A joint econometric model of macroeconomic and term structure dynamics*

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Abstract

We construct and estimate a joint model of macroeconomic and yield curve dynamics. A small-scale backward/forward-looking rational expectations model describes the macroeconomy. Bond yields are affine functions of the state variables of the macromodel, and are derived assuming absence of arbitrage opportunities and using a flexible price of risk specification. While maintaining the tractability of the affine set-up, our approach provides a way to interpret yield dynamics in terms of macroeconomic fundamentals; time-varying risk premia are also associated with the fundamental sources of risk in the economy. The model is able to capture salient features of the German term structure and its forecasting performance matches that of available models based on latent factors. The model has also some success in accounting for the empirical failure of the expectations hypothesis.

Keywords: Affine term-structure models, policy rules, new neo-classical synthesis

1 Introduction

Understanding the term structure of interest rates has long been a topic on the agenda of both financial and macro economists, albeit for different reasons. Financial economists have an indirect interest, in that the primary objective is to be able to price interest rate related securities. Macro economists on the other hand, have focused on understanding the relationship between interest rates and monetary policy, exchange rates and the business cycle. Combining these lines of research seems

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fruitful, in that there are potential gains going both ways. If macroeconomic theory has some empirical success, it should help price securities more efficiently. Likewise, if some tenets of financial economics, such as the requirement of arbitrage-free markets, are empirically important, taking them into account should help to explain the response of the yield curve to macroeconomic developments. Yield curve information could also help sharpening forecasts of future economic activity and inflation.

This paper aims at presenting a unified empirical framework where a small structural model of the macro economy is combined with a set of bond-yields of different maturities in an arbitrage-free fashion. In doing so, we build on the work of Piazzesi (2001) and Ang and Piazzesi (2003), who introduce macroeconomic variables into the standard affine term structure framework based on latent factors – e.g. Duffie and Kan (1996) and Dai and Singleton (2000). The main innovative feature of our paper is that we employ a macroeconomic model with both forward and backward-looking components, rather than relying on a bivariate VAR of inflation and output. This allows us to relax Ang and Piazzesi's restriction that inflation and output be independent of the policy interest rate, and furthermore facilitates an economic interpretation of the results. Our framework is similar in spirit to that in Wu (2002), who prices bonds within a calibrated rational expectations macromodel. The difference is that we estimate our model and allow a more empirically oriented specification of both the macro economy and the parametrization of the market price of risk.

We start from a fairly standard macroeconomic model, whose core is based on the so-called new neoclassical synthesis (Goodfriend and King, 1997; Rotemberg and Woodford, 1997, coupled with a simple monetary policy rule. Variants of this model have been successfully employed to explain empirical macroeconomic dynamics, including those of the short term interest rate, and for policy analysis (e.g., Clarida, Galí and Gertler, 2000, Rudebusch and Svensson, 1999, Smets, 2000, Rudebusch, 2002a). Next, we use the model solution to identify the state variables, or factors, that affect the short-term interest rate. As a result, all factors correspond to shocks with a standard macroeconomic interpretation. Three of them are almost always present in small macro models: inflation, output (or "aggregate demand") and monetary policy shocks; the fourth is a time-varying inflation target, introduced to account for changes that may have occurred over relatively long periods of time. Finally, we borrow a standard assumption about the market price of risk from the finance literature. The result is an affine multifactor term structure model, which we estimate jointly with the macroeconomic system using maximum likelihood methods.

Our empirical results, based on German data, show that macroeconomic factors affect the term-structure of interest rates in different ways. Monetary policy shocks have a marked impact on yields at short maturities, and a small effect at longer maturities. Inflation and output shocks mostly affect the curvature of the yield curve at medium-term maturities. Changes in the inflation target have more lasting effects and tend to have a stronger impact on longer term yields. The impulse responses of the macro variables to these shocks are similar to comparable results in the literature.

Having established that the model provides a sensible description of macroeconomic and term structure dynamics, we turn to evaluate its performance relative to other available affine term structure models. More specifically, we focus on its in-sample and out-of-sample forecasting performance compared to models (partly) based on unobservable factors. In this exercise we use two main benchmarks: the Duffee (2002) model, which has been shown to do relatively well in forecasting, and the Ang and Piazzesi (2003) model. Furthermore, we test if our model can account for the empirical failure of the "expectations hypothesis" documented, for example, by Campbell and Shiller (1991). We replicate the analysis by Dai and Singleton (2002a), who have shown that affine models based on unobservable factors can be very successful along this dimension.

Our results show that the model does relatively well in forecasting. Its in-sample performance is comparable to that of the best available affine term structure models. Out-of-sample, it often does better than competing models, but not for long-term yields. This latter limitation of the model appears to play an important role in the tests of the expectations hypothesis, that are met by a limited degree of success.

The rest of the paper is organized as follows. Section 2 describes the innovative features of our theoretical term-structure model and provides a brief overview of our estimation method, which is standard in the finance literature. Our empirical results, including impulse response functions of the yields to macroeconomic shocks and forecast error variance decompositions, are presented next, in Section 3, that also discusses the forecasting performance of our model and its ability to explain the failure of the expectations hypothesis. Section 4 concludes.

2 The model

In recent years, the finance literature on the term structure of interest rates has made tremendous progress in a number of directions (see e.g. Dai and Singleton, 2002b). Following the seminal paper by Duffie and Kan (1996), one of the most successful avenues of research has focused on models where the yields are affine functions of a vector of state variables. Refinements of such models have made them increasingly successful in capturing important features of the dynamics of the term structure of interest rates. This literature, however, has typically not investigated the connections between term structure and macroeconomic dynamics. In the rare cases in which macroeconomic variables–notably, the inflation rate–have been included in estimated term-structure model, those variables have been modelled exogenously (e.g. Evans, 2003, Zaffaroni, 2001). The interactions between macroeconomic and term structure dynamics have also been left unexplored in the macroeconomic literature, in spite of the fact that simple "policy rules" have often scored well in describing the dynamics of the short-term interest rate (e.g., Clarida, Galí and Gertler, 2000).

An attempt to bridge this gap within an estimated, arbitrage-free framework has recently been made by Ang and Piazzesi (2003). Those authors estimate a term structure model based on the assumption that the short term rate is affected partly by macroeconomic variables, as in the literature on simple monetary policy rules, and partly by unobservable factors, as in the affine term-structure literature.[1] Ang and Piazzesi's results suggest that macroeconomic variables have an important explanatory role for yields and that the inclusion of such variables in a term structure model can improve its 1-step ahead forecasting performance. Nevertheless, unobservable factors without a clear economic interpretation still play a role in their model. Moreover, Ang and Piazzesi's two-stage estimation method relies on the assumption that the short term interest rate does not affect macroeconomic variables, an assumption which precludes any meaningful role for monetary policy and thus reduces the scope for a full understanding of the interaction between monetary policy, the macroeconomy and the term structure of interest rates.

In order to redress these shortcomings, we construct a dynamic term structure model entirely based on macroeconomic factors, and where there is an explicit

[1] In a related paper, Dewachter and Lyrio (2002) also estimate jointly a term structure model built on a macroeconomic VAR. Their paper includes a different macroeconomic specification, in that the model is not closed with a monetary policy rule but with an exogenous long-run relationship between the equilibrium values of the short term rate, inflation and the outut gap.

feedback from the short term (policy) rate to macroeconomic outcomes. The joint modelling of three key macroeconomic variables–namely, inflation, the output gap and the short term "policy" interest rate–should allow us to obtain a more accurate (endogenous) description of the dynamics of the short term rate.

In this section, we start presenting the simple backward/forward-looking model that we use to describe the economy. The structure of the model is fairly standard and we solve it through numerical methods that are also standard in the macroeconomic literature. We then show how a dynamic term structure can be attached to the model in a straightforward manner, based on the assumption of absence of arbitrage opportunities assumption and the state variables of the macroeconomic model. Finally, we sketch the methodology that we employ to estimate jointly our macroeconomic plus term structure model.

2.1 A simple backward/forward-looking model

We do not aim to provide a fully-fledged micro founded model. Rather, we present an empirically plausible structural model that we motivate by highlighting the assumptions that are typically adopted to obtain it from first principles. Towards this end, the model can be motivated by an environment with monopolistic competition, sticky prices and decreasing marginal returns to labour. Together, these assumptions produce a meaningful role for monetary policy. Monopolistic competition implies downward sloping demand curves which together with sticky prices create an uneven distribution of production among firms. Decreasing returns to labour imply a deadweight loss of that distribution and monetary policy strives to minimize the loss.

We start by characterizing a stylized version of the log-linearized model. The aggregate supply equation is given by

$$\pi_t = \mu_\pi E_t \left[\pi_{t+1}\right] + \left(1 - \mu_\pi\right) \pi_{t-1} + \delta_x x_t + \varepsilon_t^\pi .$$

This equation is the first order condition of the price-setting decision of firms. In essence, current inflation is determined as the sum of expected future inflation (due to sticky prices), lagged inflation (due to partial indexation), the output-gap (marginal cost) and a "cost push shock" (e.g. a shock to the pricing power of firms). In a flexible-price economy, monopolistic competition implies that prices will be set as a markup on marginal cost. However, with sticky prices, companies do not know when their prices will adjust next, and therefore need to maximize the sum of current and expected future profits. They therefore set their prices as a sum of expected future markups on marginal cost. This explains the presence of both current output-gap and expected future inflation, in that (abstracting from the lagged inflation rate) the expected inflation term can be recursively substituted to produce an infinite sum of expected output-gaps. The additional lagged inflation rate can be motivated by partial indexation, meaning that the firms not allowed to reoptimize their prices in the current period, adjust their prices according to lagged inflation rather than keeping them unchanged (Christiano, Eichenbaum and Evans, 2001). It has also been motivated by the presence of a set of firms that use a backward-looking rule of thumb to set prices, when they have an opportunity to do so (Galí and Gertler, 1999). These features are introduced to match the serial dependence of inflation in the data (an issue wildly debated in recent years, see e.g. Fuhrer and Moore (1995), Levin and Piger (2002)).

Next, the aggregate demand equation, resulting from rewriting the intertemporal consumption Euler equation, is

$$x_t = \mu_x E_t x_{t+1} + (1 - \mu_x) x_{t-1} - \zeta_r (r_t - E_t [\pi_{t+1}]) + \varepsilon_t^x.$$

The first term on the right-hand side is essentially Hall's random walk hypothesis which states that consumption is equal to expected consumption tomorrow (with bonds in zero net supply, consumption equals output and therefore the output-gap equals the consumption gap since those are constants in the absence of capital). His hypothesis is supplemented with two additional terms. First, a real interest rate (which he assumed to be constant) shifts the consumption profile such that a real rate increase tends to discourage current consumption. The second term is lagged consumption, whose presence can be motivated by habit persistence and/or the presence of rule of thumb consumers (Campbell and Mankiw, 1989; Fuhrer, 2000; McCallum and Nelson, 1999).

The final step is to recast the model at a monthly frequency, which we do along the lines of Rudebusch (2002a).

$$\pi_t = \mu_\pi \left(\sum_{i=1}^{12} E_t [\pi_{t+i}] \right) + (1 - \mu_\pi) \left(\sum_{i=1}^{12} \delta_{\pi i} \pi_{t-i} \right) + \delta_x x_t + \varepsilon_t^\pi$$

$$x_t = \mu_x \left(\sum_{i=1}^{12} E_t \left[x_{t+i} \right] \right) + (1 - \mu_x) \left(\sum_{i=1}^{12} \zeta_{xi} x_{t-i} \right) - \zeta_r \left(r_t - E_t \left[\pi_{t+1} \right] \right) + \varepsilon_t^x$$

Note that all variables are now expressed at the monthly frequency: notably, inflation is defined as the 1-month change of the log-price level. The inflation equation is the monthly analogue of that used by Rudebusch (2002a), if one disregards Rudebusch's assumption that variables are observed with a 1-quarter lag. In the estimation, we impose $\mu_\pi + (1 - \mu_\pi) \sum_{i=1}^{12} \delta_{\pi i} = 1$, a version of the natural rate hypothesis. As for the inflation equation, we assume that there are no observation lags.

Finally, we need an assumption about how monetary policy is conducted in order to solve for the rational expectations equilibrium. The three alternatives most often used in the literature are those of full commitment, complete discretion or commitment to a "simple rule". We follow the last approach and assume that the central bank sets the nominal short rate according to

$$r_t = (1 - \rho) \left(\beta \left(\sum_{i=0}^{11} E_t \left[\pi_{t+i} \right] - \pi_t^* \right) + \gamma x_t \right) + \rho r_{t-1} + \eta_t. \tag{1}$$

where π_t^* is the unobservable inflation target and η_t is a "monetary policy shock". This is consistent with the formulation in Clarida, Galí and Gertler (1998, henceforth CGG), which is a natural benchmark for comparison because the rule has been estimated for Germany, the country which we focus on in the empirical implementation. The term in square brackets is a typical Taylor-type rule (in this case forward looking), where the rate responds to deviations of expected inflation from the time-varying inflation target. The second part of the rule is motivated by interest rate smoothing concerns, which seem to be an important empirical feature of the data.

The main difference with respect to the rule estimated by CGG is that we also allow for a time-varying, rather than constant, inflation target π_t^*. We adopt this formulation because the Bundesbank modified its "medium term price norm" over the sample period used in our analysis. The modifications were public knowledge, since they were announced every year as an input in the derivation of the monetary targets. The time-varying inflation target π_t^* should therefore capture such changes, as perceived by the markets and reflected in equilibrium bond yields. This formulation allows us to exploit the full available sample period, without having to assume a break in the policy rule at some point in the late seventies, as done by CGG.

Finally, we need to specify the stochastic processes followed by the unobservable variables of the model. In affine term structure models, latent variables are normally assumed to follow a VAR(1) process. Some restrictions must then be imposed on the parameters of the VAR to ensure econometric identification (Dai and Singleton, 2000). While having the advantage of flexibility, this approach would not be consistent with our macroeconomic assumptions. If our macromodel is indeed assumed to provide an accurate description of the dynamics of inflation, the output gap and the policy interest rate, then it must also capture all the serial correlation in the data and the three macro disturbances must be white noise. We therefore assume that our 3 macro shocks are normally distributed with constant variance.[2]

The only factor that we allow to be serially correlated is the unobservable inflation target, which will follow an AR(1) process

$$\pi_t^* = \phi_\pi \pi_{t-1}^* + u_{\pi,t}$$

where $u_{\pi,t}$ is a normal disturbance with constant variance. Ideally, we would expect the inflation to be constant *in expected terms*, even if it is not deterministically constant. Contrary to the case of the other shocks, we therefore expect to obtain a high serial correlation for the inflation target process, so as to ensure that it approximately constant in expected terms ($E_t\left[\pi_{t+i}^*\right] \simeq \pi_t^*$ for $i = 1, 2, ...$).[3]

Finally, since our shocks are structural, we assume that they are mutually uncorrelated.

