Whilst it is easy to allow for an infinite series of reset probabilities less than one, we will mainly deal with the finite case where there is a final reset probability of one after F periods, although in later sections we will look at cases with infinite F.

Corresponding to the idea of a hazard rate is that of the *survival probability,* the probability at birth that the price survives for at least i periods, with $\Omega_1 = 1$ and for $i > 1$

$$\Omega_i = \Pi_{\kappa=1}^{i-1}(1 - \omega_\kappa)$$

and we define the sum of survival probabilities Σ_Ω and its reciprocal $\bar{\omega}$:

$$\Sigma_\Omega = \sum_{i=1}^F \Omega_i \quad \bar{\omega} = \Sigma_\Omega^{-1}$$

[3]In Demography, this is given the acronym SAD.

[4]In some theoretical applications such as the Calvo model of pricing, there may be infinite lifetimes. The analysis presented is consistent with that, although for all practical applications a finite maximum is required.

[5]Since the maximum length is F, without loss of genersality we set $\omega_F = 1$. If $\omega_i = 1$ for some $i < F$, then i is the maximum duration and subsequent hazard rates become irrelevant. This leads to trivial non-uniqueness. We therefore define F as the *shortest* duration with a reset probability of 1.

Clearly, we can invert (1): we have $F-1$ equations. Hence:

Observation 1 given $\boldsymbol{\omega} \in [0,1]^{F-1}$, there exists a unique corresponding age profile $\boldsymbol{\alpha}^s \in \Delta_M^{F-1}$ given by:

$$\alpha_i^s = \bar{\omega}\Omega_i \quad i = 1...F.$$

Given the flow of new contracts $\bar{\omega}$, the proportion surviving to age i is $\Omega_i : \bar{\omega} = \Sigma_\Omega^{-1}$ ensures adding up. From the definition of hazard rates and Observation 1 we can move from an age distribution $\boldsymbol{\alpha}^s \in \Delta_M^{F-1}$ to the hazard profile and vice versa.[6]

2.1 The Distribution of Completed Durations across Firms.

Given a steady-state age distribution $\boldsymbol{\alpha}^s \in \Delta_M^{F-1}$, we can ask what is the corresponding distribution of *completed* durations or lifetimes across firms $\boldsymbol{\alpha} \in \Delta^{F-1}$. Note, we are asking for the distribution *across firms (DAF).* There is a unit interval of firms: each firm sets one price. When we measure the population shares α_i, we are measuring across firms, just as we do when we take the age distribution. We are seeking to answer the question "what is the distribution of and average length of a completed across the population of firms". Recall, the population of firms does not vary over time, and that whilst some firms change price frequently and some infrequently, each individual firm over time has an average contract length, and the average in the economy is the average over the stock of firms. It is this that corresponds to the concept of price-stickiness.

It is important to note that this is a very different question from the one we ask when we treat each individual contract as an entity and look at the distribution of contract lengths over time. From this perspective, we do not identify which firm sets the price: the total population is the total number of contracts in existence over time. To illustrate the differences, consider a world with two firms that lasts for two periods. One firm sets its price in both periods (single period contracts). The other sets the price for two periods. Now if we take the firm based view, we would say that 50% of firms set

[6] This relationship is one of the building blocks of Life Tables (Chiang 1984), which are put to a variety of uses by demographers, actuaries and biologists.

1−period contracts, and 50% set two period contracts: the average contract is 1.5 periods. That is the approach taken in this paper. However, if we take the contract-based approach, we say that in the two periods there were 3 contracts: two were 1−period contracts, and 1 was 2−periods, so that the average contract length is $1\frac{1}{3}$. Both statements are correct, but both answer different questions: the average is taken over a different population. This issue does not arise when we look at the age distribution in steady state. In this case we are taking a cross-section: since each firm sets only one price, there average across contracts and firms is exactly the same.

It is easy to see that taking the average contract length over firms will give a longer mean contract length than when you take the average over contracts. This is known as the issue of *length-biased sampling*. This is a familiar problem in unemployment measurement: if you want to find the average duration of unemployment across entrants, then looking at the average completed duration of the stock of unemployed will overestimate it[7]. For understanding pricing, however, we have exactly the opposite bias. Since we want to measure the average contract length across firms, if you take the average across contracts you will get the problem of oversampling of short contracts: there are simply more of them. Existing studies have focussed on the issue of average duration across contracts (see for example, Bils and Klenow 2004, Barhad and Eden 2003) rather than the mean duration of prices set by firms (the average across firms). *They have thus tended to significantly underestimate the degree of price stickiness.*

We can seek to show how we can move from the distribution of ages to the distribution of completed contract lengths across firms:.

Proposition 1 *Consider a steady-state age distribution $\boldsymbol{\alpha}^s \in \Delta_M^{F-1}$. There exists a unique distribution of lifetimes across firms $\boldsymbol{\alpha} \in \Delta^{F-1}$ which corresponds to $\boldsymbol{\alpha}^s$, where*

$$\begin{aligned} \alpha_1 &= \alpha_1^s - \alpha_2^s \\ \alpha_i &= i\left(\alpha_i^s - \alpha_{i+1}^s\right) \\ &.. \\ \alpha_F &= F\alpha_F^s \end{aligned} \tag{2}$$

[7] "picking an individual from the unemployed stock and observing his completed duration is non-randomly sampling the duration of entrants...We have in fact what is often called length-biased sampling of complete durations in which the probability that a spell will be sampled is proportional to its length" Lancaster (1992), p.95.

All proofs are in the appendix. Since there is a 1-1 mapping from age to lifetimes, we can compute the distribution of lifetimes from ages:

Corollary 1 Given a distribution of steady-state completed lifetimes across firms, $\boldsymbol{\alpha} \in \Delta^{F-1}$, there exists a unique $\boldsymbol{\alpha}^s \in \Delta_M^{F-1}$ corresponding to $\boldsymbol{\alpha}$

$$\alpha_j^s = \sum_{i=j}^{F} \frac{\alpha_i}{i} \qquad j = 1...F \tag{3}$$

The intuition behind Proposition 1 and the Corollary is clear. In a steady state, each period must look the same in terms of the distribution of ages This implies that if we look at the i period contracts, a proportion of i^{-1} must be renewed each period. Thus if we have 10 period contracts, 10% of these must come up for renewal each period. Otherwise, the proportion being renewed would not be constant across periods. This implies that the proportion of contracts coming up for renewal each period (which have age 1) is:

$$\alpha_1^s = \sum_{i=1}^{\infty} \frac{\alpha_i}{i}$$

The proportion of contracts aged 2 is the set of contracts that were reset last period (α_1^s), less the ones that only last one periods (α_1) and so on. The set of all possible steady state distributions of durations can be characterized either by the set of all possible age distributions: $\boldsymbol{\alpha}^s \in \Delta_M^{F-1}$ or the set of all possible lifetime distributions across firms $\boldsymbol{\alpha} \in \Delta^{F-1}$. They are just two different ways of looking at the same thing.

Proposition 1 and its corollary show that there is an exhaustive and 1-1 relationship between steady state age distributions and lifetime distributions. We can go from any age distribution and find the corresponding age distribution and vice versa. Now, since we know that there is also a 1-1 relation between Hazard rates and age distributions, we can also see that there will be a 1-1 relationship between completed contract lifetimes and hazard rates. First, we can ask what distribution of completed contract durations corresponds to a given vector of hazard rates. We can simply take observation 1 to transform the hazards into the age distribution, and then apply Proposition 1.

Corollary 2 let $\boldsymbol{\omega} \in [0,1]^{F-1}$. The distribution of lifetimes across firms corresponding to $\boldsymbol{\omega}$ is:

$$\alpha_i = \bar{\omega}.i.\omega_i.\Omega_i\text{: } i = 1...F \tag{4}$$

The flow of new contracts is $\alpha_1^s = \bar{\omega}$ each period. To survive for exactly i periods, you have to survive to period i which happens with probability Ω_i, and then start a new contract which happens with probability ω_i. Hence from a single cohort $\bar{\omega}.\omega_i.\Omega_i$ will have contracts that last for exactly i periods. We then sum over the i cohorts (to include all of the contracts which are in the various stages moving towards the their final period i) to get the expression. The mean completed contract length $\bar{T}$ generated by $\boldsymbol{\omega}$ is simple to compute directly:

$$\bar{T} = \sum_{i=1}^{F} i.\alpha_i = \bar{\omega} \sum_{i=1}^{F} i^2.\omega_i.\Omega_i$$

We can also consider the reverse question: for a given distribution of completed contract lengths $\boldsymbol{\alpha}$,what is the corresponding profile of hazard rates? From Corollary 2, note that (4) is a recursive structure relating α_i and ω_i: α_i only depends on the values of ω_s for $s \leq i$.

Corollary 3 [8]Consider a distribution of contract lengths across firms given by $\boldsymbol{\alpha} \in \Delta^{F-1}$. The corresponding hazard profile that will generate this distribution in steady state is given by $\boldsymbol{\omega} \in [0,1]^{F-1}$ where:

$$\omega_i = \frac{\alpha_i}{i} \left(\sum_{j=i}^{F} \frac{a_j}{j} \right)^{-1}$$

For completeness, we can also ask for a distribution of contracts across firms $\boldsymbol{\alpha} \in \Delta^{F-1}$ what is the distribution of durations taken across the total population of *contracts* $\boldsymbol{\alpha}^d \in \Delta^{F-1}$:

$$\alpha_i^d = \frac{\alpha_i}{i.\bar{\omega}} \tag{5}$$

Clearly, the distribution of durations across contracts is the same as the distribution across firms resetting prices. The more frequent price setters

[8]Although this is called a corollary, it has equal status to Proposition 1. We could have stated this result first as a Proposition and the former result would then become a corollary. The two proofs are independent and self-contained.

(shorter contracts) have a higher representation relative to longer contracts. Note that the *rhs* denominator is the product of the contract length and the proportion of firms resetting price. For the values of $i < \bar{\omega}^{-1}$, the share of the duration i is greater across contracts than firms: for larger $i > \bar{\omega}^{-1}$ the share across contracts is less than the share across firms. Using equation (5), we can move simply between the distributions across contracts and across firms. Note, that if we take the mean length across all contracts $\bar{d}$, from (5) we get

$$\bar{d} = \sum_{i=1}^{F} i.\alpha_i^d = \bar{\omega}^{-1}$$

That is, the mean contract length taken over all contracts in steady state is the reciprocal of the proportion of firms resetting price. This is precisely the estimate of mean contract length that has been commonly used empirically and also for calibration purposes in models of price-stickiness. However, it should be clear that this is the mean of the wrong distribution: it is the mean across all contracts, not firms. *There is clear length biased sampling: the shorter contracts are oversampled.*

There are now several studies using micro data: in particular the *Inflation Persistence Network* (IPN) across the Eurozone has been particularly comprehensive[9]. These studies have data on individual product prices at specific locations. In Veronese et al (2005), the authors attempt to measure the average duration of prices for individual products, which is much closer to our definition of the duration across firms. They find that the average duration across products is much higher than the average inferred from the reciprocal of the proportion of resetters which is what we would expect.

In the study of unemployment, each spell of unemployment is treated as an observation and the identity of the person involved is irrelevant. Hence the focus in the unemployment literature on the duration of unemployment has been on the flow of new spells of unemployment and how long they will last, rather than on the stock of unemployed[10]. In demography and evolutionary biology, each duration corresponds to a single individual, hence since people only live once, the distribution across people is exactly the same

[9] See Dhyne et al (2005) for a summary of the IPN's findings.

