Macro factors and the Term Structure of Interest Rates

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Abstract

This paper presents an essentially affine model of the term structure of interest rates making use of macroeconomic factors and their long-run expectations. The model extends the approach pioneered by Kozicki and Tinsley (2001) by modeling *consistently* long-run inflation expectations *simultaneously* with the term structure. This model thus avoids the standard pre-filtering of long-run expectations, as proposed by Kozicki and Tinsley (2001). Application to the U.S. economy shows the importance of long-run inflation expectations in the modelling of long-term bonds. The paper also provides a macroeconomic interpretation for the factors found in a latent factor model of the term structure. More specifically, we find that the standard "level" factor is highly correlated to long-run inflation expectations, the "slope" factor captures temporary business cycle conditions, while the "curvature" factor represents a clear independent monetary policy factor.

Keywords: Essentially affine term structure model, macroeconomic factors, long-run market expectations, monetary policy rule.

J.E.L.: E43, E44, E52.

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1 Introduction

Standard models of the term structure of interest rates successfully model the entire yield curve by means of a limited number of underlying factors. The models, however, fail to provide a clear economic interpretation of the factors.[1] In this paper, we construct and estimate a model of the yield curve based on factors with a clear macroeconomic interpretation.

Even though there is strong empirical evidence that the term structure of interest rates is linked to macroeconomic factors[2], it has been proven difficult to find a model that actually reproduces these links. In fact, it has been shown in the literature that term structure models based on macroeconomic variables perform poorly relative to the existing latent factor models. Although macro factors explain to a large extent the dynamics of the short-term interest rates, e.g. by standard Taylor rule specifications, the yields of bonds with longer maturities are not fitted accurately. Ang and Piazessi (2002), for instance, find that even though macroeconomic factors clearly affect the short end of the yield curve, they do not account for the long-run end of the yield curve. They close their model by allowing another latent factor to model the long end of the curve. Dewachter et al. (2001), using Ang and Piazessi's framework, also show that the misfit of the long end of the term structure can be quite substantial. Large and highly persistent pricing errors (of up to 6% p.a.) are found and clearly suggest the existence of additional factors. Kozicki and Tinsley (2001a,b) suggest that a missing factor may have a macroeconomic interpretation. Specifically, filtering the missing factor from the yield curve, they show that this factor may be related to the long-run inflation expectations of agents. This finding clearly paves the way for a structural macroeconomic interpretation of the term structure. That is, by adding long-run expectations of macro variables and, in particular, inflation expectations to the set of (macroeconomic) state variables, the misfit of the long-term yield curve, inherent in the standard macro-models of the yield curve, could be resolved in an intuitively plausible way.

The aim of this paper is twofold. First, we propose a model for the term structure of interest rates modeling long-run expectations *consistently.* Second, we relate the resulting factors to the ones obtained from a latent factor model, as usually employed in the finance literature. This allows us to interpret the latent factors in terms of macroeconomic variables.

Regarding the first aim of the paper, the method proposed improves significantly on the approach taken in the literature to use long-run expectations of macroeconomic variables in order to fit the yield curve. Currently, one takes a two step approach where in a first step these long-run expectations (endpoints) are filtered from the data using some statistical

[1] In the financial literature, it has become standard to label the factors by the way they affect the term structure, i.e. a "level", a "slope" and a "curvature" factor. This is naturally more of a description of the effects of the factors on the bond yields than a unambiguous economic interpretation.

[2] See, for instance, the extant empirical literature on the predictability of the term structure for future business cycle or inflation dynamics (e.g. Mishkin (1990), Estrella and Mishkin (1997)).

procedure. These filtered expectations are then subsequently used to fit the term structure. A drawback of this method is that not all information available is used to filter the long-run expectations. Typically, these long-run expectations are filtered on a subset of the data series. Another disadvantage of this approach is that these filtered expectations are not necessarily consistent with the notion of the expected long-run value.[3] To combine rational expectations with shifting endpoints, a necessary condition for a variable to be a candidate endpoint is that it follows a (possibly degenerate) random walk model under the empirical probability measure. However, by imposing the random walk condition on the long-run expectation, one introduces a random walk in all of the state variables driving the term structure. While this conforms well with the macroeconomic literature on the unit root behavior of inflation, and possibly interest rates, it generates a crucial problem in modeling the term structure: when state variables contain unit roots, term structure models based on the expectation hypothesis are ill defined (see, for instance, Campbell et al. (1997)). The two views on the dynamics of the economy, i.e. the (non-stationary) macroeconomic and the (stationary) finance view, can, however, be reconciled by noting that the two views are defined in a possibly different probability measure. Macroeconomic models focus on the empirical probability measure while term structure models are based on the risk neutral probability measure.[4] Hence, we can match the two views by imposing unit roots under the historical probability measure while generating mean-reverting macroeconomic variables under the risk neutral probability measure. This reconciliation of the two views is accomplished by allowing for time-varying risk premia along the line of the class of essentially affine models introduced by Duffee (2001). This has an important informational advantage as it allows us to filter the long-run expectations of macroeconomic variables using both the information contained in observable macroeconomic variables, such as output and inflation, as well as in the term structure of interest rates. Using this framework, we do find that the entire yield curve can be explained with reasonable accuracy by macroeconomic factors.

The second objective of the paper is to relate the latent factors typically found in the finance literature to a set of macroeconomic variables, observable as well as non-observable (filtered). A typical approach in the literature has been to simply decompose the correlation structure among the yields of different maturities into a number of factors. A standard result is that three factors are sufficient to capture the bulk of these correlations. No economic interpretation for these factors is usually provided. The filtered factors are usually labeled

[3]Kozicki and Tinsley (2001) propose three types of models: a constant fixed endpoint, a moving average rule in the macroeconomic variable, and a shifting model. These models are not necessarily consistent. To the extent that the dynamics of the endpoints deviate from a random walk, one cannot interpret the filtered value as a long-run expectation (since the endpoint is expected to move) unless one allows for some expectational biases on the part of the agents.

[4]Naturally, under the pure expectations hypothesis the risk neutral and the historical probability measure coincide.

according to their effect on the yield curve: a "level" factor, affecting equally the entire yield curve; a "slope" factor, affecting the yield spread; and a "curvature" factor, describing the additional curvature in maturities typically between 3 and 7 years. In this paper, we find that each of the different term structure effects, i.e. latent factors, has a macroeconomic source. More in particular, we find that the "level" effect can be linked to long-run inflation effects, that the "slope" factor correlates well with the predictable inflation and business cycle components, and that the "curvature" effect is related to the current stance of monetary policy, i.e. to real interest rate movements not related to the standard macroeconomic conditions.

The remainder of the paper is organized as follows. In section 2, we present a continuous-time model for the macroeconomy and the term structure that integrates in a consistent way the macroeconomic and the finance view. The mathematical properties of the model ensure both a proper interpretation of the filtered factors as long-run expectations and a well-defined term structure model based on these factors. Section 3 implements the model by estimating the macroeconomic dynamics and filtering the long-run expectations by means of the Kalman filter. Section 4 discusses the relation of the macroeconomic state variables to those of a latent factor Vasicek model and verifies the appropriateness of the filtered expectations by contrasting them to survey expectations. Finally, section 5 concludes and discusses possible routes for further research.

2 An affine term structure model with macro factors

In this section, we present a continuous-time model of the term structure of interest rates which incorporates both macroeconomic factors and their long-run expectations. In the first subsection, we set out the dynamics of the model under the historical probability measure and introduce the definition of long-run expectations of the macroeconomic variables. The basis is a simple non-stationary model for the output gap, inflation, and real interest rate dynamics. Next, we model the dynamics under the risk neutral probability measure and impose the necessary conditions for stationarity under this measure. As mentioned in the introduction, this is made possible through the adoption of time-varying risk premia. Implications for the term structure are also discussed.

2.1 Incorporating long-run macroeconomic expectations

The failure of the standard macroeconomic approach (see, for instance, Ang and Piazessi (2002)) to model the long end of the yield curve suggests some misspecification in the modeling of the long-run expectation of the (short-run) interest rate process. As shown by Kozicki and Tinsley (2001), the misspecification can be attributed to the failure to incorporate time-varying attractors (endpoints) for the short-run interest (inflation) rate. Unlike these authors, we do not filter these time-varying attractors for the interest rate from the term structure.[5] Instead, we construct these attractors from the macroeconomic and term structure dynamics. We start assuming the following dynamics in continuous time for the output gap, $y(t)$, inflation, $\pi(t)$, and the instantaneous real interest rate, $\rho(t)$:

$$\mathrm{d}y(t) = \left[\kappa_{yy}\left(y(t)-y^{*}(t)\right)+\kappa_{y\pi}\left(\pi(t)-\pi^{*}(t)\right)+\kappa_{y\rho}\left(\rho(t)-\rho^{*}(t)\right)\right]dt+\sigma_{y}dW_{y}(t), \tag{1}$$

$$\mathrm{d}\pi(t) = \left[\kappa_{\pi y}\left(y(t)-y^{*}(t)\right)+\kappa_{\pi\pi}\left(\pi(t)-\pi^{*}(t)\right)+\kappa_{\pi\rho}\left(\rho(t)-\rho^{*}(t)\right)\right]dt+\sigma_{\pi}dW_{\pi}(t), \tag{2}$$

$$\mathrm{d}\rho(t) = \left[\kappa_{\rho y}\left(y(t)-y^{*}(t)\right)+\kappa_{\rho\pi}\left(\pi(t)-\pi^{*}(t)\right)+\kappa_{\rho\rho}\left(\rho(t)-\rho^{*}(t)\right)\right]\mathrm{d}t+\sigma_{\rho}\mathrm{d}W_{\rho}(t), \tag{3}$$

where $W_i(t)$, $i=\{y,\pi,\rho\}$, denotes independent Wiener processes defined on the probability space $(\Omega,\mathcal{F},P)$ with filtration $\mathcal{F}_t$. As such, we can interpret the shocks $dW(t)$ as structural shocks to the output gap, inflation, and the real interest rate, respectively. The variables $y^{*}(t)$, $\pi^{*}(t)$ and $\rho^{*}(t)$ can be interpreted as long-run macroeconomic attractors, or endpoints, if two conditions are satisfied: First, if the market does not expect any change in these variables ($E_t dx(t)=0,\ x(t)=y^{*}(t),\pi^{*}(t),\rho^{*}(t)$) and, second, if the variables in the system (1)-(3) converge to their respective central tendencies. Formally, we only allow deviations from these central tendencies to determine the short-run dynamics of the respective macroeconomic variables. In this way, we actually ensure that the exogenous central tendency variables act as long-run attractors in this system:[6]

$$\begin{aligned}&\lim\nolimits_{s\to\infty}E_t\left(y(s)\mid y^{*}(t)\right)=y^{*}(t),\\&\lim\nolimits_{s\to\infty}E_t\left(\pi(s)\mid \pi^{*}(t)\right)=\pi^{*}(t),\\&\lim\nolimits_{s\to\infty}E_t\left(\rho(s)\mid \rho^{*}(t)\right)=\rho^{*}(t),\end{aligned} \tag{4}$$

where E_t denotes the mathematical expectation operator defined under the historical probability measure P. Following the recent literature on linear policy rules, we impose that the central bank uses a linear policy rule for the real interest rate. More specifically, we assume that the central bank follows a rule based on long-run expectations for both the output gap and inflation. This policy rule is formalized as:

$$\rho^{*}(t)=\gamma_0+\gamma_{y^{*}}y^{*}(t)+\gamma_{\pi^{*}}\pi^{*}(t). \tag{5}$$

[5] In their shifting model, Kozicki and Tinsley (2001) filter the endpoint of the short-run interest rate by taking the average of expected shor rates between five and ten years ahead. This approach yields the actual endpoint only if, as they assume, risk premia do not depend on long-run expectations and if the convergence process of interest rates is sufficiently fast to converge in five years time to the long-run endpoint. In our approach, we do not have to make either of these assumptions.

[6] Note that these long-run central tendencies can only serve as long-run attractors if the dynamics are stable. In the estimation of the system, we impose stability of the factors and thus the long-run attracting property of the exogeneous central tendencies.