An important feature of this macroeconomic model is that its structure is not affected by the dynamics of the yields. We can therefore solve it without prior knowledge of the equilibrium prices of nominal bonds. We do this numerically following Söderlind (1999), who proposes a solution algorithm based on the Schur decomposition. The result are two matrices $\mathbf{M}$ and $\mathbf{C}$ such that

$$\begin{aligned} \mathbf{X}_{1,t+1} &= \mathbf{M}\mathbf{X}_{1,t} + \boldsymbol{\xi}_{1,t+1} \\ \mathbf{X}_{2,t+1} &= \mathbf{C}\mathbf{X}_{1,t+1} \end{aligned}$$

[2] As a robustness check, we also estimated the model assuming that all shocks follow independent AR(1) processes. The estimated autocorrelation parameters are small, but non-zero, and the difference between the two models is statistically significant (a likelihood ratio test would reject the restricted model at the 5% confidence level). Nevertheless, in all other respects–namely, the impulse response functions, forecast error variance decompositions and forecasting performance–the two models are virtually identical. We therefore select the restricted model based on economic grounds.

[3] Imposing a unit root would violate the assumption of stationarity of the factors, an assumption maintained throughout the affine term structure literature.

where $\mathbf{X}_1$ is a vector of predetermined variables, $\mathbf{X}_2$ includes the variables which are not predetermined and $\boldsymbol{\xi}_1$ is a vector of shocks (see the appendix for more details).[4]

2.2 Adding the term structure to the model

Given the solution above, the short term interest rate can be written as a linear function of all predetermined variables $\mathbf{X}_1$, which in turn follow a first order Gaussian VAR. To build the term structure, we only need to impose the assumption of absence of arbitrage opportunities, which guarantees the existence of a risk neutral measure, and to specify a process for the stochastic discount factor. In this respect, we follow closely Ang and Piazzesi (2003).

Specifically, the nominal pricing kernel m_{t+1} which prices all nominal bonds in the economy is defined as

$$m_{t+1} = \exp\left(-r_t\right) \frac{\psi_{t+1}}{\psi_t},$$

where ψ_{t+1} is the Radon-Nikodym derivative, which is assumed to follow a log-normal process according to

$$\psi_{t+1} = \psi_t \exp\left(-\frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\boldsymbol{\xi}_{1,t+1}\right),$$

and where λ_t is the vector of market prices of risk associated with the underlying sources of uncertainty $\left(\boldsymbol{\xi}_{1,t+1}\right)$ in the economy. The market prices of risk, in turn, have commonly been assumed to be constant (in the case of Gaussian models) or proportional to the factor volatilities (e.g. Dai and Singleton (2000)), but recent research has highlighted the clear benefits in allowing for a more flexible specification of the risk prices (e.g. Duffee (2002), Dai and Singleton (2002a)). We therefore assume that the market prices of risk are affine in the state vector $\mathbf{X}_{1,t}$,

$$\lambda_t = \lambda_0 + \lambda_1 \mathbf{X}_{1,t},$$

so that the market's required compensation for bearing risk can vary with the state of the economy. More precisely, λ_t will only price contemporaneous factors, and elements in λ_0 and λ_1 which correspond to lagged variables are set to zero.

[4]The presence of non-predetermined variables in the model implies that there may be multiple solutions for some parameter values. We constrain the system to be determinate in the iterative process of maximizing the likelihood function.

Given the definition of ψ_{t+1}, and rewriting the short-term interest rate in (1) as $r_t = \mathbf{\Delta}'\mathbf{X}_{1,t}$, the pricing kernel can be written as

$$
\begin{aligned}
m_{t+1} &= \exp\left(-r_t\right)\frac{\psi_{t+1}}{\psi_t} \\
&= \exp\left(-\mathbf{\Delta}'\mathbf{X}_{1,t}\right)\exp\left(-\frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\boldsymbol{\xi}_{1,t+1}\right) \\
&= \exp\left(-\mathbf{\Delta}'\mathbf{X}_{1,t} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\boldsymbol{\xi}_{1,t+1}\right).
\end{aligned}
$$

The pricing kernel allows us to price zero-coupon bonds, through the fundamental asset pricing relation $E_t\left[m_{t+1}R_{t+1}\right] = 1$, where R_{t+1} denotes gross returns. Since the state vector and the short-term interest rate are affine functions of $\mathbf{X}_{1,t}$, we know from Duffie and Kan (1996) that bond prices will be exponential-affine functions of the state variables,

$$p_t^n = \exp\left(\bar{A}_n + \bar{B}_n'\mathbf{X}_{1,t}\right),$$

where $\bar{A}_n$ and $\bar{B}_n$ are parameters which depend on the maturity n. Taken together with the pricing kernel relation, this can be used to identify the structure of the bond pricing relation. For example, the price of a one-period bond at time t is given by

$$
\begin{aligned}
p_t^1 &= E_t\left[m_{t+1}\right] \\
&= \exp\left(-r_t\right) \\
&= \exp\left(-\mathbf{\Delta}'\mathbf{X}_{1,t}\right),
\end{aligned}
$$

but it can also be expressed as an exponential affine functions of $\mathbf{X}_1$ according to

$$p_t^1 = \exp\left(\bar{A}_1 + \bar{B}_1'\mathbf{X}_{1,t}\right).$$

This allows us to identify $\bar{A}_1 = 0$ and $\bar{B}_1 = -\mathbf{\Delta}$. Similarly, longer maturity bonds can be obtained recursively using

$$p_t^{n+1} = E_t\left[m_{t+1}p_{t+1}^n\right],$$

and the corresponding $\bar{A}_n$ and $\bar{B}_n$ parameters can be identified as following the recursions

$$\begin{aligned}\bar{A}_{n+1} &= \bar{A}_n - \bar{B}_n'\Sigma\lambda_0 + \frac{1}{2}\bar{B}_n'\Sigma\Sigma'\bar{B}_n,\\ \bar{B}_{n+1}' &= \bar{B}_n'\left(\mathbf{M} - \Sigma\lambda_1\right) - \boldsymbol{\Delta}'.\end{aligned}$$

Finally, the continuously compounded yield y_t^n on a n-period zero coupon bond is given by

$$\begin{aligned}y_t^n &= -\frac{\ln\left(p_t^n\right)}{n}\\ &= -\frac{\bar{A}_1}{n} - \frac{\bar{B}_1'}{n}\mathbf{X}_{1,t}\\ &\equiv A_n + B_n'\mathbf{X}_{1t}.\end{aligned}$$

Stacking the available yields in a vector $\mathbf{Y}_t$, we write the above equations jointly as $\mathbf{Y}_t = \mathbf{A} + \mathbf{B}\mathbf{X}_{1t}$.

2.3 Maximum likelihood estimation

We are interested in maximizing the likelihood function

$$\ell\left(\boldsymbol{\theta}\right) = \prod_{t=2}^{T} f\left(\mathbf{X}_{1t}^o, \mathbf{Y}_t, \overline{\mathbf{X}}_{2t} | \mathbf{X}_{1t-1}^o, \mathbf{Y}_{t-1}\right)$$

where $\mathbf{X}_{1t}^o$ is the vector of observable predetermined variables, $\mathbf{Y}_t$ is the vector of yields and $\overline{\mathbf{X}}_{2t}$ are the observable components of the vector of non-predetermined variables (current output gap and inflation rate).

To construct the likelihood, we adopt an approach which is common in the finance literature and which involves solving for the unobservable factors from a vector of yields (Chen and Scott, 1993). In our case, the observation equations of the state-space system include not just the yields, $\mathbf{Y}_t$, but also the non-predetermined variables, $\overline{\mathbf{X}}_{2t}$. We therefore assume that as many yields plus non-predetermined variables as unobservable states are measured exactly, and that the remaining yields are measured with error. To account for the measurement error, we rewrite the yields equation as $\mathbf{Y}_t = \mathbf{A} + \mathbf{B}\mathbf{X}_{1t} + \mathbf{B}^m\mathbf{u}_t^m$, where the $\mathbf{u}_t^m$ vector of white noise shocks has zero elements corresponding to the yields measured exactly. We then partition the state vector into observable and unobservable components and rewrite the yield equation and the equation for the non-predetermined variables as $\mathbf{Y}_t = \mathbf{A} + \mathbf{B}^o\mathbf{X}_{1t}^o + \mathbf{B}^u\mathbf{X}_{1t}^u + \mathbf{B}^m\mathbf{u}_t^m$ and $\overline{\mathbf{X}}_{2t} = \overline{\mathbf{C}}^o\mathbf{X}_{1t}^o + \overline{\mathbf{C}}^u\mathbf{X}_{1t}^u$, respectively. These equations can be used to back out the unobservable states, $\mathbf{X}_{1t}^{u}$, and the measurement shocks, $\mathbf{u}_t^m$.

Using the assumption of orthogonality of measurement error shocks and shocks to the unobservable states, we show in the appendix that the log-likelihood function to maximize takes the form

$$\begin{aligned}\ln\left(\pounds\left(\boldsymbol{\theta}\right)\right) &= -\left(T-1\right)\left(\ln|J| + \frac{1}{2}\ln\left|\Sigma\Sigma'\right| + \frac{1}{2}\ln\sum_{i=1}^{n_m}\sigma_i^2 + \frac{n_1+n_m}{2}\ln\left(2\pi\right)\right) \\ & -\frac{1}{2}\sum_{t=2}^{T}\left(\mathbf{X}_{1t}-\mathbf{M}\mathbf{X}_{1t-1}\right)'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\left(\mathbf{X}_{1t}-\mathbf{M}\mathbf{X}_{1t-1}\right) - \frac{1}{2}\sum_{t=2}^{T}\sum_{i=1}^{n_m}\frac{\left(u_{t,i}^m\right)^2}{\sigma_i^2}\end{aligned}$$

where $J \equiv \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^o & \mathbf{B}^u & \mathbf{B}^m \\ \overline{\mathbf{C}}^o & \overline{\mathbf{C}}^u & \mathbf{0} \end{bmatrix}$, n_m is the number of measurement errors and n_1 is the number of predetermined variables.

Ang and Piazzesi (2003) assumed no feedback from interest rates to the macro variables. Therefore, the model could be estimated in a two-step procedure, first estimating the macro VAR and then, fixing those parameters, the term-structure part. In our case, this is not possible because we do not assume independence between the macro variables and the interest rates. Furthermore, we back out our unobservable macroeconomic variables from the yields. We must therefore estimate the whole system jointly. In theory, this is of course preferable. The problem is that the parameter space is quite large and therefore the optimization problem of maximizing the likelihood function is non-trivial and time consuming. We employ the method of simulated annealing, introduced to the econometric literature by Goffe et al. (1994). The method is developed with an aim towards applications where there may be a large number of local optima.[5]

3 An application to the German term structure

The macroeconomic model described in section 2 is a closed economy model. Ideally, we would like to test it on the euro area, which is relatively large to make the closed economy assumption approximately valid. Unfortunately, however, euro area data are only available for a short sample period. In macroeconomic applications

[5] We are in the process of verifying that our estimates are robust to large perturbations in the starting values.

researchers have therefore relied on synthetic data, i.e. data constructed as weighted averages of national data over the pre-EMU period (e.g. Smets, 2000). Relying on synthetic data, however, appears to be particularly unsatisfactory if one is interested in testing a dynamic term structure model. These models are constructed on the assumption of absence of arbitrage opportunities and rely on the existence of a liquid and well functioning market. It is not clear that a euro area term structure of interest rates constructed as a weighted average of national term structures would indeed be arbitrage free. For this reason, we restrict our attention to the largest country in the area, Germany, under the assumption that the closed economy assumption is approximately valid also for the German economy. Our data set runs from January 1975 to December 2002.

In more detail, the term structure data consists of monthly German zero-coupon yields for the maturities 1 and 3 months, as well as 1, 3, and 10 years. We assume that the 1-month rate and the 1-year yield are perfectly observable, while the 3-month rate and the 3 and 10-year yields are subject to measurement error. Yields up to the one year maturity are observed DEM (EUR after December 1998) interbank interest rates, whereas the longer yields are fitted estimates based on observed market prices. Specifically, for the longer yields we use a monthly time-series of parameter estimates for Svensson's (1994) extension of the Nelson and Siegel (1987) model obtained from the BIS. These parameters have been estimated by fitting the model to the prevailing German yield curve at the end of each month, i.e. to available money market and government bond data. These parameters allow us to obtain zero-coupon bond prices and yields for the maturities we are interested in during the sample period, i.e. a time-series of German term structures.

Concerning the macro data, we construct the inflation series using a seasonally adjusted CPI (all items). For the output gap, we follow CGG and detrend the log of total industrial production (excluding construction) using a quadratic trend. Both series refer to unified Germany from 1991 onwards and to West Germany prior to this date. The macroeconomic and term-structure series are shown in Figure 1.

We estimate the model over the pre-EMU sample (February 1976 to December 1998, taking the lags structure into account) and use the EMU years for the out-of-sample forecasting exercise. Given the obvious possibility of a structural break occurring in 1999, with the start of EMU, the forecast represents a very strong test of the stability of our model. Given our results, it would not appear unreasonable to assume that, in the first years of EMU, the policy rule followed by the ECB was consistent with that of the Bundesbank.