[10] However, Akerlof and Main (1981) did suggest using the average duration across all the unemployed as an important indicator (see the ensuing debate Carlson and Horrigan 1983, Akerlof and Main 1983).

as the distribution across individual durations[11]: hence the focus here is also on cohort studies, exploring the distribution of ages, hazard rates and lifetimes across people born at the same time. The key difference in this paper arises because we are interested in the pricing behaviour of *firms*: hence whilst the flow of new contracts is of interest, the average duration of contracts across the stock of firms is what determines price stickiness. What this section has provided is a transparent framework that enables us to move between the different concepts.

2.2 Examples.

In this section we provide five examples. In the first column we state the rest probabilities (hazard rates) $\{\omega_i\}$ in the second, in the second and third the corresponding distribution of ages$\{\alpha_i^s\}$ and lifetimes $\{\alpha_i\}$ over firms, and in the fourth the distribution $\{\alpha_i^d\}$ over contracts. In the bottom row we compute the proportion of new contracts $\bar{\omega}$, the average age of contracts $\bar{s}$ and the average lifetime $\bar{T}$ across firms in steady state, and $\bar{d}$ the average contract length across contracts.

Example 1

$$\begin{array}{llll}
\omega_1 = \frac{9}{10} & \alpha_1^s = \frac{37}{40} & \alpha_1 = \frac{9}{10} & \alpha_1^d = \frac{36}{37} \\
\omega_2 = 0 & \alpha_2^s = \frac{1}{40} & \alpha_2 = 0 & \alpha_2^d = 0 \\
\omega_3 = 0 & \alpha_3^s = \frac{1}{40} & \alpha_3 = 0 & \alpha_3^d = 0 \\
\omega_4 = 1 & \alpha_4^s = \frac{1}{40} & \alpha_4 = \frac{1}{10} & \alpha_4^d = \frac{1}{37} \\
\bar{\omega} = \frac{37}{40} & \bar{s} = \frac{23}{20} & \bar{T} = \frac{13}{10} & \bar{d} = \frac{40}{37}
\end{array}$$

In this example, there are two lengths of contracts: 90% are 1 period and 10% 4 periods. Note that $\bar{d} < \bar{s} < \bar{T}$: because of the proliferation of short contracts, the mean *lifetime* across contracts is even less than the average *age* across firms (in all the other examples, $\bar{d} > \bar{s}$).

Example 2

$$\begin{array}{llll}
\omega_1 = \frac{1}{4} & \alpha_1^s = \frac{8}{17} & \alpha_1 = \frac{2}{17} & \alpha_1^d = \frac{1}{4} \\
\omega_2 = \frac{1}{2} & \alpha_2^s = \frac{6}{17} & \alpha_2 = \frac{6}{17} & \alpha_2^d = \frac{3}{8} \\
\omega_3 = 1 & \alpha_3^s = \frac{3}{17} & \alpha_3 = \frac{9}{17} & \alpha_3^d = \frac{3}{8} \\
\bar{\omega} = \frac{8}{17} & \bar{s} = \frac{29}{17} & \bar{T} = \frac{41}{17} & \bar{d} = \frac{17}{8}
\end{array}$$

[11] Likewise unless individuals are unemployed more than once, the number of spells of unemployment equals the number of people who experience unemployment.

This example has a gently rising reset probability, with the shares of completed contracts across firms increasing with length of contract, as do the shares across contracts.

Example 3

$$
\begin{array}{llll}
\omega_1 = \frac{1}{4} & \alpha_1^s = \frac{32}{71} & \alpha_1 = \frac{8}{71} & \alpha_1^d = \frac{1}{4} \\
\omega_2 = \frac{1}{2} & \alpha_2^s = \frac{24}{71} & \alpha_2 = \frac{24}{71} & \alpha_2^d = \frac{3}{8} \\
\omega_3 = \frac{3}{4} & \alpha_3^s = \frac{12}{71} & \alpha_3 = \frac{27}{71} & \alpha_3^d = \frac{27}{96} \\
\omega_4 = 1 & \alpha_4^s = \frac{3}{71} & \alpha_4 = \frac{12}{71} & \alpha_4^d = \frac{3}{32} \\
\bar{\omega} = \frac{32}{71} & \bar{s} = \frac{128}{71} & \bar{T} = \frac{185}{71} & \bar{d} = \frac{71}{32}
\end{array}
$$

This is similar to example 2, with a rising hazard over four periods. The shares across firms and contracts both peak at period 3 with a small 4-period share.

Example 4: Simple Taylor 4.

$$
\begin{array}{ccc}
\omega_1 = 0 & \alpha_1^s = \frac{1}{4} & \alpha_1 = \alpha_1^d = 0 \\
\omega_2 = 0 & \alpha_2^s = \frac{1}{4} & \alpha_2 = \alpha_2^d = 0 \\
\omega_3 = 0 & \alpha_3^s = \frac{1}{4} & \alpha_3 = \alpha_3^d = 0 \\
\omega_4 = 1 & \alpha_4^s = \frac{1}{4} & \alpha_4 = \alpha_4^d = 1 \\
\bar{\omega} = \frac{1}{4} & \bar{s} = \frac{5}{2} & \bar{T} = \bar{d} = 4
\end{array}
$$

A simple lesson can be derived from example 4. When completed contracts have the same length, the distribution across contracts equals the distribution across firms and hence has the same mean.

Example 5: Taylor's US economy We can now consider an example starting from an empirical distribution of completed contract lengths we can derive the corresponding GC. Taylor's US economy represents the distribution of completed contract lengths (in quarters) in the third column. We can represent this in terms of the distribution of ages and duration dependent hazard rates (both to 4 Decimal places), distribution over contracts and the resultant averages.

$\omega_1 = 0.2017$	$\alpha_1^s = 0.3470$	$\alpha_1 = 0.07$	$\alpha_1^d = 0.2017$
$\omega_2 = 0.3430$	$\alpha_2^s = 0.2770$	$\alpha_2 = 0.19$	$\alpha_2^d = 0.2738$
$\omega_3 = 0.4213$	$\alpha_3^s = 0.1820$	$\alpha_3 = 0.23$	$\alpha_3^d = 0.1825$
$\omega_4 = 0.4986$	$\alpha_4^s = 0.1052$	$\alpha_4 = 0.21$	$\alpha_4^d = 0.1513$
$\omega_5 = 0.5682$	$\alpha_5^s = 0.0528$	$\alpha_5 = 0.15$	$\alpha_5^d = 0.0865$
$\omega_6 = 0.5849$	$\alpha_6^s = 0.0228$	$\alpha_6 = 0.08$	$\alpha_6^d = 0.0384$
$\omega_7 = 0.6038$	$\alpha_7^s = 0.0095$	$\alpha_7 = 0.04$	$\alpha_7^d = 0.0165$
$\omega_8 = 1$	$\alpha_8^s = 0.0037$	$\alpha_8 = 0.03$	$\alpha_8^d = 0.0108$
$\bar{\omega} = 0.3470$	$\bar{s} = 2.365$	$\bar{T} = 3.730$	$\bar{d} = 2.8818$

It is interesting to note that here, unlike examples 1-4, we can really see the difference between the distribution across contracts and across firms: for the durations 1 and 2 are really boosted - we see a lot of shorter contracts. All the other durations are reduced, and in particular the longer contract lengths are much less common in the distribution across contracts and across firms. The resultant mean duration is 77% of the mean across firms.

One thing to note is that the average age in these examples is always much smaller than the average lifetime across firms.. In example 3, the average completed duration is almost 50% larger than the average age. If we want to compare a particular GC with a particular GTE, we should at least equate the average contract length, if not the exact distribution. Taking the average age and comparing it to a simple Taylor model with the same completed contract length would be as mistaken here as it is in the simple Calvo model.

A second point is to see how we might interpret data based on the proportion of prices that change in each period, $\bar{\omega}$. If we used Taylor's US economy and data and assumed a simple Calvo model with a constant hazard, we would get an estimate of $\hat{\omega} = 0.347$, so that $\hat{T} = 4.8$,when in fact $\bar{T} = 3.6$. This shows how dangerous it is to use the simple data on the proportion of firms changing price in a single period to infer the average lifetime of contracts. Likewise, in examples 1-4, the simple procedure of taking $\bar{\omega}$ to be the simple Calvo reset probability results in long average lifetimes - in examples 2 and 4, the resultant estimate of average lifetime is longer than the longest completed duration!

3 Pricing Models with steady state distributions of durations across firms.

Having derived a unified framework for understanding the set of all possible steady state distributions of durations across firms, we can now see how this can be used to understand commonly used models of pricing behaviour.

3.1 The Generalised Taylor Economy GTE

Using the concept of the Generalised Taylor economy GTE developed in Dixon and Kara (2005a), any steady-state distribution of completed durations across firms $\boldsymbol{\alpha} \in \Delta^{F-1}$can be represented by the GTE with the sector shares given by $\boldsymbol{\alpha} \in \Delta^{F-1}: \ GTE(\boldsymbol{\alpha})$. In each sector i there is an $i-$period Taylor contract, with i cohorts of equal size (since we are considering only uniform $GTEs$). The sector share is given by α_i. Since the cohorts are of equal size and there as many cohorts as periods, there are $\alpha_i.i^{-1}$ contracts renewed each period in sector i. This is exactly as required in a steady-state. Hence the set of all possible $GTEs$ is equivalent to the set of all possible steady-state distributions of durations.

It is simple to verify that the age-distribution in a GTE is given by (3). If we want to know how many contracts are at aged j periods, we look at sectors with lifetimes at least as large as j, $i = j...F$. In each sector i, there is is a cohort of size $\alpha_i.i^{-1}$ which set its price j periods ago. We simply sum over all sectors $i \geq j$ to get (3).

3.2 The Generalised Calvo model (GC): duration dependant reset probabilities.

The Calvo model most naturally relates to the hazard rate approach to viewing the steady state distribution of durations. The simple Calvo model has a constant reset probability ω (the hazard rate) in any period that the firm will be able to review and if so desired reset its price. This reset probability is exogenous and does not depend on how long the current price has been in place. We can think about a sequence of uninterrupted periods without any review as the "contract length". If all firms have the same reset probability there will emerge a steady state distribution of "ages" of contracts. As is well known, the distribution of ages of contracts is the following

$$\alpha_s = \omega (1-\omega)^{s-1} : \; s = 1...\infty$$

which has mean

$$\bar{s} = \sum_{s=1}^{\infty} \alpha_s . s = \omega^{-1}$$

Applying Proposition 1 gives us the result:

Dixon and Kara 2006a With a constant hazard rate ω, the steady-state distribution of completed contract lengths $i$across firms is given by:

$$\alpha_i = \omega^2 i (1-\omega)^{i-1} \; \; i = 1...\infty \tag{6}$$

which has mean $\bar{T} = \frac{2-\omega}{\omega}$

Observation Note that for the simple Calvo model, the distribution of ages is the same as the distribution across contracts: substituting (6) into (5) yields $\alpha_i^s = \alpha_i^d \; i = 1...\infty$, so that the mean age of contracts across firms equals the mean lifetime across contracts and is the reciprocal of the reset probability.

As was discussed in Dixon and Kara (2006a), there has been a confusion in the existing literature between the average age of contracts in the Calvo model and the average lifetime. The two are indeed equal when we take the mean over the population of contracts, as noted in the observation[12]. However, there is a difference between taking the average over firms and the average over contracts. When we want to understand pricing behaviour, the key concept to understand this is the average contract length over the population of firms. Researchers have instead taken the existing results in the literature that take the average completed duration over contracts. For example, when comparing a simple Taylor model with 4 periods (i.e. the average lifetime is 4 periods), it has been thought that the corresponding reset probability is $\omega = 0.25$. However, this is the mean across contracts

[12]This is a very special property. In general, the mean contract length across all contracts can be less than mean age (exapmple 1) or greater (examples 2-5).

which is not the relevant population. We should instead take the average lifetime across firms corresponding to $\omega = 0.25$,which is not 4 periods, but 7 periods. It is not surprising that authors have found the Calvo model to be more persistent when its average contract length is nearly twice the comparitor Taylor economy!