The model is closed with the following definition for the instantaneous interest rate $r(t)$:

$$r(t) \equiv \pi(t) + \rho(t). \tag{6}$$

The dynamics of the above system conform well to the standard macroeconomic view. We allow each of the observable economic variables, output gap and inflation, to be affected through three channels: the (instantaneous) real interest rate (ρ), the other economic variable (output gap or inflation), and a mean reverting component modeling the possible inertia in the adjustment process. Central tendencies of output and inflation are assumed to be strictly exogenous and independent processes. The adoption of a Gaussian (Vasicek) type of model reflects our intention to offer complete flexibility with respect to the magnitudes and sizes of the conditional and unconditional correlations among the factors. This specification, moreover, fulfills the admissibility conditions specified in Dai and Singleton (2000). The costs associated with this choice are twofold: the lack of flexibility in fitting the interest rate volatility, since we assume constant conditional variances for the factors; and the possibility of negative interest rates.

The above representation of the dynamics of the economy can be restated in matrix notation. Denoting n as the number of factors in the model, five in our case, we define the vectors of n factors and shocks and an n x n diagonal matrix $\mathbf{S}$ as:

$$\mathbf{f}(t) \equiv \begin{pmatrix} y(t) \\ \pi(t) \\ \rho(t) \\ y^*(t) \\ \pi^*(t) \end{pmatrix}, \quad \mathrm{d}\mathbf{W}(t) \equiv \begin{pmatrix} \mathrm{d}W_y(t) \\ \mathrm{d}W_\pi(t) \\ \mathrm{d}W_\rho(t) \\ \mathrm{d}W_{y^*}(t) \\ \mathrm{d}W_{\pi^*}(t) \end{pmatrix}, \text{ and} \tag{7}$$

$$\mathbf{S} \equiv diag\left(\sigma_y, \sigma_\pi, \sigma_\rho, \sigma_{y^*}, \sigma_{\pi^*}\right).$$

The dynamics of the economy can, therefore, be restated as follows:

$$\mathrm{d}\mathbf{f}(t) = \left(\boldsymbol{\psi} + \mathbf{K}\mathbf{f}(t)\right)\mathrm{d}t + \mathbf{S}\mathrm{d}\mathbf{W}(t), \tag{8}$$

where

$$\mathbf{K} = \begin{bmatrix} \kappa_{yy} & \kappa_{y\pi} & \kappa_{y\rho} & -\kappa_{yy} - \kappa_{y\rho}\gamma_{y^*} & -\kappa_{y\pi} - \kappa_{y\rho}\gamma_{\pi^*} \\ \kappa_{\pi y} & \kappa_{\pi\pi} & \kappa_{\pi\rho} & -\kappa_{\pi y} - \kappa_{\pi\rho}\gamma_{y^*} & -\kappa_{\pi\pi} - \kappa_{\pi\rho}\gamma_{\pi^*} \\ \kappa_{\rho y} & \kappa_{\rho\pi} & \kappa_{\rho\rho} & -\kappa_{\rho y} - \kappa_{\rho\rho}\gamma_{y^*} & -\kappa_{\rho\pi} - \kappa_{\rho\rho}\gamma_{\pi^*} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\boldsymbol{\psi} = \left(-\kappa_{y\rho}\gamma_0, -\kappa_{\pi\rho}\gamma_0, -\kappa_{\rho\rho}\gamma_0, 0, 0\right)'.$$

The macroeconomic dynamics is then modeled by positing two stochastic trends, y^* and π^*, and three cointegrating relationships, relating the macroeconomic variables to these stochastic trends. This construction implies a non-stationary framework for the macroeconomic dynamics, as often corroborated in the macroeconomic literature, where unit roots in inflation cannot be rejected.

2.2 The term structure of interest rates

One problematic feature of the above specification of the central tendencies is the presence of unit roots in the time series of the modeled variables. Unit roots are traditionally avoided when modeling the term structure. Unit roots in conjunction with the expectation hypothesis yield unrealistic descriptions of the term structure. Specifically, limiting forward rates tend to decrease without bounds, eventually becoming more and more negative (see, for instance, Campbell et al. (1997)). We reconcile the presence of unit roots in observed macroeconomic series with a well-behaved term structure model by augmenting the pure expectation hypothesis with time-varying risk premia. As long as the factor dynamics are stationary under the risk neutral measure, the term structure of interest rates, meaning the limiting forward rate, is still well defined.

Equations (6) and (8) completely specify the instantaneous interest rate and the dynamics of the macroeconomic variables. This system must, therefore, also determine (up to some risk premium component) the term structure of interest rates and its dynamics. Absence of arbitrage opportunities implies that the price at time t of a zero-coupon default-free bond maturing at time T is defined as:

$$p(t,T) = E_t^Q\left(\exp\left(-\int_t^T r(u)\,du\right)\right), \tag{9}$$

where Q denotes the risk-neutral probability measure. In general, this risk-neutral probability is unknown and can only be specified by assuming some specification for the prices of factor risk. Following Duffee (2001), time variability in the prices of risk can be captured by specifying prices of risk as an affine function of each of the factors. The vector containing the time-varying prices of risk $\boldsymbol{\xi}$ is defined as:

$$\boldsymbol{\xi}(t) = \mathbf{S}\boldsymbol{\Lambda} + \mathbf{S}^{-1}\boldsymbol{\Xi}\mathbf{f}(t), \tag{10}$$

where $\boldsymbol{\Lambda} \equiv (\lambda_y, \lambda_\pi, \lambda_\rho, \lambda_{y^*}\lambda_{\pi^*})'$ and $\boldsymbol{\Xi}$ is an n x n matrix containing the sensitivities of the prices of risk to the levels of the state space factors. Changing probability measures is then easily done by means of the Girsanov theorem:

$$\mathrm{d}\mathbf{W}(t) = \mathrm{d}\tilde{\mathbf{W}}(t) - \boldsymbol{\xi}(t)\,\mathrm{d}t, \tag{11}$$

where $\tilde{W}_i(t)$ constitutes a martingale under measure Q. The macroeconomic state space dynamics can be restated in terms of this risk-neutral metric Q as:

$$\mathrm{d}\mathbf{f}(t) \quad = \quad \left(\tilde{\boldsymbol{\psi}} + \tilde{\mathbf{K}}\mathbf{f}(t)\right)\mathrm{d}t + \mathbf{S}\mathrm{d}\tilde{\mathbf{W}}(t), \tag{12}$$

$$\tilde{\mathbf{K}} = \mathbf{K} - \mathbf{\Xi}, \tag{13}$$

$$\tilde{\boldsymbol{\psi}} = \boldsymbol{\psi} - \mathbf{S}^2\mathbf{\Lambda}. \tag{14}$$

Note that the dynamics under the risk neutral probability measure are the relevant ones in pricing bonds, and not the ones under the historical probability measure. As such, non-stationary historical dynamics can be combined with stationary risk neutral dynamics by imposing appropriate conditions on the $\tilde{\mathbf{K}}$ matrix. More specifically, it is sufficient to impose that all eigenvalues of $\tilde{\mathbf{K}}$ are negative to ensure a stable dynamics under Q and a well-defined term structure.[7] Only under the pure expectation hypothesis, i.e. where the historical and risk neutral probability measure coincide, will the historical dynamics and its non-stationarity generate ill-defined term structure models.

It is well known that given the essentially affine dynamics under Q, equation (9) has an exponentially affine solution in the factors $\mathbf{f}(t)$. Denoting the time to maturity of the bond by $\tau = T - t$, the functional form for the bond prices is given by:

$$p(t,T) = p(\mathbf{f}(t), \tau) = \exp\left(-a(\tau) - \mathbf{b}(\tau)' \mathbf{f}(t)\right), \tag{15}$$

where $\mathbf{b}(\tau)$ is an n x 1 vector. The values for $a(\tau)$ and $\mathbf{b}(\tau)$ are determined by imposing the no-arbitrage condition in the bond markets:

$$\mathcal{D}^Q\left(p(\mathbf{f}(t), \tau)\right) = r(t)\, p(\mathbf{f}(t), \tau), \tag{16}$$

where $\mathcal{D}^Q$ denotes the Dynkin operator under the probability measure Q. The intuitive meaning of the latter condition is that, once transformed to a risk-neutral world, instantaneous holding returns for all bonds are equal to the instantaneous riskless interest rate. Using Girsanov's theorem, we can infer the implications for the real world by changing from the risk-neutral measure, Q, to the historical one, P.

Equations (15) and (16) determine the solution for the functions $a(\tau)$ and $\mathbf{b}(\tau)$ in terms of the following system of coupled ordinary differential equations (ODEs) which, in the general case, can only be solved numerically:

$$\begin{aligned} \frac{\partial a(\tau)}{\partial \tau} &= \mathbf{b}(\tau)'\tilde{\boldsymbol{\psi}} - \frac{1}{2}\mathbf{b}(\tau)'\mathbf{S}^2\mathbf{b}(\tau), \\ \frac{\partial \mathbf{b}(\tau)}{\partial \tau} &= \mathbf{b}_0 + \tilde{\mathbf{K}}'\mathbf{b}(\tau). \end{aligned} \tag{17}$$

A particular solution to this system of ODEs is obtained by specifying a set of initial conditions on a and $\mathbf{b}$. Inspection of equation (15) immediately shows that the relevant initial

[7] A proof of this statement is available upon request.

conditions are: $a(0) = 0$ and $\mathbf{b}(0) = \mathbf{0}$. The vector of constants $\mathbf{b}_0$ is defined by the interest rate definition presented in equation (6) and in the setting of this paper is equal to $\mathbf{b}_0 = (0\ 1\ 1\ 0\ 0)'$.

The bond pricing solution differs in important ways from the standard (independent) multi-factor term structure literature. First, allowing for interrelations among the factors (i.e. non-zero off-diagonal elements in $\tilde{\mathbf{K}}$) generates a coupled system of ODEs instead of a set of uncoupled ODEs. The bond pricing solution for the a and $\mathbf{b}$ functions, therefore, do not reduce to the standard multi-factor result (see, for instance, de Jong (2000)). Second, not all of the factor loadings start from unity at maturity $\tau = 0$. In our case, both the output gap and the stochastic central tendencies have zero loadings in the determination of the short rate. However, since the central tendencies serve as attractors for the observable macroeconomic variables, they become more important for longer maturities. This property also makes clear why the introduction of long-run expectations solves the problems faced by standard macro models in fitting the long end of the term structure. The introduction of these long-run expectations basically does not affect the short-run yields (factor loadings are very small) but does affect the long maturities in a crucial way. Long-run expectations can, therefore, model the long end of the term structure without directly affecting its short end.

3 Estimation

The model is estimated by means of a Kalman filter. Given the Gaussian structure of the model, the filtered long-run expectations are consistently estimated. Relative to the specification presented above, we simplify the empirical model by concentrating on a single central tendency, representing the long-run inflation expectation. The output central tendency y^* is filtered out by applying a Hodrick-Prescott (HP) filter on the output data and making use of the transitory component of the series. This redefines the y variable as the output gap with a long-run expectation equal to zero ($y^* = 0$). We start discussing the data set used to estimate the model. Next, the Kalman filter specification is explained taking care of the necessary perfect up-dating condition for the observable state variables. Finally, we present and discuss the parameter estimates of the model.

3.1 Data

We base our analysis on data from McCulloch and Kwon (1993) and Bliss (1997) provided by Duffee (2001). This data set consists of month-end yields on zero-coupon U.S. Treasury bonds with maturities of 3 and 6 months and 1, 2, 5, and 10 years. We use a quarterly frequency in the construction of the time series in order to incorporate the output gap series. Our data set consists of 140 data points (1964:Q1 to 1998:Q4) for each of the series. Although the original data set starts in 1958:Q1, we decide not to use the data before 1964 since Fama and Bliss (1987) point to the unreliability of the data for long-term bonds before this date. The output and inflation series were obtained from International Financial Statistics (IFS) database provided by the International Monetary Fund (IMF). As mentioned before, a proxy for the output gap is obtained by using an HP filter on the GDP series over the sample period.[8] Inflation was constructed by taking the yearly percentage change in the CPI index, that is $\pi_t = \ln CPI_t - \ln CPI_{t-4}$. Figure (1) depicts the series for the output gap, inflation and the term structure.