3.1 Estimation results

To reduce the parameter space in our empirical application, we limit the lags of inflation and output gap included in the system. At a preliminary stage, we run unrestricted OLS regressions of inflation and the output gap on 12 lags of both variables and 1 lag of the short term interest rate, i.e. on all the observed state variables of our theoretical model. Based on the results of this regression, we drop all lags of inflation (in the inflation equation) and the output gap (in the output gap equation) that are insignificant at the 10% level. In our maximum likelihood estimation of the model, we therefore include selected lags for the macro variables.[6]

In addition, we restrict the λ_1 matrix which specifies the dependence of the prices of risk on the state vector. Consistently with the assumption of orthogonality between macroeconomic shocks, we assume that the λ_1 matrix has zero off-diagonal elements corresponding to the interaction between monetary policy, inflation and output gap shocks. As a result, the λ_1 is assumed to have the following structure

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & 0 & 0 \\ \lambda_{31} & 0 & \lambda_{33} & 0 \\ \lambda_{41} & 0 & 0 & \lambda_{44} \end{pmatrix}$$

3.1.1 Parameter estimates

Our paper appears to be the first to estimate a simple backward/forward-looking macroeconomic model for Germany. We therefore discuss our macroeconomic results separately, since they are of independent interest. The only study on Germany which we have found in the literature is the analysis of the Phillips curve by Jondeau and Le Bihan (2001), who estimate the German Phillips curve based on quarterly data, using a variety of specifications and two different estimation methods (GMM and maximum likelihood).

If we take into account that the macro model must also help to fit the dynamics of the term structure, the model does relatively well in explaining the joint evolution of inflation, the output gap and the policy interest rate. Specifically, the model parameters are estimated to be broadly in line with other available estimates solely

[6] Specifically, we use lags 1, 3, 7, 8, 11, 12 for inflation and lags 1, 2, 3, 9, 12 for the output gap.

based on macroeconomic variables. The presence of the yields in the model, and in the estimation, can apparently be accommodated without twisting the macromodel towards unreasonable parameter regions (see Table 1).

A more detailed review shows that our estimates for Germany confirm the US result that both forward looking and backward looking components tend to be necessary to explain inflation dynamics. Compared to available US results (based on quarterly data), however, we obtain a relatively high weight for forward looking elements in both the inflation and the output gap equations. Our $\mu_\pi = 0.54$, for example, compares to the value of 0.29 found by Rudebusch (2002a). Our point estimate is, nevertheless, within the range of values found by Jondeau and Le Bihan (2001) for German data.[7] Our results are also quite similar to those of Smets (2000) using annual data for the euro area ($\mu_\pi = 0.52$).

Not surprisingly, the value of μ_π is affected by the presence of a time varying inflation target in the policy rule. Indeed, if the 3-equation macromodel is estimated separately and with a constant inflation target in the policy rule, the forward looking inflation component appears to be somewhat lower ($\mu_\pi = 0.33$). Intuitively, a time-varying and highly persistent inflation target can account "endogenously" for some of the persistence of observed inflation.

For the output gap, the forward looking component appears to play a less important role ($\mu_x = 0.18$). In this case, our estimates are in the range of values reported by Fuhrer and Rudebusch (2002a) for the US (and again based on quarterly data), but lower than available European estimates. Both Smets (2000) for the euro area and Chadha, Masson and Meredith (1992) for a panel of France, Germany and Italy find a higher degree of forward-lookingness (0.56 and 0.45, respectively). Unfortunately, no estimates based on German data appear to be available in the literature.

The elasticity of inflation to the output gap is estimated to be very small ($\delta_x = 0.01$) compared to US estimates (e.g., Rudebusch's estimate on quarterly data is $\delta_x = 0.13$). However, relatively small values are not uncommon in analyses based on German data. Depending on the specification and the estimation method, Jondeau and Le Bihan (2001) find values between 0 and 0.19. The sensitivity of the output gap to the real interest rate ($\zeta_r = 0.03$) is also small compared to other available results. Rudebusch (2002a) reports a value of 0.09 for the US and Smets (2000) a value of 0.06 for the euro area. Nevertheless, a small sensitivity of the German

[7] The sample used by the Jondeau and Le Bihan (2001) runs from 1970Q1 to 1999Q4.

output gap to the real interest rate also emerges from VAR studies (see section 3.1.2).

Concerning the parameters of the monetary policy rule, our results are broadly consistent with those of CGG. We find a somewhat higher degree of interest rate smoothing (around $\rho = 0.95$, compared to the value of 0.91 reported in Table 1 of CGG) but, at the same time, a more aggressive equilibrium response to inflation deviations from target and to the output gap (our coefficients imply values of 3.24 and 0.34, respectively, compared to 1.31 and 0.25 in the baseline specification of CGG). Since the standard errors of our estimates are relatively large, however, the discrepancy is not statistically significant. The discrepancies may obviously also be due to differences in the estimation method (GMM in CGG), in the sample period (1979:4-1993:12 in CGG), in the specification of the policy rule (constant inflation target in CGG), and in the selected policy interest rate (interbank day-to-day rate in CGG). The net effect, however, is that of an essentially equivalent impact response to inflation and the output gap.

As to the other parameters, the autocorrelation coefficient of the inflation target process is very close to 1.[8] Concerning the term structure, our estimates of the standard deviations of the measurement errors are between 13 basis points for the 3-month rate and 22 basis points for the 10 year yield. These values are broadly in line with the results of models based solely on unobservable factors.

3.1.2 Impulse response functions

Our structural model allows us to compute impulse response functions of macro variables and yields to the underlying macro shocks. This is particularly interesting for the monetary policy shock, whose identification and effects are the subject of a vast literature (see Christiano, Eichenbaum and Evans, 1999).

Before discussing the impulse response functions of our model, Figure 2 shows actual year-on-year inflation and the (perceived) inflation target extracted through the model. The target is characterized by a decreasing trend over the estimation period: it goes from approximately 4 percent in 1975 to 1 percent at the end of 1998. This is quite consistent with the time pattern of the "price norm" announced every year by the Bundesbank, that also fell from 4.5 to below 2 percent over the period. The short-term dynamics of the (perceived) inflation target normally mimic those of actual inflation, even if in a less pronounced fashion. There are, however,

[8] The parameter is constrained to be strictly smaller than 1 in the estimation.

exceptions. Noticeably, the inflation target does not fall in the late eighties when inflation temporarily reaches negative values. The inflation target is also less variable than actual inflation: its sample standard deviation of 0.87 compares to 1.63 for actual inflation; the minimum and maximum of the target are 0.90 and 5.33 percent, compared to corresponding values of −0.98 and 7.12 percent for actual inflation.

Figures 3 to 6 show the impulse responses of the macroeconomic variables and the yields to the structural shocks.

We start with Figure 3, which displays the impulse responses to a shock to the inflation target, which increases on impact by approximately 0.2 percentage points. The shock is obviously very persistent due to the high serial correlation of the inflation target process. Since inflation is relatively forward-looking, it jumps upwards immediately, and then continues to increase for up to 1 year after the shock, while the policy rate is slower to catch up because of the high smoothing coefficient. Consequently, inflation initially "overshoots" the (new) target, then reverts to it from higher values. With inflation slightly above target and negative real rates, there is a progressive opening of an output gap. The 1-month rate continues to rise, although at an extremely slow pace, until monetary policy becomes contractionary. The yields jump up consistently with the anticipated tightening cycle of monetary policy. The size of the jump is increasing in maturity for maturities up to 3 years, and then decreasing in maturity, consistently with the ultimately mean reverting nature of the inflation target shocks.

Figure 4 shows the effect of a 45 basis points increase in the 1-month interest rate because of a monetary policy shock (the disturbance $u_{\eta,t}$). The effects of the shock are quite persistent over time, because of the high interest rate smoothing coefficient. The increase in the short term interest rate causes a progressive opening of a negative output gap, up to a peak of approximately 0.2 percentage points after 1.5 years. Inflation also falls after the shock, but by a tiny amount and approximately at the same time of the output gap. Both the size and the timing of the inflation response are somewhat different from those normally obtained based on US data (e.g., Christiano, Eichenbaum and Evans, 1999). Nevertheless, the inflation response is broadly consistent with the results of VAR studies based on German (monthly) data: Sims (1992), Clarida and Gertler (1997) and Bernanke and Mihov (1997) all find negligible, or statistically insignificant, responses of prices or inflation (or even evidence of a "price puzzle"). Reflecting the marked, but temporary, nature of the monetary policy shock, the response of the term structure is decreasing in the maturity of the bonds. This response is quite similar to that obtained by Evans and Marshall (1996) for the US.

The impulse responses to an inflation shock and an output shock are shown in Figures 5 and 6, respectively. Inflation shocks tend to be relatively short lived. On average, they imply an increase of year-on-year inflation up to a maximum of 0.30 percentage points after 1 year. The effects of the shock then quickly die out, as monetary policy reacts to them quite aggressively. After an initial and short-lived expansionary effect, the economy goes through a mild recession as a results of the policy tightening. The notable feature of the reaction of the yield curve is that it is almost completely reabsorbed at all maturities after 2 years.

Finally, an output shock implies an increase of the gap by approximately 1.3 percentage points. Because of the small elasticity of inflation to the gap, inflation increases by less than 0.1 percentage points in response to the shock. Consequently, the policy interest rate increases little and very slowly, up to a peak of approximately 15 basis points after 1.5 years. As a result, 1 3 and 5-year yields increase, on impact, more than the 1-month rate, while the 10-year maturity is less reactive because of the expected future return to baseline of the policy rate.

3.1.3 Macro shocks and risk premia

One of the advantages of our joint treatment of macroeconomics and term-structure dynamics is that we are able to derive the impulse response of theoretical risk premia to macro shocks, including the monetary policy shock.

As a preliminary step, we analyze the weights of each of the shocks of the model on the yields at various maturities, i.e. the B_n matrices (see Figure 7). These are interesting to compare to the results often obtained from models based on 3 unobservable factors, whose interpretation is typically given in terms of level, slope and curvature of the yield curve.

Figure 7 shows that the inflation target affects the maturities beyond 2 years almost uniformly, while it has a smaller effect on the short end of the curve. The inflation target therefore plays a mixed role of level factor for the long end of the curve and of slope factor for the short end. The role of level factor for long maturities is intuitively appealing from a macroeconomic perspective, as it identifies the inflation target with the nominal anchor of the economy.

At short maturities, the level and slope of the yield curve are crucially influenced by the monetary policy shock. This is also intuitively appealing, as it portrays the tight control of the central bank on short rates, due to liquidity effects.

Inflation shocks and output gap shocks have a hump shaped weight. They mostly affect the curvature of the yield curve at certain maturities. However, their standardized weights are small, especially for inflation shocks. This suggests that they play a minor role in the term-structure model.

The impact response of yield premia to the macroeconomic shocks are shown in Figure 8. The inflation target shock is immediately followed by an increase of the yield premium along the whole curve, with a peak effect of just over 10 basis points at the 3-year maturity. Such increase in the yield premium is highly significant, from an economic viewpoint. For maturities up to 3 years, it accounts for approximately half of the yield response displayed in Figure 3.

The monetary policy shock gives rise to a fall in whole term structure of yield premia. The effect is small, compared to that on the yield levels, for maturities up to 3 years. For longer maturities, however, it significantly reduces the size of the response of the yields. The small impact response of the 10-year yield to the monetary policy shock, for example, would almost double in size if yield premia were set equal to a constant.

Although limited in absolute magnitude, the impact response of yield premia to inflation and output shocks is also large when compared to the response of the yield levels. As in the case of monetary policy shocks, premia tend to move in the opposite direction of the yields themselves. A positive output gap shock, for example, gives rise to a fall of the yield premium, on impact, of up to 5 basis points for the 3-year maturity. At the same time, this shock generates an increase of the yield levels, on impact, by up to 5 basis points, again for the 3-year maturity.

We conclude that yield premium dynamics have a nonnegligible effect on the impulse responses of yields to all macroeconomic shocks. They amplify the response of the term structure to inflation target shocks and moderate the response to the other shocks. An interpretation of the yield responses based on the expectations hypothesis may therefore be significantly biased.

Figure 9 shows the estimated yield premia over time. Two general features emerge. First, the premia tend to be decreasing over the sample in parallel to the fall in inflation. Second, the volatility of the premia is increasing in maturity up to the 3-year yield, but it is relatively low for the 10-year yield.

3.1.4 Forecast error variance decomposition

Our model attributes to macroeconomic factors all yield curve and macro-variable movements. A decomposition of the forecast error variance can give us information on which factors play the most important role in this respect. The forecast error variance decomposition for all the variables in our model is presented in Table 2.

The most striking pattern of the table is that, no doubt due to the near unit-root behavior of this variable, inflation forecast movements explain the predominant part of the forecast error variance of all variables at long horizons (5 years and beyond). This result is most intuitive for the forecast error variance of inflation. It also applies to the output gap, over medium and long term horizons, because of the expansionary (contractionary) effects of inflation target increases (falls) through the short-term real interest rate.

The predominant role of inflation target shocks in explaining the forecast error variance of long term yields is suggestive of a limitation of the model at these maturities. In spite of the presence of 4 macroeconomic shocks, for long maturities the model collapses *de facto* to a 1-factor model. In the next section, we will see that this result corresponds to a limited ability of our model to forecast long term yields.

The variance decomposition of macroeconomic variables shows a hump-shaped effect of monetary policy shocks on inflation and the output gap. Over very short horizons, however, the latter variables are mostly affected by inflation and output gap shocks, respectively.

After the inflation target shock, the most important explanatory variable of the forecast error variance of the yield curve at short maturities/horizons are monetary policy shocks. Their effect, however, is relatively short-lived: for 3-year yields, the contribution of monetary policy shocks is negligible already at the 3-year horizon.

Inflation and output gap shocks also have hump-shaped effects on yields, with peaks at horizons between 1 and 2 years. The contribution of output gap shocks reaches levels of almost 20% and almost 10% for 1 and 3-year yields, respectively. The contribution of inflation shocks, on the contrary, is negligible at all horizons for yields with maturities of 3 years and beyond.

3.2 Forecasting

The forecasting performance is a particularly interesting test of our macroeconomic-based term-structure model. Due to the relatively large number of parameters that needs to be estimated, the model may be beaten by more parsimonious representations of the data. In fact, the random walk model has been shown to provide yield forecasts that are particularly difficult to beat (Duffee, 2002). We therefore present in this section, results of the forecasting performance of our model compared to the random walk.

In addition to the random walk, we also consider forecasts based on two alternative models. The first is a canonical $A_0(3)$ essentially affine model based on unobservable factors[9]. Provided that risk premia are specified to be linear functions of the states, Duffee (2002) finds this model most successful in the class of admissible affine three factor models in terms of forecasting US yields. Apart from providing a benchmark for comparison, our results on the $A_0(3)$ model are of independent interest, since they highlight the performance of this model on a different data-set.