We now consider generalising the Calvo model to allow for the reset probability (hazard) to vary with the age of the contract (duration dependent hazard rate). This we will denote the *Generalised Calvo Model* GC. A GC is defined by a sequence of reset probabilities: as in the previous section this can be represented by any $\omega \in [0,1]^{F-1}$ where F is the shortest contract length with $\omega_F = 1$. From observation 1, given any possible GC there is a unique age profile $\alpha^s \in \Delta_M^{F-1}$ corresponding to it and a unique distribution of completed contract lengths from Proposition 1. Again, from observation 3, if we have a distribution of completed contract lengths, there is a unique GC which corresponds to it. Thus, the two approaches to modelling pricing: the GTE and the GC are comprehensive and coextensive, both being consistent with any steady-state distribution of durations.

Note that an alternative parameterization of the duration dependent hazard rate model is to specify not the hazard rate at each duration, but rather the probability of the completed contract length at birth (see for example Guerrieri 2004). The probability at birth f_i of a contract lasting exactly i periods is simply the probability it survives to period i and then resets at $i : f_i = \Omega_i \omega_i$.

3.3 The Multiple Calvo Model (MC).

We now use the framework to address the issue of *aggregation* over Calvo processes. Alvarez et al (2005) argue that an aggregate hazard rate declines over time and that this can be attributed to the heterogeneity of hazard rates. We can define a multiple Calvo process MC as $MC(\boldsymbol{\omega}, \boldsymbol{\beta})$ where $\boldsymbol{\omega} \in (0,1)^n$ gives a sector specific hazard rate[13] $\bar{\omega}_k$ for each sector $k = 1, ...n$ and $\boldsymbol{\beta} \in \boldsymbol{\Delta}^{n-1}$ is the vector of shares β_k.

Firstly, we can model this as a GTE immediately by choosing the Calvo-GTE weights for each individual Calvo process and then summing them using the MC sector weights. Hence α_{ki} is the proportion of i period contracts in

[13] The notation here should not be confused: the substrcripts k are sectoral: none of the sectoral calvo reset probabiltities are duration dependent.

the k^{th} Calvo sector, α_i the proportion of i period contracts across the whole economy.

$$\begin{aligned}\alpha_i &= \bar{\omega}_k i \left(1-\bar{\omega}_k\right)^{i-1} \\ \alpha_i &= \sum_{k=1}^{n} \beta_k \alpha_{kT}\end{aligned}$$

From this distribution of completed durations, we can construct the corresponding GC using corollary 3. The flow of new contracts is $\bar{\omega}$ and the aggregate period i hazard is ω_i which we can either define in terms of the distribution of lifetimes or the underlying sectoral reset probabilities:

$$\begin{aligned}\bar{\omega} &= \sum_{i=1}^{\infty} \frac{\alpha_i}{i} = \sum_{i=1}^{\infty} \frac{\sum_{k=1}^{n} \beta_k \alpha_{ki}}{i} = \sum_{i=1}^{\infty} \sum_{k=1}^{n} \beta_k \bar{\omega}_k \left(1-\bar{\omega}_k\right)^{i-1} \\ \omega_i &= \frac{\alpha_i}{i} \left(\sum_{j=0}^{\infty} \frac{a_{i+j}}{i+j} \right)^{-1} = \frac{\sum_{k=1}^{n} \beta_k \bar{\omega}_k \left(1-\bar{\omega}_k\right)^{i-1}}{\sum_{j=0}^{\infty} \sum_{k=1}^{n} \beta_k \bar{\omega}_k \left(1-\bar{\omega}_k\right)^{i+j-1}}\end{aligned}$$

Proposition 2: The aggregate GC model corresponding to MC model has a declining hazard rate. In the limit as $i \to \infty$, the hazard rate in the GC tends to the lowest hazard rate in the MC.

Clearly, the aggregate hazard in the GC corresponding to an MC is decreasing over time: $\omega_i > \omega_{i+1}$. The way to understand this is that sectors with higher $\bar{\omega}_k$ tend to change contract sooner. So, for a given cohort, the relative share of sectors with higher $\bar{\omega}_k$ tends to go down. At any duration i, the share of type k contracts increases if the reset probability is below average or decreases if it is above average. The reset probability gradually declines and asymptotically reaches the lowest reset probability. In the long-run, the type with the lowest reset probability comes to dominate and the GC tends to this lowest vale. Clearly, the MC is a subset of possible GCs. It is also an example of a GC with $F = \infty$.

4 The Typology of Contracts and Aggregation.

In terms of contract structure, we can say that the following relationships hold:

- $GC = GTE = SS$. The set of all possible steady state distributions of durations is equivalent to the set of all possible GTEs and the set of all possible GCs.

- $C \subset MC \subset GC$. The set of distributions generated by the Simple Calvo is a special case of the set generated by MC which is a special case of GC.

- $ST \subset GTE = GC$ Simple Taylor is a special case of GTE, and hence also of GC.

- $ST \cap MC = \varnothing$. Simple Taylor contracts are a special case of GC, but not of MC.

Figure 1: The typology of Contracts

This is depicted in Fig 1. The GC and the GTE are coextensive, being the set of all possible steady-state distributions (Propositions 1 and corollary 3). The Simple calvo C (one reset probability) is a strict subset of the Multiple Calvo process MC which is a strict subset of the GC. The simple Taylor ST and the MC are disjoint. The ST is a strict subset of the GTE. The size of the distributions is reflected by the Figure: ST has elements corresponding to the set of integers and is represented by a few dots; Calvo is represented by the unit interval; MC by the unit interval squared.

We can now ask the question: if we aggregate over two contract structures, what is the type of contract structure that results? This is an important question: if we believe that the economy is heterogenous, we should not represent it with a contract type which is not closed under aggregation. We can think of this in terms of giving each contract structure a strictly positive proportion of the total set of contracts; for example 50%. We can define the ST in terms of contract length, under the assumption that each cohort is of equal size.

$$ST(k) + ST(j) = GTE\left((0.5, 0.5), (k, j)\right)$$

Clearly, if we aggregate over Standard Taylor contracts with different contract lengths $k > j$, we no longer have a Standard Taylor contract but GTE.

Similarly, if we aggregate over simple *Calvo* contracts with different reset probabilities, we do not get a C contract but a multiple Calvo MC:

$$C(\omega_1) + C(\omega_2) = MC((0.5, 0.5), (\omega_1, \omega_2))$$

By definition, If we aggregate over MCs, we still have an MC. We can say that a type of contract structure is closed under the operation of aggregation if we aggregate two different contracts of that type and the resultant contract structure is also of the same type. Clearly, neither the ST or C are closed under aggregation. However, MC, GC, and GTE are all closed under aggregation:

Observation MC, GC, and GTE are all closed under aggregation.

Consider the case of GTE. Suppose we have two $GTEs$ with maximum contract lengths F_1 and F_2 respectively and w.l.o.g. $F_1 \leq F_2$. We can choose to represent the number of sectors to be the maximum of the contract lengths (with $\alpha_{1i} = 0$ for $i > F_1$. The corresponding vector of sector shares is then $\boldsymbol{\alpha}_j \in \Delta^{F_2-1}$, $j = 1, 2$. If we combine the two $GTEs$, we get another GTE with the sector shares being the average of the other two.

$$GTE(\boldsymbol{\alpha}_1) + GTE(\boldsymbol{\alpha}_2) = GTE\left(\frac{1}{2}\boldsymbol{\alpha}_1 + \frac{1}{2}\boldsymbol{\alpha}_2\right)$$

Hence $GTEs$ are closed under aggregation. Similarly ,since GCs can be represented as equivalent $GTEs$, the closure of $GTEs$ implies the closure of GC under aggregation.MC are closed, since we can simply combine the different ω_i from each MC and reweight on a 50-50 basis. We have illustrated the proof using two distributions with a 50-50 combination. The idea obviously generalises to a convex combination of any number of MC, GTE and GCs.

The importance of this result is that if we really do believe that contract structures are heterogeneous, we should use contract types that are closed under aggregation. The simple Calvo and Taylor models are only applicable if there is one type of contract and no heterogeneity in the economy. If we believe the Calvo model, but that reset probabilities are heterogenous across price or wage setters, then the MC makes sense. If we believe that the Calvo model is not a good one, then the GC or GTE is appropriate.

5 Pricing behaviour compared.

We have developed a general framework for understanding steady-state distributions of durations across firms and how they are related in terms of pricing models. In this section we consider how pricing models differ when we control for the distribution of durations, for example by requiring that mean contract ages are the same, or even the exact distribution. This is a useful exercise, because it enables us to isolate the differences in the pricing behaviour per se, rather than mixing them up with different distributions as has been done in much previous work.

Let us recap the comparison between the simple Calvo model and the equivalent GTE (named the Calvo-GTE) made in Dixon and Kara (2005). The Calvo-GTE is a model with exactly the same distribution of completed contract lengths as in the Calvo model. The main difference is that in the Calvo GTE , when firms set their prices, they know the exact duration of the contract, as in the standard Taylor model. In the Calvo setting, the firms do knot know how long the contract will last: all firms have the same probability distribution over durations and set the same price. In the Calvo-GTE by contrast, in each sector the firms know how long the contract will last, so the rest price in each sector can be different. Hence in the Calvo-GTE there is a range of reset prices set (one for each contract duration), whereas in the Calvo model there is only one reset price (firms do not know how long their contract will run when they reset the price). Furthermore, the "forward lookingness" of the Calvo-GTE is less: on average a greater weight is put on nearer dates. We will now perform the same comparison between GC and corresponding GTE.

Let us suppose that there is an optimal price P^* (or target variable in general) in each period starting from t: $\left\{P^*_{t+s}\right\}_{s=0}^{\infty}$. In a GC with $\boldsymbol{\omega} \in [0,1]^{F-1}$, the proportion of firms resetting price at time t is $\bar{\omega}$ and they all set the same reset price X_t^{GC}. Ignoring discounting, the reset wage and forward lookingness[14] FL are:

$$X_t^{GC} = \bar{\omega}\sum_{j=0}^{F-1}\Omega_j P^*_{t+j-1} = \sum_{j=0}^{F-1}\alpha_j^s P^*_{t+j-1}$$
$$FL^{GC} = \bar{\omega}\sum_{s=1}^{F-1} s\Omega_s = s\alpha_j^s = \bar{s}$$

hence in the GC as in the simple Calvo model, the weights on the future periods are simply the same as the age shares α_j^s. Hence the forward lookingness is the average age $\bar{s}$.

In the corresponding GTE there are F sectors, with contract lengths $i = 1...F$ with corresponding sector shares:

$$\alpha_i = \bar{\omega}.i.\omega_i.\Omega_i = i.\omega_i\alpha_i^s$$

In each sector i, a proportion i^{-1} reset their prices every period: hence a total of $\bar{\omega}$ reset their prices, since from Corollary 3

$$\bar{\omega} = \sum_{i=1}^{F}\frac{\alpha_i}{i} \tag{7}$$

The important thing to note about (7) is that the longer contract lengths are under-represented amongst resetters relative to the population, since they reset prices less often. Now, if we look at the sector i resetters, the reset price will be

$$X_{it} = \frac{1}{i}\sum_{j=0}^{i-1}P^*_{t+j}$$

[14] Note that Forward Lookingness is not in general equal to the expected duration of the contract when the price is set (life expectation at birth). They are equal in the simple Calvo model because it has the special property that life expectancy at birth equals the average age. For example, in the simple Taylor model, $FL = \bar{s}$, but life expectation at birth equals the full length of the contract (average completed contract length $\bar{T}$).

