

Endogenous Fluctuations in Open Economies: The Perils of Taylor Rules Revisited ^{*}

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Abstract

Can active Taylor rules (i.e. monetary rules where the nominal interest rate responds more than proportionally to inflation) deliver global equilibrium uniqueness in small open economies? By studying the local and global dynamics of a standard small open economy we point out the misleading results and policy advices that one would derive from a standard local analysis. We show that rules that guarantee a local unique equilibrium may actually lead the economy into liquidity traps, cycles and chaos. More importantly we find that there is an interesting interaction between the relative risk aversion coefficient and the degree of openness that determines the nature of the global dynamics of the aforementioned economy. In particular, given the relative risk aversion coefficient, we show that the more open the economy is, the more likely is that a contemporaneous rule will drive the economy into a liquidity trap. On the other hand, the more closed the economy is, the more likely is that the same rule will lead to cycles and chaotic dynamics around the inflation target. In contrast for forward-looking rules we find that given the relative risk aversion coefficient, it is more likely that these rules will lead the economy into cycles and chaos, the higher the degree of openness of the economy is.

Although the perils of Taylor rules are evident, the monetary authority can still play a role by at least eliminating cyclical equilibria without giving up its local stability properties. This can be achieved by targeting a high enough inflation level and by being “not too aggressive” with respect to this target, with such relative levels being functions of the “cash dependency” of the economy.

Through a simple calibration exercise, we provide a quantitative evaluation of how feasible and relevant our analytically derived results are for the design of monetary policy. In this sense the theoretical results of this paper might provide some warning for small open economies moving to inflation targeting regimes through interest rates feedback rules and Ricardian fiscal rules.

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1 Introduction

In recent years there has been a revival of theoretical and empirical literature aimed at understanding the macroeconomic consequences of implementing diverse monetary rules in the small open economy. Some examples of this literature are the works by Ball (1999), Svensson (2000), Clarida, Gali and Gertler (1998, 2001), Gali and Monacelli (2002), and Kollmann (2002).¹

In this literature the study of interest rate rules whose interest rate response coefficient to inflation is greater than one has received particular attention. These rules also known as Taylor rules or active rules, imply that in response to a one percent in inflation, the government raises the nominal interest rate by more than one percent leading to an increase in the real interest rate.² To some extent the importance given to these rules in the small open economy literature is just a consequence of some of the benefits that the closed economy literature has claimed for these rules. For instance, Leeper (1991), Bernanke and Woodford (1997) and Clarida, Gali and Gertler (2000) have argued that active interest rate rules guarantee a unique rational expectations equilibrium whereas rules whose interest rate response coefficient to inflation is less than one, also referred as passive rules, induce aggregate instability in the economy by generating multiple equilibria. Although this is an important argument that supports the implementation of active interest rate rules in closed economies, it is not exempt from some drawbacks. In particular Benhabib, Schmitt-Grohé and Uribe (2001a) have pointed out that this argument relies on a local determinacy of equilibrium analysis, that is, on small fluctuations around the inflation target and depends on how money is introduced in the model and on the interaction between fiscal and monetary policy. In addition Benhabib, Schmitt-Grohé and Uribe (2001b, 2002) have also noticed that previous analyses of interest rate rules in closed economies have not taken into account the fact that nominal interest rates are bounded below by zero. Once this zero bound is considered and a non-linear analysis is pursued, they have shown that active interest rate rules may also induce aggregate instability in closed economies by generating cycles, chaotic dynamics or liquidity traps (deflationary paths).

Taking into consideration these results of the literature for active interest rate rules in closed economies, it is possible to argue that its counterpart for open economies has been overlooking two important elements of the analysis. First, it has disregarded the fact that active rules may also lead to aggregate instability in the open economy by generating local multiple equilibria under conditions that are not a simple extension of the conditions in the closed economy literature. In other words this literature has paid little attention to the fact that depending on some particular features of the open economy, active rules may embark the open economy on fluctuations that are determined not only by fundamentals but also by self-fulfilling expectations. Second, the observation emphasized by Benhabib et al. for the closed economy literature of active Taylor rules also applies to the open economy literature. In other words, the studies for open economies have restricted their analysis to local dynamics and not to global dynamics, and some of the works have not considered the zero bound on the nominal interest rate.