Insert Figure 1

In Table 1 we give some descriptive statistics of the sample series. The average term structure series display an increasing yield curve and the observed variance of the term structure tends to decrease in the maturity. There is strong evidence against normality in most series in terms of skewness and excess kurtosis (both decreasing with maturity) and in terms of a summary Jarque-Bera statistic (p-values are reported in the table). Also, strong autocorrelation is observed in all series over the sample period. Most interestingly, however, is the correlation matrix showing extreme correlation among the various bonds and significant but more moderate correlations between bonds on the one side and output gap or inflation on the other side. The output gap is positively correlated with the term structure up to 2-year yields and negatively correlated afterwards, while inflation is positively correlated with the entire term structure. The output gap and inflation are positively correlated with each other.

3.2 The Kalman filter

In order to estimate the model's parameters and filter its central tendencies (long-run perceptions), we derive the discrete-time implications of the continuous-time model so as to match the observation frequency of the sample. The discrete time dynamics, consistent with, i.e. derived from, the continuous-time model can then be used in a Kalman filter procedure to generate the unobservable central tendencies of the macroeconomic factors. Under the historical probability measure, the dynamics of the state vector $\mathbf{f}(t)$ are fully defined as:

$$\mathbf{f}(s) = \exp^{\mathbf{K}(s-t)} \boldsymbol{\psi}(s-t) + \exp^{\mathbf{K}(s-t)} \mathbf{f}(t) + \int_t^s \exp^{\mathbf{K}(s-u)} \mathbf{SdW}(u) \tag{18}$$

The stochastic properties of the state vector are thus (in discrete) time also fully known. More specifically, $\mathbf{f}(s)$ is conditionally normally distributed with mean and variance equal to[9], respectively:

$$E_t[\mathbf{f}(s)] = \exp^{\mathbf{K}(s-t)} \boldsymbol{\psi}(s-t) + \exp^{\mathbf{K}(s-t)} \mathbf{f}(t) \tag{19}$$

[8] We use a standard "lambda" in the filtering procedure equal to 1600.

[9] In practice, since the matrix $\mathbf{K}$ is in general not diagonal, the computation of the conditional mean and variance of the factors are not easily done. Dewachter et al. (2001) and Fackler (2000) provide equivalent procedures to compute the conditional mean and variance of the factors.

Table 1: **Summary statistics for the data used (1964:Q1-1998:Q4)**

	yield_{1q}	yield_{2q}	yield_{1yr}	yield_{2yr}	yield_{5yr}	yield_{10yr}	y	π
Mean (%)	6.522	6.778	7.009	7.252	7.564	7.770	0.034	4.776
Std. (%)	2.624	2.660	2.615	2.525	2.405	2.324	1.642	2.844
Min (%)	2.780	2.878	3.090	3.822	4.055	4.157	-4.689	1.128
Max (%)	15.241	15.924	15.911	16.107	15.696	15.065	3.879	13.502
Auto	0.986	0.987	0.989	0.992	0.995	0.997	0.869	0.993
Skew	1.333	1.316	1.201	1.175	1.098	0.917	-0.331	1.199
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.110)	(0.000)
Kurt	4.745	4.764	4.382	4.247	3.950	3.543	3.413	3.769
	(0.000)	(0.000)	(0.001)	(0.003)	(0.022)	(0.189)	(0.318)	(0.063)
JB	59.188	58.550	44.782	41.284	33.365	21.351	3.555	36.971
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.169)	(0.000)
				correlations				
yield_{1q}	1.000							
yield_{2q}	0.995	1.000						
yield_{1yr}	0.982	0.994	1.000					
yield_{2yr}	0.959	0.973	0.990	1.000				
yield_{5yr}	0.900	0.914	0.943	0.979	1.000			
yield_{10yr}	0.852	0.866	0.899	0.947	0.991	1.000		
y	0.190	0.177	0.144	0.068	-0.033	-0.078	1.000	
π	0.680	0.683	0.661	0.617	0.558	0.533	0.078	1.000

The bond yield data are based on spliced data from McCulloch and Kwon (1993) and Bliss (1997) provided by Duffee (2001) and concern U.S. Treasury bonds with maturities of 3 and 6 months and 1, 2, 5, and 10 years. Output gap (y) and inflation (π) data are constructed as mentioned in the text. The data series cover the period from 1964:Q1 until 1998:Q4, totalling 140 quarterly time series observations. *Mean* denotes the sample arithmetic average, expressed as *p.a.* percentage, *Std* standard deviation, *Min* minimum, *Max* maximum, *Auto* the first order quarterly autocorrelation, *Skew* and *Kurt* stand for skewness and kurtosis, respectively, while underneath these statistics are the significance levels at which the null of no skewness and the null of no excess kurtosis may be rejected. *JB* stands for the Jarque-Bera normality test statistic with the significance level at which the null of normality may be rejected underneath it.

$$E_t\left[\left(\mathbf{f}\left(s\right)-E_t\mathbf{f}\left(s\right)\right)^2\right] \quad = \quad \int_t^s \exp^{\mathbf{K}(s-u)}\mathbf{SS}'(\exp^{\mathbf{K}(s-u)})'\mathbf{d}u. \tag{20}$$

In order to estimate the parameters of the model and filter the central tendencies, we make use of a measurement equation including both the observable macroeconomic variables, output gap and inflation, and the observed yield curve. In this way, we ensure that the filtered central tendencies are consistent with both the macroeconomic dynamics and the term structure of interest rates. Consider a set of observed yields, $\hat{y}\left(\tau_i,t\right), i=1,...,m$. Given that the yield is defined as $-\ln p\left(\tau_i,t\right)/\tau_i$, the model implies a linear relation between the theoretical yields and the factors, i.e. $\hat{y}\left(\tau_i,t\right)=a\left(\tau_i\right)/\tau_i+\mathbf{b}'\left(\tau_i\right)/\tau_i\mathbf{f}\left(t\right)$. The measurement equation can then be written as:

$$\begin{pmatrix} \hat{y}_1(t,\tau_1) \\ \vdots \\ \hat{y}_m(t,\tau_m) \\ y(t) \\ \pi(t) \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{B} \\ \mathbf{e}_1' \\ \mathbf{e}_2' \end{pmatrix} \begin{pmatrix} y(t) \\ \pi(t) \\ \rho(t) \\ y^*(t) \\ \pi^*(t) \end{pmatrix} + \boldsymbol{\varepsilon}_t \ , \tag{21}$$

where $\mathbf{e}_i$ is a $(n \times 1)$ column vector of zeros with a one on the ith row, $\boldsymbol{\varepsilon}_t$ is an $(m{+}2)$ x 1 vector of measurement errors and

$$\mathbf{a} \ = \ (a(\tau_1)/\tau_1, \ldots, a(\tau_m)/\tau_m)', \tag{22}$$

$$\mathbf{B} \ = \ \begin{pmatrix} \mathbf{b}(\tau_1)'/\tau_1 \\ \vdots \\ \mathbf{b}(\tau_n)'/\tau_m \end{pmatrix}. \tag{23}$$

We can rewrite the measurement equation more concisely as:

$$\begin{gathered} \mathbf{z}(t) = \mathbf{c}_z + \mathbf{Hf}(t) + \boldsymbol{\varepsilon}(t), \\ E_t(\boldsymbol{\varepsilon}(t)\boldsymbol{\varepsilon}'(t)) = \mathbf{R}, \end{gathered} \tag{24}$$

where $\mathbf{z}(t)$ denotes the left-hand side of (21). Finally, we impose zero measurement errors for the output gap and inflation. This ensures that the actual observed output gap measure and inflation rate are in the information set of the agents. By imposing a zero measurement error on these variables, we make sure that agents update their inflation and output gap assessments to the actual observed ones. Technically speaking, this perfect updating is obtained by imposing zero variance-covariance structure on the $m+1$-th and $m+2$-th rows and columns of the variance-covariance matrix $\mathbf{R}$.

Given this measurement equation and the conditional normality of the factors, we can apply the standard Kalman filter algorithm. Defining $\boldsymbol{\Phi}(s) = \exp^{\mathbf{K}(s)}$, $c(s) = \exp^{\mathbf{K}(s)}\boldsymbol{\psi}s$ and $\mathbf{Q}(s) = \int_0^s \exp^{\mathbf{K}(s-u)}\mathbf{SS}'(\exp^{\mathbf{K}(s-u)})'\mathbf{d}u$, we can express the necessary prediction equations to be used in the Kalman filter as:

$$\begin{aligned} \hat{\mathbf{f}}_{t+\Delta t|t} &= \mathbf{c}(\Delta t) + \boldsymbol{\Phi}(\Delta t)\hat{\mathbf{f}}_{t|t}, \\ \hat{\mathbf{P}}_{t+\Delta t|t} &= \boldsymbol{\Phi}(\Delta t)\hat{\mathbf{P}}_{t|t}\boldsymbol{\Phi}'(\Delta t) + \mathbf{Q}(\Delta t). \end{aligned} \tag{25}$$

The updating equations for the filtered factors and variance-covariance matrix are then equal to:

$$\hat{\mathbf{f}}_{t+\Delta t\,|\,t+\Delta t} = \hat{\mathbf{f}}_{t+\Delta t\,|\,t} + \hat{\mathbf{P}}_{t+\Delta t\,|\,t}\mathbf{H}'\left(\mathbf{H}\hat{\mathbf{P}}_{t+\Delta t\,|\,t}\mathbf{H}' + \mathbf{R}\right)^{-1}\left(\mathbf{z}\left(t+\Delta t\right) - \mathbf{c}_z - \mathbf{H}\,\hat{\mathbf{f}}_{t+\Delta t\,|\,t}\right),$$

$$\hat{\mathbf{P}}_{t+\Delta t\,|\,t+\Delta t} = \hat{\mathbf{P}}_{t+\Delta t\,|\,t} - \hat{\mathbf{P}}_{t+\Delta t\,|\,t}\mathbf{H}'\left(\mathbf{H}\hat{\mathbf{P}}_{t+\Delta t\,|\,t}\mathbf{H}' + \mathbf{R}\right)^{-1}\mathbf{H}\hat{\mathbf{P}}_{t+\Delta t\,|\,t}\,. \tag{26}$$

Finally, given the non-stationary nature of the central tendencies, the standard full maximum likelihood procedure is infeasible. To circumvent the problem, we estimate additionally the first observation of the central tendency of inflation, $\pi^*(1)$, and assume that this is known such that the initial variance-covariance matrix $\hat{\mathbf{P}}_{1|1} = \mathbf{0}$. Conditional on these additional assumptions, we estimate the parameters of the model, collected in the vector ζ, by means of a maximum likelihood procedure, where the likelihood function is given by:

$$\ell\left(\mathcal{Z} \mid \zeta, \mathbf{f}_{1|1}, \hat{\mathbf{P}}_{1|1}\right) = \sum_{t=1}^{nd} -\tfrac{1}{2} ln \left|\mathbf{H}\hat{\mathbf{P}}_{t+\Delta t\,|\,t}\mathbf{H}' + \mathbf{R}\right|$$

$$-\sum_{t=1}^{nd} \tfrac{1}{2}\left(\mathbf{z}\left(t+\Delta t\right) - \mathbf{c}_z - \mathbf{H}\,\hat{\mathbf{f}}_{t+\Delta t\,|\,t}\right)'\left(\mathbf{H}\hat{\mathbf{P}}_{t+\Delta t\,|\,t}\mathbf{H}' + \mathbf{R}\right)^{-1}\left(\mathbf{z}\left(t+\Delta t\right) - \mathbf{c}_z - \mathbf{H}\,\hat{\mathbf{f}}_{t+\Delta t\,|\,t}\right), \tag{27}$$

and nd stands for the number of observations in the data set.