The average reset price is thus

$$\begin{aligned} X_t^{GTE} &= \frac{1}{\bar{\omega}} \sum_{i=1}^{F} \frac{\alpha_i}{i} X_{it} = \sum_{j=1}^{F} b_s P^*_{t+j-1} \\ b_j &= \sum_{i=j}^{F} \frac{\omega_i . \Omega_i}{i} = \sum_{i=j}^{F} \frac{\alpha_i^s}{i} \\ FL^{GTE} &= \sum_{j=1}^{F} j b_j \end{aligned}$$

It is illuminating to write the GTE weights in terms of the GC weights. In the GTE, each price setter knows the exact length of the contact: hence when setting price it ignores what happens after the end of the contract. In contrast, in the GC the price-setter is uncertain of the contract length and must always consider the possibility of lasting until the longest duration F. As we identified in Dixon and Kara (2005), this results in the fact that comparing the GTE to the GC weights, weight is "passed back" from longer durations to shorter making the GTE more myopic:

$$b_j = \frac{\omega_j}{\bar{\omega}} \alpha_j^s - \frac{j}{j+1} \frac{\omega_j}{\bar{\omega}} \alpha_j^s + \sum_{i=j+1}^{\infty} \frac{\omega_i}{\bar{\omega}} \frac{\alpha_i^s}{i} \tag{8}$$

Hence there are three components of b_j in (8): the corresponding α_j^s, the weight passed back to shorter contracts and thirdly the wight it receives from longer contracts. To translate from α_j^s to b_j, you need to correct by a factor of $\omega_j / \bar{\omega}$ for each duration j: in the simple Calvo model this is unity and the equation reverts to Dixon and Kara (2005). This means that the $b_j s$ put a greater weight on the immediate future and less on the more distant future than the corresponding GC. We can see this if we look at the cumulative weights: looking at the sum of weights up to q periods ahead, the sum of $b_j s$ is the sum of $C_j s$ plus the weights passed back from the periods in the future to each of the b_j where $j \leq q$.

$$\sum_{j=1}^{q} b_j = \sum_{j=1}^{q} \frac{\omega_j}{\bar{\omega}} \alpha_j + \sum_{k=0}^{q} \sum_{i=k}^{\infty} \frac{\omega_i}{\bar{\omega}} \frac{\alpha_i}{i+1}$$

We can illustrate the differences in forward lookingness using the some of the examples we considered before in section 3.1. Since the GC weights are just the age-shares α_i^s and were given previously as fractions in section 2.2, we state them here as decimals to 4 decimal places. The b_j coefficients are give as both exact fractions and decimals.

Example 1

$$\begin{array}{rr} \alpha_1^s = 0.925 & b_1 = 0.9521 = \frac{457}{480} \\ \alpha_2^s = 0.025 & b_2 = 0.0271 = \frac{13}{480} \\ \alpha_3^s = 0.025 & b_3 = 0.0146 = \frac{7}{480} \\ \alpha_4^s = 0.025 & b_4 = 0.0063 = \frac{1}{160} \\ FL^{GC} = 1.15 & FL^{GTE} = \frac{486}{480} = 1.075 \end{array}$$

Example 2.

$$\begin{array}{rr} \alpha_1^s = 0.4706 & b_1 = \frac{9}{16} = 0.5625 \\ \alpha_2^s = 0.3530 & b_2 = \frac{5}{16} = 0.3125 \\ \alpha_3^s = 0.1765 & b_3 = \frac{2}{16} = 0.125 \\ FL^{GC} = 1.706 & FL^{GTE} = \frac{25}{16} = 1.5625 \end{array}$$

Example 3

$$\begin{array}{rr} \alpha_1^s = 0.4507 & b_1 = \frac{71}{128} = 0.5547 \\ \alpha_2^s = 0.3380 & b_2 = \frac{39}{128} = 0.3047 \\ \alpha_3^s = 0.1690 & b_3 = \frac{15}{128} = 0.1172 \\ \alpha_4^s = 0.0423 & b_4 = \frac{3}{128} = 0.0234 \\ FL^{GC} = 1.803 & FL^{GTE} = 1.609 \end{array}$$

Clearly, in all three examples the GTE corresponding to a GC puts a much greater weight on the current period and less on the subsequent periods, resulting in a less forward looking pricing decision. In example 1, the weight is still greater on the second period, but falls off rapidly.

Lastly, we can consider the case of a MC process. In this case, the forward lookingness of the MC is simply the average of the Forward lookingnesses of the individual Calvo processes weighted by sector shares β_k

$$FL^{MC} = \sum_{k=1}^{n} \beta_k \bar{\omega}_k = \sum_{j=1}^{k} \beta_k \bar{s}_k = \bar{s}$$

where $\bar{s}_k$ is the average age in steady state in the $k-$sector, and $\bar{s}$ the average age in the population. Since by construction the MC and the equivalent GC have the same distribution of ages in steady state and hence average age in the population, the two different constructions have the same forward-lookingness, $FL^{GC} = FL^{MC}$. Furthermore, we can see that the average reset price at time t will be equal.

In the MC, there will be different reset prices, one for each $\bar{\omega}_k$. Hence the reset price of firms with hazard $\bar{\omega}_k$ is:

$$X_{kt}^{MC} = \sum_{j=1}^{\infty} \alpha_{kj}^{s} P_{t+j-1}^{*}$$

where $\left\{\alpha_{kj}^{s}\right\}_{j=1}^{\infty}$ is the steady-state age-distribution for those with hazard $\bar{\omega}_k$. The average reset price is then

$$\begin{aligned} X_t^{MC} &= \sum_{k=1}^{n} \beta_i X_{kt}^{MC} = \sum_{k=1}^{n} \sum_{j=1}^{\infty} \beta_k \alpha_{kj}^{s} P_{t+j-1}^{*} \\ &= \sum_{j=1}^{\infty} \alpha_j^{s} P_{t+j-1}^{*} \end{aligned}$$

since $\sum_{k=1}^{n} \beta_k \alpha_{kj}^{s} = \alpha_j^{s}$.

Hence the average reset price in the MC is the same as the equivalent GC setting. This means that when modelling an economy with heterogeneous Calvo contracts as in the MC model, it may well be the most parsimonious to use the GC framework. The degree of forward lookingness and the average reset price are the same. The only difference is that in MC there are as many reset prices as hazard rates, whereas in the GC there is only one.reset price in any one period.

6 Price Data: an application to the Bils-Klenow Data set.

In this section, we apply our theoretical framework to the Bils-Klenow Data set (Bils and Klenow 1994). This data is the micro-price data collected monthly for the US CPI over the period 1995-7. The BK data covers 350 categories of commodities comprising 68.9% of total consumer expenditure. They focus on the proportion of prices that change in a month in each category (sector). They then derive the distribution of durations across contracts on the assumption that there is a sector specific Calvo reset probability in continuous time[15].

In this section I use the BK data to construct the distribution of contracts across firms. Each sector has a sector-specific average proportion of firms resetting their price per month over the period covered. We can interpret this as a Calvo reset probability in *discrete* time. We adopt the discrete time approach in order to be consistent with the pricing models which are in discrete time. The first approach we adopt is to model this as a Multiple Calvo process $BK - MC$. The second is to model the resulting distribution across all sectors. Within each sector we have the Calvo distribution of contract lengths as derived in Dixon and Kara (2006a): using the sectoral weights we can then aggregate across all sectors. This gives us the following distribution of contract lengths:

Fig 1: the BK Distribution of Contract lengths Across Firms

Note that the mean in our distribution is larger than is reported in BK. This is because we are looking at the mean duration across firms rather than contracts and hence we are more likely to observe longer contracts. With the aggregate distribution of contract lengths we can model this as either a GTE or a GC as well as an MC. We therefore have three different pricing models of the same distribution of contract lengths derived from the BK dataset.

6.1 Pricing Models Compared.

We will see how the different models of pricing differ in terms of their impulse-response. We adopt the model of price or wage setting developed in Dixon and Kara (2005a, 2006): the details are set out in the appendix. We consider two monetary policy shocks: in the first case there is a one off permanent shock in the level of the money supply; in the second an autoregressive

[15] The use of continuous time leads to a lower expected expected duration at birth. If the proportion resetting price is $\bar{\omega}$, the expected duration at birth is $-1/In(1-\bar{\omega})$. This is less than the discrete time expectation $1/\bar{\omega}$. The difference gets proportinatley larger as $\bar{\omega}$ gets larger. When $\bar{\omega} = 0.8$ the discrete time estimator is over twice the continuous time estimator. The analysis in this paper is in discrete time becuase that is how the pricing models are employed in the literature, and also it provides spreadsheet simplicity and trnasparency.

process. Expressed as log deviations form steady state, money follows the process

$$
\begin{aligned}
m_t &= m_{t-1} + \varepsilon_t \\
\varepsilon_t &= \nu \varepsilon_{t-1} + \xi_t
\end{aligned}
$$

where ξ_t is a white noise error term. We consider the case of $\nu = 0$ and the autoregressive case, where we set $\nu = 0.5$. The other key parameter γ captures the sensitivity to the flexible price to output. The optimal flexible price at period t in any sector p_t^* is given by

$$p_t^* = p_t + \gamma y_t$$

where (p_t, y_t) are aggregate price and output (all in log-deviation form). We allow for two values of $\gamma = \{0.01, 0.2\}$: a high one and a low one as discussed in Dixon and Kara (2006b).

Fig 2: Responses to a one-off monetary Shock.

In Figure 2, we depict the responses of output, the reset price, the general price level and inflation to a one-off shock with $\gamma = 0.2$. Looking at all the graphs, it is striking that the three models of pricing have fairly similar impulse-responses: none of them are far apart. However, in all cases the MC and the GC are close together and the GTE is farther away, particularly towards the end. To understand this, we can look at the IR for the average reset price and the general price level. In the GTE case, the reset price rises less on impact than the MC or GC. This reflects the greater myopia: those cohorts resetting prices look less far ahead on average, so that they do not raise prices as much as in the MC or GC case. At about 10 months however, the situation is reversed: the GTE reset price exceeds the MC and GC case: whilst the latter are slowing down price increases in anticipation of the approaching steady state, the GTE maintains momentum for longer. This comparative myopia of the GTE explains why the output response starts off above both the MC and GC, but ends up after 15 months below both.

Fig 3: Serial Correlation in Monetary growth $\nu = 0.5$

In Figure 3 we consider the autoregressive monetary policy shock and concentrate on the IR for output and inflation for both the high and the low values of γ. We find that there is now a more radical difference between the GTE and the other two models. If we look at inflation we see that there is a hump shape: the peak impact on inflation appears after the initial monetary shock: with the high value of γ it happens at 3 months: with the low value at around 20 months. Both the MC and the GC are not hump shaped. This reflects the finding in Dixon and Kara 2006b that the Calvo model does not capture the characteristic "hump shaped" response indicated by empirical VARS. This feature appears tp be shared by its generalisations MC and GC.

This simple example of the IR of major variables shows how different models of pricing can yield different patterns of behaviour even though the distribution of contract lengths are exactly the same. Partly this is due to different degrees of forward lookingness. The MC and the GC do differ slightly, but are quite close, which reflects the fact that they have the same forward lookingness. It suggests that since the GC is computationally much simpler (you only have to model one pricing decision for all firms resetting price, rather than one for each sector), this model might be preferred to the MC.

