Endogenous Cycles in Optimal Monetary Policy with a Nonlinear Phillips Curve

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Abstract

There is by now a large consensus in modern monetary policy. This consensus has been built upon a dynamic general equilibrium model of optimal monetary policy with sticky prices *a la* Calvo and forward looking behavior.

In this paper we extend this standard model by introducing nonlinearity into the Phillips curve. As the linear Phillips curve may be questioned on theoretical grounds and seems not to be favoured by empirical evidence, a similar procedure has already been undertaken in a series papers over the last few years, e.g., Schaling (1999), Semmler and Zhang (2004), Nobay and Peel (2000), Tambakis (1999), and Dolado et al. (2004). However, these papers were mainly concerned with the analysis of the problem of inflation bias, by deriving an interest rate rule which is nonlinear, leaving the issues of stability and the possible existence of endogenous cycles in such a framework mostly overlooked.

Under the specific form of nonlinearity proposed in our paper (which allows for both convexity and concavity and secures closed form solutions), we show that the introduction of a nonlinear Phillips curve into a fully deterministic structure of the standard model produces significant changes to the major conclusions

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regarding stability and the efficiency of monetary policy in the standard model. We should emphasize the following main results: (i) instead of a unique fixed point we end up with multiple equilibria; (ii) instead of saddle–path stability, for different sets of parameter values we may have saddle stability, totally unstable and chaotic fixed points (endogenous cycles); (iii) for certain degrees of convexity and/or concavity of the Phillips curve, where endogenous fluctuations arise, one is able to encounter various results that seem interesting. Firstly, when the Central Bank pays attention essentially to inflation targeting, the inflation rate may have a lower mean and is certainly less volatile; secondly, for changes in the degree of price stickiness the results are not are clear cut as in the previous case, however, we can also observe that when such stickiness is high the inflation rate tends to display a somewhat larger mean and also higher volatility; and thirdly, it shows that the target values for inflation and the output gap (π_t^*, x^*), both crucially affect the dynamics of the economy in terms of average values and volatility of the endogenous variables — e.g., the higher the target value of the output gap chosen by the Central Bank, the higher is the inflation rate and its volatility — while in the linear case only the π_t^* does so (obviously, only affecting in this case the *level* of the endogenous variables). Moreover, the existence of endogenous cycles due to chaotic motion may raise serious questions about whether the old dictum of monetary policy (that the Central Bank should conduct policy with discretion instead of commitment) is not still very much in the business of monetary policy.

Keywords: Optimal monetary policy, Interest Rate Rules, Nonlinear Phillips Curve, Endogenous Fluctuations and Stabilization

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1 Introduction

Since the early 1990s we have witnessed an increasing consensus in the conduct of modern monetary policy. Goodfriend and King (1997) have labelled this new consensus as "The New Neoclassical Synthesis and the Role of Monetary Policy", while Clarida et al. (1999) called it the "The Science of Monetary Policy: A New Keynesian Perspective". This new framework is a natural extension of the seminal idea developed by Taylor (1993), in which the central bank should conduct monetary policy through an aggressive and publicly known rule with commitment. In fact, this emerging consensus besides turning upside down the basic prescriptions of monetary and fiscal policies of the old Neoclassical Synthesis of the 60's and 70's, has also led to a standard DGEM so successful that, as Laurence Ball has recently put it "the model is so hot that the Keynesians and Classicals fight over who gets credit for it" (2005, 265).

The standard model in this new strand of literature is the old IS/LM model extended with micro foundations, forward looking expectations and price rigidity following the seminal work of Calvo (1983). It has a conventional IS curve, derived from an optimal control problem in which families evaluate the trade-off between consumption vs saving and between leisure *vs* labour, in which present output is a function of real expected interest rates (negatively), and positively of future expected output and of aggregate demand shocks, these taken as e.g. government expenditures.

The second major element of this new synthesis is an aggregate supply function expressed in terms of a traditional Phillips Curve, but now also derived from microeconomic principles, in which in any period of time a significant proportion of firms cannot adjust prices instantaneously due to market imperfections. In this new Phillips curve, it has been largely assumed that the rate of inflation is a linear positive function of the output gap, for a given level of expected inflation, with an added supply shock.

Finally, the model is closed by stipulating the behavior of the central bank. If this bank controls the short term interest rate, then it has to somehow present a rule such that inflation and output (or output gap) are ingredients to be taken into account in the maximization of social welfare. For practical reasons, it has been assumed that the central bank has quadratic loss preferences over inflation and the output gap, such that its objective is to minimize the squared errors of these two variables with respect to their target values.

This standard model has been extended or refined in several ways over the last few years. The list would be so extensive that we mention here only what we consider to be more relevant as far as the basic structure of the model is concerned. Albanesi (2000, 2005) has introduced heterogeneous agents into the IS function. Endogenous price rigidity has been studied by Carboni and Ellison (2003) and Siu (2005), and an asymmetric loss function by the central bank has been considered, e.g., by Surico (2004). Different forms of learning have also been taken into account by Svensson (1999a), Semmler, Greiner and Zhang (2002), and Orphanides and Williams (2002), while adding investment to the model has been recently analyzed by Sveen and Weinke (2005). The model with an open economy framework can be found in Razen and Yuen (2001) or in McCallum and Nelson (2001).

In this paper we extend this standard model by introducing nonlinearity into the Phillips curve. As the linear Phillips curve may be questioned on theoretical grounds and seems not to be favoured by empirical evidence, a similar procedure has already been undertaken in a series papers over the last few years, e.g., Schaling (1999), Semmler and Zhang (2004), Nobay and Peel (2000), Tambakis (1999), and Dolado et al. (2004). However, these papers were mainly concerned with the analysis of the problem of inflation bias, by deriving an interest rate rule which is nonlinear, leaving the issues of stability and the possible existence of endogenous cycles in such a framework mostly overlooked. One possible justification for this fact is the type of nonlinearity that is introduced into the standard model, because, as it is well known in the literature, quadratic preferences by the central bank with a convex Phillips Curve as the one used by most of those papers does not secure closed form solutions.

In contrast, under the specific form of nonlinearity proposed in our paper (which allows for both convexity and concavity and secures closed form solutions), we show that the introduction of a nonlinear Phillips curve into a fully deterministic structure of the standard model produces significant changes to the major conclusions regarding stability and the efficiency of monetary policy in the standard model. We should emphasize the following main results: (i) instead of a unique fixed point we end up with multiple equilibria; (ii) instead of saddle–path stability, for different sets of parameter values we may have saddle stability, totally unstable and chaotic fixed points (endogenous cycles); (iii) for certain degrees of convexity and/or concavity of the Phillips curve, where endogenous fluctuations arise, one is able to encounter various results that seem interesting. Firstly, when the Central Bank pays attention essentially to inflation targeting, the inflation rate may have a lower mean and is certainly less volatile; secondly, for changes in the degree of price stickiness the results are not are clear cut as in the previous case, however, we can also observe that when such stickiness is high the inflation rate tends to display a somewhat larger mean and also higher volatility; and thirdly, it shows that the target values for inflation and the output gap (π_t^*, x^*), both crucially affect the dynamics of the economy in terms of average values and volatility of the endogenous variables — e.g., the higher the target value of the output gap chosen by the Central Bank, the higher is the inflation rate and its volatility — while in the linear case only the π_t^* does so (obviously, only affecting in this case the *level* of the endogenous variables).

The paper is organized as follows. In section 2, the nonlinear Phillips curve is discussed in some detail. Section 3 deals with the dynamic optimization analysis of the monetary policy model, while in section 4 is concerned with the study of the local and global properties of the dynamics. In section 5 we discuss some possible implications for monetary policy from the main results of the nonlinear model and section 5 concludes.

2 The Nonlinear Phillips Curve

> "Though analytically convenient, the linear model ignores much of the historical context underlying the original split between classical and Keynesian economics: under conditions of full employment, inflation appeared to respond strongly to demand conditions, while in deep recessions, it was relatively insensitive to changes in activity" Eric Schaling, 1998, p.3.

The Phillips curve can be presented in a general form as

$$\pi_t = F(x_t) + \beta E_t \pi_{t+1} + u_t \tag{1}$$

where $F(x_t)$ can be either a linear or a nonlinear function and the symbols stand for: π_t is the inflation rate; x_t is the output gap, E_t is an expectations operator, u_t is an autoregressive process of order lower than one and defined as $u_t = \rho u_{t-1} + \widehat{u}_t$, with $0 < \rho < 1$ and $\widehat{u}\tilde{\ } iid(0, \sigma_u^2)$. The new consensus takes this curve as a linear function, but some other shapes have also been proposed over the last decade. The basic rationale behind a linear Phillips curve consists of assuming that changes in the inflation rate, caused by a certain change in the output gap, are always of the same magnitude independently of the level of the output gap, that is, independently whether the economy is in a boom or in a deep recession.

