Money Creation in a Random Matching Model*

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Abstract

I study money creation in versions of the Trejos-Wright (1995) and Shi (1995) models with indivisible money and bounded individual holdings. I work with the same class of policies as in Deviatov and Wallace (2001), who study money creation in that model. However, I consider two alternative notions of implementability — the ex ante pairwise core and the ex post pairwise core. I compute a set of numerical examples to determine whether money creation is beneficial in my model. I find that if the ex post pairwise core is the notion of implementability, then examples where money creation is beneficial are easily found, while I find no such examples if the notion of implementability is the ex ante pairwise core.

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1 Introduction

A minimum test for the usefulness of a monetary model seems to be its ability to study lump-sum money creation. Among such models there seems to be a sharp contrast in results depending on whether there is heterogeneity in asset holdings. Representative agent models tend to yield results which are in line with what has become known as the Friedman rule: the optimal monetary policy is not creation, but destruction financed by lump-sum taxes. Models which make use of heterogeneity do not give a general answer: in some of these models the optimal monetary policy is contractionary, in some other models it is expansionary. Examples of models where it is expansionary include Imrohoroglu (1992), Levine (1991) and a generalization by Kehoe, Levine and Woodford (1992), Molico (1997) and Deviatov and Wallace (2001).

Although these settings differ in details, in all of them one role of money is to provide insurance against idiosyncratic shocks faced by the agents. Thus, Imrohoroglu (1992) is an income shock model; Levine (1991) and Kehoe, Levine and Woodford (1992) use a preference shock model; Molico (1997) and Deviatov and Wallace (2001) work with the random matching model of Trejos-Wright (1995) and Shi (1995). In the latter setting, idiosyncratic uncertainty takes the form of potentially long runs of production and consumption opportunities.

In all of these models, there is a potentially beneficial role for expansionary policy. An expansionary policy, consisting of equal per capita lump-sum transfers, reduces the dispersion in money holdings of individuals. Because it may be hard to get trades to occur between rich producers and poor consumers, the reduction in dispersion may be beneficial. If the parameters fall into the right region (e.g. as in Deviatov and Wallace (2001) if individuals are patient enough), this beneficial external margin effect outweighs the usual harmful internal margin effect of money creation.

An important difference among these models is that they use different notions of implementable outcomes. Imrohoroglu (1992), Levine (1991) and Kehoe, Levine and Woodford (1992) use competitive outcomes; Molico (1997) uses a particular bargaining solution; Deviatov and Wallace (2001) allow for any outcome which satisfies ex post individual rationality in meetings.

Here I study the model used in Deviatov and Wallace (2001). However, I work with two alternative notions of implementability — the ex post pairwise core and the ex ante pairwise core. The core restrictions (as opposed to individual rationality alone) allow to eliminate outcomes which are subject to defections by pairs in meetings. Because closed-form solutions cannot be obtained, I proceed numerically.

My main finding is that the choice of a notion of implementability seems to matter: if the ex post pairwise core is the notion of implementability, then there are examples where money creation is beneficial; if the ex ante pairwise core is the notion of implementability, then there seem to be no such examples. This second result is striking because it stands in contrast with a long list of examples (above) where money creation is beneficial.

The rest of the paper is organized as follows. In the next section I describe the environment; in section 3 I define implementable allocations; in section 4 I discuss some general properties of implementable allocations; in section 5 I describe the algorithm; in section 6 I discuss examples; section 7 concludes.

2 Environments

The background environment is a simple random matching model of money due to Shi (1995) and Trejos and Wright (1995). Time is discrete and the horizon is infinite. There are $N \geq 3$ perishable consumption goods at each date and a $[0, 1]$ continuum of each of N types of agents. A type n person consumes only good n and produces good $n + 1$ (modulo N). Each person maximizes expected discounted utility with discount parameter $\beta \in (0, 1)$. Utility in a period is given by $u(y) - c(x)$, where y denotes consumption and x denotes production of an individual ($x, y \in \mathbb{R}_+$). The function u is strictly concave, strictly increasing and satisfies $u(0) = 0$, while the function c is convex with $c(0) = 0$ and is strictly increasing. Also, there exists $\hat{y} > 0$ such that $u(\hat{y}) = c(\hat{y})$. In addition, u and c are twice continuously differentiable. At each date, each agent meets one other person at random.

There is only one asset in this economy which can be stored across periods: fiat money. This money is indivisible and no individual can have more than B units of money at any given time, where $2 \leq B < \infty$. Agents cannot commit to future actions (except commitment to outcomes of randomized trades when the ex ante pairwise core notion of implementability is assumed). Finally, each agent's specialization type and individual money holdings are observable within each meeting, but the agent's history, except as revealed by money holdings, is private.

3 Implementable allocations and the optimum problem

The pairwise meetings, the inability to commit, the privacy of individual histories, and the perishable nature of the goods imply that any production must be accompanied by a positive probability of receiving money. In every meeting of a potential producer with i units of money and a potential consumer with j units, there is a set, denoted $\mathcal{K}_{ij}$, of feasible money transfers from the consumer to the producer, transfers which are consistent with each person's money holdings being in the set $\{0,1,...,B\}$: $\mathcal{K}_{ij} = \{0,1,...\min(j, B-i)\}$. A *trade meeting* is one where $\mathcal{K}_{ij}^{+} \equiv \mathcal{K}_{ij}\backslash\{0\}$ is nonempty. For each trade meeting between a producer with i and a consumer with j units of money, trade is represented by a probability measure μ_{ij} on $\mathbb{R}_{+} \times \mathcal{K}_{ij}$ with the interpretation that if (y,k) is randomly drawn in accordance with μ_{ij}, then (y,k) is the suggested trade in that meeting. Let $\boldsymbol{\mu}$ be the collection of measures μ_{ij} corresponding to trade meetings.

For any measure μ_{ij} it is convenient to consider the collection of conditional measures $\mu_{ij}^{k}(A) = \mu_{ij}(A \,|\, k)$, $k \in \mathcal{K}_{ij}$, and their supports Ω_{ij}^{k}.[1] Then μ_{ij} can be expressed as $\mu_{ij}(A) = \sum\limits_{k \in \mathcal{K}_{ij}} \lambda_{ij}^{k} \mu_{ij}^{k}(A)$, where $\lambda_{ij}^{k} \equiv \mu_{ij}(\Omega_{ij}^{k})$, is the probability that k units of money are offered in a meeting. Finally, let p_i be the fraction of agents in each specialization type who start a date with i units of money and let $\mathbf{p} = (p_0, ..., p_B)$. Then, in terms of p_i and λ_{ij}^{k}, an arbitrary off-diagonal element of the transition matrix T for $\mathbf{p}$ is given by:

$$\pi_{mn} = \begin{cases} \frac{1}{N} \sum\limits_{i=0}^{B-m+n} p_i \lambda_{im}^{m-n} \text{ if } m > n \\ \frac{1}{N} \sum\limits_{j=n-m}^{B} p_j \lambda_{mj}^{n-m} \text{ if } m < n \end{cases} \tag{1}$$

where π_{mn} is the probability of a trade that results in transition from having m units of money to having n units. Note that since T is a transition matrix, its diagonal elements are given by $\pi_{mm} = 1 - \sum\limits_{s \neq m} \pi_{ms}$.

In addition to trades there is lump-sum money creation. I use the same kind of policy that was studied by Deviatov and Wallace (2001). The policy

[1] Recall that if μ is a probability measure, the support of μ, denoted $\operatorname{supp}\mu$, is the smallest closed set A such that $\mu(A) = 1$.

is a probabilistic version of the proverbial helicopter drops of money. The timing of events in a period is the following. First there are meetings and trades. Next, each person receives one unit of money with probability α. (Those who are at the upper bound and receive a unit must discard it.) Then each unit of money disintegrates with probability δ.

This policy has a close resemblance with the standard policy (expansion at a rate) which is followed by proportional reduction (normalization, see e.g. Lucas and Woodford (1994)) in individual holdings. The standard policy shifts the distribution of money holdings towards the mean and makes money less desirable to acquire because poor producers are less willing to produce for money (because they get a transfer without production) and rich consumers are more willing to part with money (because they lose some of its value). The (α, δ)-policy of Deviatov and Wallace (2001) has these effects as well.

Similar to trades, creation and destruction parts of the policy yield a pair of transition matrices for money holdings, denoted A and D respectively. According to my description of the policy, A is a two-diagonal matrix where the probability of getting a unit of money, α, is next to and above the main diagonal, and the probability of getting no transfer, $1-\alpha$, is on the main diagonal. Matrix D is lower-triangular where the first i entries in the i-th line comprise the binomial distribution of order i. Thus, the elements of A and D are:

$$a_{ij} = \begin{cases} 1-\alpha, & \text{if } j = i \\ \alpha, & \text{if } j = i+1 \\ 0, & \text{otherwise} \end{cases} \qquad d_{ij} = \begin{cases} \binom{i}{j}\delta^{i-j}(1-\delta)^j, & \text{if } j \le i \\ \\ 0, & \text{otherwise} \end{cases}$$

The stationarity requirement is $\mathbf{p}TAD = \mathbf{p}$.

It is convenient to express individual rationality and pairwise core constraints in terms of discounted expected utilities. For an allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$, that is stationary, discounted expected utility of an agent who ends up with i units of money at the end of the period, denoted v_i, is constant. Then vector $\mathbf{v} \equiv (v_0, ..., v_B)$ satisfies the following $B+1$-equation system of Bellman equations:

$$\mathbf{v}' = \beta(\mathbf{q}' + TAD\,\mathbf{v}'), \tag{2}$$

where $\mathbf{q}$, the vector of (expected) one period returns from trade, is given by:

$$q_l = \sum_{i=0}^{B-1} \frac{p_i}{N} \sum_{k \in \mathcal{K}_{il}} \lambda_{il}^k \int\limits_{\Omega_{il}^k} u(y) d\mu_{il}^k - \sum_{j=1}^{B} \frac{p_j}{N} \sum_{k \in \mathcal{K}_{lj}} \lambda_{lj}^k \int\limits_{\Omega_{lj}^k} c(y) d\mu_{lj}^k \tag{3}$$

and where $l \in \{0, ..., B\}$. Note that an individual with no money can only expect to be a producer, an agent with B units can only be a consumer, and anyone else can be either a consumer or a producer.