6.2 Alternative Interpretations of the BK Data set.

The previous analysis was based on assuming that the true distribution within each sector is generated by the sector specific discrete time Calvo distribution. However, this is just one hypothesis about the underlying distribution of contract lengths generating the proportion of firms resetting their price per month. Let us look at the class of GTEs that are consistent with a particular reset proportion. Again, let us consider the set of GTEs with maximum contract lengths F, $\boldsymbol{\alpha} \in \Delta^F$: we can define the subset which yield a particular reset proportion $\bar{\omega}$: define the mapping $A(\bar{\omega}) : [0,1] \rightarrow \Delta^F$

$$A(\bar{\omega}) = \left\{ \boldsymbol{\alpha} \in \Delta^F : \sum_{i=1}^{F} \frac{\alpha_i}{i} = \bar{\omega} \right\}$$

Note that since $A(\bar{\omega})$ is defined by a linear restriction on the sector shares $\boldsymbol{\alpha}$. We assume that $\bar{\omega} \geq F^{-1}$ in what follows., so that $A(\bar{\omega})$ is non-empty[16]. Hence $A(\bar{\omega}) \subset \Delta^{F-2}$ and $A(\bar{\omega})$ is closed and bounded.

[16] This is a purely technical assumption. If we assumed a value $\bar{\omega} < F^{-1}$, then even if all contracts were at the maximum duration, there would be too many firms resetting prices. Since we are dealing with empirically relevant values, $\bar{\omega} > F^{-1}$ is automatically satisfied.

The average length of contracts is $\bar{T}(\boldsymbol{\alpha}) = \sum_{i=1}^{F} i.\alpha_i$. Let us consider the following problems: what is the lowest (highest) average contract length consistent with a particular reset proportion $\bar{\omega}$. Mathematically, we know that since $A(\bar{\omega})$ is non-empty, closed and bounded and $\bar{T}(\boldsymbol{\alpha})$ is continuous, both a maximum and a minimum will exist. Turning to the minimization problem first: we have to choose $\alpha \in \Delta^{F-1}$ to solve

$$\min \ \bar{T}(\boldsymbol{\alpha}) \quad s.t. \ \alpha \in A(\bar{\omega}) \tag{9}$$

Proposition 3 Let $\boldsymbol{\alpha}^* \in \Delta^{F-1}$ solve (9) to give the shortest average contract length $\bar{T}(\boldsymbol{\alpha})$.

(a) No more than two sectors i have values greater than zero

(b) If there are two sectors $\alpha_i > 0$, $\alpha_j > 0$ then will be consecutive integers ($|i-j| = 1$).

(b) There is one solution iff $\bar{\omega}^{-1} = k \in Z_+$. In this case, $\alpha_k = 1$.

We can sum up the proposition by saying that *the shortest average contract length consistent with a given proportion of resetters is the simplest GTE that can represent it.* It is either a pure simple Taylor, or an only slightly less simple GTE with two sectors of consecutive lengths. Note that whilst Proposition 3 is derived for a GTE, under the equivalence established by Proposition 1 and corollary 3, it will also hold across all GCs. It is a distribution which minimises mean contract length, and this distribution can be seen as either a GTE or a GC.

We can also ask what is the *maximum* average contract length consistent with a proportion of resetters:

$$\max \ \bar{T}(\boldsymbol{\alpha}) \quad s.t. \ \alpha \in A(\bar{\omega}) \subset \Delta^{F-1}$$

Proposition 4 Given the longest contract duration F, the distribution of contracts that maximises the average length of contract subject to a given proportion of firms resetting price $\bar{\omega} \geq F^{-1}$ is

$$\begin{aligned} \alpha_F &= \frac{F}{F-1}(1-\bar{\omega}) \\ \alpha_1 &= \frac{F}{F-1}\bar{\omega} - \frac{1}{F-1} \end{aligned}$$

with $\alpha_i = 0$ for $i = 2...F-1$. The maximum average contract length is

$$\bar{T}^{\max} = F(1-\bar{\omega}) + 1$$

There will be one contract length with $\alpha_F = 1$ if $\bar{\omega} = F^{-1}$, and $\alpha_1 = 1$ if $\bar{\omega} = 1$. For all intermediate values, $\alpha_F.\alpha_1 > 0$.

This proposition implies that as $F \to \infty$, so does the maximum contract length. If $F = \infty$ as in the Calvo model, there is no upper bound to the average contract length.

Let us return to the BK data set in the light of the preceding two propositions. First, we can ask the what is the shortest average contract length which is consistent with the BK data set. The BK data set gives us for each sector k the proportion of firms changing price: $\bar{\omega}_k$ in any month. Following Bils and Klenow themselves, it is natural to interpret this as a MC, that within each sector there is a Calvo process. This was the assumption we used to generate Figure 2. In terms of the mean contract length in the $BK - MC$, this is calculated as

$$\bar{T}^{MC} = \sum_{k=1}^{n} \alpha_k \frac{2-\bar{\omega}_k}{\bar{\omega}_k} = 4.4$$

Where durations are in Quarters unless otherwise specified. The minimum average contract length within each sector is simply $\bar{\omega}_k$, so that the minimum average contract length in the BK data is[17]:

$$\bar{T}^{\min} = \sum_{k=1}^{n} \alpha_k \frac{1}{\bar{\omega}_k} = 2.7$$

Clearly, for any data set like this (based on the proportion of firms changing price in a period), the shortest average contract length can be achieved using the Taylor model: using the Multiple Calvo yields an average duration nearly twice as long, with the linear relation:

$$\bar{T}^{MC} = 2\bar{T}^{\min} - 1.$$

[17] Note that this exceeds the level reported by Bils and Klenow! That is becuase they interpret the proportion as arising from a continuous time process. We are adopting the discrete time approach.

The maximum average contract length is not meaningful. There are some sectors with very low percentages of price changing: coin operated laundry, for example has 1.2% of prices changing a month implying a *minimum*[18] mean contract length in that sector of 83 months (over 6 years). However, the nature of the BK data set says nothing about the distribution within the sectors, so there is no meaningful upper bound on the average length of contracts.

Lastly, let us consider what might happen if we aggregate over all sectors by taking the mean proportion of firms changing price,

$$\hat{\omega} = \sum_{k=1}^{n} \alpha_k \bar{\omega}_k$$

This gives us an estimate of average duration $\hat{T}$:

$$\hat{T} = \frac{2}{\hat{\omega}} - 1 = 2.47$$

Now, clearly, since contract length is a convex function of $\bar{\omega}_k$, by Jensen's inequality $\hat{T} \leq \bar{T}^{MC}$. If we use the actual BK data, we have $\hat{\omega} = 0.209$ per *month*, yielding the reported estimate of 2.47 quarters which is just over one half of the "true" MC value. This shows that aggregating over sectors in this way can be extremely misleading and will considerably underestimate the "true " value even if one believes the Calvo story[19]. However, the MC might not be the true model. Even with this degree of disaggregation, there may well be intra-sectoral variation. Ultimately, what is needed is the individual price data. Certainly, *using data of the proportions changing price in a period is of limited value and might not even get you into the right ballpark.* That is certainly what the BK data set tells us.

7 Conclusion

In this paper we have developed a consistent and comprehensive framework for analyzing different pricing models which generate steady state distributions of durations which can be used to understand dynamic pricing models. In particular, the d*istribution of completed contract lengths across firms*

[18] Recall, we have the lower bonnd on F since $F \geq (\bar{\omega}_k)^{-1}$.

[19] This is also a finding emphasised on European data: see Dhyne E. et al (2005, 13).

(DAF) is a key perspective which is fundamental to understanding and comparing different models. Any steady state distribution of durations can be looked at in terms of completed durations, which suggests it can be modelled as a GTE; it can also be thought of in terms of Hazard rates which suggests the GC approach. Both the GC and the GC are comprehensive: they can represent all possible steady states. Furthermore, they are closed under aggregation. Unlike the simple Calvo and Taylor models, they are consistent with heterogeneity in the economy.

Once we have controlled for the distribution of contract durations, we can compare different pricing models. The concept Forward Lookingness is useful in comparing the way different pricing rules put weights on the future periods. We find that the GC is less myopic than the corresponding GTE, echoing the finding of Dixon and Kara (2005a) comparing Calvo and the Calvo-GTE. We then illustrate this by using a standard macromodel using the Bils-Klenow data set, which we interpret (following Bils and Klenow) as an MC process. We see even though the distributions are identical, the three pricing models are different. The GC and MC are close to each other and had the same forward lookingness. The GTE is more myopic and has a different impulse-response in relation to a monetary shock. In particular, for particular parameterization, the GTE can display a hump-shaped inflation response, whereas the GC and MC never have a hump.

The analysis also has implications for how we interpret the data on the proportion of firms setting a price in a particular period. The minimum average length consistent with this is given by the simplest GTE. There is no reasonable upper bound unless we have an upper bound on the maximum contract length possible. Certainly, there are severe problems of aggregation which indicate that using such data might lead very inaccurate estimates of average contract length. The analysis also indicates that existing estimates of price-stickiness are biased downwards and that in reality prices are stickier than some have maintained.

Our analysis has looked at one type of wage or price setting behaviour: the contract consists in the setting of a nominal price or wage that persists throughout the contract. As we show in Dixon and Kara (2006b), other types of price and wage setting can be dealt with in this framework. For example, we can have Fischer contracts, where firms or unions set a trajectory of nominal prices over the life of the contract. This is essentially the approach taken by Mankiw and Reis (2002). There are other possibilities such as indexation during the contract once the initial level has been fixed. We can have any model of pricing so long as it is consistent with a steady state distribution of durations. The main class of pricing models that do not give a steady state distribution are of course the state-dependant pricing models (menu cost models), such as Dotsey, King and Wolman. (1999). Here the duration or contracts depends on the macroeconomic environment. However, this paper makes an attempt to improve our understanding of the steady state, which will in turn provide a firmer foundation for understanding non-steady state phenomena.

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9 Appendix.

9.1 Proof of Proposition 1 and Corollaries.

Proof. The proportion of firms that have a contract that last for exactly 1 period are those that are born (age 1) and do not go on to age 2. The proportion of firms that last for exactly i periods in any one cohort (born at the same time) is given by those who attain the age i but who do not make it to $i+1$: this is $\left(\alpha_i^s - \alpha_{i+1}^s\right)$ per cohort and at any time t there are i cohorts containing contracts that will last for i periods.

Clearly, since α_j^s are monotonic, $\alpha_i \leq 1$, and

$$
\begin{aligned}
\sum_{i=1}^{F} \alpha_i &= \sum_{i=1}^{F} i\left(\alpha_i^s - \alpha_{i+1}^s\right) \\
&= \left(\alpha_1^s - \alpha_2^s\right) + 2\left(\alpha_2^s - \alpha_3^s\right) - 3\left(\alpha_3^s - \alpha_4^s\right) .... \\
&= \sum_{i=1}^{F} \alpha_i^s = 1
\end{aligned}
$$

Hence $\boldsymbol{\alpha} \in \Delta^{F-1}$.