Our second benchmark for comparison, is the Ang and Piazzesi (2003) model, which we reestimate on our data-set[10]. Based on Ang and Piazzesi's results, we use their favorite "Macro model" in this exercise, i.e. a model in which the interest rate responds to current inflation and output gap, as well as to 3 unobservable factors. A potentially important difference in our application of their model, however, is that we use inflation and the output gap directly in the estimation, rather than the principal components of real and nominal variables employed by Ang and Piazzesi (2003). This is arguably a more theory-based choice and it facilitates the comparison to our results.

For all models, we report in-sample forecasting performances (in terms of RMSE) based on the February 1975 - December 1998 period. Concerning the out-of-sample results, we perform a series of 1 to 12 step ahead forecasts for all yields used in the estimation over the period January 1999 to December 2002. Each month, we update the information set, but we *do not reestimate* the model. Instead, we rely on the estimates up until end-1998. We choose this approach to limit the computational burden of the exercise. All results are therefore based on the same estimated parameters. Since the 4 years from 1999 to 2002 are EMU years, this test is particularly interesting, as it gives suggestions as to whether the EMU has or has not represented a structural break.

The results of the two forecast evaluation exercises are summarized in Tables 3 and 4, for the in-sample and out-of-sample cases, respectively. Lower values of the RMSE denote better forecasts. The best forecast at each maturity/horizon is

[9] For a definition of the $A_0(3)$ class of affine models, see Dai and Singleton (2000).

[10] We use year-on-year inflation rates in the estimation of the Ang-Piazzesi model, rather than month-on-month rates, to make our results more comparable to those of Ang and Piazzesi (2003).

highlighted in bold.

The in-sample exercise shows that, with two exceptions, forecasts based on affine models beat the random walk at all horizons and maturities. This represents further evidence, based on German data, of the good performance of the essentially affine class models (with or without the inclusion of macro variables).

Within the affine class, the performance is mixed. Our model tends to do better for shorter maturities/horizons; the Ang and Piazzesi (2003) model performs better at longer forecasting horizons; the $A_0(3)$ model works best for the long term maturities, regardless of the forecasting horizon. No model appears to stand out.

The out-of-sample results broadly confirm the indications of the in-sample ones, but with an interesting twist. Our model emerges as the best performer in the vast majority of cases, and seems to perform particularly well for shorter maturities. The Ang and Piazzesi (2003) model, on the other hand, works best for the longest maturities. The $A_0(3)$ model does reasonably well for most maturities/horizons, but it is always (with one exception) dominated by one of the macro-based models.

We conclude that the joint modelling of macroeconomic and term-structure dynamics is possible without incurring serious costs in terms of forecasting performance. By and large, the out-of-sample results from our model are also suggestive of the absence of a structural break in the data after EMU.

3.3 Expectations hypothesis tests

According to the expectations hypothesis, the yield on an n-period zero-coupon bond should increase when the spread between the same yield and the short term rate (the "slope of the yield curve") widens. A number of empirical tests of this implication of the theory have, however, found a negative relationship. This pattern, which represents an expectations puzzle, appears to be particularly clear for United States data, where the relevant regression coefficient can be as big as -5 for 10-year bonds, according to e.g. Campbell and Shiller (1991), while the expectations hypothesis predicts a value of one for all maturities. An interpretation of these results is that yields incorporate time varying risk premia of significant magnitudes, rather than just a constant one as permitted by the expectations hypothesis.

Prompted by this consideration, Dai and Singleton (2002a) investigate whether the predictions of dynamic affine term structure models are consistent with the observed expectations puzzle. Dai and Singleton (2002a) show that two tests of a dynamic term structure model are particularly informative in this respect. The first, defined as LPY(i), is a test of the capacity of the model to replicate the historical behavior of yields. To be successful, the model should be capable of generating the negative intercept coefficient of Campbell and Shiller-type regressions. The second, defined as LPY(ii), is a test of the realism of the specification of model risk premia. The dynamics of risk premia in a realistic model should be such that a Campbell and Shiller-type regression based on risk premium adjusted yields would recover the coefficient of unity consistent with the expectations hypothesis.

Dai and Singleton (2002a) show that an affine 3-factor model with Gaussian innovations and including a risk-premium specification of the type suggested by Duffee (2002) scores extremely well in terms of both LPY(i) and LPY(ii). Since our model also has Gaussian innovations and a specification of the risk premium as in Duffee (2002), in this section we also evaluate its capacity to pass the two LPY tests.

Since the evidence on Campbell and Shiller-type regressions based on European data is less compelling than for the US (e.g. Hardouvelis, 1994, Gerlach and Smets, 1997, Bekaert and Hodrick, 2001), we start by replicating Campbell and Shiller's analysis on our data. The results of the regression of the yield changes $y_{t+1}^{n-1} - y_t^n$ on a constant and the yield spread $(y_t^n - r_t) / (n-1)$ are shown in Figures 10 and 11 under the label "Sample". Consistently with the puzzle, the estimated intercept coefficient is always negative and highest at the longest maturities included in the regression. We confirm, however, that the puzzle is less severe for German yield data: the estimated coefficient hovers around -1, compared to a value of less than -4 reported by Dai and Singleton (2002a) for 10-year yields.

In figure 10 we show the results of the LPY(i) test. Following Dai and Singleton (2002a), we examine both the model-implied, theoretical population coefficients and their small sample counterparts. Some correction for small sample bias is desirable because of the persistent nature of yields. We therefore generate 1000 samples of the same length of our data (287) and report in the figure the mean estimate of the intercept coefficients (labelled "Model-implied MC" in Figure 10), together with 95% confidence bands of its small-sample distribution.

Our model appears to do relatively well for maturities up to 3 years, but then increasingly worse as the maturity lengthens. In particular, the model only captures to a limited extent the characteristic downward sloping pattern of the projection coefficient seen in the data. This result appears to reflect the relatively worse performance of the model over longer term maturities, consistent with the results of the forecasting exercises in the previous section. The coefficients based on the Monte Carlo simulations appear to confirm the existence of a small sample bias and also bring the model closer to the data. However, the 95% confidence band based on these simulations would still fail to include the sample coefficients for maturities beyond 4 years.

Figure 11 shows the results of the LPY(ii) tests. In this case, the model has very limited success in reproducing the unit coefficient required by the theory. Although the risk-premium correction generally goes in the right direction, the best the model can do is to generate a coefficient of 0.5 for 3 to 4 year maturities. For longer maturities the model fails to capture the persistence of the market risk premia.

To summarize, our model does less well than the essentially affine $A_0(3)$ class in tests of the expectations hypothesis. The results of LPY(i) are relatively good, in that the model can replicate the estimated coefficient of Campbell and Shiller-type regressions at shorter maturities. The test of LPY(ii), however, appear to suggest that the restrictions implied by the macroeconomic model, coupled with the Duffee (2002) specification of risk premia, are somewhat too strong to be supported by the data. The current model probably features too low persistence in the macroeconomic factors beyond the inflation target, which reduces the potential to explain the dynamics of long term bond yields, and in particular of premia incorporated in these yields.

4 Conclusions

This paper presents and estimates a joint model of output, inflation and term structure dynamics. The model extends the term-structure literature, since it derives bond prices using no-arbitrage conditions and based on an explicit structural macroeconomic model that includes both forward-looking and backward-looking elements. At the same time, we extends the macroeconomic literature by studying the term structure implications of a standard macro model within a dynamic no-arbitrage framework.

Our results show that there are synergies to be exploited from current advances in macroeconomic and term-structure modelling. The two approaches can be seen as complementary and, when used jointly, give rise to sensible results. Notably, we show that our estimates of macroeconomic parameters, that are partly determined by the term-structure data, are reasonable and intuitively appealing. At the same time, our model's explanatory power for the term-structure is comparable to that of term-structure models based only on unobservable variables. Our model also does reasonably well in forecasting.

Future work will address the deeper interactions between macroeconomics and the term structure by allowing the output gap equation to respond directly to the yields, rather than only to the short term rate. This extension will hopefully improve further the understanding of the joint dynamics of macro variables and bond prices.

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A Appendix

A.1 State-space form

Following Söderlind, define

$$\mathbf{X}_{1t} = [x_{t-1}, ..., x_{t-12}, \pi_{t-1}, ..., \pi_{t-12}, \pi_t^*, \eta_t, \varepsilon_t^\pi, \varepsilon_t^x, r_{t-1}]'$$

$$\mathbf{X}_{2t} = [E_t x_{t+11}, ..., E_t x_{t+1}, x_t, E_t \pi_{t+11}, ..., E_t \pi_{t+1}, \pi_t]'$$

$$\mathbf{X}_t = \begin{bmatrix} \mathbf{X}_{1,t} \\ \mathbf{X}_{2,t} \end{bmatrix}$$

$$\mathbf{F} = \left[\underset{1x24}{\mathbf{0}}, \beta\,(1-\rho), -1, 0, 0, -\rho, \underset{1x11}{\mathbf{0}}, -\gamma\,(1-\rho), -\beta\,(1-\rho), \underset{1x12}{\mathbf{1}}\right]$$

$$\mathbf{B}_1 = \begin{bmatrix} \underset{28x1}{\mathbf{0}} \\ 1 \end{bmatrix}$$

$$\mathbf{B}_2 = \begin{bmatrix} \frac{\zeta_r}{\mu_x} \\ \underset{23x1}{\mathbf{0}} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

$$\mathbf{\Omega} = \begin{bmatrix} \phi_\pi & & & & \\ & \phi_\eta & & \mathbf{0} & \\ & & \phi_\varepsilon^\pi & & \\ & \mathbf{0} & & \phi_\varepsilon^x & \\ & & & & 0 \end{bmatrix}$$

$$\mathbf{A}_{11} = \begin{bmatrix} \underset{1x11}{\mathbf{0}} & 0 & \underset{1x11}{\mathbf{0}} & 0 & \underset{1x5}{\mathbf{0}} \\ \underset{11x11}{\mathbf{I}} & 0 & \underset{11x11}{\mathbf{0}} & 0 & \underset{1x5}{\mathbf{0}} \\ \underset{1x11}{\mathbf{0}} & 0 & \underset{1x11}{\mathbf{0}} & 0 & \underset{1x5}{\mathbf{0}} \\ \underset{11x11}{\mathbf{0}} & 0 & \underset{11x11}{\mathbf{I}} & 0 & \underset{1x5}{\mathbf{0}} \\ \underset{5x11}{\mathbf{0}} & 0 & \underset{1x11}{\mathbf{0}} & 0 & \mathbf{\Omega} \end{bmatrix}$$

$$\mathbf{A}_{12} = \begin{bmatrix} \underset{1x11}{\mathbf{0}} & 1 & \underset{1x11}{\mathbf{0}} & 0 \\ \underset{11x11}{\mathbf{0}} & \underset{11x1}{\mathbf{0}} & \underset{11x11}{\mathbf{0}} & \underset{11x1}{\mathbf{0}} \\ \underset{1x11}{\mathbf{0}} & 0 & \underset{1x11}{\mathbf{0}} & 1 \\ \underset{16x11}{\mathbf{0}} & \underset{16x1}{\mathbf{0}} & \underset{16x11}{\mathbf{0}} & \underset{16x1}{\mathbf{0}} \end{bmatrix}$$

$$\boldsymbol{\zeta}_x = [\zeta_{x1}, ..., \zeta_{x12}]'$$

$$\boldsymbol{\delta}_\pi = [\delta_{\pi 1}, ..., \delta_{\pi 12}]'$$

$$\mathbf{A}_{21}=\begin{bmatrix} -\frac{1-\mu_x}{\mu_x}\boldsymbol{\zeta}_x' & \underset{1x12}{\mathbf{0}} & \underset{1x2}{\mathbf{0}} & 0 & -\frac{1}{\mu_x} & 0 \\ \underset{11x12}{\mathbf{0}} & \underset{11x12}{\mathbf{0}} & \underset{11x2}{\mathbf{0}} & \underset{11x1}{\mathbf{0}} & \underset{11x1}{\mathbf{0}} & \underset{11x1}{\mathbf{0}} \\ \underset{1x12}{\mathbf{0}} & -\frac{1-\mu_\pi}{\mu_\pi}\boldsymbol{\delta}_\pi' & \underset{1x2}{\mathbf{0}} & -\frac{1}{\mu_\pi} & 0 & 0 \\ \underset{11x12}{\mathbf{0}} & \underset{11x12}{\mathbf{0}} & \underset{11x2}{\mathbf{0}} & \underset{11x1}{\mathbf{0}} & \underset{11x1}{\mathbf{0}} & \underset{11x1}{\mathbf{0}} \end{bmatrix}$$

$$\mathbf{A}_{22}=\begin{bmatrix} -\underset{1x11}{\mathbf{1}} & \frac{1}{\mu_x} & \underset{1x10}{\mathbf{0}} & -\frac{\zeta_r}{\mu_x} & 0 \\ \underset{11x11}{\mathbf{I}} & \underset{11x1}{\mathbf{0}} & \underset{11x10}{\mathbf{0}} & \underset{11x1}{\mathbf{0}} & \underset{11x1}{\mathbf{0}} \\ \underset{1x11}{\mathbf{0}} & -\frac{\delta_x}{\mu_\pi} & -\underset{1x10}{\mathbf{1}} & -1 & \frac{1}{\mu_\pi} \\ \underset{10x11}{\mathbf{0}} & \underset{10x1}{\mathbf{0}} & \underset{10x10}{\mathbf{I}} & \underset{10x1}{\mathbf{0}} & \underset{10x1}{\mathbf{0}} \\ \underset{1x11}{\mathbf{0}} & 0 & \underset{1x10}{\mathbf{0}} & 1 & 0 \end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

$$\boldsymbol{\xi}_{1,t+1}=\left[\underset{1x24}{\mathbf{0}}, u_{\pi,t+1}, u_{\eta,t+1}, u^{\pi}_{\varepsilon,t+1}, u^{x}_{\varepsilon,t+1}, 0\right]'$$

$$\boldsymbol{\xi}_{t+1}=\left[\boldsymbol{\xi}_{1,t+1}', \underset{1x24}{\mathbf{0}}\right]'$$

Then the system can be written as

$$\mathbf{X}_{t+1}=\mathbf{A}\mathbf{X}_t+\mathbf{B}r_t+\boldsymbol{\xi}_{t+1}$$

and the policy rule as

$$r_t=-\mathbf{F}\mathbf{X}_t$$

We use Paul Söderlind's routine to solve

$$\mathbf{X}_{t+1}=\left(\mathbf{A}-\mathbf{B}\mathbf{F}\right)\mathbf{X}_t+\boldsymbol{\xi}_{t+1}$$

with solution

$$\mathbf{X}_{1,t+1}=\mathbf{M}\mathbf{X}_{1,t}+\boldsymbol{\xi}_{1,t+1}$$
$$\mathbf{X}_{2,t+1}=\mathbf{C}\mathbf{X}_{1,t+1}$$