With respect to the first element, Zanna (2003a) and Airaudo and Zanna (2003) have pursued local equilibrium analyses for interest rate rules in small open economies. They have shown that conditions under which active interest rate rules induce multiple equilibria in the small open economy depend not only

¹See also Ghironi (2002), Ghironi and Rebucci (2001), Devereux and Lane (2003), and Lubik and Schorfheide (2003).

²See Taylor (1993) and Henderson and McKibbin (1993).

on the interest rate response coefficient to inflation but also on some specific characteristics of the open economy that are not present in the closed counterpart. In particular Zanna(2003a) finds that some of these characteristics are the degree of openness of the economy and the degree of exchange rate pass-through.³ He argues that more open economies and economies with a higher degree of exchange rate pass-through are prone to suffer of aggregate instability due to the presence of multiple equilibria generated by active interest rate rules that respond to the CPI-inflation. On the other hand, Airaudo and Zanna (2003) have shown that forward-looking interest rate rules may generate endogenous fluctuations in the small open economy due to Hopf bifurcations. In their model the bifurcation parameter corresponds to the interest rate response coefficient to the weighted average of expected future CPI-inflation. However they find that there exists an interesting interaction between this coefficient, the weight that the monetary authority puts on expected future inflation in the rule and the degree of openness of the economy. This interaction determines how likely Hopf bifurcations are in their model.

The second missing element of the analysis of active interest rate rules in open economies is what motivates the present paper. In fact this paper is one of the first attempts of the open economy literature to understand how interest rate rules may lead to global endogenous fluctuations. We pursue a global and non-linear equilibrium analysis for a traditional small open economy model with traded and non-traded good, whose government follows an active Taylor rule with respect to the CPI-inflation. We show that the global equilibrium dynamics of the model induced by this rule varies with the level of some structural parameters of the economy such as the degree of openness, measured as the share of traded goods, and the relative risk aversion coefficient. In particular, we find that under both contemporaneous and forward looking Taylor rules the economy might display monotonic deflationary paths, cycles and chaotic dynamics around both the active and the passive steady state. These dynamics are possible, even for rules that under a local analysis guarantee a unique equilibrium. With respect to the closed economy work of Benhabib et al. (2002) we obtain a richer set of dynamics. For instance, given the coefficient of relative risk aversion, we show that the more open the economy is, the more likely is that a contemporaneous active rule will drive the economy into a liquidity trap. On the other hand, the more closed the economy is, the more likely is that the same rule will lead to cycles and chaotic dynamics around the inflation target. In contrast for forward-looking rules we find that given the relative risk aversion coefficient, it is more likely that these rules will lead the economy into cycles and chaos, the more open the economy is.

Although the perils of Taylor rules are evident, the monetary authority can still play a role by at least eliminating cyclical equilibria without giving up its local stability properties. This can be achieved by targeting a high enough inflation level and by being “not too aggressive” with respect to this target, with such relative levels being functions of the “cash dependency” of the economy. In this sense monetary policy can be used as the only tool to completely eliminate endogenous fluctuations without resorting to specific fiscal rules. The latter might instead be used to avoid the risk of deflationary paths.

In principle more “cash dependent” economies that follow the appropriate contemporaneous rule might be able to completely eliminate endogenous fluctuations. However this contrasts with less “cash dependent” economies that follow forward-looking rules in which the appearance of cycles and chaotic dynamics seems to be pervasive.

³See also Linnemann and Schabert (2002) and De Fiore and Liu (2003) that also discuss the importance of the degree of openness of the economy in the determinacy of equilibrium analysis.

Through a simple calibration exercise, we provide a quantitative evaluation of how feasible and relevant our analytically derived results are for the design of monetary policy. Furthermore we also discuss how changing the target of inflation from the CPI-inflation to the non-traded goods inflation affects the previous results.

We believe the results of our paper may be interesting for two reasons. First, as is well known there exists an unanimous consensus about the benefits of the framework of inflation targeting through interest rate feedback rules. In this sense the theoretical results of this paper might provide some warning about some possible negative consequences for small open economies moving to this framework. The message that we want to convey is that some specifications of the aforementioned rules may generate endogenous fluctuations and therefore aggregate instability in the economy. This implies that further research in this area is needed.