3.3 Parameter estimates

In this subsection we discuss the estimation results of the model.[10] This includes the analysis of the dynamic properties of the estimated factors and an assessment of both the long- and short-term real interest rate policy rule adopted by the central bank. The parameter estimates can be found in Table 2 and we concentrate on its more important implications.

By far the most important feature of the estimates is the finding that all of the macroeconomic variables, i.e. $y(t)$, $\pi(t)$ and $\rho(t)$ exhibit statistically significant reversion towards their stochastic trend. This can be inferred from the statistically significant negative diagonal elements of the $\mathbf{K}$ matrix. This finding corroborates the idea that each of the macroeconomic series has a trend-cycle decomposition.

Since we impose a random walk model on the long-run inflation expectation under the empirical probability measure, the system is, under this measure, non-stationary. Under the risk neutral probability measure, however, the state space dynamics is stable, *i.e.* the real parts of the eigenvalues of the matrix $\tilde{\mathbf{K}}$ are all negative. Moreover, $\mathbf{K}$ has imaginary eigenvalues, indicating an oscillating impulse response for the dynamic system. Figure 2

[10] The full model is estimated in a single step procedure. Optimization was performed using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm with a convergence tolerance for the gradient of the estimated coefficients equal to 1E-04. The robustness of the "optimum" reported is verified by checking convergence from an array of starting points.

Table 2: **Maximum likelihood estimates (1964:Q1-1998:Q4)**

	$\mathbf{y}$	$\boldsymbol{\pi}$	$\boldsymbol{\rho}$	$\boldsymbol{\pi}^*$		
$\boldsymbol{\kappa}_{y,\cdot}$	-0.9888	-0.4496	0.0835			
	(0.1347)	(0.0648)	(0.0488)			
$\boldsymbol{\kappa}_{\pi,\cdot}$	1.1313	-0.3646	-0.3889			
	(0.1064)	(0.0958)	(0.1237)			
$\boldsymbol{\kappa}_{\rho,\cdot}$	0.2744	-1.1707	-1.9107			
	(0.1503)	(0.2498)	(0.2910)			
$\boldsymbol{\gamma}_0$			0.0038			
			(0.0041)			
$\boldsymbol{\gamma}_\cdot$				0.3251		
				(0.1374)		
$\boldsymbol{\sigma}^{\mathbf{2}}_\cdot$	0.000339	0.000107	0.000850	0.000043		
	(0.000056)	(0.000014)	(0.000121)	(0.000015)		
$\boldsymbol{\lambda}_\cdot$	71.2743	-242.6286	40.2630			
	(26.5016)	(147.5687)	(18.2616)			
$\boldsymbol{\Xi}_{\cdot,\pi^*}$	1.3038	-3.1465	4.4159	0.1040		
	(0.5343)	(1.0096)	(0.9333)	(0.0425)		
$\boldsymbol{\Xi}_{\rho,\cdot}$	1.5258	-1.1236	-2.0459			
	(0.1932)	(0.2603)	(0.3159)			
$\mathbf{R}_{1q}$	13.7340					
$\mathbf{R}_{2q}$	14.5482	22.8339				
$\mathbf{R}_{1yr}$	5.8979	18.0036	19.4094			
$\mathbf{R}_{2yr}$	-2.4018	9.1557	12.7532	10.0375		
$\mathbf{R}_{5yr}$	-11.1427	-2.3571	2.1555	2.3205	0.0006	
$\mathbf{R}_{10yr}$	-7.4120	-0.9746	2.1784	2.2424	0.0755	0.0000

Maximum likelihood estimates with robust standard errors between brackets. The values in the measurement error covariance matrix ($\mathbf{R}$) are multiplied by 10^6. Total likelihood is equal to 5929.4434 or 42.3532 on average (excluding constant in the likelihood).

presents the filtered time series for the four factors involved: two observable ones (output gap, y, and inflation, π) and two non-observable ones (the real interest rate, ρ, and the long-run inflation expectation, π^*). The time series "STrule" also presented in this figure is discussed below.

Insert Figure 2

Next to the macroeconomic dynamics, Table 2 also provides important information with respect to the estimated prices of risk. Due to identification issues, we only estimate a subset of the prices of risk.[11] Note that the prices of risk are estimated with relatively high precision.

[11] We face identification problems when we allow for the estimation of all market prices of risk. The chosen set of market prices of risk avoids identification problems and gives enough flexibility in the determination of the time-varying risk premia. Duffee (2001) and Dai and Singleton (2001) also estimate restricted sets of the market prices of risk due to identification problems.

Only one of the parameter estimates is not significant at the five percent level. More striking, however, is the observation that the inflation expectations enter significantly in the prices of risk attached to any of the state variables. Inflation expectations thus tend to function as a general factor of the size of all of the risk premia, indicating that market prices of risk attached to any of these factors tend to increase with the level of the expected long-run inflation.

3.3.1 The real interest rate rule

The above economic framework estimates implicitly a Taylor-type rule. In order to make this rule explicit, we re-arrange equation (3) to obtain a real interest rate dynamics that conforms to standard Taylor rule specifications (see Clarida et al. (1998)). More specifically, we construct what we call the central bank short-term target rate, ρ_s^*, and make future expected real interest rate movements conditional on the gap between the observed and the target real interest rate:

$$E_t \mathrm{d}\rho\,(t) = \kappa_{\rho\rho}\,(\rho - \rho_s^*(t))\,\mathrm{d}t,$$

$$\rho_s^* = \underbrace{\gamma_0 - \frac{\kappa_{\rho y}}{\kappa_{\rho\rho}} y\,(t) - \frac{\kappa_{\rho\pi}}{\kappa_{\rho\rho}} \pi\,(t) + \left(\gamma_{\pi^*} + \frac{\kappa_{\rho\pi}}{\kappa_{\rho\rho}}\right) \pi^*\,(t)}_{\text{central bank short-term target}}\,.$$

Based on the estimates reported in Table 2, this rule can be written as (see also Figure 2):

$$\rho_s^* = \underset{(0.004)}{0.004} + \underset{(0.083)}{0.144}\; y\,(t) - \underset{(0.117)}{0.613}\; \pi\,(t) + \underset{(0.184)}{0.938}\; \pi^*\,(t)\,.$$

The above rule shows the importance of both the observed level of inflation and of its long-run expectation. In special, one observes the almost one-to-one relation between the central tendency of inflation and the short-run real interest rate target. The negative loading on observed inflation may at first seem strange. However, it can be interpreted as an indication of a forward looking Taylor rule specification. Recall that the dynamic specification we use models future expected dynamics for the output gap, inflation and the real interest rate in terms of deviations form the stochastic trend. Expressing this rule in terms of the state factors separately is, therefore, not very appropriate. After some algebra, equation (3) can also be written in terms of a forward looking Taylor-type rule. In this form, the real interest rate reacts to the expected change in both the output gap and inflation, as well as to a mean reverting term:

$$E_t \mathrm{d}\rho\,(t) = \underset{(0.307)}{-1.455}\,(\rho\,(t) - \rho_s^*(t))\,\mathrm{d}t,$$

$$\rho_s^* = \rho^* + \underset{(0.308)}{0.968}\; E_t\left(\frac{dy(t)}{dt}\right) + \underset{(0.312)}{1.013}\; E_t\left(\frac{d\pi(t)}{dt}\right)\,. \tag{28}$$

An expected increase in either the output gap or inflation triggers an expected increase in the real interest rate. Interestingly, we find an expected real interest rate increase that is approximately one-to-one with both expected output and inflation changes.

Finally, there is the third feature of the real interest rate rule, which is related to the interest rate smoothing, *i.e.* to the mean reversion properties of the real interest rate to its short-term target. We find a relatively strong mean reversion for the real interest rate, *i.e.* a high value for $\kappa_{\rho\rho}$. This results in a halving time of the deviation from the short-term target (equation 28) equal to approximately half year. This mean reversion suggests that this policy reaction factor exerts more influence on the short end of the term structure than on the long end. In other words, the policy reaction factor represents a "slope" factor and not a "level" factor for the term structure. This is in line with the conjecture of Knez *et al.* (1994), Evans and Marshall (1998) and Wu (2000).

4 A macroeconomic interpretation of the yield curve

Based on the empirical estimates of the previous section, we now proceed to explain the term structure from a macroeconomic perspective. As shown above, we use four macroeconomic factors: the output gap, inflation, the real interest rate, and the long-run inflation expectation. First, we explain the roles played by each of these variables in shaping the yield curve. Subsequently, we compare the term structure fit to that of a latent factor, and, finally, we try to relate the traditional latent factors to the estimated macro factors.

4.1 Decomposing the yield curve

Figure 3 depicts the factor loadings on each of the macroeconomic factors for the yield curve. A first important observation is that, unlike the standard (latent factor) literature, we do not find evidence of a direct macroeconomic level effect. Instead, we find evidence of two types of factors driving the term structure. First, factors that primarily determine the range of small maturities, being inflation, the real interest rate, and the output gap. The factor loadings of these factors typically dampen quite fast and are relatively small for the longer end of the term structure. The second type of factor is the long-run inflation expectation. This type of factor primarily models the long end of the term structure while not affecting the short end.

Insert Figure 3

Even though this decomposition is quite different from the standard decomposition obtained in the latent factor literature, the macro model performs reasonably well when compared to a standard latent factor model. Table 3 gives some diagnostic statistics which allow us to compare the performance of the macroeconomic model with the benchmark of a three-factor latent model. Keeping in mind that the three-factor latent model is sufficiently flexible to model the term structure, we still find the macro model to be competitive. Looking at the mean and autocorrelation of the fitting errors, the latent factor model seems to be modeling better the short end of the term structure while the macro model seems to do better in the range of longer maturities. The lack of flexibility of the macro model becomes clear by the size and standard deviation of the measurement errors, which are especially large for the short end of the yield curve. Nevertheless, when compared to the latent factor model the macro model displays somewhat lower autocorrelation in the measurement errors for maturities above two quarters. Figure 4 displays the fit of the macro model for the term structure of interest rates.

Table 3: **Term structure fitting erros**

	Macro model			**3-factor latent model**		
	mean (%)	std.dev. (%)	autocorr.	mean (%)	std.dev. (%)	autocorr.
yield_{1q}	0.13	0.62	0.32	-0.0002	0.34	0.15
yield_{2q}	0.21	0.67	0.35	0.14	0.32	0.25
yield_{1yr}	0.18	0.64	0.38	0.17	0.29	0.45
yield_{2yr}	0.15	0.53	0.37	0.15	0.28	0.44
yield_{5yr}	0.09	0.35	0.31	0.10	0.20	0.41
yield_{10yr}	0.07	0.26	0.31	0.08	0.17	0.46

autocorr. stands for one-lag autocorrelation.

Insert Figure 4

An alternative method to test for the adequacy of the model is to run regressions of the actual yields (or changes in yields) on the implied yields based on the macro model. Model adequacy is then tested by the null hypothesis that the implied yields are unbiased predictors of the actual yields. Running a regression in levels, we are not able to reject the null hypothesis of unbiased implied yields (see Table A1 in the Appendix). For all maturities, more than 93% of the variation of the yields is explained. So, in general, the macro model gives a reasonable description of the yield curve. Performing the regression on yield changes, however, reveals some biases. The regression coefficients tend to be significantly different from 1. Nevertheless, the fit is still reasonable since for all maturities more than 60% of the variation in the yield changes is explained. In short, on a general level we have a good macroeconomic description of the yield curve. The restrictions imposed on the macro model by fitting the yield curve only by means of macro factors do, however, restrict its flexibility which shows up in the fit of the changes in the yield curve.