Nevertheless, despite its extensive use in the recent surge of optimal IS/LM models, the linear version of the Phillips curve shows three major shortcomings, at least as the standard model of the new synthesis is concerned. Firstly, it is apparently mainly a result of analytical convenience as the consideration of some form of nonlinearity introduces large analytical complications to the basic structure of the model. Secondly, there is little evidence of linearity in the Phillips curve, and thirdly, the linear version seems to be extremely unrobust. The model's results are dramatically altered once one introduces no more than mild convexity or mild concavity into the Phillips curve. Therefore, apparently the basic results of the model behind the emerging new consensus remain valid as long as one is ready to accept an elasticity in the output-inflation trade-off exactly equal to one.

Given this result, it looks somewhat surprising that the nonlinear version of the Phillips curve has apparently been relegated to a secondary role in the recent surge of New Keynesian models based on forward looking expectations and microeconomic principles. And this seems surprising mainly for four basic reasons.

Firstly, the nonlinear version has a large theoretical appeal, as almost all original Keynesian models have arrived at shapes which are intrinsically nonlinear in both types of models, either those built upon microeonomic principles in the 80's and 90's or those following the old style developed upon ad hoc assumptions. That was in fact the original result of Phillips (1958) that has become a traditional presence in almost all macroeconomic textbooks, as well as the new Keynesian models developed more recently such as in Ball et al. (1988), Ball and Mankiw (1994), Stiglitz (1997), Eisner (1997), Sargent (1999), and Akerlof et al. (1996, 2001), to name just a few.

Secondly, the nonlinear Phillips curve may have large empirical relevance. For example, it may explain the inflationary bias of monetary policy as showed by Macklem (1995), and may also explain the asymmetric effects of monetary policy in several countries, as documented, e.g., by Cover (1992), Karras (1996), Karras and Stokes (1999), Kaufman (2001), Weise (1999) and Peersman and Smets (2001). Moreover, a nonlinear shape can also explain another crucial fact of modern business cycles: benefice ratios of inflation are smaller than the sacrifice ratios of deflation (see Filardo, 1998).

Thirdly, empirical evidence based upon econometric tests somewhat favours the nonlinear shape in detriment of the linear version. In fact, tests have encountered evidence of a slightly convex function in various studies. This is the case of Clark et al. (1996) who found significant nonlinearity for the US, Debelle and Laxton (1997) arrive at similar conclusions for Canada and the United Kingdom, while Dupasquier and Ricketts (1998) analyzed the data for Canada and US and concluded that there is stronger evidence of nonlinearity for the US Phillips curve. Other studies on the shape of the Phillips curve include Eliasson (2001), who estimates the relation between inflation and unemployment for Australia, Sweden and the United States, and finds evidence of nonlinearity (nevertheless, without making any strong statement about a convex or concave shape) which seems stronger for Australia and Sweden than for the US. In contrast, Eisner (1997) and Stiglitz (1997) found evidence pointing to a concave Phillips curve for the US data.

Despite the fact that the evidence apparently favours the nonlinear shape, there are voices in the opposition. For instance, Gordon (1997) and Blinder (1998) emphasize that the Phillips curve is linear for the US macroeconomic data, while Aguiar and Martins (2002) find evidence of a linear Phillips curve for the euro zone. Finally, Yates (1998b) found no significant non-linearity for the UK for the period between 1966 and 1994 (quarterly) and for various G7 countries for the period 1800-1938.

Finally, the policy implications of the nonlinear shape seem to be of significant importance and it is worthwhile to mention some of them here. For example, take the case of a convex Phillips curve. In order to have an effective control inflation (low and stable), the Central Bank has to prevent inflation from leaving the part of the curve in which the nonlinearity is relatively mild. Secondly, in order to maximize social welfare, inflation should be prevented from taking off, as the costs of deflation are larger than the benefits of inflation. Thirdly, the efficacy of monetary policy is largely asymmetrical: excess supply (negative output gap) has a much lower impact on inflation than excess demand (positive output gap). On the other hand, if the curve is concave, then the risks of taking more aggressive policy measures in order to reduce unemployment might be lower than if the curve were both convex or linear.

2.1 Nonlinear functions

In the literature there are various versions of a nonlinear shape of the Phillips curve (for a survey see Dupasquier and Ricketts, 1998). One framework that leads to a convex shape is price setting with *capacity constraints*. As Filardo (1998) and Semmler and Zhang (2004) explain, these constraints mean that the ability of firms to expand output tends to decline with accumulated capacity, and thus, increased demand tends progressively to be translated into higher inflation rather than higher output. Moreover, when demand declines firms prefer to accumulate stocks instead of large reductions in prices. This reasoning has also been shared by Clark et al. (1996), Schalling (1999), Laxton et al. (1999), Zhang and Semmler (2003), Dolado et al. (2004) and Tambakis (2004), among others, who explore the economic consequences of a convex aggregate supply function.

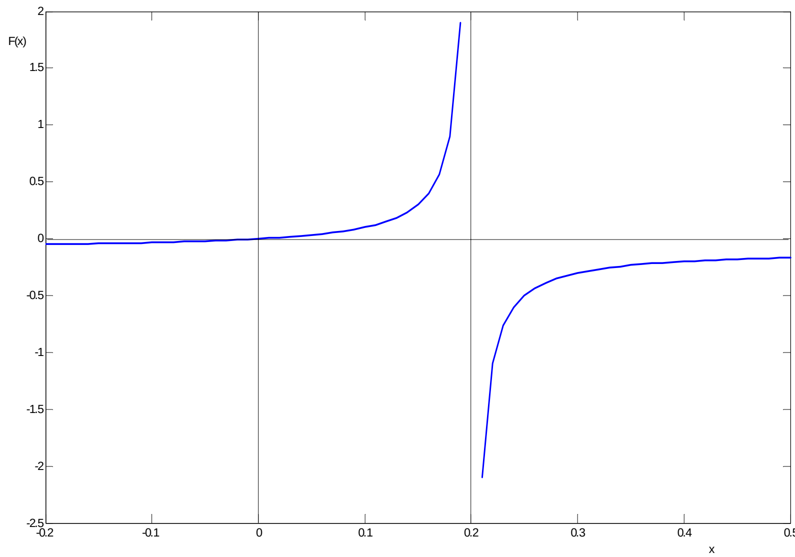

Figure 1: A convex Phillips curve, for $\lambda = 0.5$ and $\theta = 10$.

As far as the convex version of the Phillips curve is concerned, almost all studies assume the type of function firstly introduced by Schalling (1999)

$$F(x_t) = \frac{\lambda x_t}{1 - \lambda \theta x_t}, \theta \geq 0 \tag{2}$$

where $0 < \lambda < 1$ is the degree of price flexibility (the level of the inflation/output elasticity, a low value for this parameter represents a high level of price rigidity) and θ is a convexity parameter. The larger the value of the latter parameter, the larger is the degree of convexity of the function; for $\theta = 0$ we return to the linear new Keynesian Phillips curve.[1] For instance, assuming $\lambda = 0.5$ and $\theta = 10$, the function displays as in figure 1.

The capacity constraints argument has been challenged by Eisner (1997) and Stiglitz (1997), who put forward arguments in favor of a concave Phillips curve. Their explanations are also based upon microeconomic foundations, in particular, their results arise due to monopolistic competition. Firms act as monopolistic entities and thus they tend to

[1] Note that function (2) has economic meaning only for $x_t < 1/(\lambda\theta)$. Zhang and Semmler (2003) address this issue. They only consider the segment of the curve for which $F'' < 0$; this is precisely the part of the curve where the referred condition holds.

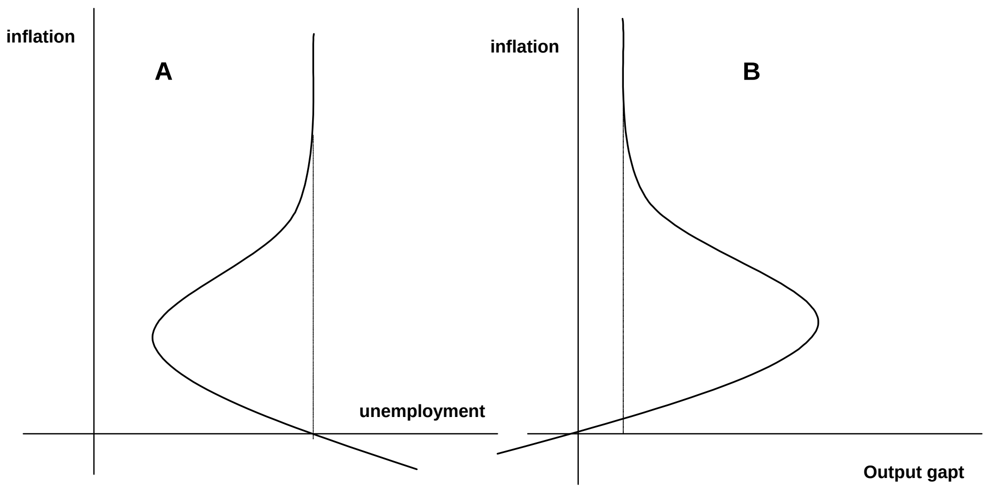

Figure 2: The nonlinear Phillips Curve following Akerlof et al. (2001)

react to increased demand with progressively higher increases on output rather than by causing inflationary pressures. In other words, monopolistic firms do not have to respond necessarily with an increase in prices when demand rises; if they want to expand their market share, rising prices is not the best response, if other options are available. Equation (2) continues to be a possible representation of the Phillips curve when its shape is concave, all that is needed is to impose $\theta < 0$.