The relationship between the distribution of ages and lifetimes can be depicted in terms of matrix Algebra: in the case of $F = 4$:

$$
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \alpha_1^s \\ \alpha_2^s \\ \alpha_3^s \\ \alpha_4^s \end{bmatrix}
$$

Clearly, the 4×4 matrix is a mapping from $\Delta^3 \rightarrow \Delta^3$: since the matrix is of full rank, the mapping from α^s to α is $1-1$. Clearly, this holds for any F.
■

9.1.1 Proof of Corollary 1:

Proof. To see this, we can rewrite (2):

$$\begin{aligned}\alpha_1 &= \alpha_1^s - \alpha_2^s \\ \frac{\alpha_2}{2} &= (\alpha_2^s - \alpha_3^s) \\ \frac{\alpha_i}{i} &= \left(\alpha_i^s - \alpha_{i+1}^s\right) \\ \frac{\alpha_F}{F} &= \alpha_F^s\end{aligned}$$

hence summing over all possible durations $i = 1...F$ gives

$$\sum_{i=1}^{F} \frac{\alpha_i}{i} = \sum_{i=1}^{F-1} \left(\alpha_i^s - \alpha_{i+1}^s\right) + \alpha_F^s = \alpha_1^s$$

So that by repeated substitution we get:

$$\begin{aligned}\alpha_2^s &= \alpha_1^s - \alpha_1 = \sum_{i=2}^{F} \frac{\alpha_i}{i} \\ \alpha_j^s &= \sum_{i=j}^{F} \frac{\alpha_i}{i} \qquad j = 1...F\end{aligned}$$

■

9.1.2 Corollary 3.

Proof. Rearranging the $F - 1$ equations (4) we have:

$$\frac{\alpha_1}{\bar{\omega}} = \omega_1; \frac{\alpha_2}{2\bar{\omega}} = \omega_2 (1 - \omega_1) ... \frac{\alpha_i}{i.\bar{\omega}} = \omega_i \Omega_i; ... \frac{\alpha_F}{F\bar{\omega}} = \Omega_F$$

By repeated substitution starting from $i = 1$ we find that

$$\omega_i = \frac{\alpha_i}{i} \left(\bar{\omega} - \sum_{j=1}^{i-1} \frac{\alpha_j}{j} \right)^{-1} \tag{10}$$

$$\Omega_i = \frac{1}{\bar{\omega}} \left[\bar{\omega} - \sum_{j=1}^{i-1} \frac{\alpha_j}{j} \right]$$

Since we know that $\omega_F = 1$, from (10)this means that:

$$1 = \frac{\alpha_F}{F}\left(\bar{\omega} - \sum_{i=1}^{F-1}\frac{\alpha_i}{i}\right)^{-1} \Rightarrow \bar{\omega} = \sum_{i=1}^{F}\frac{\alpha_i}{i}$$

Substituting the value of $\bar{\omega}$ into (10) establishes the result. ■

9.2 Proof of Proposition 2.

Without loss of generality let $\bar{\omega}_1 < \bar{\omega}_2 < .... < \bar{\omega}_K$. We then have

$$\frac{\alpha_i}{i} = \sum_{k=1}^{n}\beta_k\bar{\omega}_k^2\left(1-\bar{\omega}_k\right)^{T-1} = \sum_{k=1}^{n}\beta_k\frac{\alpha_{ki}}{i}$$

the period i hazard rate is the ω_i is the average of the hazard rates taken over the survivors to i. Note that k and j subscripts refer to the sectoral calvo reset probabilities: ω_i the aggregate duration dependent reset, which is a weighted sum of the sectoral reset probabilities, the weights being given by ϕ_{ki}, the share of sector k survivors in all survivors:

$$\omega_i = \sum_{k=1}^{n}\phi_k\bar{\omega}_k$$

$$\phi_{ki} = \frac{\beta_k\bar{\omega}_k\left(1-\bar{\omega}_k\right)^{i-1}}{\sum_{k=1}^{n}\beta_k\bar{\omega}_k\left(1-\bar{\omega}_k\right)^{i-1}} \text{ share of k in survivors}$$

I now divide up the proof into three steps.

Lemma 1 The share of survivors of type k at duration i is increasing (decreasing) if the hazard rate $\bar{\omega}_k$ is less than (greater than) the average hazard ω_i

Proof.

$$\begin{aligned}
\phi_{ki+1}-\phi_{ki} &= \frac{\beta_k\bar{\omega}_k\left(1-\bar{\omega}_k\right)^i}{\sum_{j=1}^n\beta_j\bar{\omega}_j\left(1-\bar{\omega}_j\right)^i}-\frac{\beta_k\bar{\omega}_k\left(1-\bar{\omega}_k\right)^{i-1}}{\sum_{j=1}^n\beta_j\bar{\omega}_j\left(1-\bar{\omega}_j\right)^{i-1}} \\
&= \frac{\beta_k\bar{\omega}_k\left(1-\bar{\omega}_k\right)^{i-1}\left[\sum_{j=1}^n\beta_j\bar{\omega}_j\left(1-\bar{\omega}_j\right)^{i-1}\left(\bar{\omega}_j-\bar{\omega}_k\right)\right]}{\left(\sum_{j=1}^n\beta_j\bar{\omega}_j\left(1-\bar{\omega}_j\right)^i\right)\left(\sum_{j=1}^n\beta_j\bar{\omega}_j\left(1-\bar{\omega}_j\right)^{i-1}\right)} \\
&= \frac{\beta_k\bar{\omega}_k\left(1-\bar{\omega}_k\right)^{i-1}\left(\omega_i-\bar{\omega}_k\right)}{\sum_{j=1}^n\beta_j\bar{\omega}_j\left(1-\bar{\omega}_j\right)^i} \\
&= \phi_{ki+1}\frac{\omega_i-\bar{\omega}_k}{1-\bar{\omega}_k}
\end{aligned}$$

which establishes Lemma 1. ■

Lemma 2 The hazard rate decreases with duration $\omega_{i+1}<\omega_i$.

Proof.

$$\begin{aligned}
\omega_{i+1}-\omega_i &= \sum_{k=1}^n\bar{\omega}_k\left(\phi_{ki+1}-\phi_{ki}\right) \\
&= \sum_{k=1}^n\bar{\omega}_k\phi_{ki+1}\frac{\omega_i-\bar{\omega}_k}{1-\bar{\omega}_k} \\
&= \sum_{k=1}^n\frac{\bar{\omega}_k}{1-\bar{\omega}_k}\phi_{ki+1}\left(\omega_i-\bar{\omega}_k\right)<0
\end{aligned}$$

Since the higher vales of $\bar{\omega}_k$ have higher weights, this sum is negative unless all values of $\bar{\omega}_k=\omega$, in which case the hazard is constant. ■

Lemma 3 In the limit as $i\to\infty$, $\omega_i\to\min\left[\bar{\omega}_k\right]=\bar{\omega}_k$, $\phi_{1i}\to 1$.

Problem 2 ***Proof.*** *By definition,*

$$\omega_i\geq\bar{\omega}_k$$

The share of type 1's in the survivors is

$$\begin{aligned}
\phi_{1i} &= \frac{\beta_1 \omega_1 (1-\bar{\omega}_k)^{i-1}}{\sum_{k=1}^{n} \beta_k \bar{\omega}_k (1-\bar{\omega}_k)^{i-1}} \\
&= \left[1 + r_2 \left(\frac{1-\bar{\omega}_2}{1-\bar{\omega}_1}\right)^{i-1} \dots + r_n \left(\frac{1-\bar{\omega}_n}{1-\bar{\omega}_1}\right)^{i-1}\right]^{-1} \\
\textit{where } r_k &= \frac{\beta_k \bar{\omega}_k}{\beta_1 \bar{\omega}_1} \\
\underset{i\to\infty}{Lim} \phi_{1i}^{-1} &= \underset{i\to\infty}{Lim} \left[1 + \sum_{k=2}^{n} \left[r_k \left(\frac{1-\bar{\omega}_k}{1-\bar{\omega}_1}\right)^{i-1}\right]\right]^{-1} = 1 \\
\textit{since } \bar{\omega}_1 &< \bar{\omega}_k \textit{ for } k = 2 \dots n. \\
\textit{Hence } \underset{i\to\infty}{Lim} \phi_{ki}^{-1} &= 0 \textit{ for all } k > 1.
\end{aligned}$$

■

9.3 Proof of Proposition 3.

Proof. Firstly we will prove (a) and (b). We do this by contradiction. Let us suppose that the solution $\boldsymbol{\alpha}$ such that $\alpha_k > 0$ and $\alpha_j > 0$ and $k - j \geq 2$ We will then show that there is another feasible GTE $\boldsymbol{\alpha}'$ with $\alpha_j > 0$ and $\alpha_{j+1} > 0$ which generates a shorter average contract length.

Let us start at the proposed solution α, and in particular the two sectors k and j, whose sector shares must satisfy the two relations:

$$\begin{aligned}
\alpha_k + \alpha_j &= \rho = 1 - \sum_{i=1, i\neq j,k}^{F} \alpha_i & (11) \\
\frac{\alpha_k}{k} + \frac{\alpha_j}{j} &= \eta = \bar{\omega} - \sum_{i=1, i\neq j,k}^{F} \frac{\alpha_i}{i}
\end{aligned}$$

ρ is the total share of the two sectors: if there are only two sectors then $\rho = 1$; if there are more than two sectors with positive shares then ρ is equal 1 minus the share of the sectors other than j and k. Likewise, η is the sum of the contribution of these two sectors to $\bar{\omega}$ less the contribution of any sectors other than j and k. Note that since $k > j$,

$$\frac{\rho}{\eta} > j \qquad (12)$$

We can rewrite (11) as

$$\begin{aligned}\alpha_j &= \frac{kj}{k-j}\eta - (k-j)\,\rho \\ \alpha_k &= \rho\,(1+k-j) - \frac{kj}{k-j}\eta\end{aligned} \tag{13}$$

What we show is that we can choose $\left(\alpha_j', \alpha_{j+1}\right)$ which satisfies the two relations above (and hence is feasible) but yields a lower average contract length. Specifically, We choose α_{j+1}', α_j' such that

$$\begin{aligned}\alpha_j' &= j\,(j+1)\,\eta - \rho \\ \alpha_{j+1}' - \alpha_{j+1} &= 2\rho - j\,(j-1)\,\eta\end{aligned} \tag{14}$$

Define $\Delta\alpha_{j+1} = \alpha_{j+1}' - \alpha_{j+1}$. What we are doing is redistributing the total proportion ρ over durations j and $j+1$ so that the aggregate proportion of firms resetting the price is the same: $\boldsymbol{\alpha}' \in A\,(\bar{\omega})$,since (14) is equivalent to (13) implies

$$\begin{aligned}\Delta\alpha_{j+1} + \alpha_j' &= \rho \\ \frac{\Delta\alpha_{j+1}}{k} + \frac{\alpha_j'}{j} &= \eta\end{aligned} \tag{15}$$

Lastly, we show that $\boldsymbol{\alpha}'$ has a lower average contract length. Since we leave the proportions of other durations constant, their contribution to the average contract length is unchanged. From (13) the contribution of durations k and j is given by

$$\begin{aligned}T_k &= ka_k + j\alpha_j \\ &= \rho\left(k + (k-j)^2\right) - kj\eta\end{aligned}$$

Likewise the contribution with α' is given by

$$\begin{aligned}T_j &= (j+1)\,\Delta\alpha_{j+1} + j\alpha_j' \\ &= \rho\,(j+2) - (j+1)\,j\eta\end{aligned}$$

Now we show that

$$T_k - T_{j+1} = \rho\left(k - (j+1) + (k-j)^2 - 1\right) - \eta\,(kj - (j+1)\,j)$$

Noting strict inequality (12) we have

$$\begin{aligned} T_k - T_{j+1} &> \eta \left[j \left(k - (j+1) + (k-j)^2 - 1 \right) - kj + (j+1) j \right] \\ &> \eta \left[j \left(k - j - 1 \right) \right] > 0 \end{aligned}$$

Hence

$$\bar{T} (\boldsymbol{\alpha}) - \bar{T} (\boldsymbol{\alpha}') = T_k - T_{j+1} > 0$$

the desired contradiction.

Hence, the GTE with the minimum contract length consistent with the observed $\bar{\omega}$ cannot have strictly positive sector shares which are not consecutive integers. There are at most two strictly positive sector shares.

To prove (c) for sufficiency, if $\bar{\omega}^{-1} = k \in Z_+$, then if $\alpha_k = 1 \in A(\bar{\omega})$. If $\alpha_k < 1$ any other element of $A(\bar{\omega})$ must involve strictly positive α_j and α_i with $j - i \geq 2$, which contradicts the parts (a) and (b) of the proposition already established.