4.2 A macroeconomic interpretation of the latent factors

Even though the macroeconomic interpretation derived from the macro model is by definition more informative than the results obtained from a standard latent factor model, it is important to find out the link between the two representations. This would allow us to give the latent factors an economic interpretation beyond the standard "level", "slope" and "curvature" labels traditionally attached to them. Here we perform such an exercise and show that there is a clear link between the latent and macro factors. A first analysis consists of performing a standard projection of the macro factors on the latent factors. Table 4 shows the regression results for each of the latent factors. Consistent with the above analysis, the macro model is able to track the latent factors relatively well. Most importantly, the "level" factor is fitted quite well with an $\overline{R}^2$ of 83%. This result stands in sharp contrast to the standard macro models (e.g. Ang and Piazessi (2002)), that fail exactly to fit the "level" factor. The success of our proposed model can be attributed almost completely to the introduction of the long-run inflation expectation. The "slope" and "curvature" factors are fitted less accurately. Nevertheless, the macro factors are able to explain around 55% of the variation of the latent factors. However, based on the reported regressions, a clear macroeconomic interpretation of the latent factors does not emerge immediately.

Table 4: **Regression of latent factors upon macro factors**

	"Level" factor	"Slope" factor	"Curvature" factor
cte	-0.0142	0.0042	0.0064
	(0.0031)	(0.0038)	(0.0029)
y	0.0134	0.4583	-0.3820
	(0.0590)	(0.0715)	(0.0559)
π	-0.0397	0.4326	0.4543
	(0.0498)	(0.0604)	(0.0472)
ρ	-0.0074	0.0523	0.7066
	(0.0579)	(0.0702)	(0.0549)
π^*	1.8714	-0.6095	-0.9583
	(0.1016)	(0.1232)	(0.0963)
$\overline{R}^2$	0.832	0.564	0.548

The three latent factors are regressed upon the filtered factors of the macro model presented above. Standard errors between brackets.

In order to get an idea of the macroeconomic meaning of the "slope" and "curvature" factors, we perform an OLS regression on orthogonalized components of the macroeconomic factors. This allows us to assess the contribution of the independent components of the macroeconomic variables for each of the latent factors. We orthogonalize the macroeconomic factors by means of a Cholesky factorization based on the following ordering: π^*, $\pi - \pi^*$, y, and $\rho - \rho^*$. The regression results of the latent factors on the orthogonalized factors are displayed in Table 5. Again, we find that the "level" factor is basically explained by the filtered long-run inflation expectation. This variable explains about 84 % of the total variation of the "level" factor. The "slope" factor is mainly explained by the second and third Cholesky factors, respectively inflation shocks orthogonal to long-run inflation expectations, and output gap effects not attributable to inflation components. More than 96% of the explained variablity of the "slope" factor is due to these two factors. Also, we find the inflation component (cholfac2) to be more important than the (inflation-independent) output gap component (cholfac3), in terms of $\overline{R}^2$ contribution. Finally, given the positive regression coefficients for both the second and third Cholesky factors, we can interpret the "slope" factor as a business cycle factor. That is, the "slope" factor tends to correlate positively with both demand inflation shocks and the output gap, both indicating a clear link to the business cycle. The third latent factor, the "curvature" factor, is mainly explained by the last Cholesky factor. More than 99% of the total explained variation of this factor is due to the fourth Cholesky factor. By construction, this factor represents real interest movements orthogonal (independent from) all other macroeconomic variables. As such, this factor represents an independent monetary policy factor that we label here as the "monetary stance" factor. The Cholesky factor regression coefficient is positive, meaning that an increase in the "curvature" is positively related to a tougher monetary policy stance.

Table 5: **Regression of latent factors upon ortogonalized macro factors**

	"Level" factor			"Slope" factor			"Curvature" factor		
	coefficient	$\overline{R}^2$	$\Delta\overline{R}^2$	coefficient	$\overline{R}^2$	$\Delta\overline{R}^2$	coefficient	$\overline{R}^2$	$\Delta\overline{R}^2$
cte	-0.0142			0.0044			0.0091		
	(0.0031)			(0.0038)			(0.0030)		
cholfac1	1.8265	0.835	0.835	-0.2408	0.019	0.019	-0.1449	0.009	0.009
	(0.0694)			(0.0841)			(0.0657)		
cholfac2	-0.0340	0.835	0.000	0.4464	0.394	0.375	0.0022	0.001	-0.008
	(0.0336)			(0.0407)			(0.0318)		
cholfac3	0.0102	0.834	-0.001	0.4806	0.565	0.171	-0.0806	0.002	0.001
	(0.0536)			(0.0650)			(0.0508)		
cholfac4	-0.0074	0.832	-0.002	0.0523	0.564	-0.001	0.7066	0.548	0.546
	(0.0579)			(0.0702)			(0.0549)		

The three latent factors are regressed upon orthogonalized macro factors. The Cholesky decomposition was done with the macro factors in the following order: $\pi^*, \pi - \pi^*, y - y^*, \rho - \rho^*$. Standard errors between brackets.

In summary, we find, within the limitations of the fit for the latent "slope" and "curvature" factors, a clear interpretation of the latent factors in macroeconomic terms. The standard "level" factor is a macroeconomic expectations effect, clearly related to long-run inflation expectations. The "slope" factor represents business cycle conditions while the "curvature" factor is related to the monetary policy stance held by the central bank. In order to track the responses across the yield curve of the respective Cholesky factors, we transform the obtained loadings on the macro factors to those relevant for the Cholesky factors. The following transformation is used:

$$\mathbf{B}'_{chol} = \mathbf{B}'\mathbf{Z}^{-1}\mathbf{L}', \text{ with } \mathbf{L}'\mathbf{L} = \mathbf{Z}E\left(\mathbf{f}\left(t\right)\mathbf{f}'\left(t\right)\right)\mathbf{Z}'$$
$$\left(\pi^*\left(t\right), \pi\left(t\right) - \pi^*\left(t\right), y\left(t\right), \rho\left(t\right) - \rho^*\left(t\right)\right)' = \mathbf{Zf}\left(t\right). \tag{29}$$

Figure 5 presents the factor loadings $\mathbf{B}_{chol}$. As can be seen, the first Cholesky factor, i.e. π^* exerts an important effect throughout the yield curve. Although we still do not recover a full "level" factor, it comes close. A shock to inflation expectations is transmitted through the entire yield curve. Note also that there is not a one-to-one relation. Interestingly, we find that interest rates respond more than one-to-one to long-run inflation expectations. We recover a sensitivity of the short-term interest rates to long-run inflation expectations equal to 1.44, which is close to the 1.5 often posited in the Taylor-rule related literature. The second and third factors, i.e. the business cycle conditions, are basically important for the short end of the term structure, as expected from temporary effects. The fourth Cholesky factor loadings show a much smaller mean reversion and also exerts some effect on the longer maturities. This is exactly what makes it a "curvature" factor. Consistent with the above interpretation, we find that a tougher monetary stance increases short and intermediate interest rate levels. Figures 6 to 8 give a graphical display of the latent factors together with the fit of the retained Cholesky factors. As can be observed, the fit is especially relevant for the first latent factor, in terms of π^*. Note also that the second latent factor has a strong business cycle component and that the fit based on the second and third Cholesky factors do track these cycles pretty well. Finally, we assess the third latent factor which we interpret as a monetary stance variable. Note that according to this interpretation the stance of monetary policy was rather loose in the seventies and then abruptely changed in the beginning of the eighties when the monetary stance became particularly strong. This interpretation of the latent factor in terms of toughness of the monetary policy stance is corroborated by the historical switches in inflation fighting policies.

Insert Figures 5 to 8

5 Cross-validation of long-run inflation expectations

The introduction of long-run inflation expectations plays a crucial role in the above analysis. Notwithstanding the fact that these inflation expectations are formally modeled as stochastic trends, and thus form consistent long-run attractors, and the fact that in the statistical analysis significant mean reversion towards these long-run expectations was found, the series itself still remains a *filtered* representation of long-run inflation expectations. In order to assess the plausibility of this filtered expectation series, we conduct two experiments. First, we relate the filtered inflation expectations to survey data for long-run inflation expectations.

We then relate them to long-run inflation expectations obtained from a univariate structural break analysis, as proposed by Kozicki and Tinsley (2001).

Figure 9 contrasts the filtered long-run inflation series (more specifically the ten-year average inflation forecast implied by the model) with the market long-run inflation expectation as recorded in the Survey of Professional Forecasters provided by the Federal Reserve Bank of Philadelphia. This figure provides supporting evidence for the filtered long-run expectations as the two series are highly correlated. In fact, regression results provided in Appendix C do not allow us to reject the one-to-one relation between the survey and the filtered expectation. Note that these results are particularly strong since the survey inflation expectation series was not used whatsoever in the filtering procedure.

Insert Figure 9

A second method to assess the plausibility of the filtered long-run inflation expectation consists of a break analysis for the long-run endpoints, as proposed by Kozicki and Tinsley (2001a)[12]. In order to construct a market inflation expectation, we follow Kozicki and Tinsley in assuming that expectations are an average of heterogenous inflation expectations built on an information lag that agents use to update their long-run expectations:

$$\pi_M = \sum_{i=1}^{N} \omega_i \, \pi_{M,i} \tag{30}$$

where π_M denotes the market long-run expectation, ω_i is the fraction of agents using an i year updating lag in their structural analysis, and $\pi_{M,i}$ denotes the long-run expectation of agents with an i year updating lag. More details can be found in Kozicki and Tinsley (2001a) and Bai and Perron (1998). Figure 10 shows the results from the structural break analysis for an agent with zero updating lag. We find evidence of three major break points in structural (long-run inflation). In a second step, we calculate the inferred structural inflation estimates for an agent with an i year information lag, where we allow i=1,...,10. The weights on each of the information lags is then obtained by a standard GMM procedure imposing positivity ($\omega_i \geq 0$) of our filtered long-run inflation on the alternative structural inflation endpoints $\pi_{M,i}$. Figure 11 presents the optimal fit based on the mean squared error (MSE) criteria. As can be seen from this figure, the fit is reasonable, implying that also from this structural break point of view we can account for the filtered inflation expectation. Figure 12 finally gives the weights attached to each information lag. As can be inferred from this figure, one interpretation of our filtered long-run inflation expectation is that it is an average of individual market expectations (i.e. different information lags) which is rather diffuse. A substantial part of the agents adapts structural inflation expectations quite rapidly, i.e. they

[12] Unlike these authors, we use the methodology proposed by Bai and Perron (1998).

use a relatively small information lag, say one to four years. Nevertheless, the regression results also indicate that a substantial part of the agents is only willing to adjust structural inflation expectations after a sufficient amount of strong and protracted information indicates this, i.e those classes of agents that use long information lags. Our results are again in line with Kozicki and Tinsley (2001).

Insert Figures 10 to 12

6 Conclusions

In this paper we have proposed a macro model that consistently and simultaneously models the rate of inflation, the output gap and the term structure of interest rates. This approach extends and corroborates the research by Kozicki and Tinsley (2001a), which shows the importance of long-run inflation expectations in the modelling of the yield curve. We extend their approach by focussing on a consistent model and by including time-varying prices of risk specifications. This allows us to avoid the two-step approach employed by these authors and to replace it by a fully consistent and relatively powerful one-step approach based on the standard Kalman filter technique.

Our results reinforce the basic message that inflation expectations are important in modeling the term structure dynamics from a macroeconomic point of view. The proposed technique for filtering long-run inflation expectations is based on both macroeconomic as well as term structure information. This technique delivers quite plausible estimates of these expectations. First, we obtain statistical evidence that the filtered inflation expectation serves as a stochastic trend both for the term structure and inflation. Second, the filtered inflation expectations conform well both to survey expectations and to some aggregated inflation expectation recovered from a structural break technique.

We find, moreover, that four macroeconomic factors model both the term structure as well as the macroeconomic dynamics rather well. While inflation expectations play a crucial role for the long-term maturities, actual macro variables such as inflation, output gap and the real interest rate are of primary importance for the short-term maturities. This macroeconomic decomposition allows us to interpret the standard "level", "slope" and "curvature" effect factors typically found in the standard finance literature. We find the "level" factor to be closely linked to the long-run inflation expectation, the "slope" factor to be an aggregate series for the business cycle condition, and the "curvature" factor to be related to the monetary stance of the central bank. As such, we have an interesting, although incomplete, macroeconomic interpretation of the standard latent factors.

Next to filtering long-run inflation expectations, we also allow for time-varying prices of risk. The time variability is assumed to be captured by the macroeconomic variables.