A.2 Likelihood function

We are interested in

$$\ell\left(\boldsymbol{\theta}\right)=\prod_{t=2}^{T} f\left(\mathbf{X}^o_{1t}, \mathbf{Y}_t, \overline{\mathbf{X}}_{2t} | \mathbf{X}^o_{1t-1}, \mathbf{Y}_{t-1}\right)$$

Recall that $\mathbf{Y}_t=\mathbf{A}+\mathbf{B}^o\mathbf{X}^o_{1t}+\mathbf{B}^u\mathbf{X}^u_{1t}+\mathbf{B}^m\mathbf{u}^m_t$ and that $\overline{\mathbf{X}}_{2t}=\overline{\mathbf{C}}^o\mathbf{X}^o_{1t}+\overline{\mathbf{C}}^u\mathbf{X}^u_{1t}$ (where $\overline{\mathbf{X}}_{2t}$ are selected variables from the $\mathbf{X}_{2t}$ vector), so that

$$\begin{bmatrix} \mathbf{Y}_t \\ \overline{\mathbf{X}}_{2t} \end{bmatrix}=\begin{bmatrix} \mathbf{A} \\ \mathbf{0} \end{bmatrix}+\begin{bmatrix} \mathbf{B}^o \\ \overline{\mathbf{C}}^o \end{bmatrix}\mathbf{X}^o_{1t}+\begin{bmatrix} \mathbf{B}^u \\ \overline{\mathbf{C}}^u \end{bmatrix}\mathbf{X}^u_{1t}+\begin{bmatrix} \mathbf{B}^m \\ \mathbf{0} \end{bmatrix}\mathbf{u}^m_t$$

so that

$$\begin{bmatrix} \mathbf{X}_{1t}^{o} \\ \mathbf{Y}_{t} \\ \overline{\mathbf{X}}_{2t} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{A} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{I} \\ \mathbf{B}^{o} \\ \overline{\mathbf{C}}^{o} \end{bmatrix} \mathbf{X}_{1t}^{o} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}^{u} \\ \overline{\mathbf{C}}^{u} \end{bmatrix} \mathbf{X}_{1t}^{u} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}^{m} \\ \mathbf{0} \end{bmatrix} \mathbf{u}_{t}^{m}$$

Now stack the $\mathbf{X}_{1t}^{o}$, $\mathbf{X}_{1t}^{u}$ and $\mathbf{u}_{t}^{m}$ vectors, so that

$$\begin{bmatrix} \mathbf{X}_{1t}^{o} \\ \mathbf{Y}_{t} \\ \overline{\mathbf{X}}_{2t} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{A} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^{o} & \mathbf{B}^{u} & \mathbf{B}^{m} \\ \overline{\mathbf{C}}^{o} & \overline{\mathbf{C}}^{u} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{1t}^{o} \\ \mathbf{X}_{1t}^{u} \\ \mathbf{u}_{t}^{m} \end{bmatrix}$$

so that this can be inverted to find

$$\begin{bmatrix} \mathbf{X}_{1t}^{o} \\ \mathbf{X}_{1t}^{u} \\ \mathbf{u}_{t}^{m} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^{o} & \mathbf{B}^{u} & \mathbf{B}^{m} \\ \overline{\mathbf{C}}^{o} & \overline{\mathbf{C}}^{u} & \mathbf{0} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{X}_{1t}^{o} \\ \mathbf{Y}_{t} \\ \overline{\mathbf{X}}_{2t} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{A} \\ \mathbf{0} \end{bmatrix} \right)$$

It follows that the likelihood function can be rewritten as

$$\begin{aligned} \ell(\theta) &= \prod_{t=2}^{T} \left| \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^{o} & \mathbf{B}^{u} & \mathbf{B}^{m} \\ \overline{\mathbf{C}}^{o} & \overline{\mathbf{C}}^{u} & \mathbf{0} \end{bmatrix}^{-1} \right| f\left(\mathbf{X}_{1t}^{o}, \mathbf{X}_{1t}^{u}, \mathbf{u}_{t}^{m} | \mathbf{X}_{1t-1}^{o}, \mathbf{X}_{1t-1}^{u}\right) \\ &= \prod_{t=2}^{T} \frac{1}{\begin{vmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^{o} & \mathbf{B}^{u} & \mathbf{B}^{m} \\ \overline{\mathbf{C}}^{o} & \overline{\mathbf{C}}^{u} & \mathbf{0} \end{vmatrix}} f\left(\mathbf{X}_{1t}^{o}, \mathbf{X}_{1t}^{u}, \mathbf{u}_{t}^{m} | \mathbf{X}_{1t-1}^{o}, \mathbf{X}_{1t-1}^{u}\right) \\ &= \prod_{t=2}^{T} \frac{1}{\begin{vmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^{o} & \mathbf{B}^{u} & \mathbf{B}^{m} \\ \overline{\mathbf{C}}^{o} & \overline{\mathbf{C}}^{u} & \mathbf{0} \end{vmatrix}} f_{\mathbf{X}_1}\left(\mathbf{X}_{1t}^{o}, \mathbf{X}_{1t}^{u} | \mathbf{X}_{1t-1}^{o}, \mathbf{X}_{1t-1}^{u}\right) f_{\mathbf{u}^{m}}\left(\mathbf{u}_{t}^{m}\right) \end{aligned}$$

where the second equality come from the properties of the determinant and the third equality from the independence assumption between structural shocks and measurement errors. Defining $J \equiv \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^{o} & \mathbf{B}^{u} & \mathbf{B}^{m} \\ \overline{\mathbf{C}}^{o} & \overline{\mathbf{C}}^{u} & \mathbf{0} \end{bmatrix}$, the loglikelihood is simply

$$\ln\left(\pounds\left(\boldsymbol{\theta}\right)\right) = \sum_{t=2}^{T} \left(-\ln|J| + \ln f_{\mathbf{X}_1}\left(\mathbf{X}_{1t}^{o}, \mathbf{X}_{1t}^{u} | \mathbf{X}_{1t-1}^{o}, \mathbf{X}_{1t-1}^{u}\right) + \ln f_{\mathbf{u}^{m}}\left(\mathbf{u}_{t}^{m}\right)\right)$$

and

$$\begin{aligned}\ln\left(\pounds\left(\boldsymbol{\theta}\right)\right) &= -\left(T-1\right)\ln\left|J\right|-\frac{\left(T-1\right)n_1}{2}\ln\left(2\pi\right)-\frac{T-1}{2}\ln\left|\boldsymbol{\Sigma\Sigma}'\right| \\ &-\frac{1}{2}\sum_{t=2}^{T}\left(\mathbf{X}_{1t}-\mathbf{M}\mathbf{X}_{1t-1}\right)'\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1}\left(\mathbf{X}_{1t}-\mathbf{M}\mathbf{X}_{1t-1}\right) \\ &-\frac{\left(T-1\right)n_m}{2}\ln\left(2\pi\right)-\frac{T-1}{2}\ln\sum_{i=1}^{n_m}\sigma_i^2-\frac{1}{2}\sum_{t=2}^{T}\sum_{i=1}^{n_m}\frac{\left(u_{t,i}^m\right)^2}{\sigma_i^2}\end{aligned}$$

or, more similar to the structure of our programme,

$$\begin{aligned}\ln\left(\pounds\left(\boldsymbol{\theta}\right)\right) &= -\left(T-1\right)\left(\ln\left|J\right|+\frac{1}{2}\ln\left|\Sigma\Sigma'\right|+\frac{1}{2}\ln\sum_{i=1}^{n_m}\sigma_i^2+\frac{n_1+n_m}{2}\ln\left(2\pi\right)\right) \\ &-\frac{1}{2}\sum_{t=2}^{T}\left(\mathbf{X}_{1t}-\mathbf{M}\mathbf{X}_{1t-1}\right)'\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1}\left(\mathbf{X}_{1t}-\mathbf{M}\mathbf{X}_{1t-1}\right)-\frac{1}{2}\sum_{t=2}^{T}\sum_{i=1}^{n_m}\frac{\left(u_{t,i}^m\right)^2}{\sigma_i^2}\end{aligned}$$

where n_m is the number of measurement errors and n_1 is defined as in Söderlind.

Note that, in order to actually calculate the unobservable factors and measurement errors, it is useful to rewrite the system

$$\begin{bmatrix}\mathbf{X}_{1t}^o\\ \mathbf{Y}_t\\ \overline{\mathbf{X}}_{2t}\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{A}\\ \mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{I}\\ \mathbf{B}^o\\ \overline{\mathbf{C}}^o\end{bmatrix}\mathbf{X}_{1t}^o+\begin{bmatrix}\mathbf{0}\\ \mathbf{B}^u\\ \overline{\mathbf{C}}^u\end{bmatrix}\mathbf{X}_{1t}^u+\begin{bmatrix}\mathbf{0}\\ \mathbf{B}^m\\ \mathbf{0}\end{bmatrix}\mathbf{u}_t^m$$

in terms of perfectly observable variables, i.e. $\left[\mathbf{X}_{1t}^o, \mathbf{Y}_t^p, \overline{\mathbf{X}}_{2t}\right]$, and variables measured with error, i.e. $[\mathbf{Y}_t^m]$. This leads to

$$\begin{bmatrix}\mathbf{X}_{1t}^o\\ \mathbf{Y}_t^p\\ \mathbf{Y}_t^m\\ \overline{\mathbf{X}}_{2t}\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{A}^p\\ \mathbf{A}^m\\ \mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{I}\\ \mathbf{B}^{op}\\ \mathbf{B}^{om}\\ \overline{\mathbf{C}}^o\end{bmatrix}\mathbf{X}_{1t}^o+\begin{bmatrix}\mathbf{0}\\ \mathbf{B}^{up}\\ \mathbf{B}^{um}\\ \overline{\mathbf{C}}^u\end{bmatrix}\mathbf{X}_{1t}^u+\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \widehat{\mathbf{B}}^m\\ \mathbf{0}\end{bmatrix}\mathbf{u}_t^m$$

which, forgetting about $\mathbf{X}_{1t}^o$, can also be split into

$$\begin{bmatrix}\mathbf{Y}_t^p\\ \overline{\mathbf{X}}_{2t}\end{bmatrix}=\begin{bmatrix}\mathbf{A}^p\\ \mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{B}^{op}\\ \overline{\mathbf{C}}^o\end{bmatrix}\mathbf{X}_{1t}^o+\begin{bmatrix}\mathbf{B}^{up}\\ \overline{\mathbf{C}}^u\end{bmatrix}\mathbf{X}_{1t}^u$$

$$\mathbf{Y}_t^m=\mathbf{A}^m+\mathbf{B}^{om}\mathbf{X}_{1t}^o+\mathbf{B}^{um}\mathbf{X}_{1t}^u+\widehat{\mathbf{B}}^m\mathbf{u}_t^m$$

These two systems can be solved for $\mathbf{X}_{1t}^u$ and $\mathbf{u}_t^m$ in a recursive fashion as

$$\mathbf{X}_{1t}^u=\begin{bmatrix}\mathbf{B}^{up}\\ \overline{\mathbf{C}}^u\end{bmatrix}^{-1}\left(\begin{bmatrix}\mathbf{Y}_t^p\\ \overline{\mathbf{X}}_{2t}\end{bmatrix}-\begin{bmatrix}\mathbf{A}^p\\ \mathbf{0}\end{bmatrix}-\begin{bmatrix}\mathbf{B}^{op}\\ \overline{\mathbf{C}}^o\end{bmatrix}\mathbf{X}_{1t}^o\right)$$

$$\mathbf{u}_t^m=\left(\widehat{\mathbf{B}}^m\right)^{-1}\left(\mathbf{Y}_t^m-\mathbf{A}^m-\mathbf{B}^{om}\mathbf{X}_{1t}^o-\mathbf{B}^{um}\mathbf{X}_{1t}^u\right)$$

A.3 Risk premia

A.3.1 Holding premia

Let $e_{n,t}$ denote the one-period holding premium on an n-period bond purchased at t, defined as the expected holding return of that bond over one period, less the risk-free rate:

$$e_{n,t} = E_t\left[\ln\left(p_{t+1}^{n-1}\right) - \ln\left(p_t^n\right)\right] - r_t.$$

Using that

$$\begin{aligned} p_t^n &= \exp\left(\bar{A}_n + \bar{B}_n' \mathbf{X}_{1,t}\right), \\ p_{t+1}^{n-1} &= \exp\left(\bar{A}_{n-1} + \bar{B}_{n-1}' \mathbf{X}_{1,t+1}\right), \end{aligned}$$

and

$$r_t = \mathbf{\Delta}' \mathbf{X}_{1,t}$$

we obtain

$$\begin{aligned} e_{n,t} &= \bar{A}_{n-1} + \bar{B}_{n-1}' E_t\left[\mathbf{X}_{1,t+1}\right] - \bar{A}_n - \bar{B}_n' \mathbf{X}_{1,t} - \mathbf{\Delta}' \mathbf{X}_{1,t} \\ &= \bar{A}_{n-1} + \bar{B}_{n-1}'\left(\mathbf{M}\mathbf{X}_{1,t} + \Sigma E_t\left[\varepsilon_{1,t+1}\right]\right) - \bar{A}_n - \bar{B}_n' \mathbf{X}_{1,t} - \mathbf{\Delta}' \mathbf{X}_{1,t} \\ &= \bar{A}_{n-1} + \bar{B}_{n-1}'\left(\mathbf{M}\mathbf{X}_{1,t} + \Sigma E_t\left[\varepsilon_{1,t+1}\right]\right) - \left(\bar{A}_{n-1} - \bar{B}_{n-1}' \Sigma\lambda_0 + \frac{1}{2}\bar{B}_{n-1}' \Sigma\Sigma' \bar{B}_{n-1}\right) \\ &\quad - \left(\bar{B}_{n-1}'\left(\mathbf{M} - \Sigma\lambda_1\right) - \mathbf{\Delta}'\right)\mathbf{X}_{1,t} - \mathbf{\Delta}' \mathbf{X}_{1,t} \\ &= \bar{A}_{n-1} + \bar{B}_{n-1}' \mathbf{M}\mathbf{X}_{1,t} + \bar{B}_{n-1}' \Sigma E_t\left[\varepsilon_{1,t+1}\right] - \bar{A}_{n-1} + \bar{B}_{n-1}' \Sigma\lambda_0 - \frac{1}{2}\bar{B}_{n-1}' \Sigma\Sigma' \bar{B}_{n-1} \\ &\quad - \bar{B}_{n-1}' \mathbf{M}\mathbf{X}_{1,t} + \bar{B}_{n-1}' \Sigma\lambda_1 \mathbf{X}_{1,t} + \mathbf{\Delta}' \mathbf{X}_{1,t} - \mathbf{\Delta}' \mathbf{X}_{1,t} \\ &= \bar{B}_{n-1}' \Sigma\lambda_0 + \bar{B}_{n-1}' \Sigma\lambda_1 \mathbf{X}_{1,t} - \frac{1}{2}\bar{B}_{n-1}' \Sigma\Sigma' \bar{B}_{n-1}. \end{aligned}$$