The convex *vs* concave controversy has been significantly reconsidered by Akerlof et al. (1996, 2001). They present a thorough exposition of arguments that justify a peculiar shape of the Phillips curve (see Figure 2). In panel A, this curve shows a trade-off between unemployment and inflation which is negative for very low levels of inflation, becomes positive for moderate levels, ending up with the traditional vertical line for high levels of inflation. Translating this reasoning to the output gap *vs* inflation locus (see panel B), the curve has three branches.

The first one is convex which implies a positively sloped Phillips curve, followed by a second phase in which the curve becomes concave for higher values of inflation. Finally, for very high levels of inflation the curve turns into a vertical straight line (demand shocks are reflected only on the inflation axes). The economic intuition for this awkward shape can go as follows. When inflation is very low, inflation becomes relatively unimportant for individual actions. This occurs as the result of a negative trade-off of balancing the costs of obtaining and processing information (which are relevant in most circumstances) and the benefits from preventing against inflation (which are very low in this case due to very low inflation). That is, for low inflation, the private agents will just expect very small changes in inflation, even if there are economic signs pointing at another direction. If individuals expect low inflation when inflation is low, monetary authorities can make use of the convex part of the Phillips curve to increase the output gap and reduce unemployment.

Beyond a certain level of the inflation rate, the benefits from preventing against inflation overcome the costs of collecting and processing information, and in this case inflation becomes an important problem for private economic agents. These will take into consideration the available information to form expectations about future inflation, and attempts by the public authorities to reduce unemployment will in fact be counterproductive, leading to higher inflation and to higher unemployment.

Finally, for very high levels of inflation, the costs of collecting information are much lower than the benefits from preventing against higher inflation, inflation becomes the centre of attention of the whole economic process and any attempt from the authorities to reduce unemployment will not be successful, leading only to further inflation. In this case the curve becomes vertical.

2.2 A new nonlinear function

The functions presented above have some particular properties that turn them of little help for the purpose of this paper. Firstly, the nonlinear function that has been widely used in the literature (the one put forward by Schalling (1998)), does not allow for closed form solutions of the standard new Keynesian model, and therefore no significant conclusions could be reached under this scenario. Secondly, the view presented by Akerlof and associates seems very interesting but raises very complicated analytical problems that seem worthwhile to avoid for the moment.

Therefore, in this paper we have to revert to a somewhat different version of a nonlinear Phillips curve. The function we propose has two branches and is defined as

$$
\begin{aligned}
F(x_t) &= \lambda[(x_t - x^*)^{\phi} + (x^*)^{\phi}],\ x_0 > x^* \\
F(x_t) &= \lambda[(x^*)^{\phi} - (x^* - x_t)^{\phi}],\ x_0 < x^*.
\end{aligned}
\tag{3}
$$

Some characteristics of this function are worthwhile to mention. Firstly, ϕ is the parameter that leads to convexity, concavity or linearity in both branches. If $\phi = 1$, the two branches are identical and we are back to the linear case, where $F(x_t) = \lambda x_t$, which is totally similar to the function proposed by Schalling. Secondly, if we combine the different possible values that ϕ may assume, with the two different domains in which the

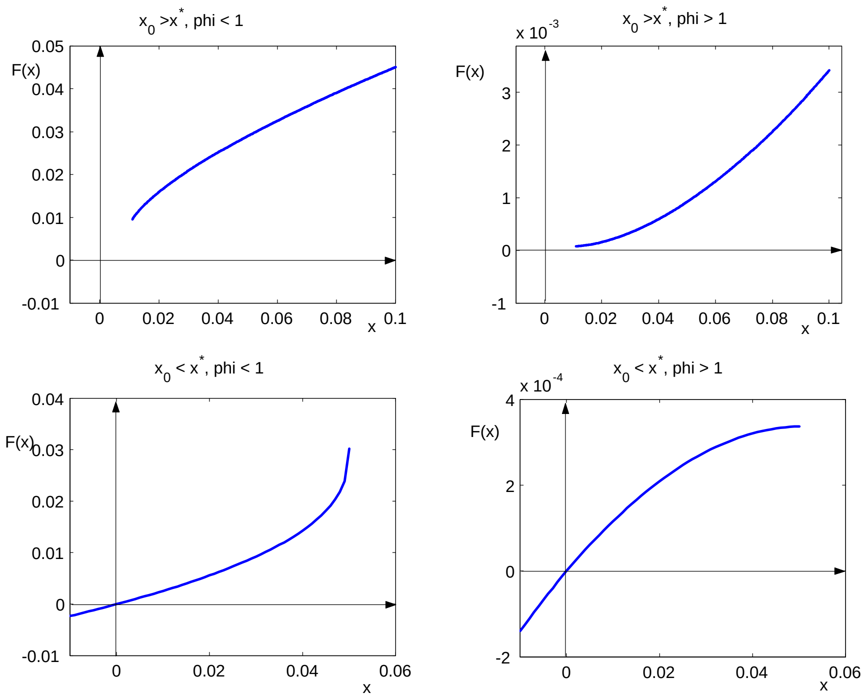

Figure 3: The nonlinear Phillips curve

branches are defined ($x_0 > x^*$ and $x_0 < x^*$), we may have a nonlinear Phillips curve that is concave for the initial values of the output gap lower than its target value ($x_0 < x^*$), and convex for the opposite case (that is, for $x_0 > x^*$). In fact, what we do with such a specification is to allow for concavity for low positive and/or for negative values of the output gap, and to allow for convexity for high values of the output gap. Figure 3 shows the four different possibilities that may occur with our specification of the nonlinear Phillips curve. The scenario that we have just described above is simply the combination of the two right hand panels, for a common level of the target value for the output gap (x^*).

The third major characteristic that we would like to mention is the fact that the function with the two branches allow us to take into consideration the " historical context underlying the original split between classical and Keynesian economics", because under conditions of full employment (that is, the classical case), inflation will strongly respond to excess demand (high values of the output gap), while in deep recessions (extreme Keynesian cases), inflation becomes relatively insensitive to changes in aggregate demand.

Finally, there is a practical issue which seems of large importance for the modelling of Central Bank behavior that can be easily accommodated by the model. When the Central Bank decides to go for a cut or an increase in short term interest rates, it does so by assuming that the present perceived condition of the economy in terms of the output gap (the initial condition of our model) is below or above its target value. Therefore, what the model will show us is that it is not irrelevant for the Central Bank to have $x_0 > x^*$ or $x_0 < x^*$ in order to try to achieve its target value for the output gap. This issue is related to

3 The Model

Following Goodfriend and King (1997), Clarida et al. (1999), Svensson (1999b), Woodford (2003) and Svensson and Woodford (2003), we assume a standard optimal monetary policy model with sticky prices and forward looking behavior. This model considers an objective function relating to Central Bank preferences, which is subject to two constraints that reflect the behavior of the economy's private sector. The underlying problem corresponds to an optimal control setup - the monetary authorities make decisions, in $t = 0$, regarding future periods. Thus, one may interpret this setup as a framework with Central Bank commitment over policy decisions. Given the well understood inflationary bias problem, commitment rather than discretion arises as one solution to avoid inefficiently high steady state inflation rates.

Consider that at the initial period, $t = 0$, the Central Bank commits to a time path for the nominal interest rate, i_t, that is optimal given its objective of minimizing through time the difference between the output gap (x_t) and the corresponding target value (x^*) and between the inflation rate (π_t) and its target value chosen by the monetary authority (π^*). Formally, the Central Bank maximizes the value of function V_0

$$V_0 = -\frac{1}{2} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\alpha (x_t - x^*)^2 + (\pi_t - \pi^*)^2 \right] \right\} \tag{4}$$

Parameter $1/2 < \beta < 1$ is an intertemporal discount factor and $0 \leq \alpha \leq 1$ represents the relative weight given by the Central Bank to the real stabilization objective in the utility function.