An interesting result is that long-run inflation expectations determine to a large extent the level of the prices of risk of all sources of risk. For instance, the price of risk attached to business cycle conditions changes with the level of the long-run inflation expectation. Long-run inflation expectations are, therefore, also a prime determinant of the level of risk premia.

The macroeconomic framework presented here allows for many extensions and applications. First, as we have incorporated a Taylor rule, the framework could be used to assess the implications of alternative (linear) feedback rules for both the macroeconomic as well as the bond market stabilization. Second, there is a vast literature on the information content of the term structure with respect to the future macroeconomic developments. Our approach gives a direct way to infer where the information content with respect to a macroeconomic variable is maximal. Finally, a straightforward extension is to include additional long-term expectations, such as for instance the structural real growth rate of the economy. In the current paper, we used the output gap obtained via HP filtering. The alternative is then to add one more stochastic factor for structural GDP and to replace the output gap by a direct measure of the production level. These issues are left for future research.

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Appendix A: Term structure fit

Table A1: Regression of yield data on model implied yields.

	Levels			First differences		
	cte.	coef.	$\overline{R}^2$	cte.	coef.	$\overline{R}^2$
yield$_{1q}$	0.002545	0.9813	0.945	0.000026	0.7717	0.665
	(0.001390)	(0.0202)		(0.000582)	(0.0465)	
yield$_{2q}$	0.002267	0.9973	0.937	0.000014	0.7728	0.609
	(0.001553)	(0.0220)		(0.000639)	(0.0526)	
yield$_{1yr}$	0.002191	0.9944	0.940	0.000012	0.7770	0.623
	(0.001548)	(0.0213)		(0.000589)	(0.0513)	
yield$_{2yr}$	0.002360	0.9878	0.956	0.000003	0.7866	0.667
	(0.001359)	(0.0181)		(0.000492)	(0.0472)	
yield$_{5yr}$	0.002138	0.9836	0.979	-0.000003	0.7957	0.744
	(0.000969)	(0.0123)		(0.000336)	(0.0397)	
yield$_{10yr}$	0.001544	0.9890	0.987	-0.000001	0.8354	0.781
	(0.000772)	(0.0096)		(0.000257)	(0.0377)	

Regression of yield data on model implied yields and a constant, both in levels and in first differences. Standard errors between brackets.

Appendix B: Latent factor model

Table B1: Estimated parameters for 3-factor latent model (1958:Q1-1998:Q4)

	"Level" factor	"Slope" factor	"Curvature" factor			
$\boldsymbol{\kappa}$	0.0120	0.8030	0.1907			
	(0.0035)	(0.0911)	(0.0415)			
θ	0.1755					
	(0.0640)					
$\boldsymbol{\sigma^2}$	0.000107	0.000436	0.000216			
	(0.000024)	(0.000110)	(0.000052)			
$\boldsymbol{\lambda}$.		-23.3985	2.4643			
		(8.9120)	(12.1550)			
$\mathbf{R}_{1q}$	0.0000					
$\mathbf{R}_{2q}$	-4.0713	0.2954				
$\mathbf{R}_{1yr}$	-11.3335	-2.5857	0.4815			
$\mathbf{R}_{2yr}$	-12.4916	-4.3614	0.0000	1.3896		
$\mathbf{R}_{5yr}$	-10.6210	-4.7770	-1.0166	0.2521	0.0753	
$\mathbf{R}_{10yr}$	-9.0145	-4.2164	-1.1756	0.0422	-0.3231	0.0000

ML estimates with standard errors underneath. The values in the measurement error covariance matrix ($\mathbf{R}$) are multiplied by 10^6. Total likelihood is equal to 5483.12 or 33.4337 on average (excluding constant in the likelihood).

Appendix C: Inflation forecasting

In this appendix, we present the results from OLS regressions of the ten-year average inflation forecasts provided by the Survey of Professional Forecasters on the forecasts computed based on our model and a constant.

Table C1: Inflation forecast regressions

	average 10-year inflation forecast
constant	-0.0058
	(0.0029)
coefficient	1.0487
	(0.0584)
$\overline{R}^2$	0.809
correlation	0.901
no.obs.	77

Note: The null hypothesis of no cointegration between the variables is rejected at the 1% confidence level. The regression in levels, as it is done, is therefore appropriate.

Figures

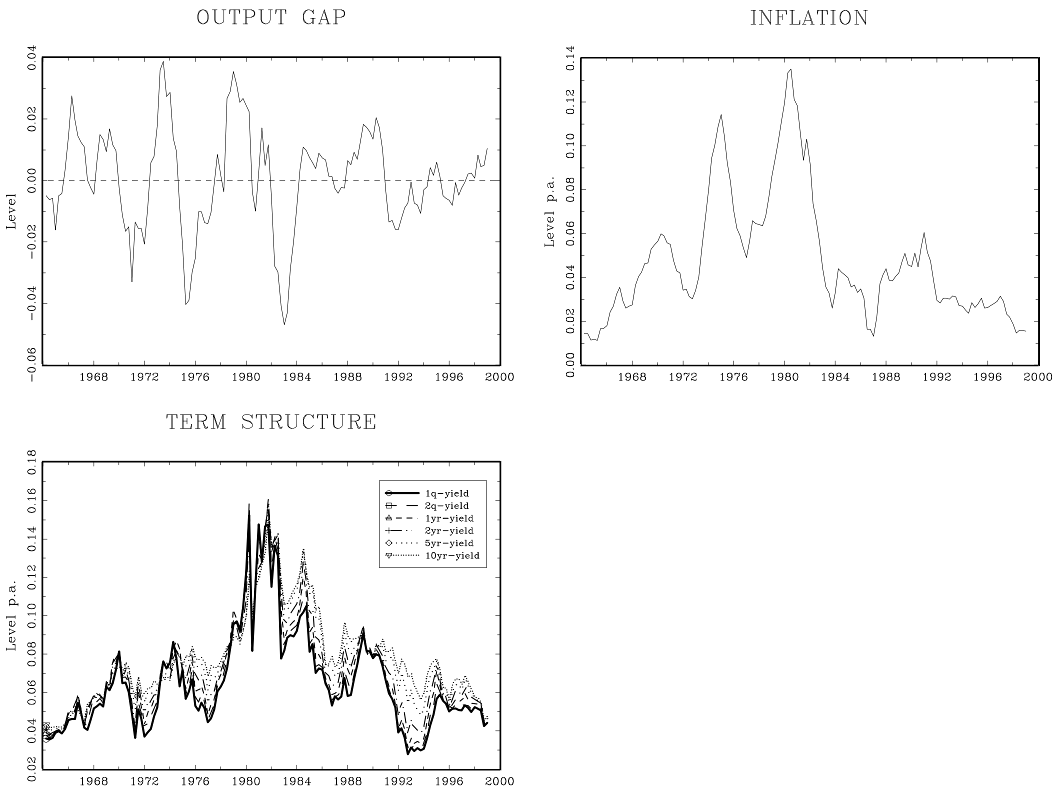

Figure 1: Data on output gap, inflation, and the term structure of interest rates (1964:Q1-1998:Q4).

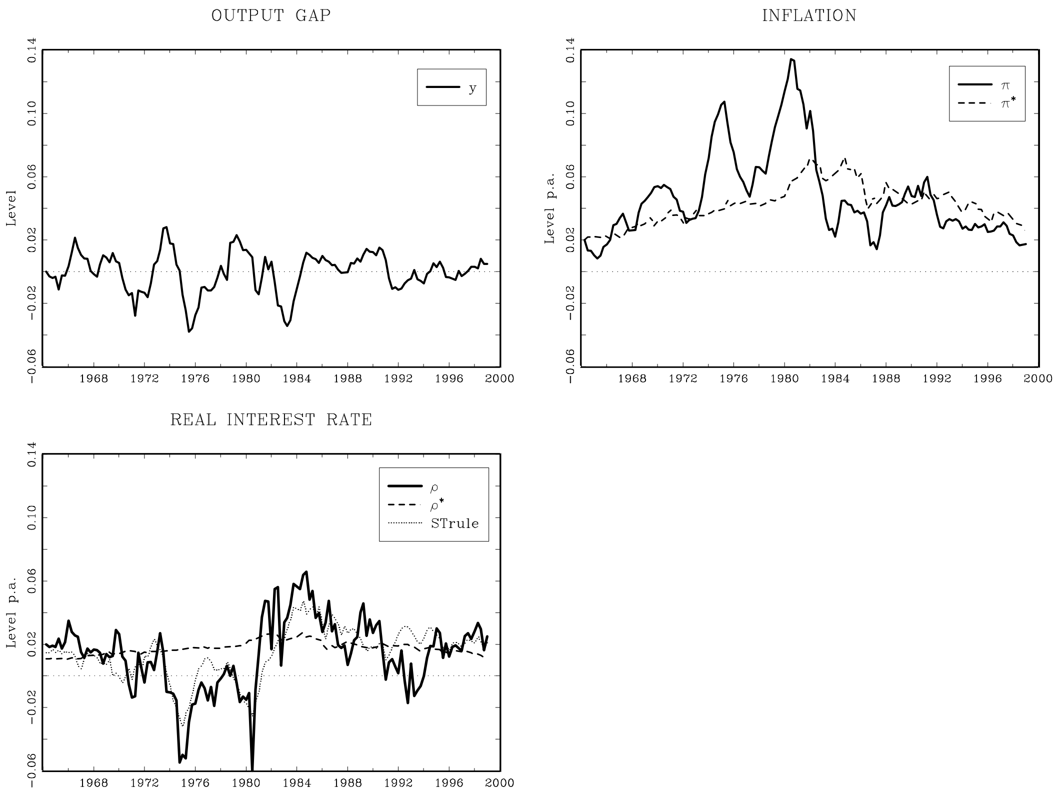

Figure 2: Macro variables and their estimated central tendencies.

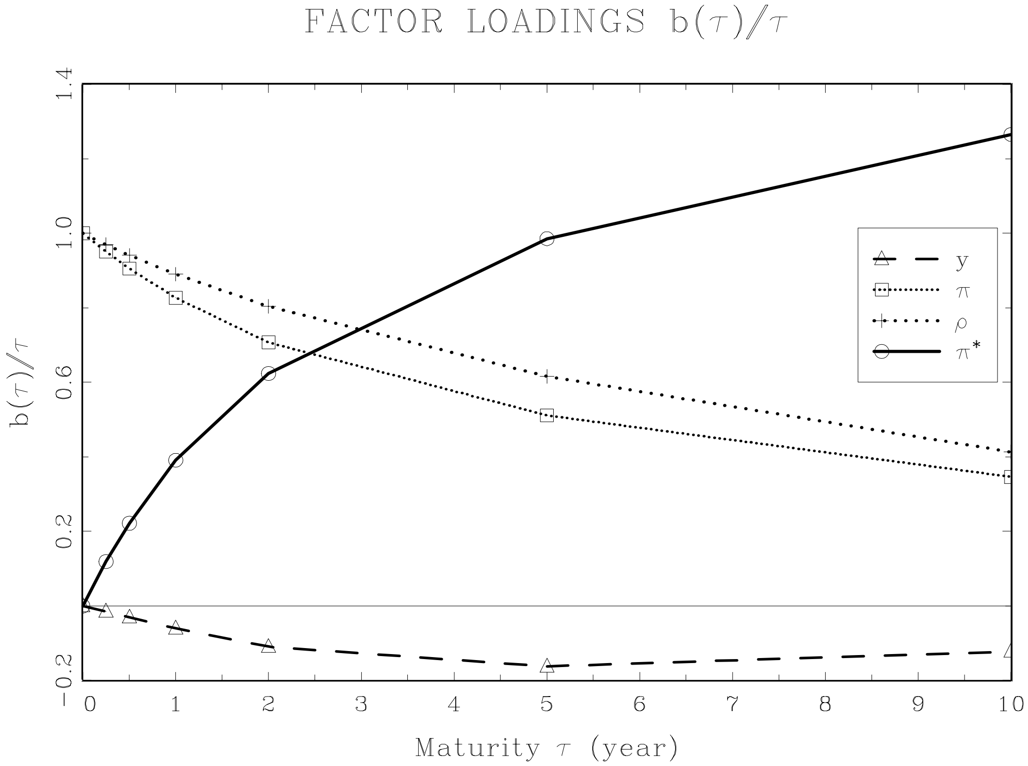

Figure 3: Estimated factor loadings.