A.3.2 Forward premia

Let $\psi_{n,t}$ denote the one-period forward premium at t for maturity n, defined as the difference between the implied one-period forward rate n periods ahead, $f_{n,t}$, less the corresponding expected one-period interest rate:

$$\psi_{n,t} = f_{n,t} - E_t\left[r_{t+n}\right].$$

The implied forward rate, expressed in one-period terms, is given by

$$\begin{aligned} f_{n,t} &= \ln\left(p_t^n\right) - \ln\left(p_t^{n+1}\right) \\ &= \bar{A}_n + \bar{B}_n' \mathbf{X}_{1,t} - \bar{A}_{n+1} - \bar{B}_{n+1}' \mathbf{X}_{1,t} \\ &= \bar{A}_n + \bar{B}_n' \mathbf{X}_{1,t} - \left(\bar{A}_n - \bar{B}_n' \Sigma\lambda_0 + \frac{1}{2}\bar{B}_n' \Sigma\Sigma' \bar{B}_n\right) - \left(\bar{B}_n'\left(\mathbf{M} - \Sigma\lambda_1\right) - \mathbf{\Delta}'\right)\mathbf{X}_{1,t} \\ &= \bar{B}_n' \Sigma\lambda_0 - \frac{1}{2}\bar{B}_n' \Sigma\Sigma' \bar{B}_n + \left[\bar{B}_n' - \bar{B}_n'\left(\mathbf{M} - \Sigma\lambda_1\right) + \mathbf{\Delta}'\right]\mathbf{X}_{1,t}, \end{aligned}$$

while the expected short rate is

$$\begin{aligned} E_t [r_{t+n}] &= \mathbf{\Delta}' E_t [\mathbf{X}_{1,t+n}] \\ &= \mathbf{\Delta}' E_t \left[\mathbf{M}^n \mathbf{X}_{1,t} + \sum_{i=1}^{n} \mathbf{M}^{n-i} \Sigma \varepsilon_{1,t+i} \right] \\ &= \mathbf{\Delta}' \mathbf{M}^n \mathbf{X}_{1,t}. \end{aligned}$$

The one-period forward premium is therefore

$$\begin{aligned} \psi_{n,t} &= f_{n,t} - E_t [r_{t+n}] \\ &= \bar{B}_n' \Sigma \lambda_0 - \frac{1}{2} \bar{B}_n' \Sigma \Sigma' \bar{B}_n + \left[\bar{B}_n' - \bar{B}_n' (\mathbf{M} - \Sigma \lambda_1) + \mathbf{\Delta}' (I - \mathbf{M}^n) \right] \mathbf{X}_{1,t}. \end{aligned}$$

The forward premium is expressed in one-period terms.

A.3.3 Yield risk premia

Let $\omega_{n,t}$ denote the n-maturity yield premium at t, defined as the sum of the forward premia up until $t + n - 1$:

$$\begin{aligned} \omega_{n,t} &= \sum_{i=0}^{n-1} \psi_{n,t} \\ &= \sum_{i=0}^{n-1} \left(\bar{B}_i' \Sigma \lambda_0 - \frac{1}{2} \bar{B}_i' \Sigma \Sigma' \bar{B}_i + \left[\bar{B}_i' - \bar{B}_i' (\mathbf{M} - \Sigma \lambda_1) + \mathbf{\Delta}' \left(I - \mathbf{M}^i \right) \right] \mathbf{X}_{1,t} \right). \end{aligned}$$

The n-maturity yield premium can be expressed in one-period terms by dividing $\omega_{n,t}$ with n.

Table 1: Parameter estimates
(Sample period: Feb 1975-Dec 1998)

Parameter	Point estimate	Standard error
ρ	0.955	0.015
β	0.144	0.080
γ	0.015	0.008
μ_π	0.544	0.062
δ_x	0.010	0.017
μ_x	0.177	0.130
ζ_r	0.031	0.043
ϕ_{π^*}	0.999	-
$\sigma_{\pi^*} \times 10^2$	0.018	0.012
$\sigma_\eta \times 10^2$	0.038	0.008
$\sigma_x \times 10^2$	0.142	0.006
$\sigma_\pi \times 10^2$	0.107	0.008
$\sigma_1^m \times 10^2$	1.129	0.040
$\sigma_2^m \times 10^2$	1.538	0.038
$\sigma_3^m \times 10^2$	1.823	0.040
$\lambda_{0,1}$	-0.022	0.115
$\lambda_{0,2}$	-0.437	0.148
$\lambda_{0,3}$	3.466	0.856
$\lambda_{0,4}$	-0.640	0.326

The standard errors are based on the asymptotic variance-covariance matrix of White (1982). The estimates of the lag coefficients for inflation and output are not reported.

$\lambda_1 \times 10^{-2}$

	π^*	η	ε^π	ε^x
π^*	0.434 (0.765)	3.893 (1.434)	0.389 (0.466)	1.519 (0.661)
η	-3.169 (1.501)	0.006 (0.006)	0	0
ε^π	9.096 (7.603)	0	7.005 (0.709)	0
ε^x	0.017 (0.057)	0	0	3.022 (3.826)

Standard errors in parentheses

Table 2: Forecast error variance decompositions

Output gap

Steps	π^*	η	π	x
	Variance due to (in %)			
1	0.03	0.06	0.00	99.90
6	1.19	2.22	0.00	96.60
12	5.90	9.74	0.08	84.27
36	58.62	41.12	0.26	0.01
60	86.11	4.83	0.02	9.03

Inflation

π^*	η	π	x
Variance due to (in %)			
0.68	0.01	99.26	0.06
87.72	0.78	8.67	2.83
66.23	0.51	32.21	1.05
98.82	0.05	1.09	0.03
99.99	0.00	0.00	0.01

Short rate

Steps	π^*	η	π	x
	Variance due to (in %)			
1	0.08	98.31	1.21	0.39
6	6.09	79.07	6.74	8.09
12	29.77	44.89	3.80	21.55
36	93.55	0.65	0.00	5.79
60	99.80	0.02	0.00	0.18

3-month rate

Steps	π^*	η	π	x
	Variance due to (in %)			
1	1.01	97.25	1.03	0.70
6	11.12	71.70	6.91	10.27
12	37.91	38.18	2.33	21.58
36	94.83	0.46	0.00	4.71
60	99.85	0.01	0.00	0.14

1-year yield

π^*	η	π	x
Variance due to (in %)			
19.24	76.20	1.14	3.42
38.60	42.47	3.19	15.73
63.72	18.35	0.32	17.61
97.69	0.11	0.00	2.20
99.94	0.01	0.00	0.05

3-year yield

Steps	π^*	η	π	x
	Variance due to (in %)			
1	81.15	16.13	0.05	2.67
6	79.84	10.26	0.33	9.57
12	89.44	3.59	0.03	6.94
36	99.50	0.00	0.00	0.50
60	99.99	0.00	0.00	0.01

10-year yield

π^*	η	π	x
Variance due to (in %)			
99.79	0.20	0.02	0.00
96.77	1.40	0.05	1.79
98.33	0.46	0.00	1.20
99.93	0.00	0.00	0.07
100.0	0.00	0.00	0.00

Table 3: In-sample yield forecast performance: RMSEs

		1 month forecast horizon		
maturity	RW	$A_0\,(3)$	AP	Struct.
1 month	0.484	**0.462**	0.473	0.464
3 months	0.450	0.452	**0.449**	**0.449**
1 year	0.393	0.390	**0.388**	0.427
3 years	**0.311**	0.373	0.366	0.316
10 years	0.265	**0.261**	0.263	0.407

		3 month forecast horizon		
maturity	RW	$A_0\,(3)$	AP	Struct.
1 month	0.739	0.686	0.694	**0.659**
3 months	0.735	0.711	0.699	**0.674**
1 year	0.722	0.713	0.700	**0.692**
3 years	**0.638**	0.650	0.650	0.639
10 years	0.480	**0.467**	0.471	0.550

		6 month forecast horizon		
maturity	RW	$A_0\,(3)$	AP	Struct.
1 month	1.134	1.039	1.009	**0.976**
3 months	1.151	1.085	1.038	**1.019**
1 year	1.114	1.078	**1.049**	**1.049**
3 years	0.957	**0.949**	0.958	0.969
10 years	0.698	**0.659**	0.673	0.704

		9 month forecast horizon		
maturity	RW	$A_0\,(3)$	AP	Struct.
1 month	1.487	1.350	**1.238**	1.254
3 months	1.508	1.403	**1.285**	1.313
1 year	1.447	1.376	**1.320**	1.349
3 years	1.202	**1.172**	1.186	1.216
10 years	0.865	**0.796**	0.821	0.841

		12 month forecast horizon		
maturity	RW	$A_0\,(3)$	AP	Struct.
1 month	1.805	1.600	**1.397**	1.504
3 months	1.813	1.654	**1.460**	1.567
1 year	1.734	1.622	**1.537**	1.619
3 years	1.442	**1.370**	1.391	1.465
10 years	1.042	**0.939**	0.973	1.014

The table shows root mean square errors (RMSEs) for in-sample forecasts between 1975:01 and 1997:12. RW denotes random walk forecasts, $A_0\,(3)$ is a canonical essentially affine Gaussian three-factor model, AP denotes the Ang-Piazzesi (2003) Macro Model (estimated using our macro data, but with inflation expressed in y-o-y terms), and "Struct." denotes our structural macro model.

Table 4: Out-of-sample yield forecast performance: RMSEs

	1 month forecast horizon			
maturity	RW	$A_0(3)$	AP	Struct.
1 month	0.208	0.215	0.298	**0.190**
3 months	**0.198**	0.208	0.294	0.217
1 year	0.212	**0.210**	0.216	0.289
3 years	0.224	0.250	0.256	**0.222**
10 years	0.194	0.193	**0.189**	0.540

	3 month forecast horizon			
maturity	RW	$A_0(3)$	AP	Struct.
1 month	0.440	0.451	0.694	**0.353**
3 months	0.447	0.433	0.648	**0.373**
1 year	0.480	0.470	0.476	**0.458**
3 years	0.473	0.468	0.449	**0.432**
10 years	0.349	0.340	**0.319**	0.562

	6 month forecast horizon			
maturity	RW	$A_0(3)$	AP	Struct.
1 month	0.754	0.731	1.229	**0.565**
3 months	0.745	0.715	1.102	**0.588**
1 year	0.776	0.739	0.732	**0.672**
3 years	0.720	0.659	**0.593**	0.643
10 years	0.533	0.496	**0.456**	0.619

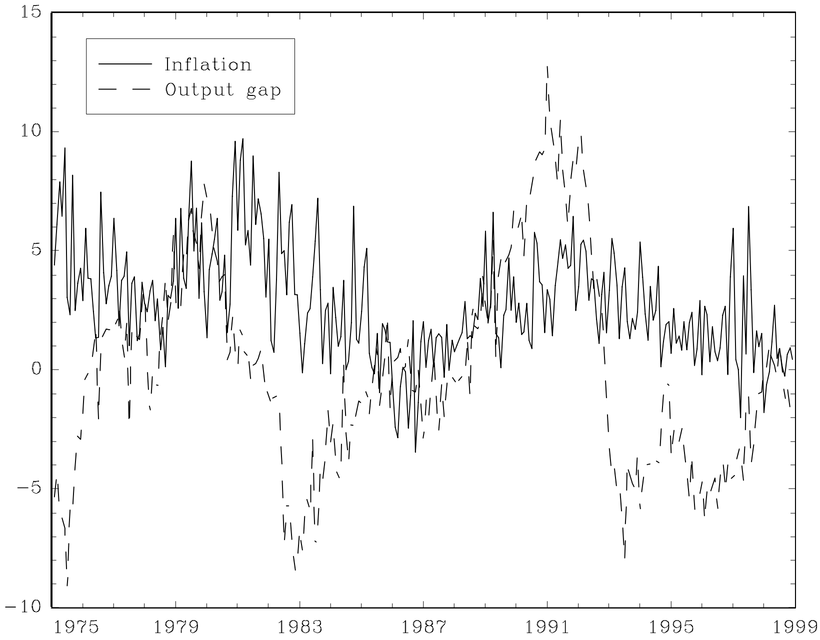
	9 month forecast horizon			
maturity	RW	$A_0(3)$	AP	Struct.
1 month	1.015	0.954	1.637	**0.718**
3 months	0.984	0.931	1.446	**0.730**
1 year	0.981	0.934	0.915	**0.801**
3 years	0.864	0.786	**0.689**	0.764
10 years	0.617	0.603	**0.545**	0.689

	12 month forecast horizon			
maturity	RW	$A_0(3)$	AP	Struct.
1 month	1.228	1.177	1.976	**0.870**
3 months	1.200	1.174	1.765	**0.891**
1 year	1.194	1.160	1.137	**0.975**
3 years	1.013	0.965	**0.849**	0.912
10 years	0.662	0.713	**0.632**	0.750

RMSEs for out-of-sample forecasts between 1999:01 and 2002:12, based on parameter estimates for 1975:02 - 1998:12. RW are random walk forecasts, $A_0(3)$ is a canonical essentially affine Gaussian three-factor model, AP denotes the Ang-Piazzesi (2003) Macro Model (estimated using our macro data, but with inflation expressed in y-o-y terms), and "Struct." denotes our structural macro model.