Because T, A, and D are transition matrices and $\beta \in (0,1)$, the mapping $G(\mathbf{x}) \equiv \beta(\mathbf{q}' + TAD\,\mathbf{x}')$ is a contraction. Therefore, (2) has a unique solution which can be expressed as

$$\mathbf{v}' = \left(\frac{1}{\beta}I - TAD\right)^{-1}\mathbf{q}' \tag{4}$$

where I is the $(B+1) \times (B+1)$ identity matrix.

Let

$$\Pi^p_{ij}(y,k) \equiv (\mathbf{e}_{i+k} - \mathbf{e}_i)\,AD\mathbf{v}' - c(y) \tag{5}$$

be the gain from trade of y units of output for k units of money for the producer with i units of money in a meeting with a consumer with j units and let

$$\Pi^c_{ij}(y,k) \equiv (\mathbf{e}_{j-k} - \mathbf{e}_j)AD\mathbf{v}' + u(y) \tag{6}$$

be the gain from the same trade for the consumer (where $\mathbf{e}_l$ is the $B+1$-component coordinate vector with indices running from 0 to B). Also, let

$$\Pi^p_{ij} \equiv \sum_{k\in\mathcal{K}_{ij}} \lambda^k_{ij} \int_{\Omega^k_{ij}} \Pi^p_{ij}(y,k)d\mu^k_{ij} \qquad \text{and} \qquad \Pi^c_{ij} \equiv \sum_{k\in\mathcal{K}_{ij}} \lambda^k_{ij} \int_{\Omega^k_{ij}} \Pi^c_{ij}(y,k)d\mu^k_{ij}$$

be the expected (before the realization (y,k) from measure μ_{ij} is announced to trading parties) gains from trade in that meeting.

The ex ante pairwise core notion of implementability gives rise to the following definition:

Definition 1. *An allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ is called ex ante pairwise core implementable if (i) $\mathbf{p}TAD = \mathbf{p}$, (ii) $\mathbf{v}$ (given by 4) is non-decreasing, (iii) the participation constraints*

$$\Pi^p_{ij} \geq 0 \qquad \textit{and} \qquad \Pi^c_{ij} \geq 0 \tag{7}$$

hold for all i and j, and (iv) there exists a vector $\boldsymbol{\theta} \in [0,1]^{B^2}$ such that for all pairs (i,j) corresponding to trade meetings, measure μ_{ij} solves

$$\max_{\mu_{ij}} \left(\Pi^p_{ij}\right)^{1-\theta_{ij}} \left(\Pi^c_{ij}\right)^{\theta_{ij}}. \tag{8}$$

where the value function, $\mathbf{v}$, is taken as given.

The ex post pairwise core notion yields the following definition:

Definition 2. *An allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ is called ex post pairwise core implementable if conditions (i) and (ii) in Definition 1 hold and if (iii) $p_i p_j > 0$ and $(y, k) \in \operatorname{supp} \mu_{ij}$ imply*

$$\Pi_{ij}^p(y, k) \geq 0 \qquad \textit{and} \qquad \Pi_{ij}^c(y, k) \geq 0,$$

and (iv) $p_i p_j > 0$ implies that there does not exist a pair (y', k') such that for every $(y, k) \in \operatorname{supp} \mu_{ij}$

$$\Pi_{ij}^p(y', k') \geq \Pi_{ij}^p(y, k) \qquad \textit{and} \qquad \Pi_{ij}^c(y', k') \geq \Pi_{ij}^c(y, k)$$

where at least one of the inequalities is strict.

Definitions 1 and 2 say that an allocation is implementable if (i) it is stationary, (ii) satisfies free disposal of money, (iii) satisfies individual rationality, and (iv) there is no incentive for defections by pairs in meetings.

Finally, our optimum problem is to maximize ex ante utility. That is, the optimum problem, denoted P, is to choose $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ from among those that are implementable to maximize $\mathbf{p}\mathbf{v}' \equiv W$.

It is useful to express the objective W in terms of returns. If I multiply (2) by $\mathbf{p}$ and use the fact that $\mathbf{p}TAD = \mathbf{p}$, then I obtain:

$$W = \mathbf{p}\mathbf{v}' = \frac{\beta}{1-\beta}\mathbf{p}\mathbf{q}'$$

Then, by writing out the product $\mathbf{p}\mathbf{q}'$, I get:

$$W = \frac{\beta}{1-\beta}\frac{1}{N}\sum_{i=0}^{B-1}\sum_{j=1}^{B}\sum_{k \in \mathcal{K}_{ij}} p_i p_j \lambda_{ij}^k \int_{\Omega_{ij}^k} z(y) d\mu_{ij}^k \tag{9}$$

where $z(y) \equiv u(y) - c(y)$. As one would expect, because for every consumer there is a producer, welfare is equal to the net expected discounted utility in all trade meetings.

4 General results

In their paper on lotteries, Berentsen, Molico and Wright (2002) give a complete characterization of the ex ante pairwise core for the case of one-unit bound on holdings. Here I use their results to show that every ex ante pairwise core implementable allocation has no randomization over output; each conditional measure μ_{ij}^k is degenerate and does not depend on k. The proof is the same as that of Proposition 3 in Berentsen, Molico and Wright (2002), so I do not reproduce it here. Degeneracy follows immediately from concavity of $u(x)$ and $-c(x)$. Independence on k follows from concavity of the Nash product (8) in Definition 1.

Degeneracy of conditional measures implies that the optimum problem P is finite dimensional. This allows me to characterize the ex ante pairwise core in terms of the necessary first order conditions for maximization of the Nash product. Because of concavity of the latter these necessary conditions are also sufficient. If an allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ has $y_{ij} > 0$ in all trade meetings,[2] then the first order conditions can be conveniently written as

$$\left[(e_{j-k} - e_j) + \frac{u'(y_{ij})}{c'(y_{ij})}(e_{i+k} - e_i)\right] AD\mathbf{v}' \begin{array}{l} \geq 0 \text{ if } \lambda_{ij}^k = \overline{\lambda}_{ij}^k \\ = 0 \text{ if } 0 < \lambda_{ij}^k < \overline{\lambda}_{ij}^k \\ \leq 0 \text{ if } \lambda_{ij}^k = 0 \end{array} \tag{10}$$

for all pairs (i, j) corresponding to trade meetings and transfers of positive amounts of money k, where $\overline{\lambda}_{ij}^k \equiv 1 - \sum_{s \in \mathcal{K}_{ij}^+ \setminus \{k\}} \lambda_{ij}^s$.

The first order conditions (10) yield a set of constraints which an ex ante pairwise core implementable allocation must satisfy in addition to the participation constraints in Definition 1. If the value function $AD\mathbf{v}'$ implied by an implementable allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ is strictly concave, then (10) has implications for the level of output in some meetings. In particular, if $\lambda_{ij}^k > 0$ and $k \geq j - i$ for some $k \in \mathcal{K}_{ij}^+$, then $y_{ij} \leq y^*$, the unconstrained maximizer of $z(y)$.[3] In the examples below, $B = 2$, so the only meetings in which output can exceed y^* are those between a producer with zero and a consumer with two units of money.

[2] A sufficient condition for this is that $AD\mathbf{v}'$, where $\mathbf{v}$ is the value function implied by an implementable allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$, is strictly increasing and that $u'(0) = \infty$ and $c'(0) = 0$.

[3] Notice that if $j \geq i - 1$, that is if the producer's holdings are one unit less than the consumer's or larger, then trade implies $\lambda_{ij}^k > 0$ for some $k \geq j - i$.

If the ex post pairwise core is the notion of implementability, then degeneracy of conditional measures μ_{ij}^k is less straightforward.[4] One conceivable approach to establishing degeneracy is to replace any nondegenerate distribution over output by its mean. While this would increase the objective (9), because it is concave, it is not evident how to show that such a non-local alternative also satisfies the ex post pairwise core restrictions. Therefore, I develop a local argument. First, I devise a way to perturb measures μ_{ij}^k in terms of a few parameters. The perturbation adjusts the endpoints of the support and creates an atom or adjusts any that exist. Second, in order to carry out the perturbations and to invoke the Kuhn-Tucker theorem's necessary conditions, the allocations under consideration have to be internal. This requirement forces me to consider a subset of allocations, those I call connected. Because this is a proper subset of all ex post pairwise core implementable allocations, I also have to argue that it is plausible that the optimum over the larger set is in fact connected.

The formal definition of connectedness is somewhat lengthy and may be difficult to follow at first. Roughly speaking, it requires that an allocation implies a value function consistent with a *willingness* to trade one unit of money in a sufficient number of meetings. Here willingness does not require that actual trades involve transfers of one unit of money, but only that trades of one unit satisfy the participation constraints implied by the allocation. A sufficient number of meetings means that these meetings can be linked into a chain that covers the entire set of money holdings. Here, by describing simple sufficient conditions for connectedness, I suggest that adding the connectedness requirement is likely to be innocuous for problem P.

Given the form of the objective (9), one would expect that any solutions to problem P would have trade in many meetings. But, requiring trade in all trade meetings is too restrictive; it may be hard to get trades between poor consumers and rich producers. Fortunately, that is not necessary for

[4] One can obtain three different characterizations of the optima which are useful in the computation of examples. First, because the meetings are pairwise, it suffices to consider allocations which have two-point-support conditional measures over output. If B is the bound on money holdings, this leads to a $4M + B + 2$ dimensional optimization problem, where $M \equiv \frac{1}{6}B(B+1)(2B+1)$. Alternatively, if free disposal of goods in meetings is allowed, then it is easily shown that randomization over output is not needed. In that case the dimensionality of the problem is $3M + B + 2$. Degeneracy result reduces it further to $2M + B + 2$ dimensions. The reduction is proportional to the cube of the bound and is, therefore, significant.

connectedness. Instead, the following is sufficient: (i) $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ implies a concave value function $\mathbf{v}$ and $\mathbf{p}$ has full support; and (ii) trade occurs in all meetings where the consumer is at least as rich as the producer. It is plausible that solutions to problem P satisfy (i) and (ii) and, hence, are connected.[5]

I use connectedness to show that every optimum over the set of ex post IR implementable (conditions (i)-(iii) in Definition 2) and connected allocations has degenerate measures μ_{ij}^k. Then, addition of the ex post pairwise core restrictions (condition (iv) in Definition 2) does not enlarge the set of feasible outcomes which implies that every optimum over the set of ex post pairwise core implementable and connected allocations also satisfies degeneracy. The formal definition of connectedness and proofs of sufficient conditions and of degeneracy result are given in the Appendix.