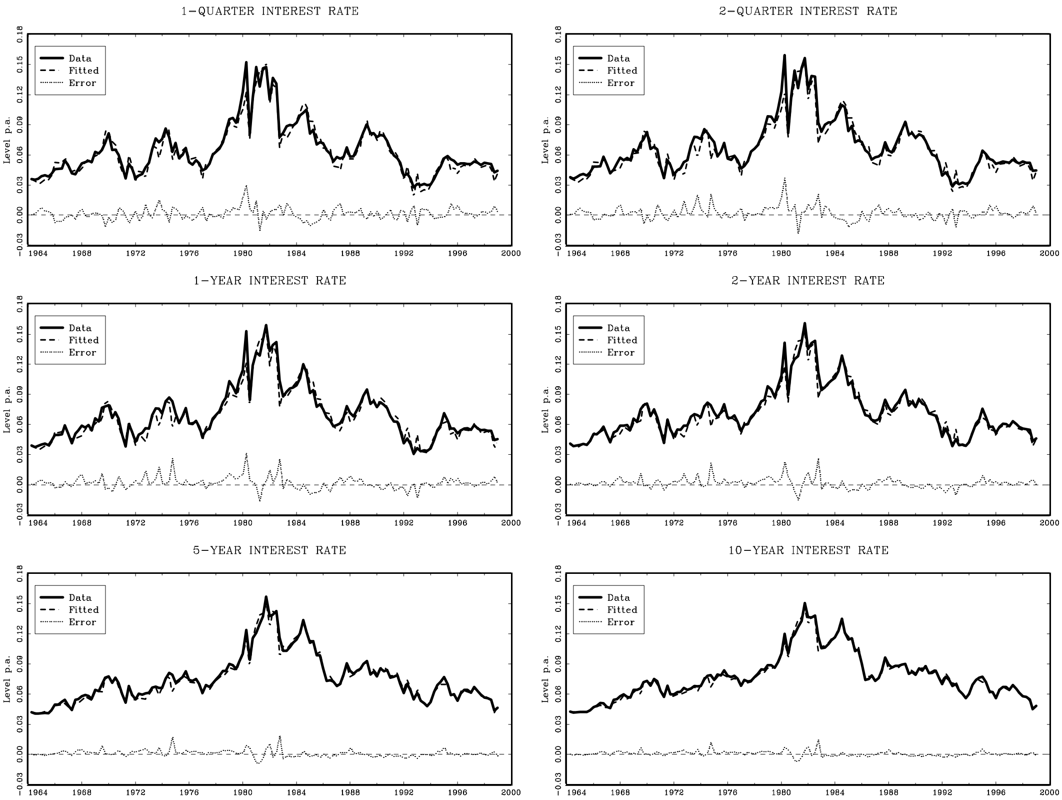

Figure 4: Model fit (model errors) of the term structure of interest rates.

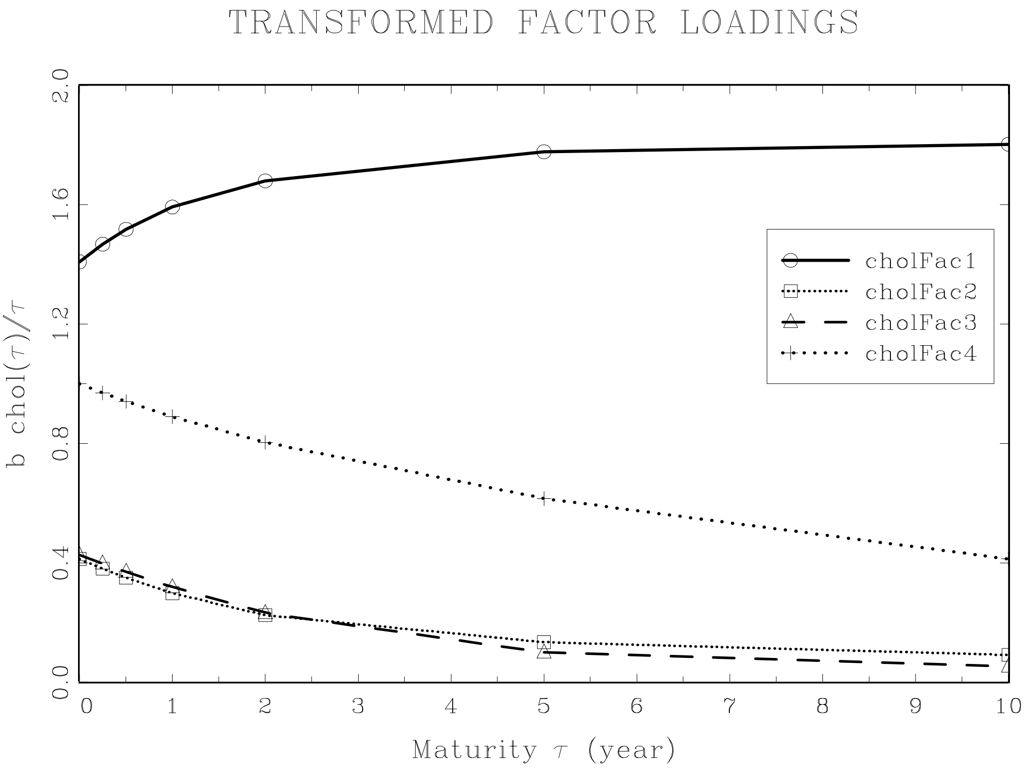

Figure 5: Transformed factor loadings.

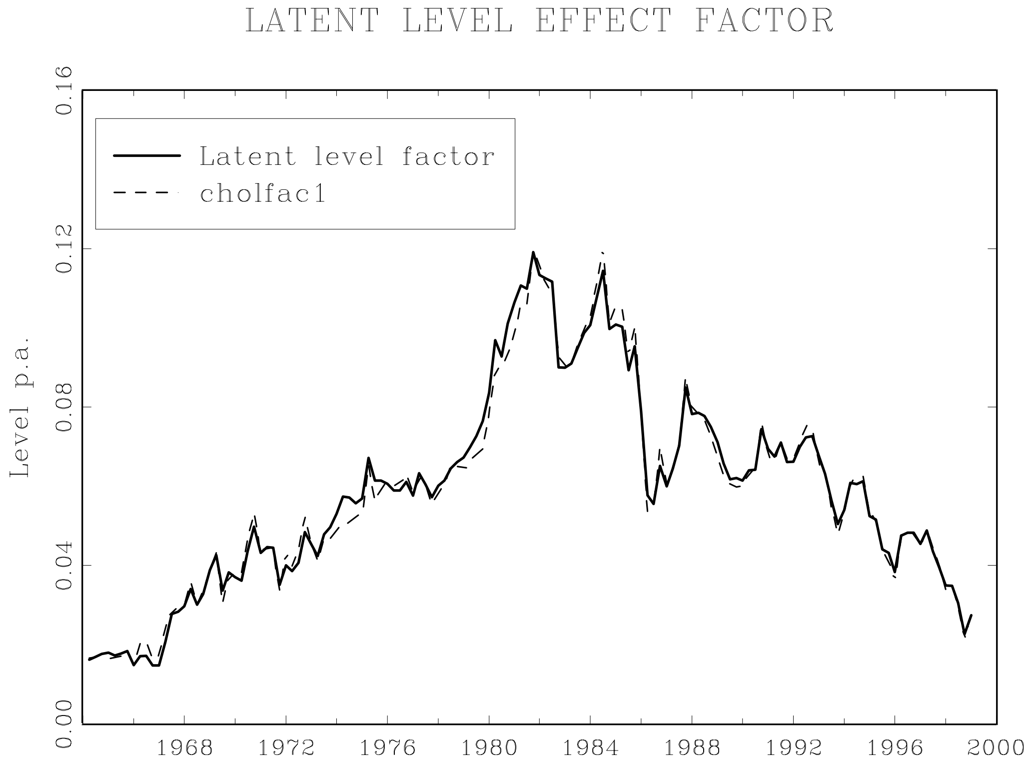

Figure 6: Latent “level” effect factor.

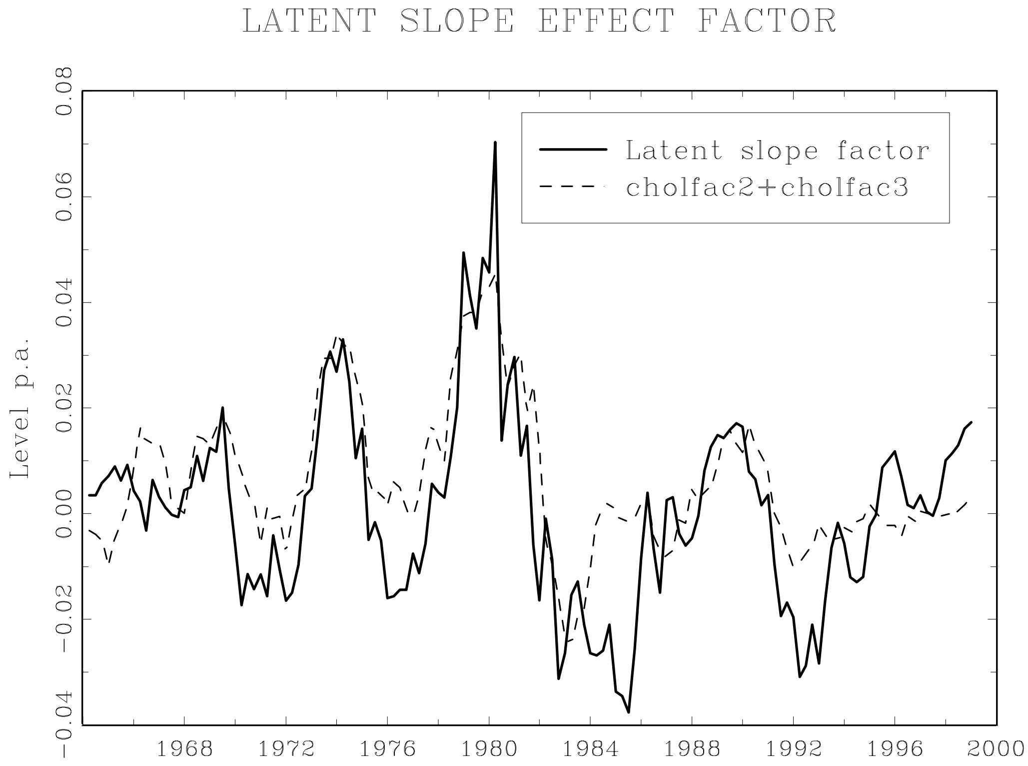

Figure 7: Latent “slope” effect factor.