In (4), the inflation rate is measured as the percentage change in the price level between two consecutive periods, while x_t corresponds to the difference between the effective level of output and its potential level, both measured in logs, and it should be noticed that potential output is the level of output correspondent with a situation where wages and prices are perfectly flexible.

The objective function (4) is subject to two constraints. The first is a forward looking IS equation in which the output gap depends on the real interest rate and the expected future levels the output gap and takes the form

$$x_t = -\varphi(i_t - E_t\pi_{t+1}) + E_t x_{t+1} + g_t, \quad x_0 \text{ given} \tag{5}$$

parameter $\varphi > 0$ is an interest elasticity, $E_t\pi_{t+1}$ and $E_t x_{t+1}$ are the private sector expectations about next period's levels of the inflation rate and the output gap, and g_t translates eventual shocks over government purchases and/or potential output. This latter variable is defined through an autoregressive Markov process: $g_t = \mu g_{t-1} + \widehat{g}_t$, $0 \leq \mu \leq 1$, $\widehat{g}_t \sim iid(0, \sigma_g^2)$.

The second constraint, which characterizes an aggregate supply function, is a new Keynesian Phillips curve largely discussed in the previous section. We should recall that in order to avoid analyzing only its linear standard form, we introduce a function in general form $(F(x_t))$ which may allow linearity and nonlinearity in the relation between inflation and output.

$$\pi_t = F(x_t) + \beta E_t \pi_{t+1} + u_t, \quad \pi_0 \text{ given} \tag{6}$$

where u_t is defined as $u_t = \rho u_{t-1} + \widehat{u}_t$, $0 \leq \rho \leq 1$, $\widehat{u}_t \sim iid(0, \sigma_u^2)$ and represents possible cost push shocks.

The intertemporal objective function (4), subject to constraints (5) and (6), can be solved using the usual tools of dynamic optimization. In our case we follow optimal control. The current value Hamiltonian function takes the form

$$\begin{aligned}
\aleph(i_t, x_t, \pi_t) &= -\frac{1}{2}[\alpha(x_t - x^*)^2 + (\pi_t - \pi^*)^2] \\
&\quad + \beta q_{t+1}\left\{\varphi\left[i_t - \frac{1}{\beta}\pi_t + \frac{1}{\beta}F(x_t) + \frac{1}{\beta}u_t\right] - g_t\right\} \\
&\quad + \beta p_{t+1}\left[\frac{1-\beta}{\beta}\pi_t - \frac{1}{\beta}F(x_t) - \frac{1}{\beta}u_t\right]
\end{aligned} \tag{7}$$

Variables q_t and p_t are shadow-prices associated with x_t and π_t, respectively. The expectations operators for next period inflation and output gap are ignored hereafter, under the hypothesis of a fully deterministic *perfect foresight* framework that we adopt from now on.

First order necessary conditions are

$$\aleph_i = 0 \Longrightarrow q_{t+1} = 0 \tag{8}$$

$$\beta q_{t+1} - q_t = \alpha(x_t - x^*) - \varphi q_{t+1} F'(x_t) + p_{t+1} F'(x_t) \tag{9}$$

$$\beta p_{t+1} - p_t = \pi_t - \pi^* + \varphi q_{t+1} - (1-\beta) p_{t+1} \tag{10}$$

$$\lim_{t \longrightarrow +\infty} q_t \beta^t x_t = \lim_{t \longrightarrow +\infty} p_t \beta^t \pi_t = 0 \tag{11}$$

Given (8), one simplifies (9) and (10)

$$p_{t+1} = -\alpha \frac{x_t - x^*}{F'(x_t)} \tag{12}$$

and

$$p_{t+1} - p_t = \pi_t - \pi^* \tag{13}$$

Combining (12) and (13), we arrive at an equation in which the major arguments are our two endogenous state variables (x_t and π_t)

$$\frac{x_{t+1} - x^*}{F'(x_{t+1})} = \frac{x_t - x^*}{F'(x_t)} - \frac{1}{\alpha\beta}[\pi_t - F(x_t) - \beta\pi^*] \tag{14}$$

The reduced form of our dynamic problem is comprised of a system of two difference equations, which are equation (14) and the Phillips curve. Recall that the solution of this system corresponds to the optimal time paths of the two endogenous variables, which are obtained when the Central Bank chooses the interest rate time trajectory that minimizes the losses relatively to the target values of inflation and the output gap.

The optimal interest rate rule (the interest rate trajectory underlying the solution of the optimal control problem) may produce stable or unstable paths for the inflation rate and the output gap. The stability concern should be vital for monetary policy evaluation, and therefore we should concentrate on the perception of the dynamics underlying the system composed by equations (6) and (14).

3.1 The linear case

As the linear case constitutes the benchmark relatively to which we will examine the implications of a non linear Phillips curve, let us begin by reviewing this case. Consider a linear function $F(x_t) = \lambda x_t$, with $0 < \lambda < 1$ the degree of price flexibility (inflation-output elasticity). A low value of this parameter means a high level of price stickiness or rigidity. For this F function, equation (6) represents the standard new Keynesian Phillips curve.

The linear model is synthesized in the following system

$$\begin{aligned} \pi_{t+1} &= \frac{1}{\beta}\pi_t - \frac{\lambda}{\beta}x_t \\ x_{t+1} &= \left(1+\frac{\lambda^2}{\alpha\beta}\right)x_t - \frac{\lambda}{\alpha\beta}(\pi_t - \beta\pi^*) \end{aligned} \tag{15}$$

which is linear and, hence, local and global dynamic results coincide. To (15) corresponds the following Jacobian matrix,

$$\begin{bmatrix} \frac{1}{\beta} & -\frac{\lambda}{\beta} \\ -\frac{\lambda}{\alpha\beta} & 1+\frac{\lambda^2}{\alpha\beta} \end{bmatrix} \tag{16}$$

Noticing that

$$1 - Det(J) = \frac{1-\beta}{\beta} > 0$$

$$1 - Tr(J) + Det(J) = -\frac{\lambda^2}{\alpha\beta} < 0$$

$$1 + Tr(J) + Det(J) = 2\frac{1+\beta}{\beta} + \frac{\lambda^2}{\alpha\beta} > 0$$

we assure that one, and only one, of the eigenvalues of J is located inside the unit circle and a saddle-path stability arises as the only possible outcome for the proposed optimal monetary policy problem. The steady state point $(\bar{\pi}, \bar{x}) = \left(\pi^*, \frac{1-\beta}{\lambda}\pi^*\right)$ is reached only if the initial point is over the saddle-path or stable trajectory, which we can compute as follows. For ε_1 the eigenvalue of J that lies inside the unit circle, the vector $P = \begin{bmatrix} 1 & \frac{1-\beta\varepsilon_1}{\lambda} \end{bmatrix}'$ is a corresponding eigenvector.

Therefore, the stable trajectory is

$$x_t - x^* = \frac{1-\beta\varepsilon_1}{\lambda}(\pi_t - \pi^*) \tag{17}$$

Since $-1 < \varepsilon_1 < 1$, the slope of the stable trajectory is positive, indicating that for a pair (π_0, x_0) over the stable arm, the convergence to the steady state succeeds and the inflation rate and the output gap will both decline/rise towards the fixed point.

The linear case allows for a single possibility regarding the qualitative behavior of the variables' evolution over time (saddle-path stability). The introduction of a non linear Phillips curve will radically change this, even for mild degrees of convexity or concavity, as we will show in the following sections.

3.2 The nonlinear case.

We consider two alternative cases for the introduction of nonlinearity into the Phillips curve, both taking into account that the target value for the output gap is likely to be a non negative value ($x^* \geq 0$).[2] The two cases can be separately analyzed and they depend essentially on whether the currently perceived output gap is lower than its target value ($x_0 > x^*$) or higher than the perceived value by the Central Bank ($x_0 < x^*$). Defining a positive parameter ϕ, the two specific nonlinear functions are the following

$$\begin{aligned} F(x_t) &= \lambda[(x_t - x^*)^\phi + (x^*)^\phi],\ x_0 > x^* \\ F(x_t) &= \lambda[(x^*)^\phi - (x^* - x_t)^\phi],\ x_0 < x^* \end{aligned}$$

Notice that the two above functions contain the properties required for our Phillips curve to assume the desired generic form. First, for $\phi = 1$, the functions are identical and we are back to the linear case. Second, the function can be either concave or convex for both cases depending on the value of ϕ and on whether the initial condition is to the left/right of the target value of the output gap.