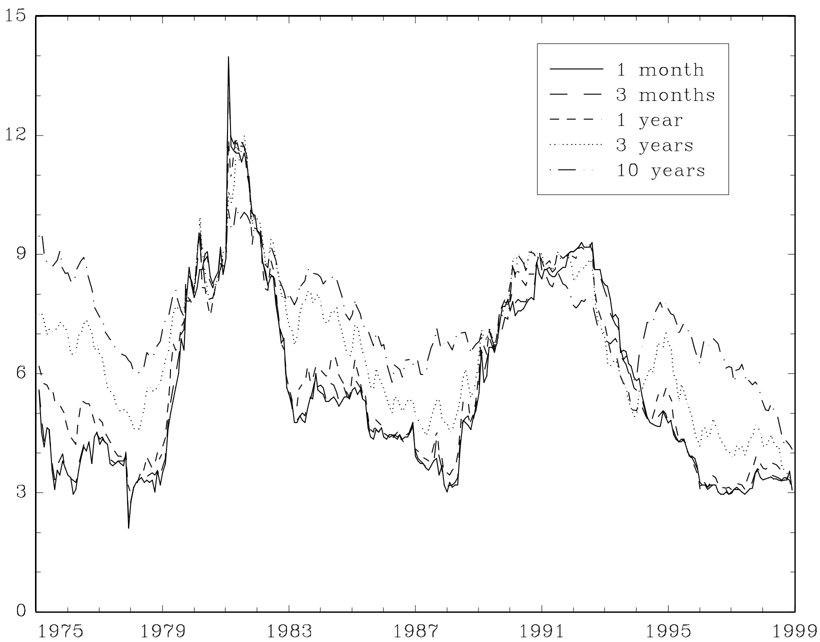
Figure 1: Data used in the estimations

(a) Macro data



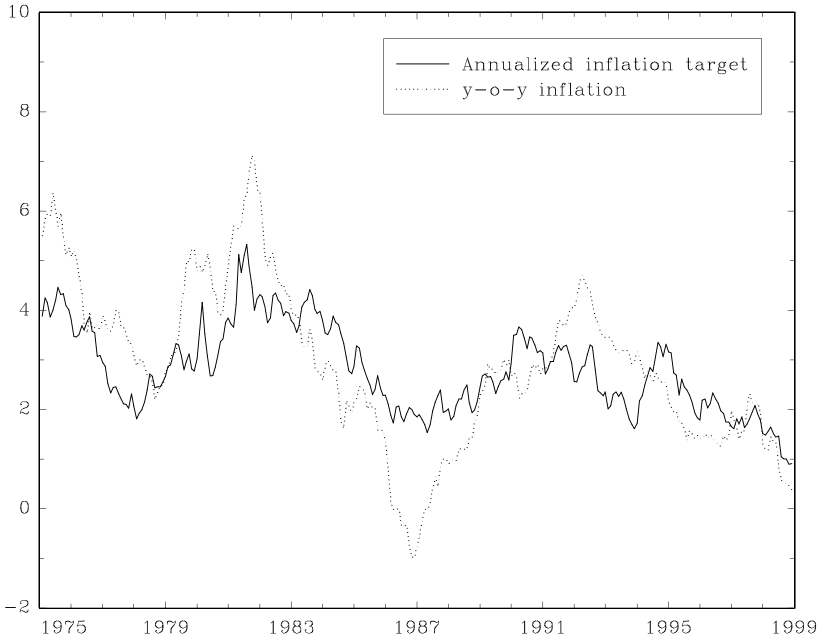
The inflation series has been multiplied by 1200 and the output gap series by 100. The sample period is January 1975 to December 1998.

(b) Yield data



German term structure data over the sample period January 1975 to December 1998 (percent per year).

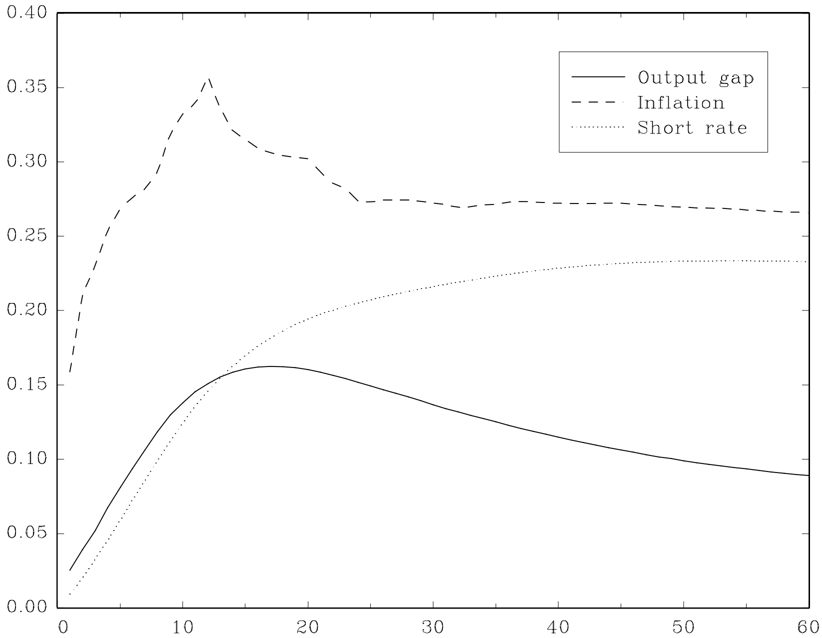
Figure 2: Estimated inflation target and actual inflation (year-on-year)



The estimated inflation target has been scaled up by 1200, and the year-on-year inflation series by 100.

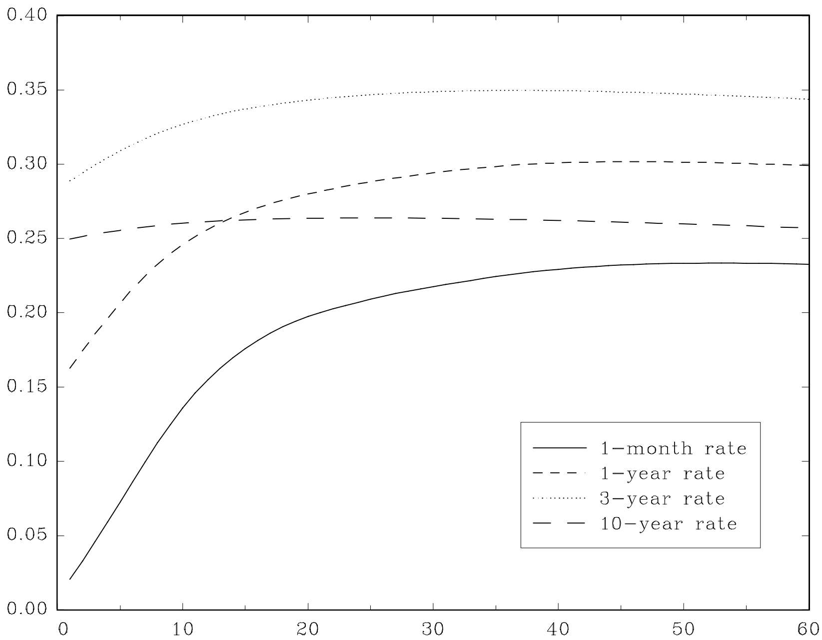
Figure 3: Impulse responses from inflation target shock

(a) Response of macro variables



All responses are expressed in percentage terms. The inflation and short rate responses have been scaled up by 12 to be expressed in annual terms. The inflation target was shocked by one standard deviation (around 0.2% p.a.).

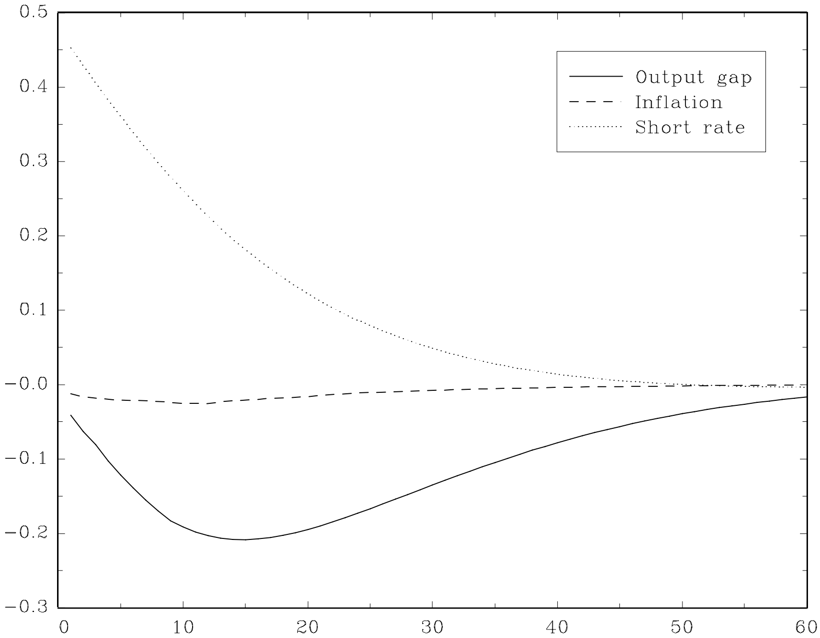
(b) Response of yields



All responses are expressed in annual percentage terms. The inflation target was shocked by one standard deviation (around 0.2% p.a.).

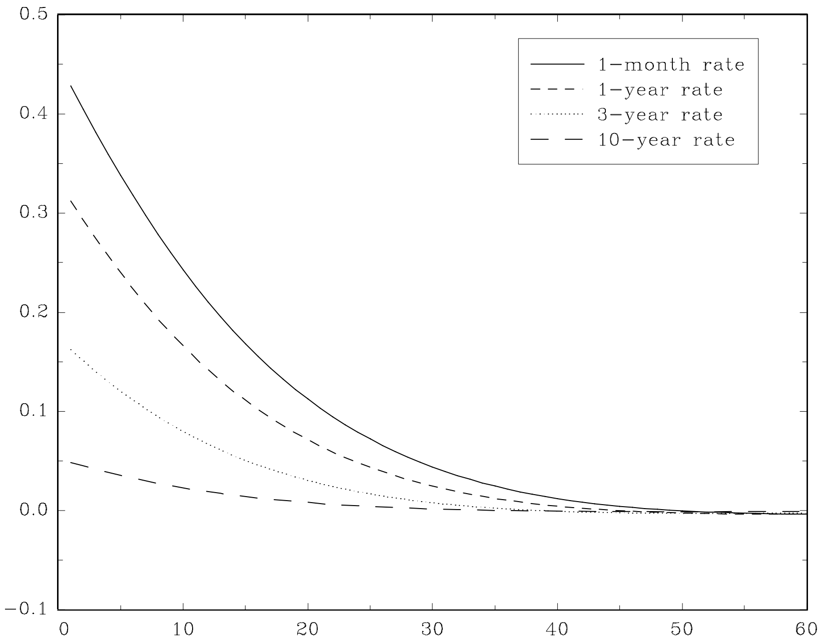
Figure 4: Impulse responses from monetary policy shock

(a) Response of macro variables



All responses are expressed in percentage terms. The inflation and short rate responses have been scaled up by 12 to be expressed in annual terms. The short-term interest rate was shocked by one standard deviation (around 0.46% p.a.).

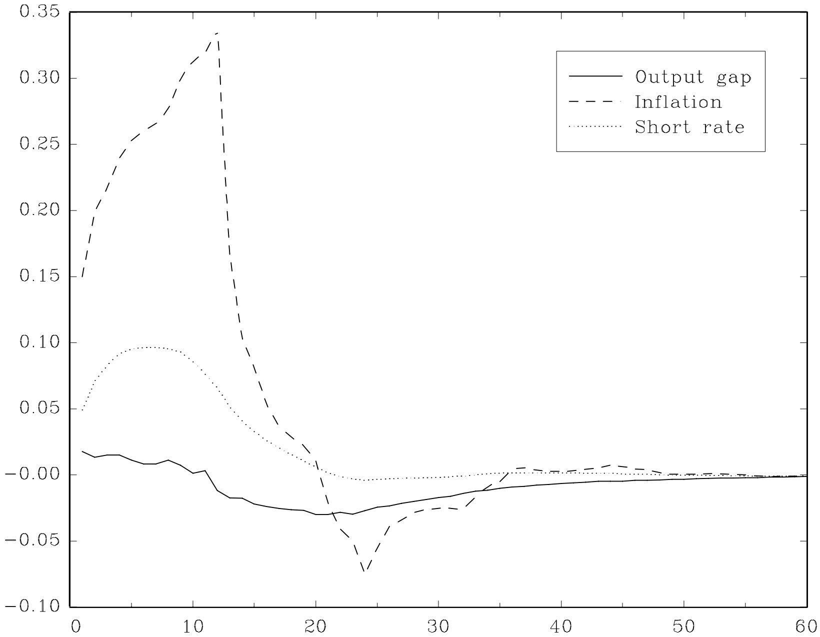
(b) Response of yields



All responses are expressed in annual percentage terms. The short-term interest rate was shocked by one standard deviation (around 0.46% p.a.).

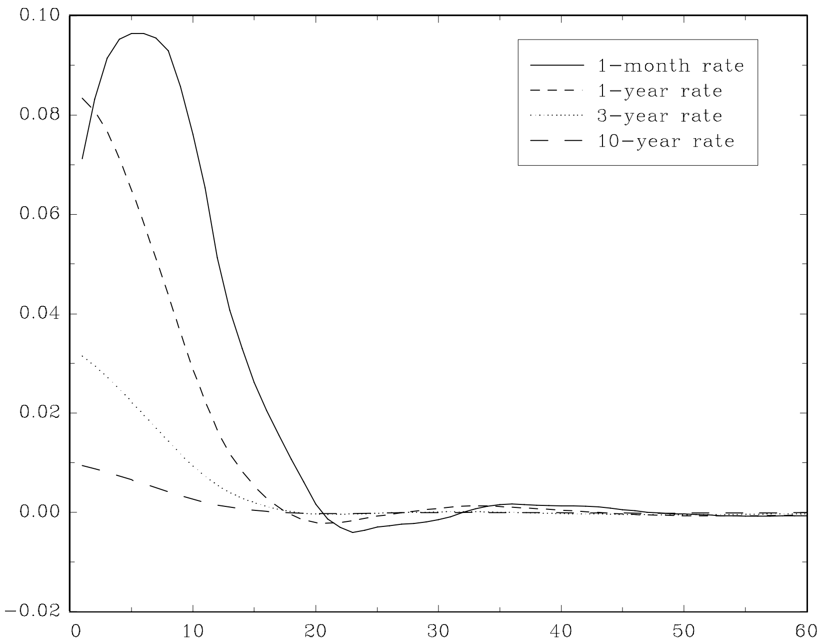
Figure 5: Impulse responses from inflation shock

(a) Response of macro variables



All responses are expressed in percentage terms. The inflation response in the figure corresponds to year-on-year inflation, while the short rate has been scaled up by 12. Inflation was shocked by one standard deviation (around 0.14% p.a.).