For necessity, note that if $\bar{\omega}^{-1} \notin Z_+$, then no solution with only one contract length can yield the observed proportion of firms resetting prices. ■

9.4 Proof of Proposition 4.

Proof. First, note that if the proportions are given by the equations, then the rest of the proposition follows. I know show that these equations are indeed the maximising ones. Assume the contrary, that there is a distribution $\boldsymbol{\alpha}$ with $\alpha_i > 0$ where $1 < i < F$ which gives the maximum contract length. I show that this proposed optimum can be improved upon. Hence the optimum must involve only durations $\{1, F\}$ and the given equations follow automatically. So, let us take the proposed solution, with $\alpha_i > 0$. Let us redistribute the weight on sector i between $\{1, F\}$. In order to ensure that we remain in $A(\bar{\omega})$ the additional weights must satisfy

$$\begin{aligned} \Delta\alpha_i + \frac{\Delta\alpha_F}{F} &= \frac{\alpha_i}{i} \\ \Delta\alpha_i + \Delta\alpha_F &= \alpha_i \end{aligned}$$

which gives us

$$\begin{aligned} \Delta\alpha_F &= \alpha_i \frac{F(i-1)}{i(F-1)} \\ \Delta\alpha_1 &= \alpha_i \frac{F-i}{i(F-1)} \end{aligned}$$

The resulting Change in the average contract length is

$$\begin{aligned} \Delta\bar{T} &= \alpha_i \left[\frac{F\,(i-1)}{i\,(F-1)}\,(F-i) - \frac{F-i}{i\,(F-1)}\,(i-1) \right] \\ &= \frac{\alpha_i\,(i-1)\,(F-1)}{i\,(F-1)}\,[F-1] > 0 \end{aligned}$$

The desired contradiction. Given that all contracts must be either 1 or F periods long, the rest of the proposition follows by simple algebra. ■

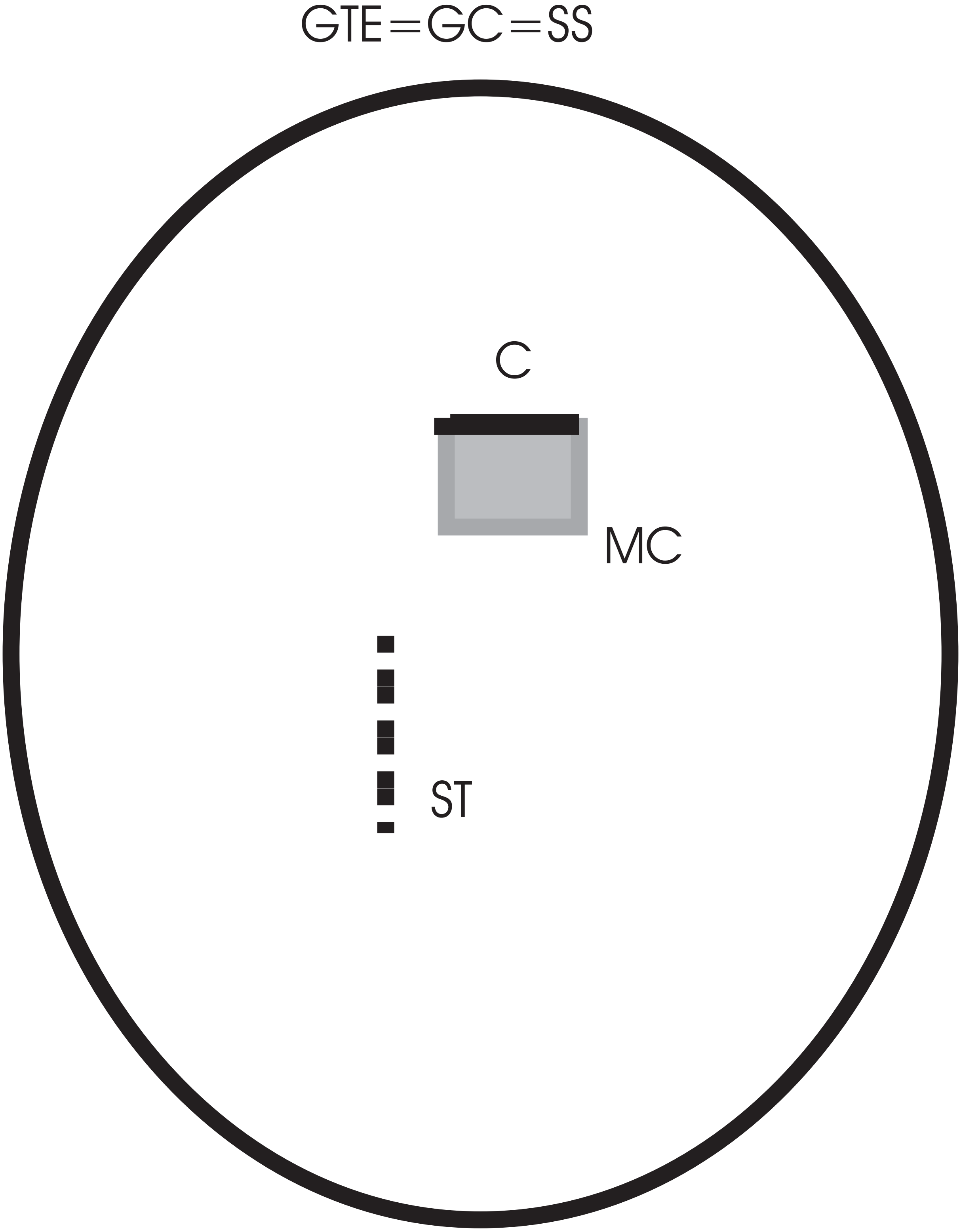


Fig.1. The Typology of contract types.

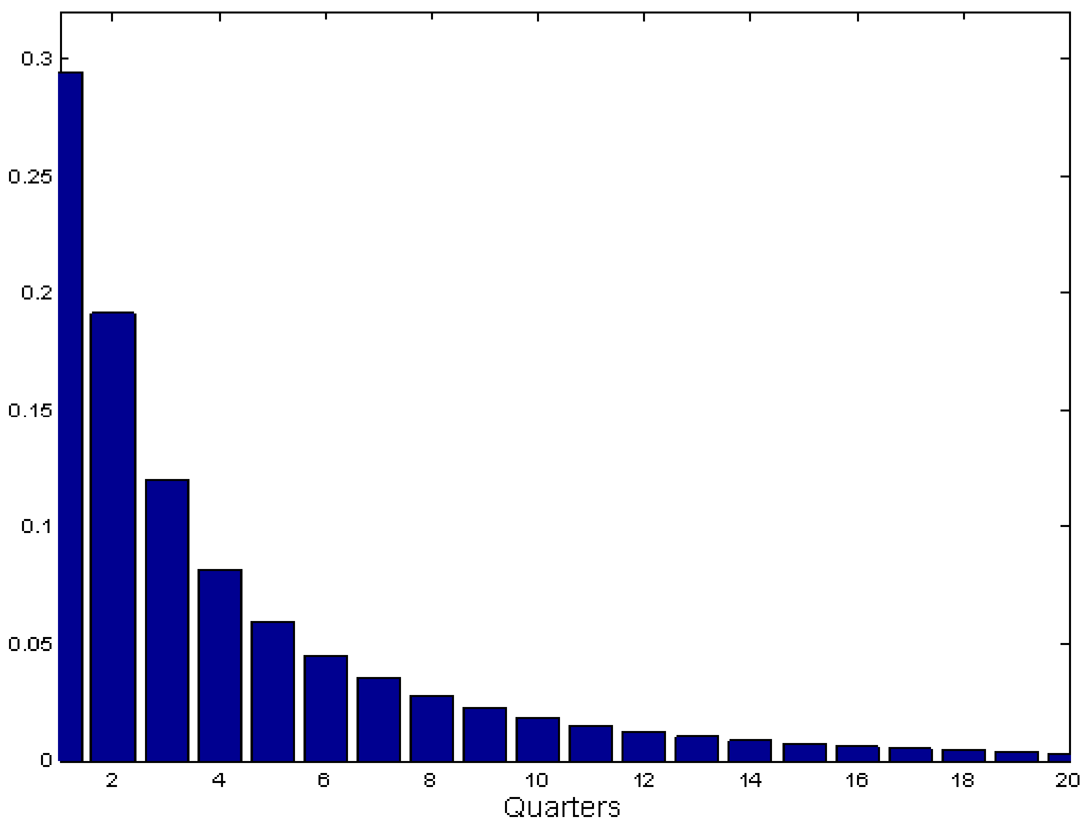


Figure 2: The BK Distribution of Contract Lenghts

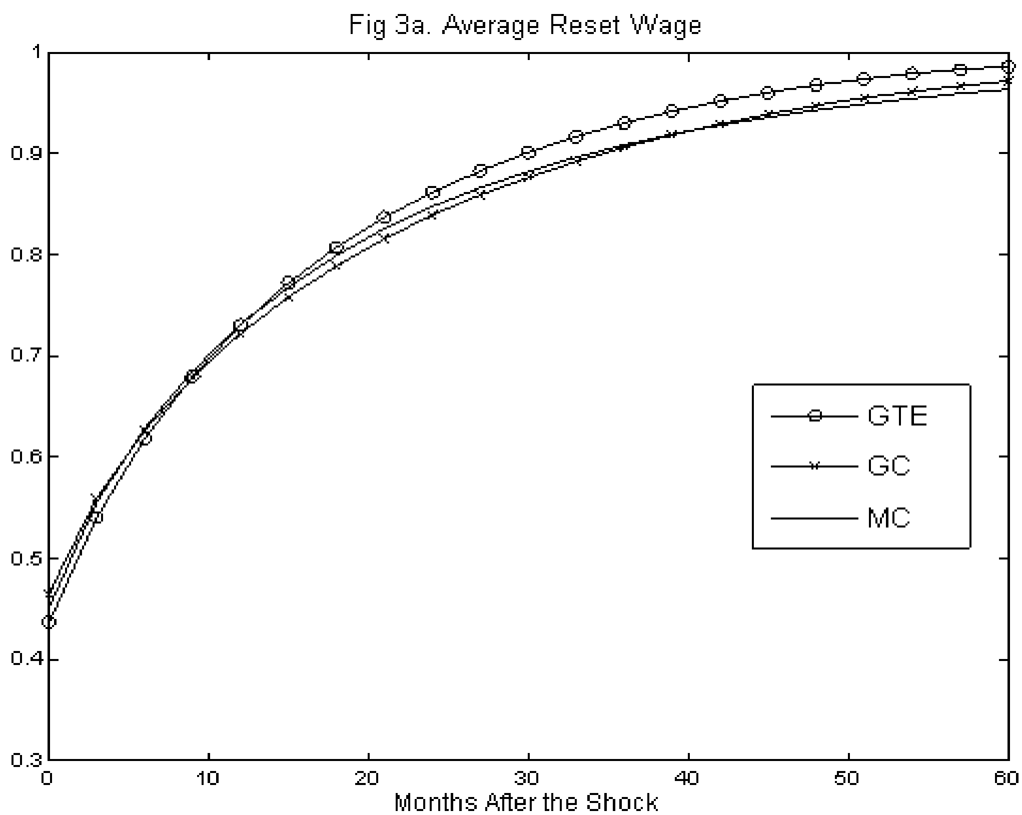


Fig 3a. Average Reset Wage
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0
10
20
30
40
50
60
Months After the Shock
GTE
GC
MC