Under the specific nonlinear functions chosen for $F(x_t)$, the following two systems should be evaluated in order to derive any kind of results potentially useful for policy implications:

i) for $x_0 > x^*$:

$$\begin{aligned} \pi_{t+1} &= \frac{1}{\beta}\pi_t - \frac{\lambda}{\beta}\left[(x_t - x^*)^\phi + (x^*)^\phi\right] \\ x_{t+1} &= x^* + \left\{(x_t - x^*)^{2-\phi} - \frac{\lambda\phi}{\alpha\beta}\left[\pi_t - \lambda((x_t - x^*)^\phi + (x^*)^\phi) - \beta\pi^*\right]\right\}^{1/(2-\phi)} \end{aligned} \tag{18}$$

i) for $x_0 < x^*$:

$$\begin{aligned} \pi_{t+1} &= \frac{1}{\beta}\pi_t - \frac{\lambda}{\beta}\left[(x^*)^\phi - (x^* - x_t)^\phi\right] \\ x_{t+1} &= x^* - \left\{(x^* - x_t)^{2-\phi} - \frac{\lambda\phi}{\alpha\beta}\left[\pi_t - \lambda((x^*)^\phi - (x^* - x_t)^\phi) - \beta\pi^*\right]\right\}^{1/(2-\phi)} \end{aligned} \tag{19}$$

Systems (18) and (19) change significantly the results of the monetary policy problem. For several sets of parameter values we will regard, along

[2] It does not seem likely that the Central Bank has a negative value for the target value of the output gap, for reasons that, we hope, are easily understood.

the following section, that deterministic endogenous cycles are generated. Chaotic motion will characterize the evolution of the output gap and of the inflation rate over time (there is sensitive dependence on initial conditions). Note also that the existence of a unique steady state locus will certainly be replaced by a scenario of multiple equilibria.

4 Local and global dynamics

In this section we present the dynamic behavior of the two models defined in (18) and (19). There are several equilibrium points real and/or complex conjugate but only one, for each model, is consequent with perfectly acceptable values for the endogenous variables. This equilibrium point is analytically determined and discussed in detail in what follows. Saddle-node bifurcations are possible and a Neimark-Sacker (or torus breakdown) bifurcation route to chaos is encountered when the parameter β is varied. Since we have power functions we have to consider positive square powers in order to ensure the existence of real iterations. This is the reason why in the numerical examples presented below we almost always assume $\phi = 1.5$ (which gives $1/(2-\phi) = 2$).

4.1 The case: $x_0 > x^*$

Proposition 1 *The dynamical system (18) has always an unstable equilibrium point given by*

$$\pi_t = \pi^*,\ x_t = x^* + \left(\frac{1-\beta}{\lambda}\pi^* - (x^*)^\phi\right)^{1/\phi}.$$

Proof. In order to compute the equilibrium points we have to solve the nonlinear algebraic system

$$\begin{aligned}
\pi_t &= \frac{1}{\beta}\pi_t - \frac{\lambda}{\beta}\left[(x_t - x^*)^\phi + (x^*)^\phi\right] \\
x_t &= x^* + \left\{(x_t - x^*)^{2-\phi} - \frac{\lambda\phi}{\alpha\beta}\left[\pi_t - \lambda((x_t - x^*)^\phi + (x^*)^\phi) - \beta\pi^*\right]\right\}^{1/(2-\phi)}
\end{aligned}$$

The first equation gives

$$\pi_t = \frac{\lambda}{1-\beta}\left((x_t - x^*)^\phi + (x^*)^\phi\right) \tag{20}$$

and from the second we can obtain

$$x_t = x^* + \left(\frac{1}{\lambda}\pi_t - \frac{\beta}{\lambda}\pi^* - (x^*)^\phi\right)^{1/\phi}. \tag{21}$$

Now, replacing (21) into (20) we calculate the fixed point

$$(\pi_t, x_t) = \left(\pi^*, x^* + \left(\frac{1-\beta}{\lambda}\pi^* - (x^*)^\phi\right)^{1/\phi}\right), \tag{22}$$

with the positivity condition

$$\frac{1-\beta}{\lambda}\pi^* - (x^*)^\phi > 0 \Rightarrow \pi^* > \frac{\lambda\,(x^*)^\phi}{1-\beta}$$

For $\phi = 1$ in (22) we obtain the equilibrium discussed for the linear case.

The stability of this fixed point is analyzed using the sufficient conditions

$$\begin{cases} 1 + trace(J) + \det(J) > 0 \\ 1 - trace(J) + \det(J) > 0 \\ 1 - \det(J) > 0 \end{cases},$$

where J is the Jacobian matrix computed at the fixed point. We have, then,

$$\begin{cases} 2 + \dfrac{2}{\beta} + \dfrac{\lambda^2\phi^2}{\alpha\beta\,(2-\phi)}u^2 > 0 \;\text{ iff }\; \phi < 2 \\ -\dfrac{\lambda^2\phi^2}{\alpha\beta\,(2-\phi)}u^2 > 0 \;\text{ iff }\; \phi > 2 \\ 1 - \dfrac{1}{\beta} > 0 \;\text{ iff }\; \beta > 1 \end{cases}$$

where

$$u = \left(\frac{(1-\beta)}{\lambda}\pi^* - (x^*)^\phi\right)^{\frac{\phi-1}{\phi}}.$$

This means that there is no ϕ such that the equilibrium is stable.

The other real fixed point can not be determined analytically, but numerical calibration for the parameters show that this fixed point corresponds to very large values and it is always unstable. ■

For the admissible values of the parameters the system has an aperiodic motion. By solving

$$\begin{cases} 1 + trace(J) + \det(J) = 2 + \dfrac{2}{\beta} + \dfrac{\lambda^2\phi^2}{\alpha\beta\,(2-\phi)}u^2 = 0 \\ 1 - trace(J) + \det(J) = -\dfrac{\lambda^2\phi^2}{\alpha\beta\,(2-\phi)}u^2 = 0 \\ 1 - \det(J) = 1 - \dfrac{1}{\beta} = 0 \end{cases}$$

we obtain, if possible, the period-doubling (a single real eigenvalue, μ, crosses the boundary of stability with $\mu = -1$), the Neimark-Saker (conjugated complex pair crosses the boundary of stability) and the Saddle-Node (a single real eigenvalue crosses the boundary of stability with

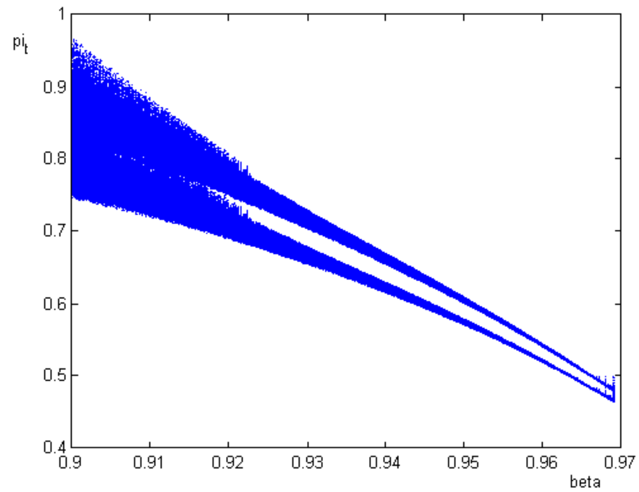

Figure 4: Bifurcation diagram for β

$\mu = 1$) bifurcation points. For

$$\pi^* = \frac{\lambda}{1-\beta}\left((x^*)^{\phi} + \left(\frac{2\alpha\,(\phi-2)\,(1+\beta)}{\lambda^2\phi^2}\right)^{\frac{\phi}{2(\phi-1)}}\right) \text{ and } \phi > 2$$

there is a period-doubling bifurcation (however, since we are mainly interested in the case $\phi < 2$, that is, not too pronounced nonlinearity, this is not relevant). For

$$\pi^* = \frac{\lambda\,(x^*)^{\phi}}{1-\beta}$$

we have a saddle-node bifurcation, and for $\beta = 1$ there is a Neimark-Sacker bifurcation.

The bifurcation diagrams of the π_t variable when the parameters β, α and λ are varied are presented in Figures 4 and 5. In all bifurcation diagram it is characteristic a pronounced increasing of the variable π_t when the parameter α it is increased and decreasing when the parameters λ and β are increased. It is also typical a several pieces attractor with tendency to join in an unique strange attractor when β and λ decrease and α increase.

A typical strange attractors and the associated time series for the π_t variable are presented in Figure 6. There are two cases, in the first one a several pieces attractor for $\beta = 0.95$ and in the second one the asymptotic behavior of the system is convergent to the one piece strange attractor for $\beta = 0.88$. The other parameters values considered here are: $\alpha = 0.5$; $\pi^* = 0.01$; $\phi = 1.5$; $\lambda = 0.8$; $x^* = 0.01$; $x(1) = 0.011$; $\pi(1) = 0.02$.