Similar to the case of the ex ante pairwise core notion above degeneracy allows to simplify the presentation of the ex post pairwise core constraints in Definition 2. Given an implementable allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$, let y_{ij}^k denote the degenerate support of (nonempty) measure μ_{ij}^k and let $\mathcal{K}_{ij}^k$ denote the set of all feasible transfers of money, k', different from k, such that $(\mathbf{e}_{i+k'} - \mathbf{e}_i) AD\mathbf{v}' - \Pi_{ij}^p(y_{ij}^k, k) \geq 0$. Using this notation, the ex post pairwise core constraints can be written as

$$\Pi_{ij}^c(y_{ij}^k, k) - (\mathbf{e}_{j-k'} - \mathbf{e}_j) AD\mathbf{v}' \geq g\left((\mathbf{e}_{i+k'} - \mathbf{e}_i) AD\mathbf{v}' - \Pi_{ij}^p(y_{ij}^k, k)\right) \quad (11)$$

for all $k' \in \mathcal{K}_{ij}^k$, all $k \in \mathcal{K}_{ij}$ such that $\lambda_{ij}^k > 0$, all pairs (i, j) corresponding to trade meetings, and where $g(x) \equiv u\left(c^{-1}(x)\right)$.

5 The algorithm

Because the beneficial external margin and harmful internal margin effects of money creation are at balance in any optimum, the optima always have some binding participation constraints. If the notion of implementability is the ex ante pairwise core notion, then, if individuals are patient enough, the optima also have randomization over how much money is transferred in

[5]Another way to get reassurance about the connectedness restriction is by way of a description of the set of allocations that are (ex post pairwise core) implementable, but not connected. They tend to be allocations which do not make full use of the set of possible money holdings. For example, for $B = 2$, any non-connected allocation can be achieved with $B = 1$ and two distinct monies (see Aiyagari, Wallace and Wright (1996) for examples of such allocations).

meetings. This implies that some of the constraints in (10) are also binding. Because these constraints are complicated functions of an allocation, closed-form solutions for the optima are out of reach even for the case of a two-unit bound on holdings. That is why I compute a set of examples.

The optimization problem P falls within the class of problems generally referred to as "nonlinear programming problems", for which many standard routines are available. However, as one can see, the constraints in (10) are discontinuous.[6] Another difficulty is that the mapping $F(\mathbf{p}) \equiv \mathbf{p}TAD - \mathbf{p}$ is ill-behaved at $\alpha = \delta = 0$.[7] This precludes application of routines which require continuous differentiability of objective and constraints, such as sequential quadratic programming. I overcome this difficulty by designing a hybrid algorithm which combines genetic and conventional smooth optimization techniques.

There are three main steps in this algorithm.

- **Step 1.** Create an initial population of allocations.
- **Step 2.** Amend the population by replacing the worst allocations by better ones.
- **Step 3.** Check if the termination criterion is satisfied for the best allocation in the population. If yes, terminate. If no, return to step 2.

In step 1 I create a matrix where each row is an allocation. Allocations in the initial population are picked randomly among those which satisfy ex ante (ex post) individual rationality. The size of the population is a parameter of the algorithm.

To amend the population in step 2 I use several genetic operators. These operators are called selection, crossover and mutation. I use standard selection and crossover operators, a subset of those described in Houck, Joines

[6] Each constraint in (10) is equivalent to

$$\left[(\mathbf{e}_{j-k} - \mathbf{e}_j) + \frac{u'(y_{ij})}{c'(y_{ij})}(\mathbf{e}_{i+k} - \mathbf{e}_i)\right] AD\mathbf{v}' + (\mathrm{sign}(\lambda_{ij}^k) - \mathrm{sign}(\overline{\lambda}_{ij}^k - \lambda_{ij}^k))\vartheta_{ij}^k = 0$$

and

$$\vartheta_{ij}^k \leq 0,$$

where $\mathrm{sign}(x)$ is the sign function, and ϑ_{ij}^k is a slack variable.

[7] See Deviatov and Wallace (2001), who study the properties of that mapping for $B = 2$.

and Kay (1996). However, I modify the standard mutation operator. The standard operator alters a single allocation (called "the parent") to produce another allocation (called "the child"). The operator I use is a composition of two independent operators.

The first one is applied only if the parent has at least one of the transfer probabilities λ_{ij}^k at its upper or lower bound or if it has $\alpha = \delta = 0$. The operator pushes a random subset of these variables into the interior. If a better allocation is produced, it replaces the parent in the population. This simple mutation deals with discontinuity of the constraints in (10) and with ill behavior of the mapping $F(\mathbf{p})$ at zero.

The second operator alters only those of the transfer probabilities and policy pairs which are already in the interior. There, because all constraints are twice continuously differentiable, application of smooth methods is possible. This leaves a range of possibilities for what this second operator can be. In particular, one can run a few iterations of a sequential quadratic routine or of the BFGS algorithm[8] (as long as these iterations remain in the interior). The operator I adopt makes use of the gradients in the following way.

First, I compute (reduced) gradients of the objective and of all active constraints. Then I compute an orthogonal projection of the gradient of the objective onto the subspace orthogonal to the one spanned by the gradients of the active constraints. After that I randomly pick a search direction in the neighborhood (small cone) of that projection. Going in that search direction is likely to improve the objective and does not violate (at least by much) the active constraints. The child is obtained from the parent by moving along the search direction. However, this procedure often leads to violation of some constraints even if the parent satisfies all the constraints. In this case the objective implied by the child is reduced by some value which is proportional to the amount by which the constraints are violated. If the penalty parameter is large, even a small violation is costly, and the child dies out of the population quickly. If the parent itself violates constraints by large amounts, then the search direction is chosen to move the child closer to the feasible region regardless of what happens to the objective. Because the initial population is chosen randomly, this is important in the beginning of search. In other words, the second operator first pushes allocations towards satisfaction of the pairwise core conditions; then it drives the population to the optimum.

[8] See Judd (1998) for further details.

The termination criterion in step 3 is based on the first order conditions for the Kuhn-Tucker theorem. If the length of the projection of the gradient of the objective onto the subspace orthogonal to that spanned by the gradients of the active constraints is less than the tolerance value, the necessary conditions for the theorem are (approximately) satisfied. Because the probability of selection of parents in the population is an increasing function of the objective, this is sufficient to guarantee that every terminal point is a (local) maximum.

6 The examples

I use the above algorithm to compute optima for examples with a two-unit bound on individual money holdings. I compute three sets of examples. In all the examples, the utility function, $u(x)$, is x^κ; the cost function, $c(x)$, is x; and the number of specialization types, N, is 3. The examples are computed for various κ and various degrees of patience, r, where $r \equiv \frac{1}{\beta} - 1$.

There are two things that are common to every example. First, there are no binding consumer participation constraints. Second, in a meeting of a producer with no money and a consumer with two units, one unit of money changes hands with probability one.

However, the most important finding is that there are no examples where money creation is beneficial if the ex ante pairwise core is the notion of implementability, while such examples are easily found if the notion of implementability is the ex post pairwise core. The former may not be merely coincidental.

Consider an ex ante pairwise core implementable allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta) \equiv x$ with $(\alpha, \delta) > 0$. Next, consider another allocation $(\mathbf{p}', \boldsymbol{\mu}', \alpha', \delta') \equiv x'$ with the same outputs and with $\alpha' < \alpha$ and $\delta' < \delta$ such that TAD and hence $\mathbf{p}$ are unchanged. (One can show that it is sufficient to adjust λ_{11}^1 alone and that there exists a unique direction in the (α, δ) plane such that x' remains in the ex ante pairwise core.) Then x and x' yield the same welfare. In addition, the replacement of x by x' tends to relax producer participation constraints and to tighten consumer participation constraints. But since the optima tend not to have binding consumer participation constraints, the replacement tends to slacken the relevant constraints. Then, if the replacement makes all of the producer participation constraints be slack, continuity implies that it is possible to find an ex ante pairwise core implementable allocation which is better than x. A formal argument along these lines is difficult because it is difficult to show that the optima do not have binding consumer participation constraints. (This, however, is not surprising because, as demonstrated in Berentsen, Molico and Wright (2002), money has no value if the gain from trade for consumers is zero.)

I take advantage of these common features to simplify presentation of examples in the tables below. I omit the probabilities of transfer of money in meetings of producers with nothing and consumers with two units (λ_{02}^1 and λ_{02}^2). I also suppress superscripts in the notation for the other transfer probabilities (λ_{01}^1, λ_{12}^1 and λ_{11}^1). I attach stars (*) to outputs which correspond to binding producer participation constraints and daggers (†) to the transfer probabilities which correspond to binding first order constraints in (10).

The first two sets of examples are examples of optima with the ex ante pairwise core notion of implementability. The first set shows how optima change with patience. Here I fix $\kappa = \frac{1}{2}$ and vary r. This choice implies that the best quantity of output, y^*, is 0.25. I compute examples for all r from 0.01 through 0.25 in increments of 0.01 and for $r \in \{0.3, 0.35, 0.4, 0.5\}$. I report a subset of these examples in Table 1. The examples are consistent with the existence of four different regions with respect to the degree of patience r. If r is small enough, then the optima have randomization over the transfers of money in all three trade meetings where transfers of only one unit are feasible. If r belongs to the second region, the optima have randomization over the transfers of money only in meetings where the consumers have one unit. In meetings of producers with one and consumers with two units, money changes hands with probability one. In the next region the optima have randomization over the transfers of money in meetings where both producers and consumers have one unit. Finally, if r is big enough, one unit of money changes hands with probability one in all trade meetings. The examples are consistent with the transfer probabilities λ_{12}, λ_{01}, and λ_{11} being decreasing functions of patience.