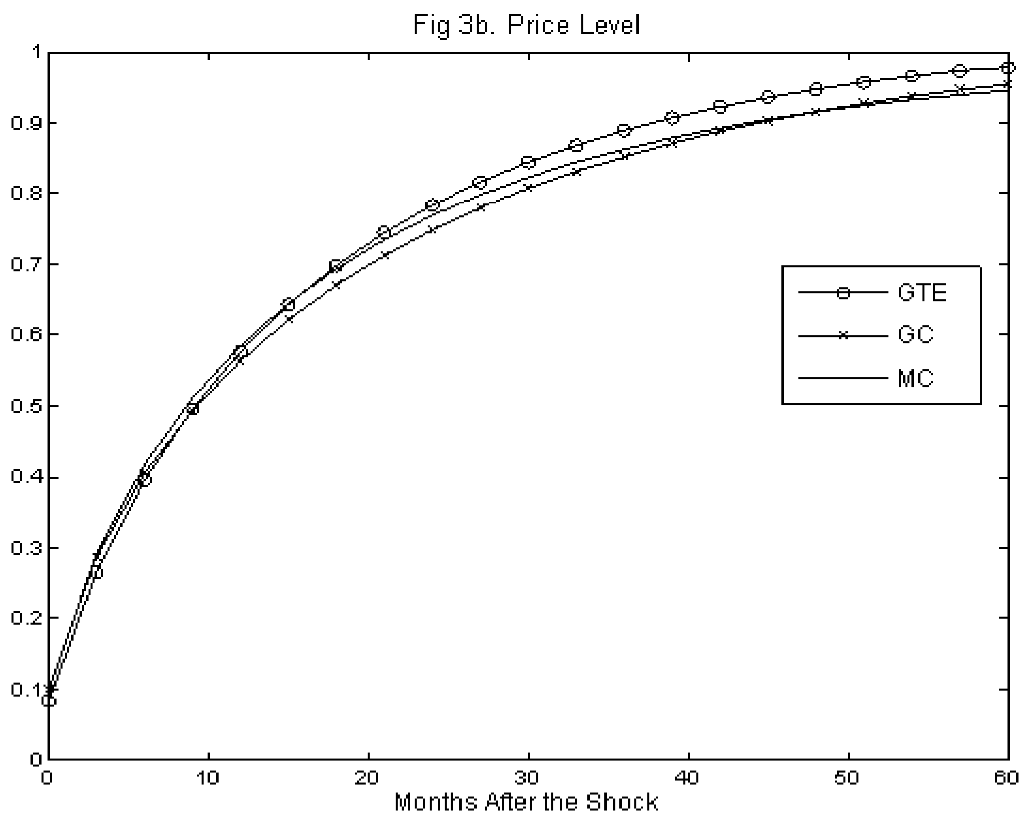
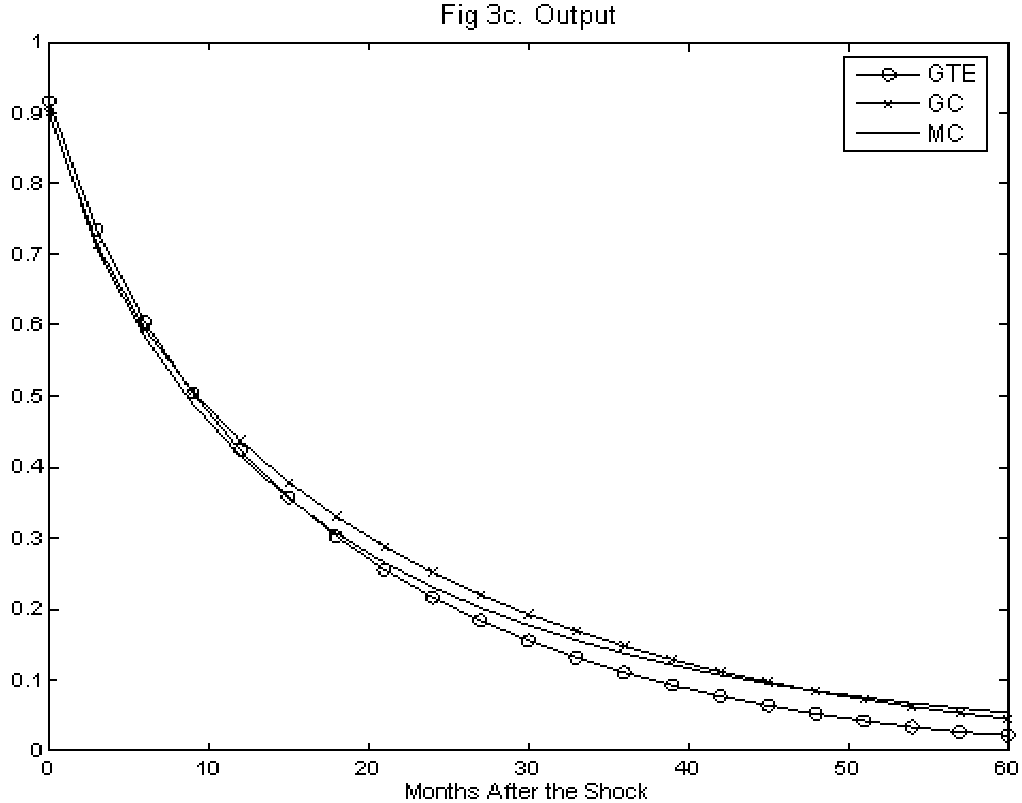


Fig 3b. Price Level
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60
Months After the Shock
GTE
GC
MC

Fig 3c. Output



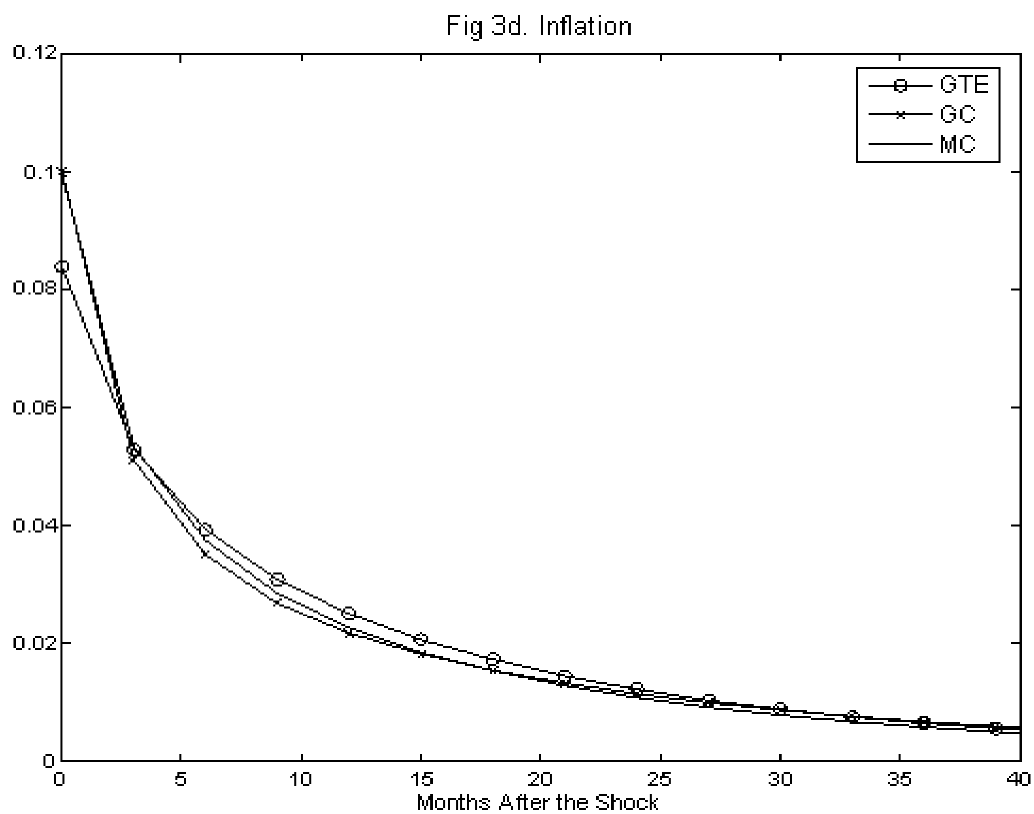


Fig 3d. Inflation
GTE
GC
MC
0.12
0.1
0.08
0.06
0.04
0.02
0
0
5
10
15
20
25
30
35
40
Months After the Shock

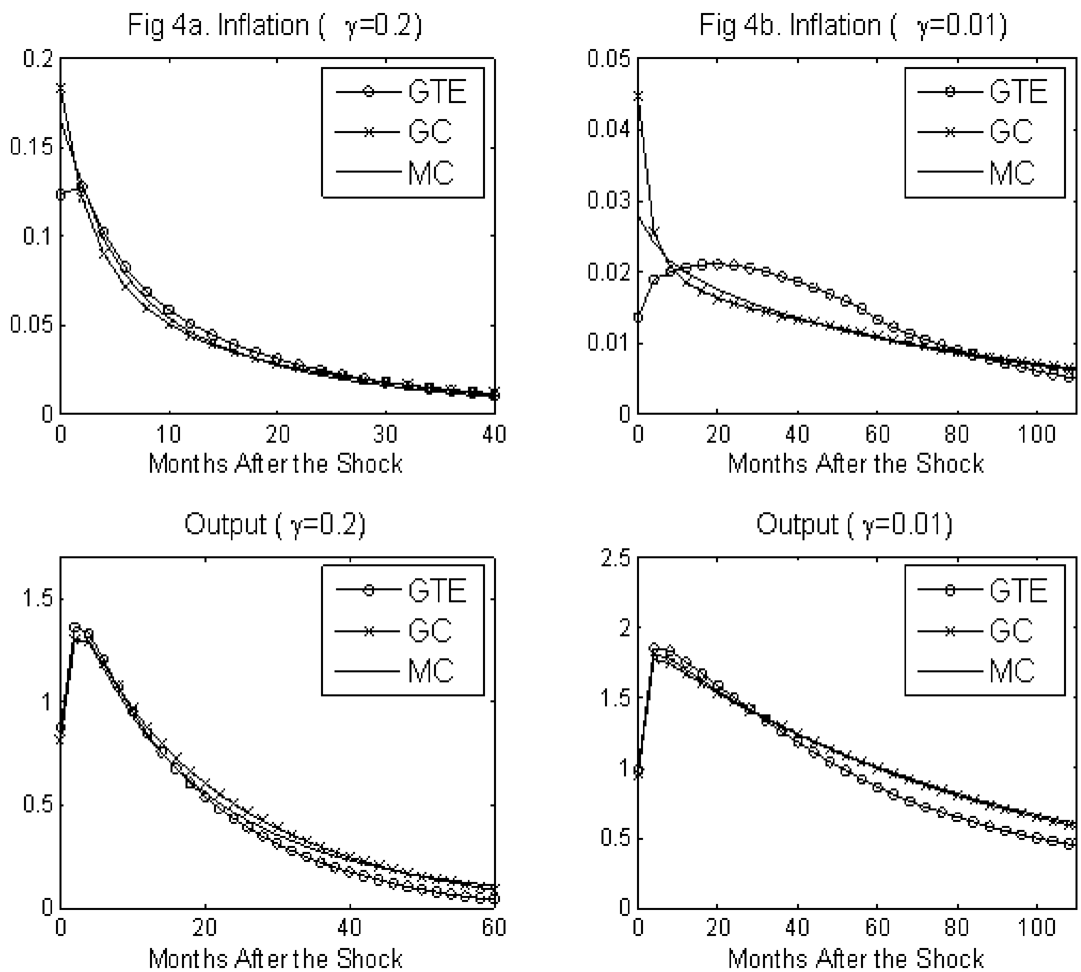


Figure 4: Serial Correlation in Monetary growth