Second our results point out the importance of considering particular features of the open economy in the design of the monetary policy. In particular this paper emphasizes the relevant role that the degree of openness of the economy plays not only in the local equilibrium analysis but also in the global equilibrium analysis. Clearly the degree of openness of the economy, measured in our model as the share of traded goods, is a characteristic of an open economy that is not present in previous closed economy models. More importantly this feature of the open economy varies among economies that follow (or followed) active contemporaneous or forward-looking interest rate rules as Table 1 shows.⁴

Table 1:

Country	Degree of Openness Imports/GDP	Type of Rule	ρ_π	Study
Germany	0.26	<i>Forward – Looking</i>	1.31	Clarida, Gali and Gertler (1998)
France	0.22	<i>Forward – Looking</i>	1.13	Clarida, Gali and Gertler (1998)
Japan	0.10	<i>Forward – Looking</i>	2.04	Clarida, Gali and Gertler (1998)
United Kingdom	0.28	Contemporaneous	1.84	Lubik and Schorfheide (2003)
Australia	0.19	Contemporaneous	2.10	Lubik and Schorfheide (2003)
Canada	0.31	Contemporaneous	2.24	Lubik and Schorfheide (2003)
New Zealand	0.28	Contemporaneous	2.49	Lubik and Schorfheide (2003)
Costa Rica	0.42	<i>Forward – Looking</i>	1.47	Corbo (2000)
Colombia	0.20	Contemporaneous	1.31	Zanna (2003b)
Chile	0.28	<i>Forward – Looking</i>	1.39	Restrepo (1999)

Note: ρ_π is the interest rate response coefficient to the CPI-inflation in the rule.

Data from IFS was used to calculate the Imports/GDP share.

The remainder of this paper is organized as follows. Section 2 presents a flexible-price model with its

⁴In this table we measure the degree of openness of the economy as the share of imported goods. We also present some of the estimates of contemporaneous and forward-looking rules that have been done for some of the economies. We borrow the estimates from Clarida, Gali and Gertler (1998), Restrepo (1999), Corbo (2000), Lubik and Schorfheide (2003), and Zanna(2003). The share of imports goods was calculated as the annual average of this share for the respective period of time used for the aforementioned estimations.

main assumptions. Section 3 defines the equilibrium concept we refer to. Section 4 pursues a local and a global equilibrium analyses for an active contemporaneous interest rate rule. Section 5 does the same analyses for an active forward looking rule. Section 6 presents a sensitivity analysis for the previous results under changes of the inflation target, the degree of aggressiveness of the rule and the importance of money in our model. Section 7 analyzes the role of cash in providing transaction services and argues that there is still some role for monetary policy to eliminate cyclical fluctuations without any help from the fiscal side. Section 8 discusses briefly the implications in terms of our previous results of adopting a backward-looking rule or of targeting the non-traded goods inflation in stead of the CPI-inflation. Finally Section 9 concludes.

2 A Flexible-Price Model

2.1 The Household-Firm Unit

Consider a small open economy populated by a large number of infinitely lived household-firm units with preferences described by the following intertemporal utility function⁵

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\left[(c_t^T)^{\alpha\gamma} (c_t^N)^{(1-\alpha)\gamma} \left(\frac{M_t^d}{P_t^T} \right)^{1-\gamma} \right]^{1-\sigma}}{1-\sigma} + \psi(1 - h_t^T - h_t^N) \right\} \quad (1)$$

where $\alpha, \beta, \gamma \in (0, 1)$, and $\psi, \sigma > 0$; E corresponds to the expectation operator, c_t^T and c_t^N denote the consumption of traded and non-traded goods in period t respectively, M_t^d denotes nominal money balances, P_t^T denotes the price level of the traded good, and h_t^T and h_t^N are the labor allocated to the production of the traded good and the non-traded good respectively. Equation (1) implies that the household-firm unit derives utility from consuming traded and non-traded goods, from the liquidity services of money and from not working in either sector.

The representative household-firm unit only requires labor for the production of traded and non-traded goods. It makes use of following instantaneous production technologies

$$y_t^T = (h_t^T)^{\theta_T} \quad \text{and} \quad y_t^N = (h_t^N)^{\theta_N} \quad (2)$$

where $0 < \theta_T < 1$ and $0 < \theta_N < 1$.