In addition, these examples are consistent with the optima having at most one nonbinding producer participation constraint, the one in meetings of producers with nothing and consumers with two units of money. In a meeting of a producer with one unit and a consumer with two, lowering the probability of handing over money raises v_2. That is helpful because it loosens producer constraint in $(i, j) = (1, 1)$ meeting, which, in turn, allows a decrease in λ_{11} and, thus, an increase in p_1 (and, thereby, in the frequency of trade). Because λ_{11} is low, the participation constraint in $(i, j) = (1, 1)$

meeting is binding and the output is low.

Likewise, a smaller probability of giving up money in $(i, j) = (0, 1)$ meeting lowers v_0 which helps to relax the producer constraint in $(i, j) = (0, 2)$ meeting. This allows a higher y_{02} which, again, pushes up v_2. This accounts for why y_{02} is so high. The same kind of effect on v_2 could be achieved with a positive λ_{02}^0, but that would reduce λ_{02}^1 and, hence, the inflow into p_1.

The second set of examples shows how optima change with risk aversion. Here I fix $r = 0.04$ and vary κ. I compute examples for all κ from 0.1 through 0.9 in the increments of 0.1. These examples are reported in Table 2. A general finding here is that the optima change with κ in a similar way as they change with patience. In particular, the transfer probabilities λ_{12}, λ_{01}, and λ_{11} are decreasing functions of risk aversion.

The third set of examples shows optima with the ex ante pairwise core notion of implementability. I compute examples for all r from 0.01 through 0.2 in increments of 0.01 and for $r \in \{0.25,\ 0.3,\ 0.4,\ 0.5,\ 1,\ 2\}$. I report some of these examples in Table 3.[9] The examples are consistent with the optimal inflation rate, α, and the welfare gain from adoption of the optimal policy being increasing functions of patience. Also, I find no examples where money creation is beneficial and the optima have take-it-or-leave-it offers in all meetings — the bargaining rule assumed by Molico (1997). Finally, all optima have transfers of one unit of money with probability one in all trade meetings. This is consistent with my conjecture that all optima are connected. I should make it clear that the transfers of one unit are optimal here because of the two-unit bound on holdings. In environments with larger bounds one should not expect that the optima will have transfers of one unit, but, as was argued above, it is plausible that they will be connected.

In Table 4 I present some comparison of the two notions of implementability. Note that, even though every allocation which satisfies ex post IR satisfies ex ante IR, there is no subset result for the allocations with ex ante and with ex post pairwise core notions of implementability.[10] Nevertheless, the ex

[9] The first three columns show examples of optima where the proof technique of Deviatov and Wallace (2001) is applicable; the next three columns show examples where money creation is still beneficial, but the proof technique of Deviatov and Wallace is not applicable; the next two columns yield examples where money creation is no longer beneficial, but the optima do not have take-it-or-leave-it offers by consumers; and the last two columns yield examples where money creation is not beneficial and where the optima have take-it-or-leave-it offers. The last row shows welfare gain compared to the optima subject to $\alpha = \delta = 0$.

[10] Examples of ex post pairwise core implementable allocations which fail to satisfy the

ante notion is in some sense weaker because it allows for randomization over the amount of money transferred in meetings. This, to some extent, mimics divisibility of money. With the ex post notion individuals agree to every realization in the support of randomized trades which makes randomization costly.

Given that pattern of trade, the only way to enlarge the set of feasible distributions is by means of the policy which accounts for the beneficial effects found in Deviatov and Wallace (2001). However, money creation never allows to achieve distributions which are concentrated around the average holdings to such an extent as those that are feasible with the ex ante pairwise core notion. That is why optima with ex ante pairwise core notion yield a considerably higher welfare (the difference is shown in the last row of Table 4).

7 Concluding remarks

The results in this paper show that there is a sharp contrast among the alternative notions of implementability. If the ex post pairwise core is the notion of implementability, then money creation can be beneficial, whereas with the ex ante pairwise core notion there are no examples where positive money creation is optimal. This disparity is due entirely to the distinction between committing or not committing to randomization. The disparity is interesting because as the bound on individual money holdings gets large, randomization plays a smaller and smaller role and, in the limit, no role. Then, the two notions of implementability coincide. The uniformity of the numerical finding of no beneficial money creation using the ex ante pairwise core notion leads me to surmise that it is the general result that will survive in the limit.

The latter (if correct) needs to be reconciled with a list of existing models with divisible money in which money creation can be beneficial (Imrohoroglu (1992), Levine (1991), Kehoe, Levine and Woodford (1992), and Molico (1997)). These are models where either competitive outcomes or particular bargaining rules are used as the notions of implementability. The outcomes, therefore, are proper subsets of the (ex ante) pairwise core. These models may give rise to a beneficial role for money creation through extensive

first order conditions for ex ante pairwise core include the best allocations under no policy described in Deviatov and Wallace (2001).

margin effects, because they impose a priori a smaller set of within-meeing (intensive margin) outcomes.

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8 Appendix

Here I show degeneracy of conditional measures μ_{ij}^k for the case of the ex post pairwise core notion of implementability. As I said, I use an argument which is based on perturbations of candidates for the optima. To apply that method, I need the candidates to be internal because otherwise they can not be perturbed and remain implementable. To assure satisfaction of this condition, I consider a subset of implementable allocations — those that satisfy a property I call connectedness. In this section I prove that every optimum over the set of ex post IR implementable and connected allocation satisfies degeneracy. This, in turn, implies that degeneracy holds for the optima over the smaller set of all ex post pairwise core implementable and connected allocations. Note that with connectedness restriction degeneracy holds for all feasible policies. That is why, to simplify the notation, I proceed with zero money creation in this section.

I start with the formal definition of connectedness. Let $(\mathbf{p}, \boldsymbol{\mu})$ be an arbitrary allocation. Let $\mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}$ be the set of all pairs (i, j), i being the producer's holdings and j the consumer's holdings, such that agents are *willing to trade* one unit of money. That is:

$$\mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})} = \{(i, j) : \exists\, y \in \mathbb{R}_+ \text{ such that } v_{i+1} - v_i \geq c(y) \text{ and } u(y) \geq v_j - v_{j-1}\} \tag{A1}$$

Next, I use $\mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}$ to define a correspondence $\Xi_{(\mathbf{p},\boldsymbol{\mu})}$ on the set of money holdings of producers, $\mathcal{I} \equiv \{0, ..., B-1\}$, which gives the post-trade holdings of consumers implied by $\mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}$. That is:

$$\Xi_{(\mathbf{p},\boldsymbol{\mu})}(i) = \left\{j - 1 : (i, j) \in \mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}\right\}$$

Next, let a subset $\mathcal{I}_l$ of $\mathcal{I}$ be called a *block* if the restriction of $\Xi_{(\mathbf{p},\boldsymbol{\mu})}$ to $\mathcal{I}_l \times \mathcal{I}_l$, denoted $\Xi^l_{(\mathbf{p},\boldsymbol{\mu})}$,[11] admits a selection, denoted $\sigma^l_{(\mathbf{p},\boldsymbol{\mu})}$, which is a permutation with a unique orbit.[12] Finally, block $\mathcal{I}_{l_n}$ is said to be *reachable* from $\mathcal{I}_{l_m}$ if it

[11]Note that $\Xi^l_{(\mathbf{p},\boldsymbol{\mu})}$ is

$$\Xi^l_{(\mathbf{p},\boldsymbol{\mu})} = \begin{cases} \Xi_{(\mathbf{p},\boldsymbol{\mu})}(i) \cap \mathcal{I}_l \text{ if } i \in \mathcal{I}_l \\ \emptyset \text{ otherwise} \end{cases}.$$

[12]Let $\sigma : A \to A$ be a permutation and let $\mathcal{R}$ be an equivalence relation on A such that $a_n \mathcal{R} a_m$ if and only if there exists an integer l such that $a_m = \sigma^l(a_n)$. Then an orbit

is possible to find a sequence of blocks $\{\mathcal{I}_{l_s}\}_{s=n}^{m-1}$ such that $\mathcal{I}_{l_s} \cap \mathcal{I}_{l_{s+1}} \neq \emptyset$ for all $s = n, ..., m-1$. I can now give the following definition:

Definition A1. *An allocation* $(\mathbf{p}, \boldsymbol{\mu})$ *is said to be connected if there exists a collection* $\{\mathcal{I}_{l_s}\}_{s=1}^{m}$ *of blocks such that every block in this collection is reachable from any other and* $\bigcup_{s=1}^{m} \mathcal{I}_{l_s} = \mathcal{I}$.

As I said, two simple conditions are sufficient for connectedness: (i) $(\mathbf{p}, \boldsymbol{\mu})$ implies a concave value function $\mathbf{v}$ and $\mathbf{p}$ has full support; and (ii) trade occurs in all meetings where the consumer is at least as rich as the producer. I now prove sufficiency of these conditions.

Lemma A1. *If* $(\mathbf{p}, \boldsymbol{\mu})$ *is ex post IR implementable and is such that (i)* $\mathbf{p}$ *has full support and the associated value function* $\mathbf{v}$ *is concave and (ii)* $\lambda_{ij}^k > 0$ *for some* $k \geq 1$ *in all meetings in which* $j \geq i$*, where* j *is money holdings of the consumer and* i *those of the producer, then* $(\mathbf{p}, \boldsymbol{\mu})$ *is connected.*

Proof. First, I show that concavity of the value function implies that trade in a meeting implies *willingness* to trade one unit in that meeting. Because $(\mathbf{p}, \boldsymbol{\mu})$ is ex post IR implementable and $\mathbf{p}$ has full support, $\lambda_{ij}^k > 0$ implies that there exists $y \geq 0$ such that

$$kc\left(\frac{y}{k}\right) \leq c(y) \leq v_{i+k} - v_i \leq k(v_{i+1} - v_i)$$

and

$$ku\left(\frac{y}{k}\right) \geq u(y) \geq v_j - v_{j-k} \geq k(v_j - v_{j-1})$$

where in each display the second inequality follows from implementability. The outer inequalities imply willingness to trade $\frac{y}{k}$ for one unit of money.