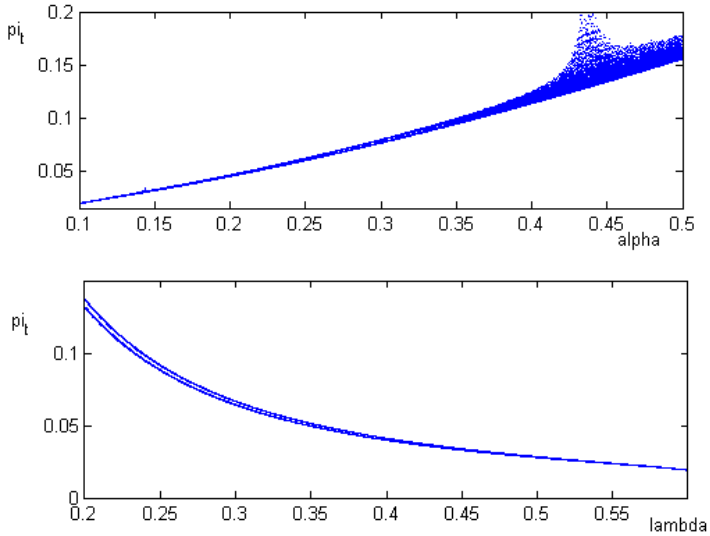

Figure 5: Bifurcation diagram for the π_t variable when the parameters α and λ are varied

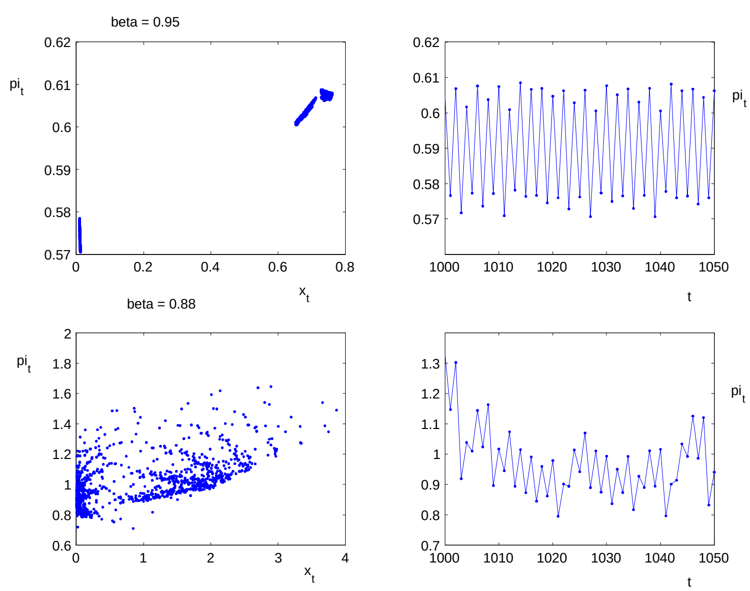

Figure 6: Strange attractors for diferent values of the parameter β.

4.2 The case: $x_0 < x^*$

Proposition 2 *The dynamical system (19) has always an unstable equilibrium given by the following point*

$$\pi_t = \pi^*,\ x_t = x^* - \left((x^*)^\phi - \frac{1-\beta}{\lambda}\pi^*\right)^{1/\phi}.$$

Proof. In order to compute the equilibrium points we have to solve the nonlinear algebraic system

$$\begin{aligned}\pi_t &= \frac{1}{\beta}\pi_t - \frac{\lambda}{\beta}\left[(x^*)^\phi - (x^* - x_t)^\phi\right] \\ x_t &= x^* - \left\{(x^* - x_t)^{2-\phi} - \frac{\lambda\phi}{\alpha\beta}\left[\pi_t - \lambda((x^*)^\phi - (x^* - x_t)^\phi) - \beta\pi^*\right]\right\}^{1/(2-\phi)}\end{aligned}$$

The first equation gives

$$\pi_t = \frac{\lambda}{1-\beta}\left((x^*)^\phi - (x^* - x_t)^\phi\right) \tag{23}$$

and from the second we obtain

$$x_t = x^* - \left((x^*)^\phi - \frac{1}{\lambda}\pi_t + \frac{\beta}{\lambda}\pi^*\right)^{1/\phi}. \tag{24}$$

Now substituting (24) into (23) we obtain the fixed point

$$(\pi_t, x_t) = \left(\pi^*, x^* - \left((x^*)^\phi - \frac{1-\beta}{\lambda}\pi^*\right)^{1/\phi}\right),$$

with

$$(x^*)^\phi > \frac{1-\beta}{\lambda}\pi^*$$

The stability of this fixed point is analyzed using the sufficient conditions

$$\left\{\begin{array}{c} 1 + trace(J) + \det(J) > 0 \\ 1 - trace(J) + \det(J) > 0 \\ 1 - \det(J) > 0 \end{array}\right.,$$

where J is the Jacobian matrix computed at the fixed point. We have then

$$\left\{\begin{array}{c} 2 + \dfrac{2}{\beta} + \dfrac{\lambda^2\phi^2}{\alpha\beta\,(2-\phi)}u^2 > 0 \ \text{ iff } \ \phi < 2 \\ -\dfrac{\lambda^2\phi^2}{\alpha\beta\,(2-\phi)}u^2 > 0 \ \text{ iff } \ \phi > 2 \\ 1 - \dfrac{1}{\beta} > 0 \ \text{ iff } \ \beta > 1 \end{array}\right.$$

where

$$u = \left((x^*)^\phi - \frac{(1-\beta)}{\lambda} \pi^* \right)^{\frac{\phi-1}{\phi}}.$$

This means that there is no stable equilibrium, independently of the value of ϕ.

The other real fixed point can not be determined analytically, but numerical calibration for the parameters shows that this fixed point is very large, always unstable or explode to infinity. ■

Analogous to the first model if we solve the conditions

$$\begin{cases} 1 + trace(J) + \det(J) = 2 + \dfrac{2}{\beta} + \dfrac{\lambda^2 \phi^2}{\alpha\beta\,(2-\phi)} u^2 = 0 \\ 1 - trace(J) + \det(J) = -\dfrac{\lambda^2 \phi^2}{\alpha\beta\,(2-\phi)} u^2 = 0 \\ 1 - \det(J) = 1 - \dfrac{1}{\beta} = 0 \end{cases}$$

we may obtain the first period-doubling bifurcation location, the Neimark-Sacker and the Saddle-Node bifurcation points. For

$$\pi^* = \frac{\lambda}{\beta - 1} \left(\left(\frac{2\alpha\,(\phi-2)\,(\beta+1)}{\lambda^2 \phi^2} \right)^{\frac{\phi}{2(\phi-1)}} - (x^*)^\phi \right) \text{ with } \phi > 2$$

there is a period-doubling bifurcation (again, the result is not relevant, because it implies a strong degree of nonlinearity). For

$$\pi^* = \frac{\lambda\,(x^*)^\phi}{1-\beta}$$

we have a saddle-node bifurcation and for $\beta = 1$ there is a Neimark-Sacker bifurcation.

When the parameter β is varied the dynamics is characterized by high order Neimark-Sacker bifurcations, breakdown of closed invariant curves, stretching and folding, and all this route leads the system to settle down in a chaotic dynamics as it is shown in Figure 7.

Moreover, when we vary the parameters α and λ in the interval $]0, 1[$ the system is always chaotic, with eigenvalues with modulus greater than 1; this means that the first bifurcations happen for parameter values outside the given intervals. Figures 8 and 9 show the complex motion of the model, where no stability windows can be observed.

If we vary the x^* parameter, the bifurcation diagram of the variable π_t is more suggestive, illustrating several stability windows, where high order Neimark-Sacker bifurcations take places. In these windows several