Before we continue with the description of the model it is worth pointing out that we have introduced money in the utility function but we have not imposed any restrictions in terms of the relationship between real money balances and consumption. In other terms denoting c as the aggregate consumption, $c = (c_t^T)^{\alpha} (c_t^N)^{(1-\alpha)}$, we will consider the case in which real money balances and consumption are Edgeworth substitutes, $U_{cm} < 0$ and the case in which they are complements, $U_{cm} > 0$. In our model these cases are in turn determined by the value of the parameter σ that corresponds to the relative risk aversion coefficient. Namely if $\sigma > 1$ ($\sigma < 1$) then real money balances and consumption are Edgeworth substitutes (complements).

⁵In this paper specific functional forms are assumed to be able to convey the main message of this paper.

In addition it is important to observe that recent studies about interest rate rules in closed economies, such as Benhabib et al. (2001a,b, 2002), have analyzed the consequences in the equilibrium analysis of introducing money-in-the-production-function (MIPF). As noted by Feenstra (1986) models of money in the production function are isomorphic to a money-in-the-utility-function (MIUF) endowment economy with $U_{cm} < 0$. Therefore we expect that our results for the MIUF in which real money balances and consumption are Edgeworth substitutes, $U_{cm} < 0$, would be similar to those derived on a MIPF set-up.

We assume that the law of one price holds for the traded good and to simplify the analysis we normalize the foreign price of the traded good to one. Therefore, the domestic currency price of traded goods (P_t^T) is equal to the nominal exchange rate (\mathcal{E}_t). That is $P_t^T = \mathcal{E}_t$. This simplification in tandem with (1) can be used to derive the consumer price index (CPI)

$$p_t = \frac{(\mathcal{E}_t)^\alpha (P_t^N)^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \quad (3)$$

Using equation (3) and defining the gross nominal devaluation rate as

$$\epsilon_t = \mathcal{E}_t / \mathcal{E}_{t-1} \quad (4)$$

it is straightforward to derive the gross CPI-inflation rate, π_t , as a weighted average of the gross nominal depreciation rate, ϵ_t , and the gross inflation of the non-traded goods, $\pi_t^N = P_t^N / P_{t-1}^N$; that is

$$\pi_t = \epsilon_t^\alpha (\pi_t^N)^{(1-\alpha)} \quad (5)$$

where the weights are related to the share of traded goods α . This share can be seen as a measure of openness of the economy. As α goes to zero ($\alpha \rightarrow 0$) we regard the economy as a closed economy; whereas if α tends to one ($\alpha \rightarrow 1$) then we consider the economy as a very open one.

We define the real exchange rate (e_t) as the ratio between the price of traded goods and the aggregate price of non-traded goods.

$$e_t = \mathcal{E}_t / P_t^N \quad (6)$$

From this definition of the real exchange rate we deduce that

$$e_t = e_{t-1} \left(\frac{\epsilon_t}{\pi_t^N} \right) \quad (7)$$

In order to avoid the “unit-root” problem in the local determinacy of equilibrium analysis we assume complete financial markets.⁶ That is household-firm units have access to a complete set of contingent claims traded internationally. In each period $t \geq 0$ the agents can purchase two types of financial assets: fiat money M_t^d and a nominal state contingent claims, D_{t+1} , that pay one unit of currency in a specified state of period $t + 1$.

⁶The “unit-root” problem arises in small open economy models by the popular assumption of making the subjective discount factor (β) constant and equal to the factor $\frac{1}{1+r_t}$, that depends on the international interest rate (r_t). This assumption introduces a random walk in equilibrium consumption making the steady state dependent on the initial stock of wealth. As a result the presence of a unit root in a dynamical system implies that it is not valid to apply the common technique of linearizing the system around the steady state and studying the eigenvalues of the Jacobian matrix in order to characterize local determinacy. See Azariadis (1993). Complete markets is one of the possible approaches to solve the “unit-root” problem. See Schmitt-Grohé and Uribe (2003).