Therefore, by hypothesis (ii) of the lemma, $\mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}$ contains all pairs of money holdings (i, j) with $j \geq i$. Now for each $i \in \{1, ..., B-1\}$ consider the set $\mathcal{I}_i \equiv \{i-1, i\}$. This is a block because $j = i+1$ and $j = i$ satisfy $j \geq i$ and because the associated permutation, $\sigma_{(\mathbf{p},\boldsymbol{\mu})}^i = \begin{pmatrix} i-1 & i \\ i & i-1 \end{pmatrix}$,

of σ is an equivalence class of relation $\mathcal{R}$. Note that an arbitrary permutation can have more than one orbit. However, if a permutation has a unique orbit, this orbit necessarily coincides with the set A.

has a unique orbit. Finally, these blocks are mutually reachable and jointly cover the set $\{0, ..., B-1\}$ of money holdings of producers. ■

Now I would like to introduce some additional notation which is used later. If $(\mathbf{p}, \boldsymbol{\mu})$ is ex post IR implementable and connected, then there are participation constraints implied by both actual trades in meetings and by willingness to trade one unit of money. In particular, implementability implies that if the probability of a transfer of k units of money in a meeting of a producer with i and a consumer with j units, $p_i p_j \lambda_{ij}^k$, is positive, then the participation constraints (7) have to hold for every y in the support of conditional measure μ_{ij}^k. Connectedness implies that another group of participation constraints holds for some y in every meeting where agents are willing to trade one unit of money. Therefore, it is convenient to define the following objects:

Definition A2. *Given an arbitrary implementable and connected allocation* $(\mathbf{p}, \boldsymbol{\mu})$, *define*

$$\mathbb{Z}^1_{(\mathbf{p},\boldsymbol{\mu})} \equiv \{(i,j,1) : (i,j) \in \mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}\},\ \mathbb{Z}^2_{(\mathbf{p},\boldsymbol{\mu})} \equiv \{(i,j,k) : p_i p_j \lambda_{ij}^k > 0\},$$

and $\mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})} \equiv \mathbb{Z}^1_{(\mathbf{p},\boldsymbol{\mu})} \cup \mathbb{Z}^2_{(\mathbf{p},\boldsymbol{\mu})}$.

Next, observe that if some triplet $(i,j,1)$ is in $\mathbb{Z}^1_{(\mathbf{p},\boldsymbol{\mu})}$ but not in $\mathbb{Z}^2_{(\mathbf{p},\boldsymbol{\mu})}$, then $p_i p_j \lambda_{ij}^1 = 0$ and the associated conditional measure, μ_{ij}^1, is empty. It is convenient to replace this empty measure by one with a support whose lower endpoint is positive. Moreover, this replacement is innocuous because $p_i p_j \lambda_{ij}^1 = 0$ which implies that the fictitious support does not affect $\mathbf{v}$ or W. Accordingly, for every $(i,j,1) \in \mathbb{Z}^1_{(\mathbf{p},\boldsymbol{\mu})} \backslash \mathbb{Z}^2_{(\mathbf{p},\boldsymbol{\mu})}$, let μ_{ij}^1 be a Dirac measure with support y, where y is any suitable output in the definition of $\mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}$.

Now, let $\underline{y}_{ij}^k$ and $\overline{y}_{ij}^k$ denote the endpoints of the support of measure μ_{ij}^k with the above replacement of empty measures in $\mathbb{Z}^1_{(\mathbf{p},\boldsymbol{\mu})} \backslash \mathbb{Z}^2_{(\mathbf{p},\boldsymbol{\mu})}$. Then, we have the following. If $(\mathbf{p}, \boldsymbol{\mu})$ is ex post IR implementable and connected, then

$$c(\overline{y}_{ij}^k) - (v_{i+k} - v_i) \leq 0 \qquad \text{and} \qquad (v_j - v_{j-k}) - u(\underline{y}_{ij}^k) \leq 0 \tag{A2}$$

hold for all $(i,j,k) \in \mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})}$.

I now concentrate on the optimum problem P_0, which is to maximize welfare W subject to $(\mathbf{p}, \boldsymbol{\mu})$ being ex post IR implementable and connected. First I use connectedness to show that P_0 has solutions. This is done by endowing the space of measures μ_{ij}^k with the weak* topology and by showing that the set of the ex post IR implementable and connected allocations is compact and that the objective W is continuous.

Proposition A1. *The optimum problem P_0 has solutions.*

Proof. To see that the set of the ex post IR implementable and connected allocations, Γ, is nonempty, observe that autarky is always in Γ. The fact that $\operatorname{supp} \mu_{ij}^k = \{0\}$ for all nonempty measures μ_{ij}^k in $\boldsymbol{\mu}$ implies that the associated value function, $\mathbf{v}$, is zero and money has no value. Then, because $y = 0$ satisfies participation constraints for all i, j, k, autarky is implementable and connected.

To demonstrate compact valuedness of Γ, it suffices to show that Γ is closed valued and that all of the supports of measures μ_{ij}^k are bounded[13]. Consider a converging net of implementable and connected allocations, $(\mathbf{p}, \boldsymbol{\mu})_r$, and let $(\mathbf{p}, \boldsymbol{\mu})$ be its limit. The choice of the topology implies that $\mathbf{p}_r \to \mathbf{p}$ and $\left(\lambda_{ij}^k\right)_r \to \lambda_{ij}^k$ for all i, j, k. This and continuity of the function $g(\mathbf{p}, \boldsymbol{\lambda}) \equiv \mathbf{p}T - \mathbf{p}$ imply that $\mathbf{p}T = \mathbf{p}$ and the limiting distribution $\mathbf{p}$ is stationary.

To show that the limit $(\mathbf{p}, \boldsymbol{\mu})$ is ex post IR implementable and connected, let us first consider all converging nets $(\mathbf{p}, \boldsymbol{\mu})_r$ such that starting from some r, $\mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})_r}$ is a constant set, denoted $\mathbb{Z}$. Then, because $\operatorname{supp} \mu_{ij}^k \subseteq \lim_r \left(\operatorname{supp} \left(\mu_{ij}^k\right)_r\right)$ and because $\mathbf{v}_r \to \mathbf{v}$, all participation constraints in (A2) hold in the limit, and $(\mathbf{p}, \boldsymbol{\mu})$ is ex post IR implementable and connected. To see that the constancy of $\mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})_r}$ is without loss of generality, consider an arbitrary converging net $(\mathbf{p}, \boldsymbol{\mu})_r$. Because for every r, $\mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})_r}$ is a subset of $\{0, ..., B-1\} \times \{1, ..., B\}^2$, which is finite, there exists some set $\mathbb{Z}$ and a subnet $(\mathbf{p}, \boldsymbol{\mu})_{r_s}$ with the property that $\mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})_{r_s}} = \mathbb{Z}$. Then, because a net converges if and only if every subnet converges to the same limit, $(\mathbf{p}, \boldsymbol{\mu})$, the constancy of $\mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})_r}$ is without loss of generality.

To demonstrate boundedness of supports, let us consider an arbitrary block $\mathcal{I}_l$ and write down incentive compatibility constraints (A2), which pertain to selection $\sigma_{(\mathbf{p},\boldsymbol{\mu})}^l$ from $\Xi_{(\mathbf{p},\boldsymbol{\mu})}$:

$$c(\overline{y}_{ij}^1) \leq v_{i+1} - v_i, \qquad \textit{and} \qquad v_j - v_{j-1} \leq u(\underline{y}_{ij}^1)$$

[13]Recall that if topology on the space of probability measures $\mathcal{P}(X)$ is the weak* topology, then $\mathcal{P}(X)$ is compact if and only if X is compact.

all $i \in \mathcal{I}_l$. Because $\sigma^l_{(\mathbf{p},\boldsymbol{\mu})}$ is a permutation and selection from $\Xi_{(\mathbf{p},\boldsymbol{\mu})}$, for each j, which shows up in the above collection of the participation constraints, it is possible to find a unique i such that $j-1 = \sigma^l_{(\mathbf{p},\boldsymbol{\mu})}(i)$. Adding up separately producer and consumer constraints and taking the latter into account, one obtains:

$$\begin{aligned} \sum_{i\in\mathcal{I}_l} c(\overline{y}^1_{i(\sigma^l(i)+1)}) &\leq \sum_{i_m\in\mathcal{I}_l} (v_{i+1} - v_i) \\ \sum_{i\in\mathcal{I}_l} \left(v_{\sigma^l(i)+1} - v_{\sigma^l(i)}\right) &\leq \sum_{i\in\mathcal{I}_l} u(\underline{y}^1_{i(\sigma^l(i)+1)}). \end{aligned} \tag{A3}$$

Because $\sigma^l_{(\mathbf{p},\boldsymbol{\mu})}$ is a permutation, the two sums of gains from trade in (A3) are equal, which yields:

$$\sum_{i\in\mathcal{I}_l} \left[u\left(\underline{y}^1_{i(\sigma^l(i)+1)}\right) - c\left(\overline{y}^1_{i(\sigma^l(i)+1)}\right)\right] \geq 0.$$

Note that by definition, $\underline{y}^k_{ij} \leq \overline{y}^k_{ij}$, which, together with the properties of utility and cost functions, yields:

$$c\left(\overline{y}^1_{ij}\right) - u\left(\underline{y}^1_{ij}\right) \leq (|\mathcal{I}_l| - 1)\left[u(y^*) - c(y^*)\right],$$

all $i \in \mathcal{I}_l$, where y^* is a unique solution to $u'(y) = c'(y)$ and $|\mathcal{I}_l|$, $|\mathcal{I}_l| \leq B$, is the size of block $\mathcal{I}_l$. Then, properties of $u(y)$ and $c(y)$ guarantee that $\overline{y}^1_{ij}$ is finite for all $i \in \mathcal{I}_l$ and, because $\mathcal{I}_l$ is arbitrary, all supports that correspond to transfer of one unit and are a part of some $\sigma^l_{(\mathbf{p},\boldsymbol{\mu})}$ are bounded. Boundedness of all other supports follows immediately from consumer constraints in (A2), from $v_n - v_m = \sum\limits_{l=1}^{n-m} (v_{n-l+1} - v_{n-l})$ and $\bigcup\limits_l \mathcal{I}_l = \mathcal{I}$, and from free disposal of money.