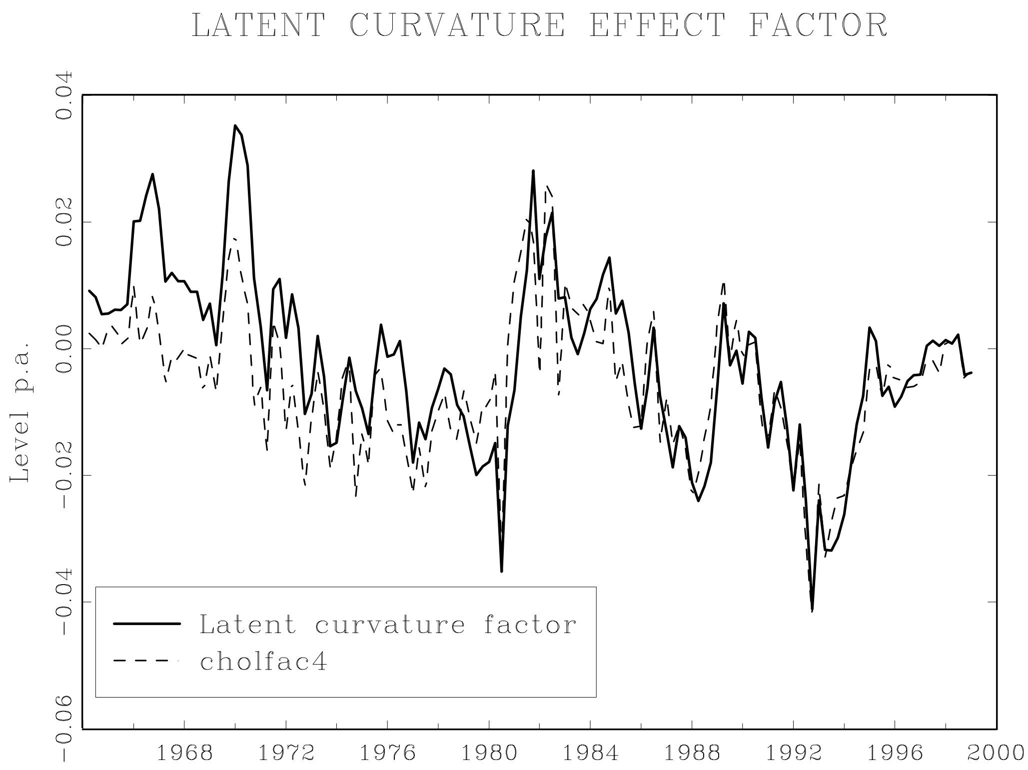

Figure 8: Latent "curvature" effect factor.

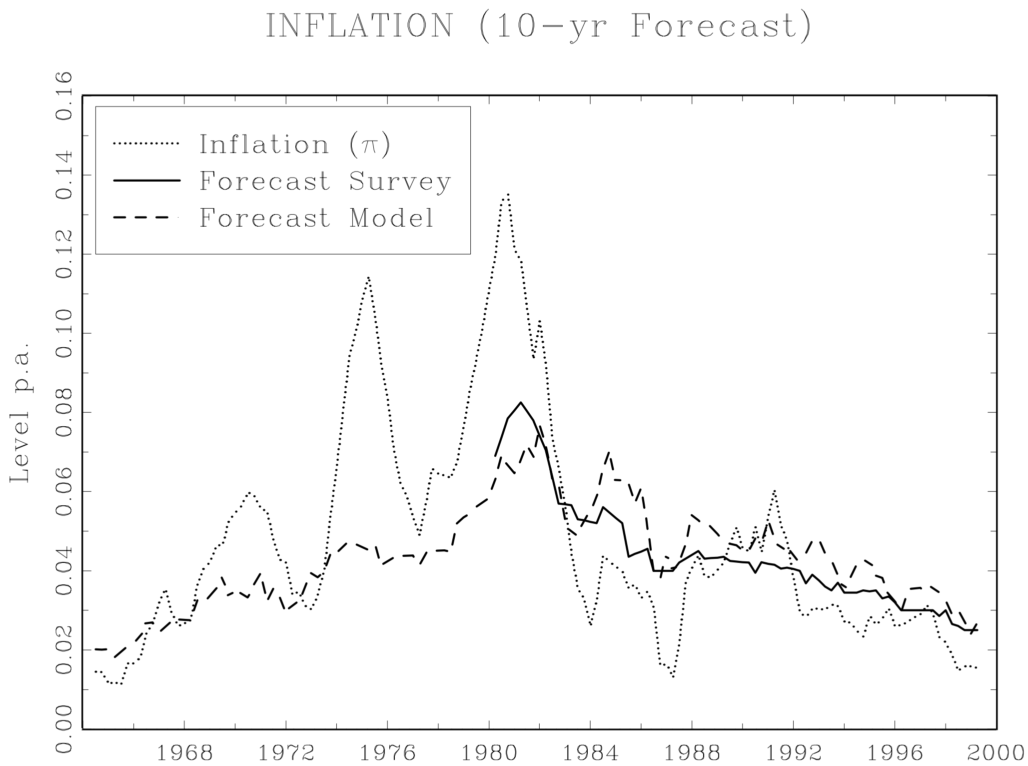

Figure 9: Comparison of average 10-year inflation forecast - Model vs. Survey of Professional Forecasters.

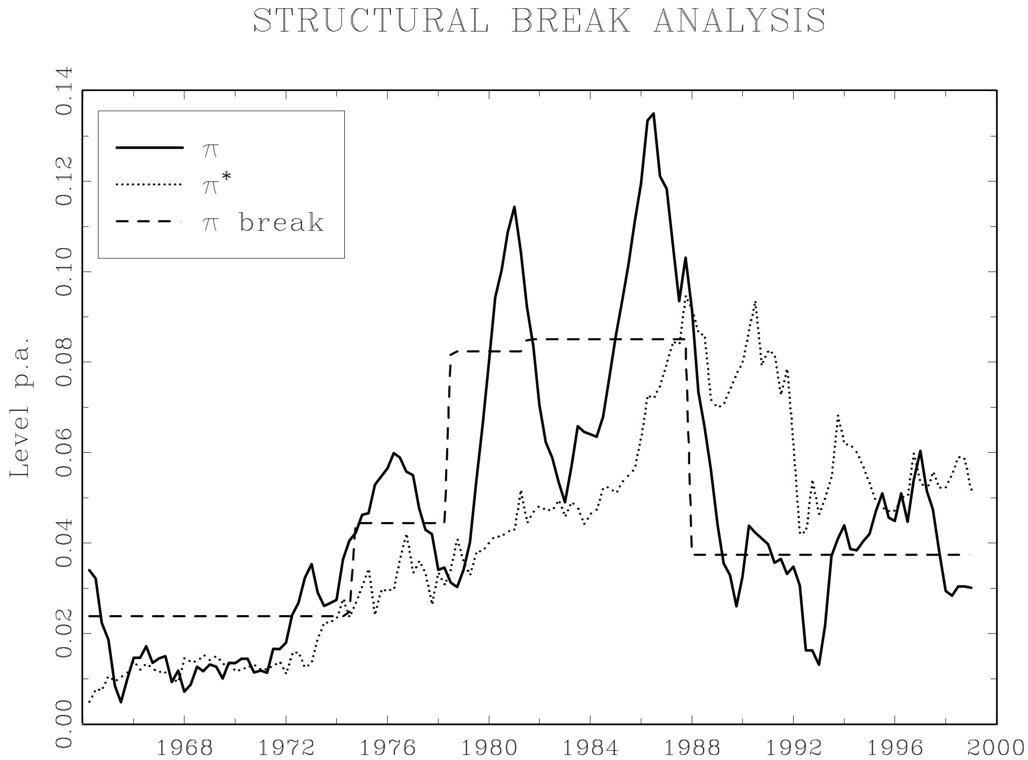

Figure 10: Break points in long-run inflation expectation.

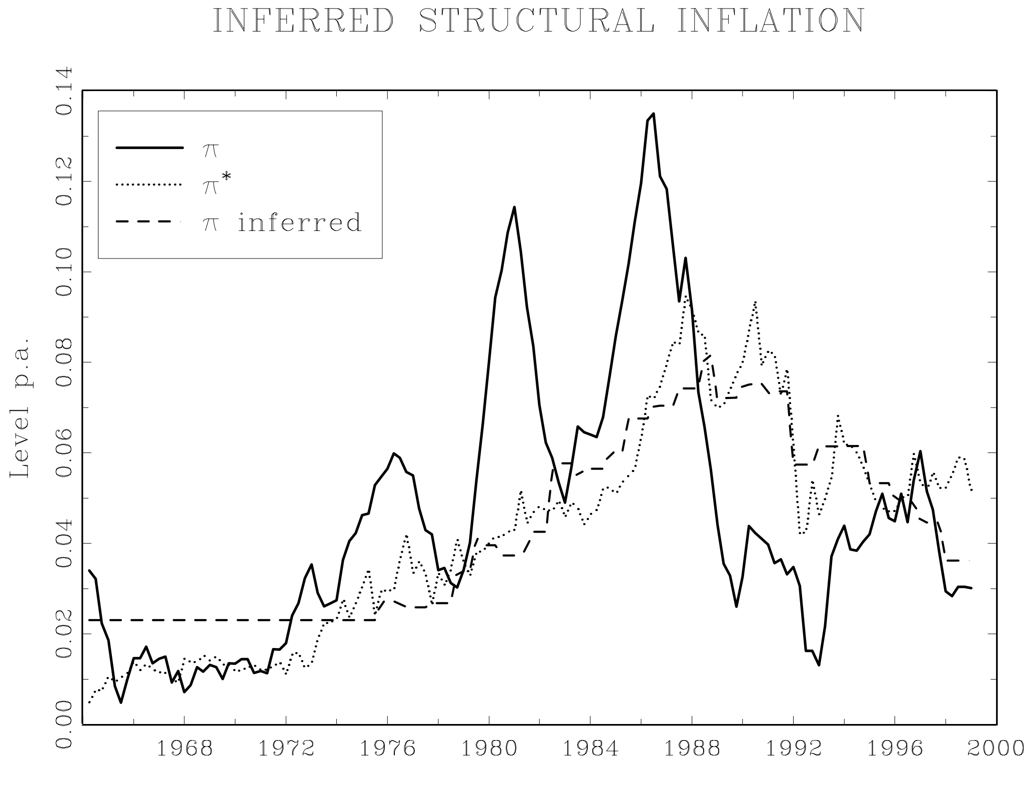

Figure 11: Inferred long-run inflation expectation.

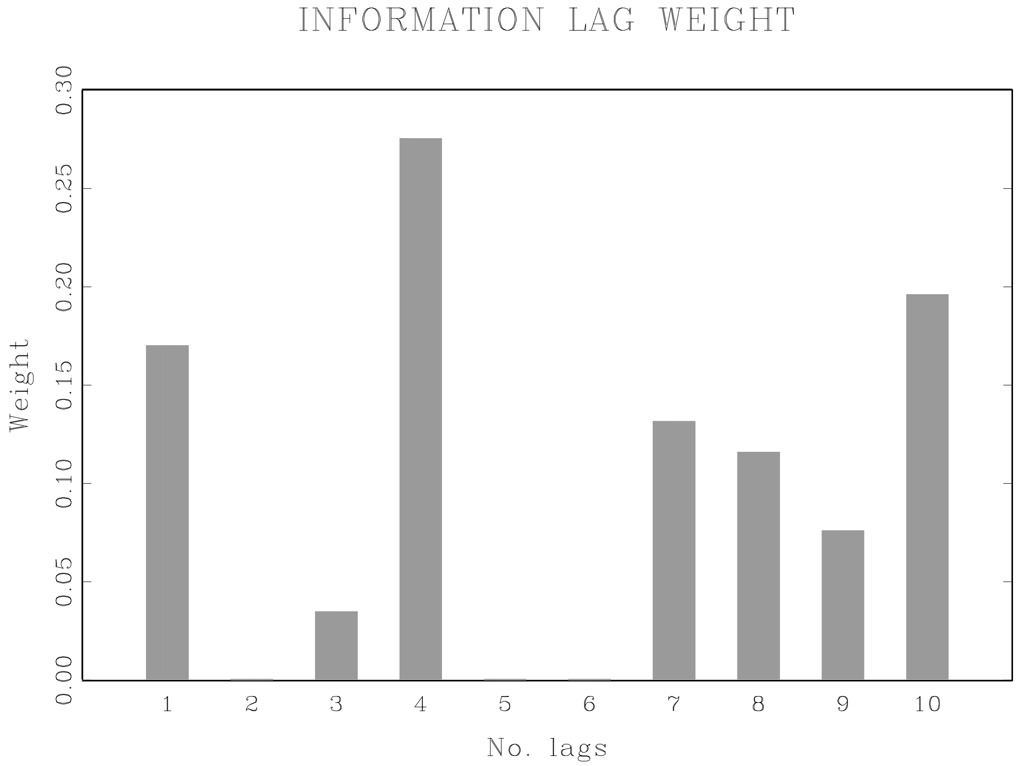

Figure 12: Distribution of weights attached to each information lag.