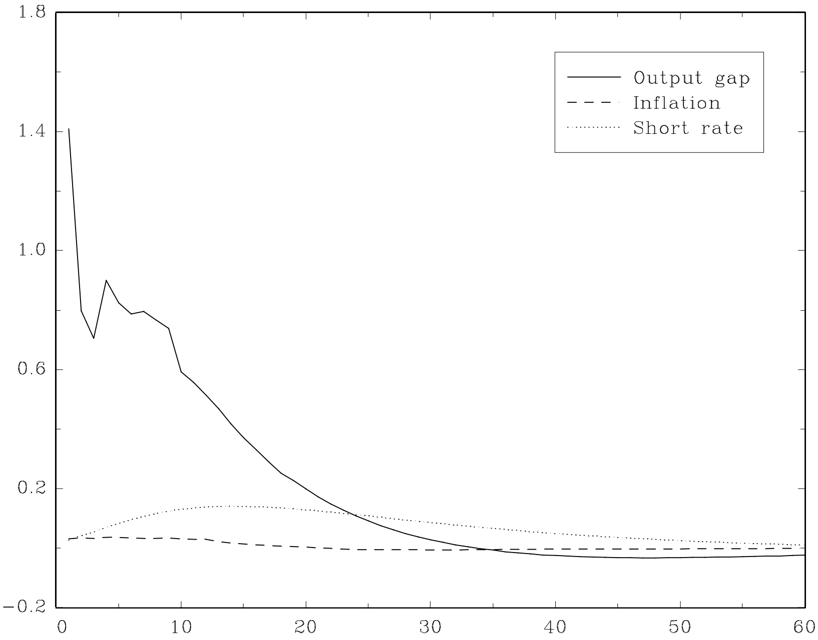
(b) Response of yields



All responses are expressed in annual percentage terms. Inflation was shocked by one standard deviation (around 0.14% p.a.).

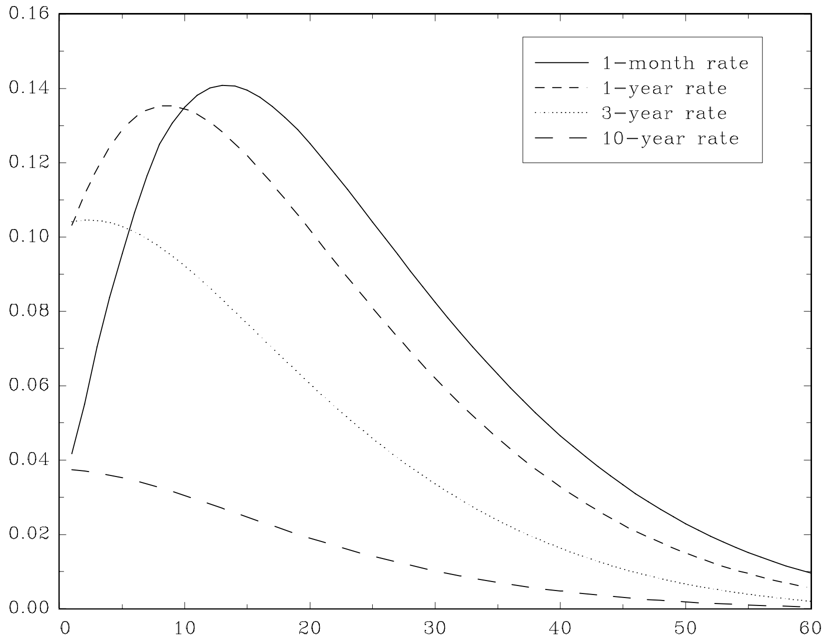
Figure 6: Impulse responses from output shock

(a) Response of macro variables



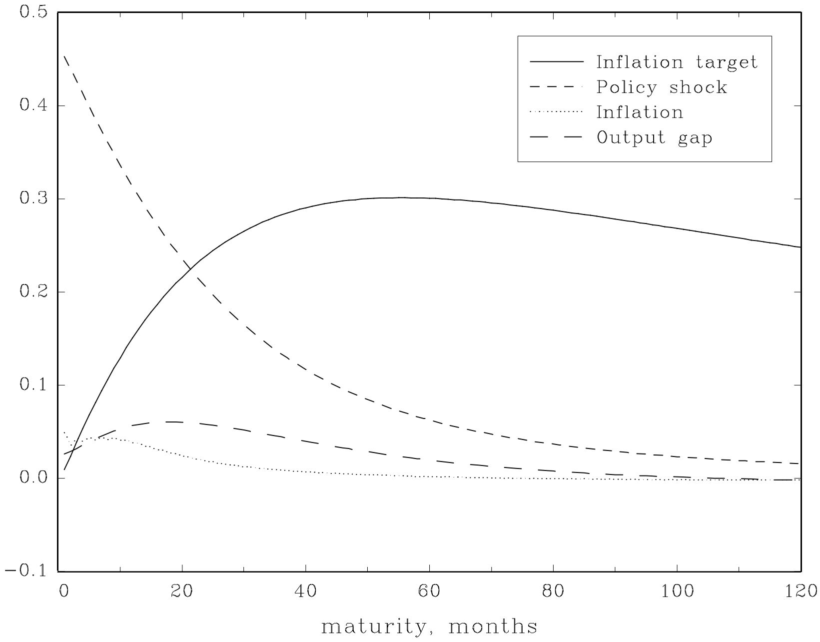
All responses are expressed in percentage terms. The inflation and short rate responses have been scaled up by 12 to be expressed in annual terms. The output gap was shocked by one standard deviation (around 1.3%).

(b) Response of yields



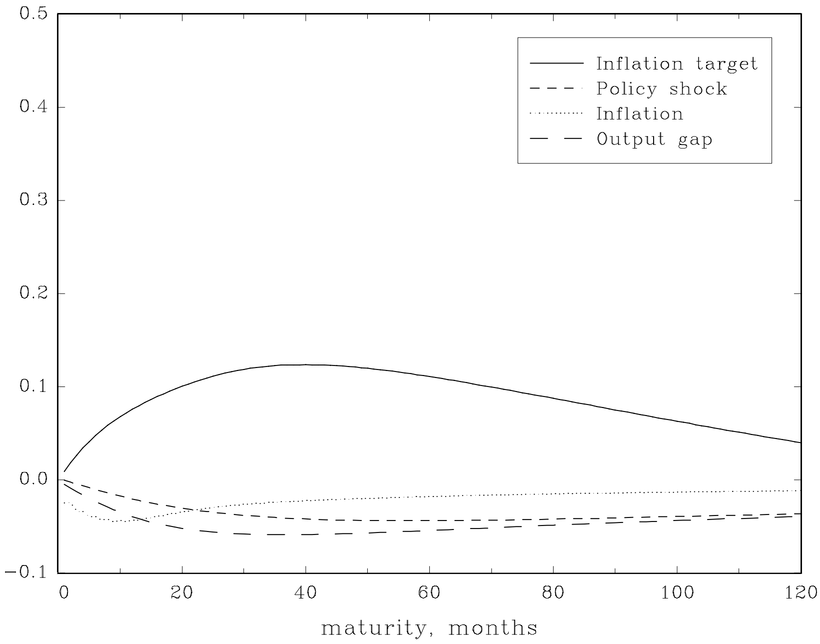
All responses are expressed in annual percentage terms. The output gap was shocked by one standard deviation (around 1.3%).

Figure 7: Factor loadings



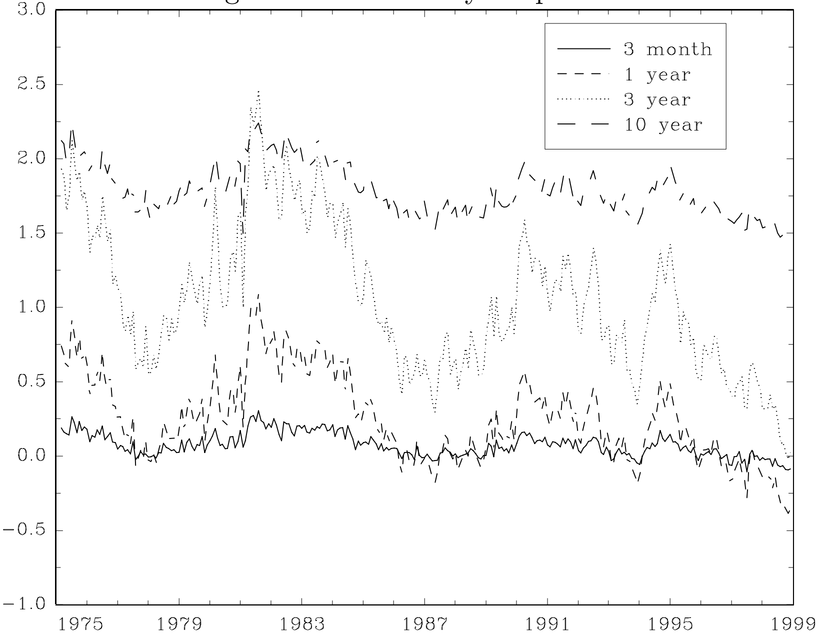
The factor loadings in the figure correspond to the B_n parameters of the four non-lagged macro factors. They have been rescaled to correspond to one standard deviation of the respective factors, expressed in annual percentage terms.

Figure 8: Initial response of yield premia to macro shocks



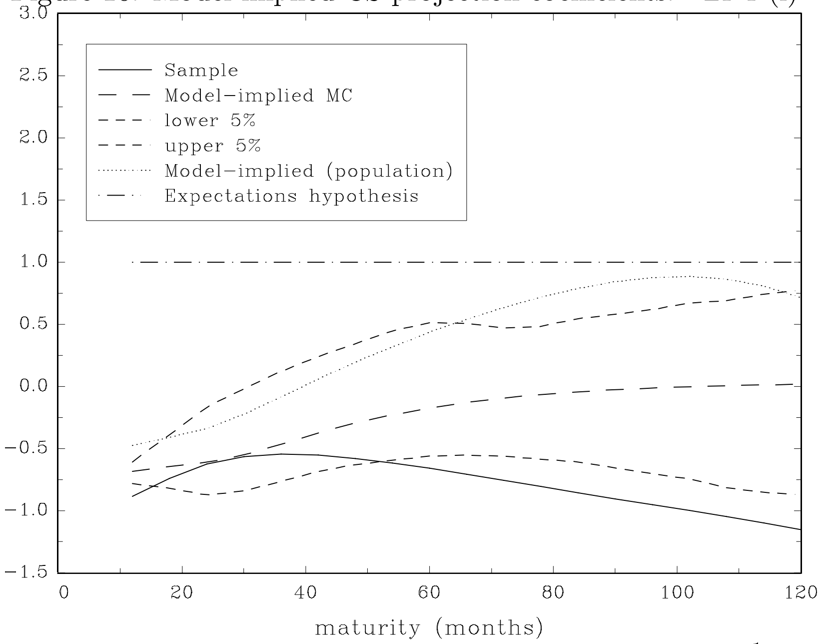
The figure shows the one-month ahead response of the yield premia ω_n, at maturities n up to 120 months, to one standard deviation shocks to the four macro factors. The premia are expressed in annual percentage terms.

Figure 9: Estimated yield premia



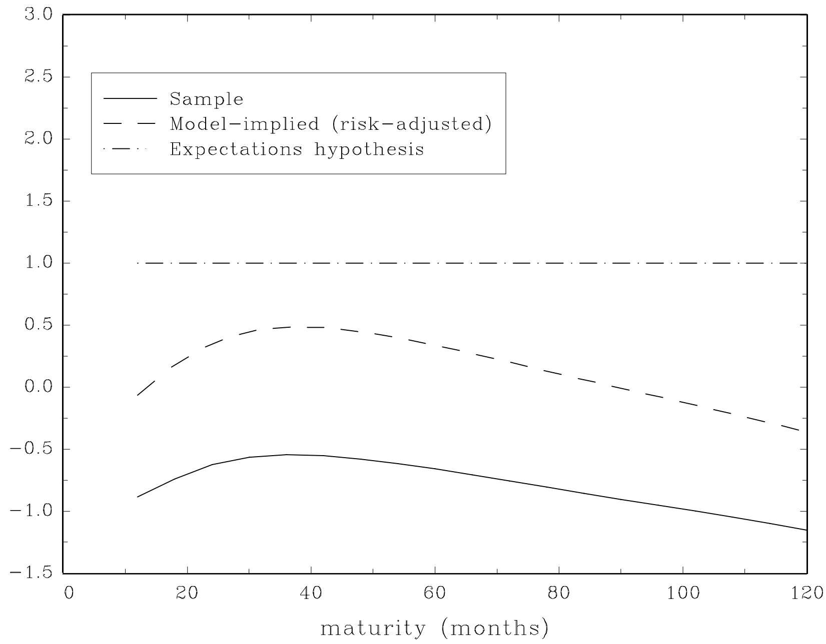
The figure shows the estimated yield premium ω_n during the sample period, for maturities n = 3, 12, 36, and 120 months. The premia are expressed in annual percentage terms.

Figure 10: Model-implied CS projection coefficients: "LPY(i)"



Empirical estimates of the CS long-rate coefficients ϕ_n in $y_{t+1}^{n-1} - y_t^n = \phi_n \left(y_t^n - r_t\right) / (n-1)$, plus corresponding model-implied coefficient values. The "population" coefficients are the theoretical values based on our estimates; the MC coefficients are the mean estimates from 1000 series of the same size as the sample, simulated from our model. The bands around the MC mean estimates are 5% confidence bands.

Figure 11: Model-implied risk-premium adjusted CS coefficients: "LPY(ii)"



The figure shows the estimates of the Campbell and Shiller (1991) long-rate coefficients ϕ_n in the regression $y_{t+1}^{n-1} - y_t^n = \phi_n (y_t^n - r_t) / (n-1)$ for our sample, along with the corresponding risk-premium adjusted model-implied cofficient values based on our parameter estimates.