Finally, recall that $u(y)$ and $c(y)$ are continuous. Because the supports, Ω^k_{ij}, are bounded and because each of the spaces of probability measures μ_{ij} on $\mathbb{R}_+ \times \mathcal{K}_{ij}$ is endowed with the weak* topology, continuity of the objective W is immediate. ■

Then, I define two classes of perturbations of non-autarkic probability measures in $\boldsymbol{\mu}$, one class for nondegenerate measures and another for degenerate ones. A measure μ^k_{ij} is called *autarkic* if it has zero support (i.e. $\operatorname{supp} \mu^k_{ij} = \{0\}$). (An allocation is autarkic if all the nonempty measures μ^k_{ij} are autarkic.) Note that autarky is defined as no production rather than no trade. The perturbations adjust measure μ^k_{ij}, but do not affect λ^k_{ij} and, hence, the distribution $\mathbf{p}$. Note that the perturbations do not affect policy parameters α and δ (which I set equal to zero in this section) as well. This is important because it accounts for why degeneracy holds for all feasible policies.

Let μ be a nondegenerate probability measure on $\mathbb{R}_+$ with a bounded support and let $\underline{y}$ and $\overline{y}$ be the endpoints of that support. Let us take six nonnegative numbers: a, b, c, d, x and ε such that $b \geq a + \frac{\overline{y}-\underline{y}}{2}$, $d \geq c + \frac{\overline{y}-\underline{y}}{2}$, $\min(a, c) \leq x \leq \max(b, d)$ and $0 \leq \varepsilon \leq 1$. Also, let us observe that μ can be tautologically written as $\mu = \mu_1 + \mu_2$, where $\mu_1 = \mu_2 = \frac{1}{2}\mu$. Then the perturbation does two things. First, it moves the endpoints $\underline{y}$ and $\overline{y}$ of μ_1 and μ_2 independently to the new positions, a and b for μ_1 and c and d for μ_2, so that the "shapes" of μ_1 and μ_2 (which are those of μ) are preserved. Second, the perturbation creates a mass point x with mass ε within the union of the perturbed supports. That is, the perturbed measure $\widetilde{\mu}$ is obtained from μ via the formula:

$$\widetilde{\mu}(A) = \varepsilon\delta_x(A) + \frac{1}{2}(1-\varepsilon)\left[\mu\left(t_1^{-1}(A)\right) + \mu\left(t_2^{-1}(A)\right)\right] \tag{A4}$$

where δ_x is a Dirac measure with support x, and t_1 and t_2 are two linear mappings on the real line defined by my requirement that t_1 maps $\underline{y}$ and $\overline{y}$ into a and b and that t_2 maps $\underline{y}$ and $\overline{y}$ into c and d.[14] Note that because I set $b > a$ and $d > c$, the mappings t_1 and t_2 are invertible.

For a measure μ which is degenerate, the perturbation splits its single-point support into two points which, however, are allowed to be the same. Each of these points gets one-half of the mass of measure μ. That is, let g and h be two nonnegative numbers. Then the perturbed measure $\widetilde{\mu}$ is given by

$$\widetilde{\mu}(A) = \frac{1}{2}\delta_g(A) + \frac{1}{2}\delta_h(A). \tag{A5}$$

Now, given an arbitrary implementable and connected allocation $(\mathbf{p}, \boldsymbol{\mu})$, I define a finite-dimensional optimization problem, denoted $\widetilde{P}_{(\mathbf{p},\boldsymbol{\mu})}$, which is to maximize W by the choice of the parameters $(a_{ij}^k, b_{ij}^k, c_{ij}^k, d_{ij}^k, x_{ij}^k, \varepsilon_{ij}^k, g_{ij}^k, h_{ij}^k)$, one eight-tuple for each nonempty non-autarkic measure μ_{ij}^k in $\boldsymbol{\mu}$, subject to

[14] That is,

$$t_1(y; a, b) = \frac{a\overline{y} - b\underline{y}}{\overline{y} - \underline{y}} + \frac{b-a}{\overline{y} - \underline{y}}\, y$$

and analogously for t_2.

$(\mathbf{p}, \widetilde{\boldsymbol{\mu}})$ being implementable and connected. If $(\mathbf{p}, \boldsymbol{\mu})$ solves P_0, then the null perturbation must solve $\widetilde{P}_{(\mathbf{p},\boldsymbol{\mu})}$. This is the basis for the proof by contradiction showing that every nonempty non-autarkic measure μ_{ij}^k in $\boldsymbol{\mu}$ must be degenerate.

Because this optimization problem is finite-dimensional, it can be analyzed by means of the Kuhn-Tucker theorem. The central hypothesis of that theorem is the constraint qualification: the rank of the Jacobian matrix should be equal to the number of active constraints. The constraint qualification is sufficient to ensure the existence of an open region $\mathcal{U}$ adjacent to the solution point in which all the constraints are relaxed. Existence of such a region allows one to claim that the solution point satisfies the first-order necessary conditions of the Kuhn-Tucker theorem. My approach is to establish existence of $\mathcal{U}$ directly, without appeal to the full rank requirement on the Jacobian matrix.

Lemma A2. *Let $(\mathbf{p}, \boldsymbol{\mu})$ be a non-autarkic solution to problem P_0. Let $\widetilde{P}^*_{(\mathbf{p},\boldsymbol{\mu})}$ be the associated perturbation problem $\widetilde{P}_{(\mathbf{p},\boldsymbol{\mu})}$ with the additional restriction that $\varepsilon_{ij}^k \equiv 0$. Let E be the set of all active constraints of problem $\widetilde{P}^*_{(\mathbf{p},\boldsymbol{\mu})}$ at $(\mathbf{p}, \boldsymbol{\mu})$ and assume that E is nonempty. Then there exists a nonempty subset E' of E and multipliers $\xi_s \geq 0$, one for each constraint in E', such that the gradient of the objective W can be written as a linear combination of the gradients of the constraints in E'.*

Proof. By assumption, $(\mathbf{p}, \boldsymbol{\mu})$ is non-autarkic, ex post IR implementable and connected. I first show that $\underline{y}_{ij}^k > 0$ for all $(i, j, k) \in \mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})}$. Suppose to the contrary, that there exists a triplet $(i, j, k) \in \mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})}$ such that $\underline{y}_{ij}^k = 0$. By (A2), it follows that in this case $v_j - v_{j-k} = 0$, which implies that $v_j - v_{j-1} = 0$. Because $(\mathbf{p}, \boldsymbol{\mu})$ is connected, there exists a block, $\mathcal{I}_l$, such that $j - 1 \in \mathcal{I}_l$. Then $v_j - v_{j-1} = 0$ implies that $\overline{y}_{i_1 j_1}^1 = 0$, where $i_1 = j - 1$ and $j_1 = \sigma_{(\mathbf{p},\boldsymbol{\mu})}^l(i_1) + 1$. From $\overline{y}_{i_1 j_1}^1 = 0$ it follows that $\underline{y}_{i_1 j_1}^1 = 0$, which implies that $v_{j_1} - v_{j_1 - 1} = 0$. Continuing this process recursively, one obtains

$$v_{\left(\sigma^l\right)^m (j-1)+1} - v_{\left(\sigma^l\right)^m (j-1)} = 0$$

for $m = 1, 2, \ldots$. Because $\sigma_{(\mathbf{p},\boldsymbol{\mu})}^l$ has a unique orbit, which spans the block $\mathcal{I}_l$, the process will cycle at $m = |\mathcal{I}_l| \leq B$, and then $v_{i+1} - v_i = 0$ for all $i \in \mathcal{I}_l$. Note that this implies that no production takes place in return for one unit of money in meetings which pertain to permutation $\sigma_{(\mathbf{p},\boldsymbol{\mu})}^l$.

If two blocks $\mathcal{I}_{l_1}$ and $\mathcal{I}_{l_2}$ overlap and one of the two has $\underline{y}_{ij}^k = 0$, then $v_{i+1} - v_i = 0$ for all $i \in \mathcal{I}_{l_1} \cup \mathcal{I}_{l_2}$. Finally, because connectedness requires that every block is reachable from any other and because these blocks jointly cover $\mathcal{I}$, the value function $\mathbf{v}$ is zero. That, in turn, implies that $(\mathbf{p}, \boldsymbol{\mu})$ is autarkic, a contradiction.

I now construct a vector $\mathbf{n}$ whose inner product with the gradients of the constraints in problem $\widetilde{P}^*_{(\mathbf{p},\boldsymbol{\mu})}$ is positive. The vector $\mathbf{n}$ is obtained by stacking vectors $\mathbf{l}_{ij}^k$, one for each $(i,j,k) \in \mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})}$. The construction of $\mathbf{l}_{ij}^k$ differs depending on whether μ_{ij}^k is or is not degenerate.