Under the assumption of complete markets the representative agent's flow constraint each period can be written as⁷

$$M_t^d + E_t Q_{t,t+1} D_{t+1} \leq W_t + \mathcal{E}_t y_t^T + P_t^N y_t^N - \mathcal{E}_t \tau_t - \mathcal{E}_t c_t^T - P_t^N c_t^N \quad (8)$$

where $Q_{t,t+1}$ refers to the period- t price of a claim to one unit of currency delivered in a particular state of period $t + 1$ divided by the probability of occurrence of that state and conditional of information available in period t . Hence $E_t Q_{t,t+1} D_{t+1}$ denotes the cost of all contingent claims bought at the beginning of period t . Constraint (8) says that the total end-of-period nominal value of the financial assets can be worth no more than the value of the financial wealth brought into the period, W_t , plus non-financial income during the period net of the value of taxes, $\mathcal{E}_t \tau_t$, and the value of consumption spending.

To derive the period-by-period budget constraint of the representative agent, it is important to notice that total beginning-of-period wealth in the following period is given by

$$W_{t+1} = M_t^d + D_{t+1} \quad (9)$$

and that $E_t Q_{t,t+1}$ corresponds to the price at period t of a claim that pays one unit of currency in every state in period $t + 1$ and represents the inverse of the risk-free gross nominal interest rate, R_t ; that is

$$E_t Q_{t,t+1} = \frac{1}{R_t} \quad (10)$$

Then we can use equations (8), (9) and (10) to derive the budget constraint of the representative agent as

$$E_t Q_{t,t+1} W_{t+1} \leq W_t + \mathcal{E}_t y_t^T + P_t^N y_t^N - \mathcal{E}_t \tau_t - \frac{R_t - 1}{R_t} M_t^d - \mathcal{E}_t c_t^T - P_t^N c_t^N \quad (11)$$

who is also subject to a Non-Ponzi game condition described by

$$\lim_{j \rightarrow \infty} E_t q_{t+j} W_{t+j} \geq 0 \quad (12)$$

at all dates and under all contingencies. The variable q_t represents the period-zero price of one unit of currency to be delivered in a particular state of period t divided by the probability of occurrence of that state, given information available at time 0. It is given by

$$q_t = Q_1 Q_2 \dots Q_t$$

with $q_0 \equiv 1$.

The problem of the representative household-firm unit is reduced to choose the sequences $\{c_t^T, c_t^N, h_t^T, h_t^N\}$ and $\{M_t^d, W_{t+1}\}_{t=0}^{\infty}$ in order to maximize (1) subject to (2), (11) and (12), and given W_0 , and the time paths for $i_t, \mathcal{E}_t, P_t^N, Q_{t+1}$ and τ_t . Note that since the utility function specified in (1) implies that the preferences of the agent display non-satiation then constraints (11) and (12) both hold with equality.

The first order conditions correspond to (11) and (12) with equality and

$$\alpha \gamma (c_t^T)^{\alpha \gamma (1-\sigma) - 1} (c_t^N)^{(1-\alpha) \gamma (1-\sigma)} \left(\frac{M_t^d}{\mathcal{E}_t} \right)^{(1-\gamma)(1-\sigma)} = \lambda_t \quad (13)$$

⁷We follow Woodford (2003) to construct the budget constraint of the representative agent under the assumption of complete markets.

$$\frac{\alpha c_t^N}{(1-\alpha)c_t^T} = e_t \quad (14)$$

$$\lambda_t \theta_T (h_t^T)^{(\theta_T-1)} = \psi \quad (15)$$

$$\lambda_t \theta_N (h_t^N)^{(\theta_N-1)} = \psi e_t \quad (16)$$

$$\frac{M_t^d}{\mathcal{E}_t} = \left(\frac{1-\gamma}{\alpha\gamma} \right) \left(\frac{R_t}{R_t-1} \right) c_t^T \quad (17)$$

$$\frac{\lambda_t}{\mathcal{E}_t} Q_{t,t+1} = \frac{\lambda_{t+1}}{\mathcal{E}_{t+1}} \beta \quad (18)$$

where λ_t/\mathcal{E}_t corresponds to the multiplier of the budget constraint.