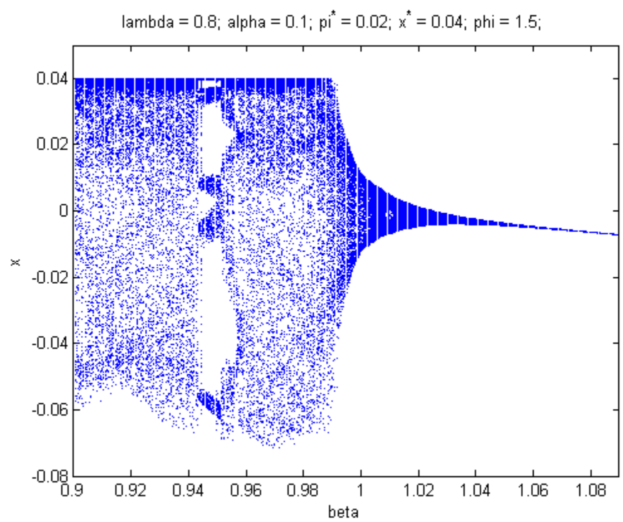

Figure 7: Bifurcation diagram for the variable x_t when β is varied

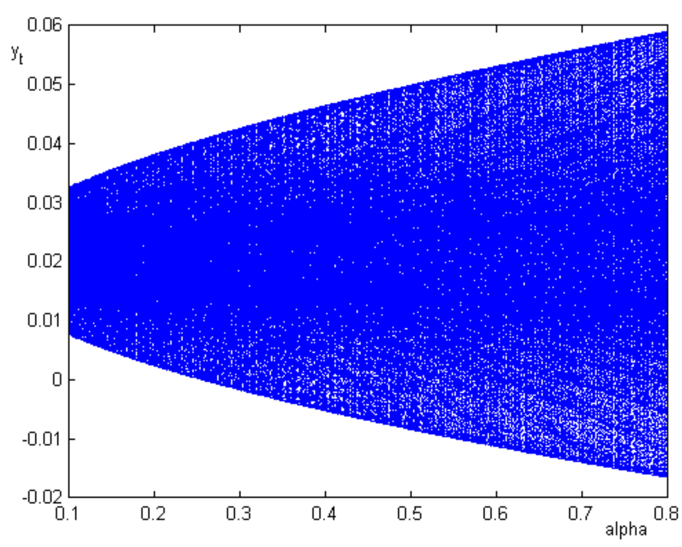

Figure 8: Bifurcation diagram when α is varied

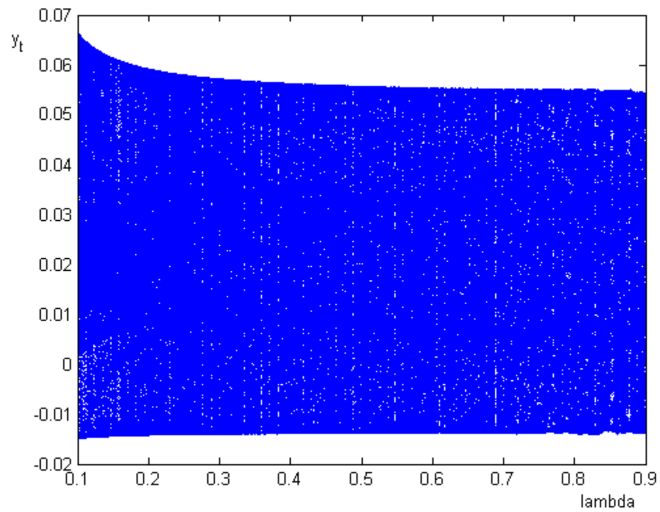

Figure 9: Bifurcation diagram when λ is varied

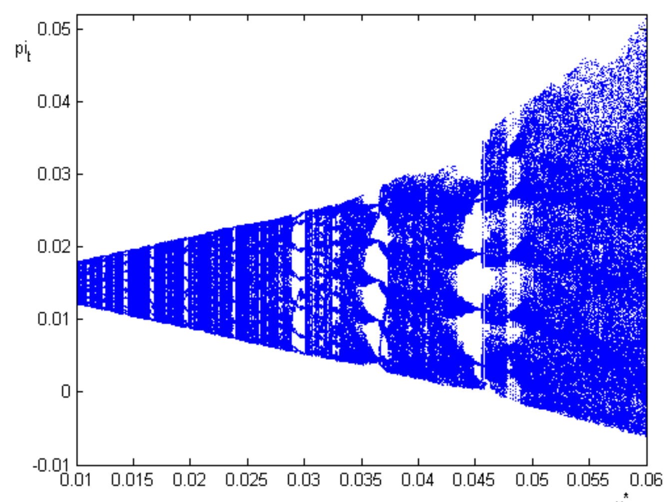

Figure 10: Stange attractor for the inflation rate produced when the target value for the output gap is changed..

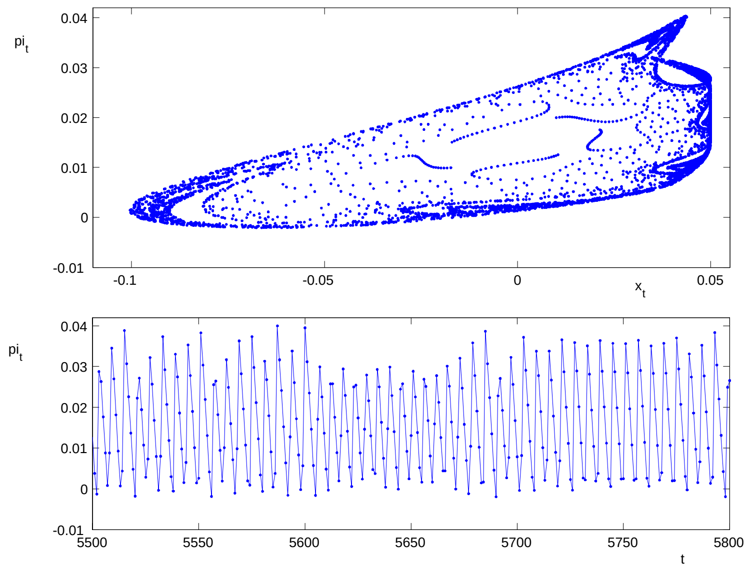

Figure 11: Strange attractor and time series of π_t variable

closed invariant curves start to stretch and fold, and after all breakdown and join in a chaotic attractor. We can also observe the increasing of the variable, when the parameter x^* it is increased.

For the strange attractor and the associated time series of variable π_t presented in Figure 11 we compute the dominant Lyapunov exponent L (for example Figure 12 shows the average value of the Lyapunov exponent) for the following parameter calibration: $\beta = 0.99$; $\alpha = 0.1$; $\pi^* = 0.015$; $\phi = 1.5$; $\lambda = 0.8$; $x^* = 0.05$.

We obtain that $L = 5.7403$ and this result confirm the chaotic nature of the system, because a positive Lyapunov exponent is a necessary condition for chaos. For the estimation of the Lyapunov exponent we used a version of the Wolf algorithm for 15.000 points.

Parameter calibration: $\beta = 0.99$; $\alpha = 0.1$; $\pi^* = 0.03$; $\phi = 1.5$; $\lambda = 0.8$; $x^* = 0.06$; $x(1) = 0.01$; $\pi(1) = 0.02$ with the following strange attractor and time series of the π_t variable (Figure 13).

5 Implications for Monetary Policy

The introduction of a nonlinear Phillips curve into the structure of a standard optimal monetary model with sticky prices, forward looking behavior and commitment by the central bank has produced the following major results:

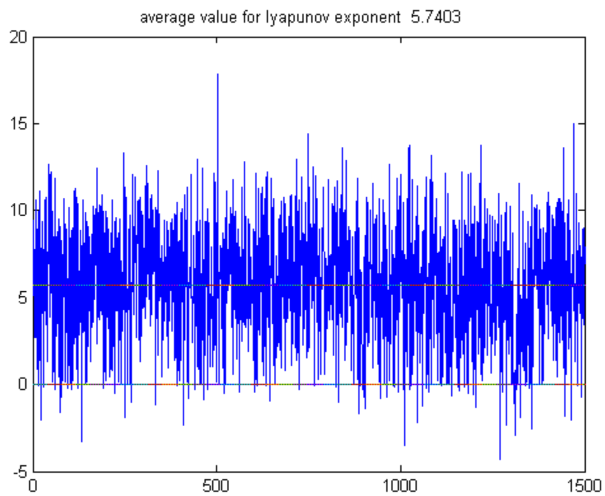

Figure 12: Dominant Lyapunov exponent

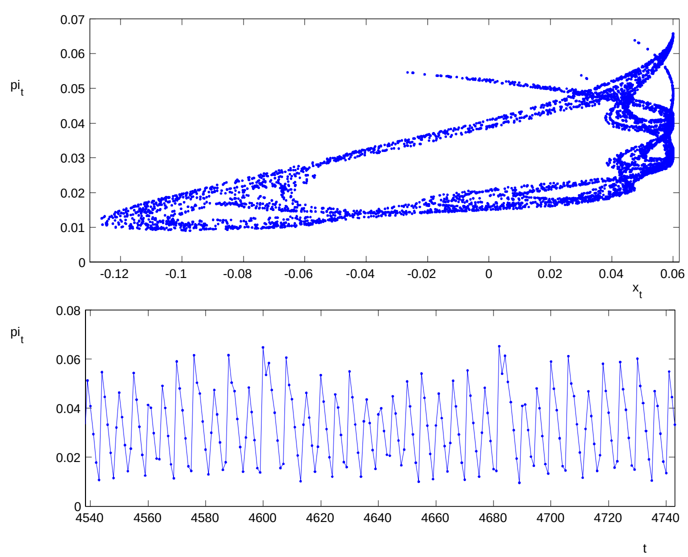

Figure 13: Strange attractor and time series of π_t

- it leads to multiple equilibria, instead of a unique fixed point as in the linear case; although some of these (real) equilibria may raise questions about their associated values for the endogenous variables;
- it allows for a very sophisticated kind of dynamics, from saddle-path stability, to total instability and even to chaotic dynamics (endogenous fluctuations) for a standard set of parameter values, while in the linear case we end up only with saddle stability;
- it shows that the parameter α (the central bank relative importance given to output-gap stabilization) is a fundamental factor in the dynamics of inflation and output, while it does not happen in the linear case as the stable trajectory is given by $x_t - x^* = \frac{1-\beta\varepsilon_1}{\lambda}(\pi_t - \pi^*)$;
- it shows that the target values for inflation and the output gap (π_t^*, x^*), both crucially affect the dynamics of the economy in terms of average values and volatility of the endogenous variables, while in the linear case only the target value for inflation does so (obviously, only affecting in this case the *level* of the endogenous variables).