Let us first consider a nondegenerate measure μ_{ij}^k. Without loss of generality, I can assume that $a_{ij}^k \le c_{ij}^k \le b_{ij}^k \le d_{ij}^k$. Then the 4×2 block of the Jacobian matrix which corresponds to the perturbation of μ_{ij}^k can be written as

$$J = \begin{bmatrix} -\left(\mathbf{e}_{i+k} - \mathbf{e}_i\right) H \frac{\partial \mathbf{q}'}{\partial a_{ij}^k}, & \left(\mathbf{e}_j - \mathbf{e}_{j-k}\right) H \frac{\partial \mathbf{q}'}{\partial a_{ij}^k} - u'(a_{ij}^k) \\ -\left(\mathbf{e}_{i+k} - \mathbf{e}_i\right) H \frac{\partial \mathbf{q}'}{\partial b_{ij}^k}, & \left(\mathbf{e}_j - \mathbf{e}_{j-k}\right) H \frac{\partial \mathbf{q}'}{\partial b_{ij}^k} \\ -\left(\mathbf{e}_{i+k} - \mathbf{e}_i\right) H \frac{\partial \mathbf{q}'}{\partial c_{ij}^k}, & \left(\mathbf{e}_j - \mathbf{e}_{j-k}\right) H \frac{\partial \mathbf{q}'}{\partial c_{ij}^k} \\ c'(d_{ij}^k) - \left(\mathbf{e}_{i+k} - \mathbf{e}_i\right) H \frac{\partial \mathbf{q}'}{\partial d_{ij}^k}, & \left(\mathbf{e}_j - \mathbf{e}_{j-k}\right) H \frac{\partial \mathbf{q}'}{\partial d_{ij}^k} \end{bmatrix}. \tag{A6}$$

Now let us take some vector $\mathbf{l} \equiv (-l_a, l_b, l_c, l_d) \in \mathbb{R}^4$. The scalar products of $\mathbf{l}$ and the columns of J are given by

$$l_d c'(d_{ij}^k) + \left(\mathbf{e}_{i+k} - \mathbf{e}_i\right) H \left(l_a \frac{\partial \mathbf{q}'}{\partial a_{ij}^k} - l_b \frac{\partial \mathbf{q}'}{\partial b_{ij}^k} - l_c \frac{\partial \mathbf{q}'}{\partial c_{ij}^k} - l_d \frac{\partial \mathbf{q}'}{\partial d_{ij}^k} \right)$$

and

$$l_a u'(a_{ij}^k) - \left(\mathbf{e}_j - \mathbf{e}_{j-k}\right) H \left(l_a \frac{\partial \mathbf{q}'}{\partial a_{ij}^k} - l_b \frac{\partial \mathbf{q}'}{\partial b_{ij}^k} - l_c \frac{\partial \mathbf{q}'}{\partial c_{ij}^k} - l_d \frac{\partial \mathbf{q}'}{\partial d_{ij}^k} \right).$$

Note that these products are positive if I can find $l_a > 0$ and $l_d > 0$ such that

$$l_a \frac{\partial \mathbf{q}'}{\partial a_{ij}^k} - l_b \frac{\partial \mathbf{q}'}{\partial b_{ij}^k} - l_c \frac{\partial \mathbf{q}'}{\partial c_{ij}^k} - l_d \frac{\partial \mathbf{q}'}{\partial d_{ij}^k} = \mathbf{0}. \tag{A7}$$

To show that such a choice of $\mathbf{l}$ is possible, let us first write out the derivatives of the vector $\mathbf{q}$ (evaluated at $a_{ij}^k = c_{ij}^k = \underline{y}_{ij}^k$ and $b_{ij}^k = d_{ij}^k = \overline{y}_{ij}^k$).

These are

$$\frac{\partial \mathbf{q}'}{\partial a_{ij}^k} = \frac{\partial \mathbf{q}'}{\partial c_{ij}^k} = \frac{\lambda_{ij}^k}{2N}\left[p_j \underline{\eta}_{ij}^k \mathbf{e}_i' - p_i \underline{\gamma}_{ij}^k \mathbf{e}_j'\right],$$
$$\frac{\partial \mathbf{q}'}{\partial b_{ij}^k} = \frac{\partial \mathbf{q}'}{\partial d_{ij}^k} = \frac{\lambda_{ij}^k}{2N}\left[p_j \overline{\eta}_{ij}^k \mathbf{e}_i' - p_i \overline{\gamma}_{ij}^k \mathbf{e}_j'\right] \tag{A8}$$

where

$$\underline{\gamma}_{ij}^k = \int_{\underline{y}_{ij}^k}^{\overline{y}_{ij}^k} u'(y)\frac{\overline{y}_{ij}^k - y}{\overline{y}_{ij}^k - \underline{y}_{ij}^k}\, d\mu_{ij}^k, \qquad \underline{\eta}_{ij}^k = \int_{\underline{y}_{ij}^k}^{\overline{y}_{ij}^k} c'(y)\frac{\overline{y}_{ij}^k - y}{\overline{y}_{ij}^k - \underline{y}_{ij}^k}\, d\mu_{ij}^k$$
$$\overline{\gamma}_{ij}^k = \int_{\underline{y}_{ij}^k}^{\overline{y}_{ij}^k} u'(y)\frac{y - \underline{y}_{ij}^k}{\overline{y}_{ij}^k - \underline{y}_{ij}^k}\, d\mu_{ij}^k, \qquad \overline{\eta}_{ij}^k = \int_{\underline{y}_{ij}^k}^{\overline{y}_{ij}^k} c'(y)\frac{y - \underline{y}_{ij}^k}{\overline{y}_{ij}^k - \underline{y}_{ij}^k}\, d\mu_{ij}^k. \tag{A9}$$

Observe that because μ_{ij}^k is nondegenerate, all four integrals in (A9) are strictly positive. Then, because the expected cost of production for producer and the expected utility of consumption for consumer show up only in the i-th and j-th entries of $\mathbf{q}$, (A7) gives rise to the following linear 2-equation system:

$$\lambda_{ij}^k \begin{bmatrix} p_j \underline{\eta}_{ij}^k & p_j \overline{\eta}_{ij}^k \\ p_i \underline{\gamma}_{ij}^k & p_i \overline{\gamma}_{ij}^k \end{bmatrix} \begin{bmatrix} l_c \\ l_b \end{bmatrix} = \lambda_{ij}^k \begin{bmatrix} p_j \underline{\eta}_{ij}^k & -p_j \overline{\eta}_{ij}^k \\ p_i \underline{\gamma}_{ij}^k & -p_i \overline{\gamma}_{ij}^k \end{bmatrix} \begin{bmatrix} l_a \\ l_d \end{bmatrix}$$

Notice that $l_c = l_a$ and $l_b = -l_d$ is a solution, which implies that $l_a > 0$ and $l_d > 0$ is possible.

If measure μ_{ij}^k is degenerate, then the analogue of (A7) is

$$l_g \frac{\partial \mathbf{q}'}{\partial g_{ij}^k} - l_h \frac{\partial \mathbf{q}'}{\partial h_{ij}^k} = \mathbf{0} \tag{A10}$$

where the derivatives of $\mathbf{q}$ are

$$\frac{\partial \mathbf{q}'}{\partial g_{ij}^k} = \frac{\lambda_{ij}^k}{2N}\left[p_j \underline{\eta}_{ij}^k \mathbf{e}_i' - p_i \underline{\gamma}_{ij}^k \mathbf{e}_j'\right] \qquad \text{and} \qquad \frac{\partial \mathbf{q}'}{\partial h_{ij}^k} = \frac{\lambda_{ij}^k}{2N}\left[p_j \overline{\eta}_{ij}^k \mathbf{e}_i' - p_i \overline{\gamma}_{ij}^k \mathbf{e}_j'\right]$$

with $\underline{\gamma}_{ij}^k = \overline{\gamma}_{ij}^k = u'(\overline{y}_{ij}^k)$ and $\underline{\eta}_{ij}^k = \overline{\eta}_{ij}^k = c'(\overline{y}_{ij}^k)$. Therefore, (A10) reduces to

$$\begin{matrix} \lambda_{ij}^k p_j \overline{\gamma}_{ij}^k (l_g - l_h) = 0 \\ \lambda_{ij}^k p_i \overline{\eta}_{ij}^k (l_g - l_h) = 0 \end{matrix}.$$

Obviously, $l_g = l_h = 1$ satisfies this equation.

Thus, we have the vector $\mathbf{n}$ whose inner product with the gradients of the constraints in problem $\widetilde{P}^*_{(\mathbf{p},\boldsymbol{\mu})}$ is positive. Because the objective and constraints are continuously differentiable, existence of $\mathbf{n}$ is equivalent to existence of an open region $\mathcal{U}$ in the space of perturbations where all the constraints in (A2) are relaxed. Because $(\mathbf{p},\boldsymbol{\mu})$ solves P_0, it follows that the gradient of the objective is in the convex hull of the gradients of the active constraints. Finally, because the number of constraints in (A2) does not exceed the number of degrees of freedom provided by perturbations, the edges of that convex hull are linearly independent. ■

The multipliers ξ_s of Lemma A2 can be used to prove the main proposition.

Proposition A2. *If* $(\mathbf{p},\boldsymbol{\mu})$ *solves problem* P_0 *and the support of* μ^k_{ij} *is nonempty, then* μ^k_{ij} *is degenerate.*

Proof. Suppose that $(\mathbf{p},\boldsymbol{\mu})$ is a solution to the optimum problem P_0 and that it has at least one nondegenerate measure μ^k_{ij}. Consider the associated perturbation problem $\widetilde{P}_{(\mathbf{p},\boldsymbol{\mu})}$ and let E be the set of active participation constraints of that problem. Let us first assume that E is nonempty.

By Lemma A2, $(\mathbf{p},\boldsymbol{\mu})$ satisfies necessary first order conditions for the Kuhn-Tucker theorem for that problem. The constraints are the participation constraints in E' and $\varepsilon^k_{ij} \in [0,1]$ and $x^k_{ij} \in [a^k_{ij}, b^k_{ij}]$. At $\varepsilon^k_{ij} = 0$, the multipliers associated with x^k_{ij} are equal to zero. Therefore, the multiplier associated with the binding constraint, $\varepsilon^k_{ij} = 0$, can be expressed as

$$\sigma = \frac{\partial W}{\partial \varepsilon^k_{ij}} - \sum_{E'} \left[\xi_{s_1} \frac{\partial \left(v_j - v_{j-k} - u(a^k_{ij}) \right)}{\partial \varepsilon^k_{ij}} - \xi_{s_2} \frac{\partial \left(v_{i+k} - v_i - c(b^k_{ij}) \right)}{\partial \varepsilon^k_{ij}} \right]$$

where ξ_{s_1} and ξ_{s_2} are the multipliers from Lemma A2.