The interpretation of the first order conditions is straightforward. Equation (13) is the usual intertemporal envelope condition that makes the marginal utility of consumption of traded goods equal to the marginal utility of wealth (λ_t). Condition (14) implies that the marginal rate of substitution between traded and non-traded goods must be equal to the real exchange rate. In addition, from equations (15) and (16) it is possible to derive the following expression

$$P_t^N \theta_N (h_t^N)^{(\theta_N-1)} = \mathcal{E}_t \theta_T (h_t^T)^{(\theta_T-1)}$$

that equalizes the value of the marginal products of labor in both sectors. Equation (17) represents the demand for real balances of money as an increasing function of consumption of traded goods and a decreasing function of the risk-free gross nominal interest rate. And finally condition (18) implies a standard pricing equation for one-step-ahead nominal contingent claims. Note that under complete markets in each period t there is one condition of this type for each possible state in period $t+1$.

2.2 The Government

The government issues two nominal liabilities: money, M_t^s , and a domestic bond, B_t^s , that pays a gross free-risk nominal interest rate R_t . We assume that it cannot issue or hold state contingent assets. It also levies taxes, τ_t , pays interest on its debt, $(R_t-1)B_t^s$, and receives revenues from seigniorage.

To derive the budget constraint of the government we proceed as follows. Let L_t^s denote the nominal government liabilities at the beginning of period t . In the financial market of period t the government issues money and bonds to finance these liabilities. Therefore

$$L_t^s = M_t^s + B_t^s$$

Using this definition and the aforementioned assumptions about the behavior of the government we can write its budget constraint as

$$L_t^s = R_{t-1} L_{t-1}^s - (R_{t-1} - 1) M_{t-1}^s - \mathcal{E}_t \tau_t \quad (19)$$

We assume that the government follows a Ricardian fiscal policy. Under this policy, it picks the path of taxes, τ_t , satisfying the intertemporal version of (19) in conjunction with the transversality condition

$$\lim_{t \rightarrow \infty} \frac{L_t^s / \mathcal{E}_t}{\prod_{k=0}^t \left(\frac{R_{t-1}}{\epsilon_t} \right)^k} = 0 \quad (20)$$

Finally we define the monetary policy as an interest rate feedback rule or Taylor rule whereby the government can set the nominal interest rate, R_t , as an increasing function of either the CPI-inflation rate between periods $t-1$ and t , π_t , or the CPI-inflation rate between periods t and $t+1$, π_{t+1} . Hence the first rule corresponds to a contemporaneous rule whereas the second one corresponds to a forward-looking rule. For analytical and computational purposes we will focus on the following specific parametrization

$$R_t = \rho(\pi_{t+j}) \equiv 1 + (R^* - 1) \left(\frac{\pi_{t+j}}{\pi^*} \right)^{\frac{A}{R^* - 1}}; \quad \text{with } j = 0, 1 \text{ and } R^* = \pi^* / \beta \quad (21)$$

where π^* corresponds to the target rate of the CPI-inflation. We will assume that the government responds aggressively to inflation. That is, at the inflation rate target the rule satisfies $\frac{\rho'(\pi^*)\pi^*}{\rho(\pi^*)} = \frac{A}{R^*} > 1$.

It is important to observe that the interest rate rule is a continuous and non-decreasing function in the CPI-inflation rate. In addition it satisfies the zero bound on the nominal interest rate, $R_t = \rho(\pi_{t+1}) > 1$.

3 The Equilibrium

In equilibrium the money market and the non-traded goods market clear. Therefore

$$M_t^d = M_t^s = M_t \quad (22)$$

and

$$y_t^N = (h_t^N)^{\theta_N} = c_t^N \quad (23)$$

We also assume free capital mobility. This implies that the following non-arbitrage condition must hold

$$Q_{t,t+1}^* = Q_{t,t+1} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \quad (24)$$

where $Q_{t,t+1}^*$ refers to the period- t foreign currency price of a claim to one unit of foreign currency delivered in a particular state of period $t+1$ divided by the probability of occurrence of that state and conditional of information available in period t . Furthermore under the assumption of complete markets we also have a similar condition to (18) in the rest of the world. That is,

$$\frac{\lambda_t^*}{P_t^{T^*}} Q_{t,t+1}^* = \frac{\lambda_{t+1}^*}{P_{t+1}^{T^*}} \beta^* \quad (25)$$

where λ_t^* represents the marginal utility of wealth for the rest of the world, $P_t^{T^*}$ is the foreign price of traded goods, that we assumed to be normalized to one, and β^* denotes the subjective discount rate of the rest of the world. Using (18), (24), (25), the Law of One Price for traded goods and the assumption that $\beta^* = \beta$ we can derive the following equation