Let us produce some short comments on each of these points, skipping the first and the potential importance of sunspots in the presence of multiple equilibria for the sake of brevity.

Saddle-path stability vs chaotic dynamics

Reconsider the linear case. We have seen, in section 3, that this case produces saddle-path stability, and in order to guarantee the absence of an explosive path for inflation, the Central Bank would have to choose the interest rate trajectory that maintains the system always on the stable trajectory. Hence, monetary policy involves, in the linear Phillips curve setup, a knife-edge result: if we fall outside the only admissible path, real and nominal economic aggregates will depart from admissible values and these cannot be recovered without breaking the commitment with an initially chosen interest rate time path. However, the story goes that once the commitment is broken, monetary policy loses credibility, and the task of maintaining price stability is seriously damaged.

Furthermore, we have encountered a unique steady state point, where the system rests after the convergence along the stable arm is completed. If the Central Bank is capable of keeping the interest rate at the level necessary to reach this fixed point, one has found that the long run inflation rate will settle down to the respective target value, that is, $\bar{\pi} = \pi^*$, and that the long term output gap will also be constant over time, $\bar{x} = \frac{1-\beta}{\lambda}\pi^*$. Note that α, that is, the parameter reflecting Central Bank preferences over output stabilization, is absent from the long run dynamics of the economy.

However, if the relation between the output gap and inflation is not linear — that is, if a nonlinear Phillips curve is assumed even with mild convexity and mild concavity — the role of monetary policy becomes dramatically changed. First, the local analysis concerning the search for a stable or unstable node or a saddle result is no longer helpful, because a global analysis of the dynamics leads to a much more sophisticated set of possible outcomes. In particular, we have found that for perfectly standard parameter values chaotic motion is present in the model. This implies that once one solves the optimal control problem by choosing the interest rate path that best serves the goals in the Central Bank objective function, long term time series regarding the endogenous variables (inflation rate and output gap) will display endogenous fluctuations.

In this case, the inflation rate will not converge to, neither diverge from, the target value, fluctuating around this value over time. This is a characteristic of most chaotic systems, and obviously the output gap will also exhibit endogenous cycles, but in this case the particular specification of the model implies that the output gap will never be above target (if $x_0 < x^*$) nor below the target (if $x_0 > x^*$). Consequently, the major result of the presented specification of the model is that the nonlinear Phillips curve is able to generate endogenous business cycles (the output gap is not a constant long run value) and endogenous price level fluctuations relatively similar to the ones that we observe in the real world we live in. Thus, we understand that if the inflation rate shows different responses to changes in the output gap for different output gap values (a consequence of the shape of the Phillips curve) the authorities are no longer able to use monetary policy in order to remain forever on the inflation rate target value.

This seems closer to what reality shows: despite the efforts of the monetary authority to keep the inflation on the selected target, the best that most Central Banks have achieved is maintaining the inflation rate fluctuating around such target value. Therefore, we conclude that a nonlinear Phillips curve may be able not only to reproduce results that fit relatively well with empirical evidence, but that it can also explain why despite the constant efforts made by monetary authorities, we continue to see that price evolution and real stabilization are far from producing fully predictable and smooth long term results.

Changes in parameter values

The four crucial parameters of the model are α, λ, x^* and π^*. We were able to study the impact of changes in parameter values, through the visualization of bifurcation diagrams. The most important results at this level are the one concerning the values of α, λ, x^*. Firstly, recall that α translates the relative concern of the Central Bank with real stabilization; the lower is its value, the more the Central Bank will "pay attention" only to inflation targeting. The result that we obtain, for several parameterizations, is admissible from an economic point of view, and it also supports the current commonly accepted view that the crucial concerns of central banks should be with the control of inflation. The argument can be easily spotted in the bifurcation diagrams: rising the value of α does not produce significant changes in terms of output gap; nevertheless, it has a growing negative impact in inflation stabilization; the higher is the value of α, the more volatile is the time series of the inflation rate. This large volatility of price changes introduces high uncertainty into the economic system, and this can be harmful in terms of potential long run real growth.

Secondly, another crucial result concerns changes in x^*. The bifurcation diagram in Figure 10 shows that the volatility of inflation increases dramatically when the target value for the output gap is increased from 1 to 6%. This result is totally different from the one of the linear case as changes in the parameter produce no impact at all in both the fixed point or in the transitional dynamics. In the nonlinear case here, if the Central Bank accepts a large value for the target output gap, the nonlinearities in the economy will lead to cycles of a large amplitude, and therefore, policy advice is maintain target values as close as possible to zero.

Thirdly, changes in the value of λ also produce some results but now not as relevant as the two previous ones. An increase in λ (that is, lower levels of price rigidity) may lead to a lower level of inflation or to a no relevant impact at all depending whether the initial condition of the economy is above or below the target value for the output gap.

6 Conclusions

We studied a totally standard optimal monetary policy problem, following the model that has become a major cornerstone in modern monetary policy, and developed, e.g. by Goodfriend and King (1997), Clarida et al. (1999), Woodford (2003) and many others. The only new ingredient was to assume that the Phillips curve should be nonlinear, allowing both for convexity and concavity, depending whether the economy (or rather, the initial condition forecasted by the Central bank) is below or above the target value for the output gap. This nonlinearity shows two advantages over the linear case. Firstly, there are sound logical reasons to expect that the Phillips curve should be nonlinear, and secondly, empirical evidence seems to favour the existence of some form of nonlinearity in the Phillips curve.

As it is well known, the standard model leads to a set of very important results, from which the most important one may include: (i) the crucial instrument of monetary policy ought to be the short term interest rate; (ii) policy should be focused on the control of inflation; (iii) inflation can be efficiently controlled by an aggressive increasing of short term interest rates; and (iv) the central bank should conduct monetary policy adopting a strategy of commitment in a forward-looking environment (instead of discretion).

The introduction of a nonlinear Phillips curve into the structure of a standard optimal monetary model with sticky prices, forward looking behavior and commitment by the central bank has produced the following major results:

- it leads to multiple equilibria, instead of a unique fixed point as in the linear case; although some of these (real) equilibria may raise questions about their associated values for the endogenous variables;

- it allows for a very sophisticated kind of dynamics, from saddle-path stability, to total instability and even to chaotic dynamics (endogenous fluctuations) for a standard set of parameter values, while in the linear case we end up only with saddle stability;

- it shows that the parameter α (the central bank relative importance given to output-gap stabilization) is a fundamental factor in the dynamics of inflation and output, while it does not happen in the linear case;

- it shows that the target values for inflation and the output gap (π_t^*, x^*), both crucially affect the dynamics of the economy in terms of average values and volatility of the endogenous variables, while in the linear case only the target value for inflation does so (obviously, only affecting in this case the *level* of the endogenous variables).

These results seem to confirm the relevance of some of the major results of the standard model. For example, if the central bank relative importance given to output-gap stabilization increases, inflation will show a higher mean and higher volatility. Moreover, a similar backing seems to come out from changes in the parameter of price rigidity. The more price rigidity we have, the higher the inflation rate will be (however, the model is not as affirmative in this case as in the previous one).

Nevertheless, we believe the nonlinear results above presented could also very well be used to question the relevance of other major results of the standard model, such as, that discretion leads to an inflation bias problem so clearly understood in the literature. There is some sense of irrationality in a consistent behavior of no–discretion at all, if the persistent objective of controlling inflation and the output gap leads to permanent cycles of some magnitude in the endogenous variables. Is not there some room for discretion (or of optimal discretion) even in fully deterministic framework? The control of chaos may show that these cycles could be easily avoided by simple and very small perturbations to the economy, without changing its fixed point!

Moreover, the nonlinear model seems to be able to give answers to some questions that the linear version could hardly provide given its strictly linearity. What happens to the economy if, for example, by some reason the target values of inflation and the output gap change? The standard model tells us that, as far as the latter one is concerned nothing happens at all; while in the former, it tells us that the optimal values of inflation and output will increase, nothing really happening to the volatility in the economy. This is not what one obtains in the case of a nonlinear Phillips curve in the standard optimal monetary model.

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