Note that optimality of $\varepsilon^k_{ij} = 0$ requires that $\sigma \geq 0$ for all $x^k_{ij} \in \left[\underline{y}^k_{ij}, \overline{y}^k_{ij} \right]$, which, because μ^k_{ij} is nondegenerate, is an interval. It follows from (A4) that:

$$\sigma = \Phi(x^k_{ij}) - \int_{\underline{y}^k_{ij}}^{\overline{y}^k_{ij}} \Phi(y) \, d\mu^k_{ij},$$

where

$$\Phi(y) = z(y) + \frac{\lambda_{ij}^k}{N}[\xi_{s_2}(\mathbf{e}_{i+k} - \mathbf{e}_i) - \xi_{s_1}(\mathbf{e}_j - \mathbf{e}_{j-k})]H\left(p_i\, u(y)\, \mathbf{e}_j' - p_j\, c(y)\, \mathbf{e}_i'\right)$$

and where $\mathbf{e}_i$ denotes i-th coordinate vector and $H = \left(\frac{1}{\beta}I - T\right)^{-1}$. Because the multipliers ξ_{s_1} and ξ_{s_2} are well-defined, $\Phi(y)$ is a continuous function. Moreover, $\Phi(y)$ is non-constant because $u(y)$ and $c(y)$ are linearly independent and because ξ_{s_1} and ξ_{s_2} can, without loss of generality, be independently scaled. Then, because μ_{ij}^k is a nondegenerate probability measure, straightforward application of the first mean value theorem for the Lebesgue integral yields existence of some $x \in \left[\underline{y}_{ij}^k, \overline{y}_{ij}^k\right]$ such that $\Phi(x) < \int_{\underline{y}_{ij}^k}^{\overline{y}_{ij}^k} \Phi(y)\; d\mu_{ij}^k$. This implies that $\sigma \geq 0$ does not hold for all choices of x_{ij}^k from $\left[\underline{y}_{ij}^k, \overline{y}_{ij}^k\right]$.

If E is empty, then all the participation constraints are slack which implies that the multipliers associated with these constraints are zeros. Then $\Phi(y) = z(y)$ and the above argument applies. ■

Note that the proof of Proposition A2 applies to any non-autarkic solution to problem P_0. Recall that autarky is an allocation where all nonempty measures μ_{ij}^k have zero supports and, thus, is degenerate. This means that Proposition A2 holds for all possible solutions to problem P_0. However, if Proposition A2 is to be of interest, it better be that there is a wide class of environments where these solutions are non-autarkic. It is easy to provide conditions for existence of non-autarkic implementable allocations. These exist if $c'(0) < \frac{1}{\left(\frac{1}{\beta}-1\right)N+1}u'(0)$.[15] Then if, as was argued above, connectedness is an innocuous restriction for problem P, then this condition is also sufficient for existence of non-autarkic solutions to problem P_0.

[15] This condition is sufficient for existence of non-autarkic implementable allocations in which the support of $\mathbf{p}$ is $\{0, B\}$, trades are limited to transfers of B units of money in meetings of producers with zero and consumers with B units, and in which consumers make take-it-or-leave-it offers to producers.

Table 1: Optima with the ex ante pairwise core notion of implementability. $u(x) = \sqrt{x}$.

r	0.01	0.07	0.11	0.12	0.15	0.19	0.20	0.30	0.40	0.50
α	0	0	0	0	0	0	0	0	0	0
δ	0	0	0	0	0	0	0	0	0	0
p_0	.2181	.2704	.2938	.3023	.3286	.3570	.3645	.4242	.4635	.4830
p_1	.5884	.4974	.4566	.4466	.4226	.3998	.3951	.3573	.3303	.3135
p_2	.1935	.2322	.2496	.2511	.2488	.2432	.2404	.2185	.2062	.2035
λ_{01}	$.2023^{\dagger}$	$.4143^{\dagger}$	$.5774^{\dagger}$	$.6291^{\dagger}$	$.7820^{\dagger}$	$.9884^{\dagger}$	1	1	1	1
λ_{12}	$.3357^{\dagger}$	$.6763^{\dagger}$	$.9478^{\dagger}$	1	1	1	1	1	1	1
λ_{11}	$.1218^{\dagger}$	$.2537^{\dagger}$	$.3516^{\dagger}$	$.3804^{\dagger}$	$.4577^{\dagger}$	$.5432^{\dagger}$	$.5613^{\dagger}$	$.7259^{\dagger}$	$.8761^{\dagger}$	1
y_{01}	$.25^{*}$	$.25^{*}$	$.25^{*}$	$.25^{*}$	$.25^{*}$	$.25^{*}$	$.2408^{*}$	$.1585^{*}$	$.1136^{*}$	$.0844^{*}$
y_{12}	$.25^{*}$	$.25^{*}$	$.25^{*}$	$.2401^{*}$	$.1870^{*}$	$.1390^{*}$	$.1301^{*}$	$.0730^{*}$	$.0452^{*}$	$.0293^{*}$
y_{11}	$.0908^{*}$	$.0938^{*}$	$.0927^{*}$	$.0913^{*}$	$.0855^{*}$	$.0755^{*}$	$.0730^{*}$	$.0529^{*}$	$.0395^{*}$	$.0293^{*}$
y_{02}	.6876	.5306	$.4331^{*}$	$.3974^{*}$	$.3196^{*}$	$.2529^{*}$	$.2408^{*}$	$.1585^{*}$	$.1136^{*}$	$.0844^{*}$

Note that for $r = 0.01$ the set of binding first order constraints (10) includes binding inequality constraint for λ^1_{02} which is not shown in the table.

Table 2: Optima with the ex ante pairwise core notion of implementability. $u(x) = x^{\kappa}$.

κ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
α	0	0	0	0	0	0	0	0	0
δ	0	0	0	0	0	0	0	0	0
p_0	.1357	.1845	.2148	.2357	.2510	.2623	.2727	.3243	.4796
p_1	.7593	.6703	.6114	.5682	.5343	.5067	.4812	.4268	.3572
p_2	.1050	.1452	.1738	.1961	.2147	.2310	.2461	.2489	.1632
λ_{01}	.0300†	.0803†	.1457†	.2232†	.3108†	.4058†	.5180†	.8995†	1
λ_{12}	.1622†	.2618†	.3514†	.4331†	.5103†	.5821†	.6640†	1	1
λ_{11}	.0247†	.0596†	.0998†	.1432†	.1888†	.2359†	.2898†	.4430†	.6134†
y_{01}	.0744*	.1337*	.1791*	.2172*	.2500*	.2789*	.3046*	.3277*	.1624*
y_{12}	.0744*	.1337*	.1791*	.2172*	.2500*	.2789*	.3046*	.3051*	.1411*
y_{11}	.0118*	.0305*	.0509*	.0717*	.0925*	.1130*	.1330*	.1352*	.0866*
y_{02}	.4792	.5326	.5549	.5726	.5859	.5969	.5881*	.3645*	.1624*

The best quantity of output varies with κ. In the table the best quantity is equal to y_{01} for all κ except $\kappa = 0.9$ for which this quantity equals 0.3487.

Table 3: Optima with the ex post pairwise core notion of implementability. $u(x) = \sqrt{x}$.

r	0.01	0.04	0.06	0.07	0.10	0.13	0.14	0.16	0.17	0.25
α	.0946	.0709	.0555	.0480	.0258	.0045	0	0	0	0
δ	.0623	.0474	.0374	.0325	.0177	.0031	0	0	0	0
p_0	.3319	.3353	.3377	.3389	.3426	.3465	.3487	.3563	.3634	.4094
p_1	.3616	.3551	.3506	.3484	.3416	.3346	.3331	.3328	.3324	.3279
p_2	.3065	.3096	.3117	.3127	.3158	.3189	.3182	.3109	.3042	.2627
λ_{01}	1	1	1	1	1	1	1	1	1	1
λ_{12}	1	1	1	1	1	1	1	1	1	1
λ_{11}	1	1	1	1	1	1	1	1	1	1
y_{01}	.2570	.2574	.2576	.2577	.2580	.2583	.2607	.2719	.2609*	.1853*
y_{12}	.1954*	.1940*	.1930*	.1924*	.1902*	.1875*	.1826*	.1612*	.1505*	.0919*
y_{11}	.1954*	.1940*	.1930*	.1924*	.1902*	.1875*	.1826*	.1612*	.1505*	.0919*
y_{02}	.3041*	.2993*	.2963*	.2949*	.2907*	.2866*	.2838*	.2729*	.2609*	.1853*
$\frac{\Delta W}{W}$	3.27%	2.29%	1.62%	1.27%	0.49%	0.04%	0%	0%	0%	0%

Table 4: Optima with ex ante versus optima with ex post pairwise core notions of implementability.

r	0.02		0.06		0.10		0.15		0.25	
α	.0866	0	.0555	0	.0258	0	0	0	0	0
δ	.0573	0	.0374	0	.0177	0	0	0	0	0
p_0	.3330	.2310	.3377	.2649	.3424	.2857	.3525	.3286	.4094	.3979
p_1	.3595	.5678	.3506	.5084	.3416	.4680	.3330	.4226	.3279	.3744
p_2	.3075	.2012	.3117	.2267	.3158	.2463	.3145	.2488	.2627	.2277
λ_{01}	1	$.2385^{\dagger}$	1	$.3802^{\dagger}$	1	$.5244^{\dagger}$	1	$.7820^{\dagger}$	1	1
λ_{12}	1	$.3948^{\dagger}$	1	$.6217^{\dagger}$	1	$.8572^{\dagger}$	1	1	1	1
λ_{11}	1	$.1441^{\dagger}$	1	$.2323^{\dagger}$	1	$.3211^{\dagger}$	1	$.4577^{\dagger}$	1	$.6262^{\dagger}$
y_{01}	.2572	$.25^{*}$	.2576	$.25^{*}$	.2580	$.25^{*}$	.2667	$.25^{*}$	$.1853^{*}$	$.1927^{*}$
y_{12}	$.1950^{*}$	$.25^{*}$	$.1930^{*}$	$.25^{*}$	$.1902^{*}$	$.25^{*}$	$.1715^{*}$	$.1870^{*}$	$.0919^{*}$	$.0959^{*}$
y_{11}	$.1950^{*}$	$.0913^{*}$	$.1930^{*}$	$.0934^{*}$	$.1902^{*}$	$.0937^{*}$	$.1715^{*}$	$.0855^{*}$	$.0919^{*}$	$.0619^{*}$
y_{02}	$.3025^{*}$	.6473	$.2963^{*}$	.5459	$.2907^{*}$	$.4763^{*}$	$.2783^{*}$	$.3196^{*}$	$.1853^{*}$	$.1927^{*}$
$\frac{\Delta W}{W}$	19.5%		14.2%		11.2%		7.62%		3.22%	

For each value of r the first column shows the optimum with ex post and the second column shows the optimum with ex ante pairwise core notions of implementability. The last row shows welfare improvement relative to optima with ex post pairwise core notion of implementability.