$$\frac{\lambda_{t+1}^*}{\lambda_t^*} = \frac{\lambda_{t+1}^*}{\lambda_t^*}$$

that holds at all dates and under all contingencies. This equation implies that the domestic marginal utility of wealth is proportional to its foreign counterpart. Then

$$\lambda_t = \xi \lambda_t^*$$

where ξ refers to a constant parameters that determines the wealth difference across countries. Since we are dealing with a small open economy, λ_t^* can be taken as an exogenous variable. To simplify the analysis we assume that λ_t^* is constant and equal to λ^* . This assumption implies that λ_t becomes a constant. That is

$$\lambda_t = \lambda = \xi \lambda^* \tag{26}$$

This result allows us to write condition (18) as

$$Q_{t,t+1} = \frac{\mathcal{E}_t}{\mathcal{E}_{t+1}} \beta = \frac{\beta}{\epsilon_{t+1}}$$

that together with (10) imply that

$$R_t = \beta^{-1} \left[E_t \frac{1}{\epsilon_{t+1}} \right]^{-1} \tag{27}$$

where E denotes the expectation operator. Note that condition (27) is very similar to the uncovered interest parity condition.

It is important to observe that to pursue the determinacy of equilibrium analysis, it is sufficient to focus on a perfect foresight equilibrium. Assuming that the representative agent forecasts correctly all the anticipated variables, we can write condition (27) as

$$R_t = \beta^{-1} \epsilon_{t+1} \tag{28}$$

that corresponds to the typical uncovered interest parity condition.

We proceed giving the definition of a perfect foresight equilibrium for a government that pursues a Ricardian fiscal policy and follows an interest rate feedback rule.

Definition 1 *Given, L_0 and π^* , a Perfect Foresight Equilibrium under a Ricardian fiscal policy is defined as a set of sequences $\{c_t^T, c_t^N, h_t^T, h_t^N, M_t, \lambda_t, L_{t+1}, \tau_t, e_t, \mathcal{E}_t, \epsilon_t, \pi_t, \pi_t^N, R_t\}_{t=0}^\infty$ satisfying definitions (4), (5), (7), the first order conditions of the representative agent (13), (14), (15), (16), (17), the intertemporal version of (19) together with (20), the rule defined by (21), the market clearing conditions (22), (23), and conditions (26), and (28).*

To pursue the equilibrium analysis the model can be further reduced. It is important to observe that we do not need to consider in the analysis equations (15), (16), (17), (19) and (20). The reasons are the followings. Under the assumption that the fiscal policy is Ricardian, we know that the intertemporal version of the government's budget constraint in conjunction with its transversality condition will be always satisfied. Moreover once we determine the paths for the risk-free gross nominal interest rate and consumption of traded goods we can use conditions (17) and (22) to obtain the sequence of the stock of money. Utilizing the market clearing condition for non-traded goods and conditions (15), (16), and (26) allows us to find out the paths for labor allocated in the traded sector and labor allocated in the non-traded sector once we determine the paths for the real exchange rate and consumption of non-traded goods.

Using definitions (5) and (7), conditions (13), (14), (16), (23), (26) and (28), and the contemporaneous version of the monetary rule (21) we can derive the following difference equation that summarizes and represents the dynamics of our model under contemporaneous interest rate rules

$$\left(\frac{R_{t+1}}{R_{t+1}-1}\right)^\chi \left(\frac{R_{t+1}-1}{R^*-1}\right)^{\frac{R^*-1}{A}} = \frac{R_t}{R^*} \left(\frac{R_t}{R_t-1}\right)^\chi \quad (29)$$

where

$$\chi = \frac{(1-\alpha)(1-\gamma)(1-\sigma)(1-\theta_N)}{\{(1-\gamma) + \gamma[\theta_N(1-\alpha) + \alpha]\}(1-\sigma) - 1} \quad (30)$$

On the other hand, using definitions (5) and (7), conditions (13), (14), (16), (23), (26) and (28), and the forward-looking version of the monetary rule (21) we can derive the following difference equation that summarizes and represents the dynamics of our model under forward-looking interest rate rules

$$\left(\frac{R_{t+1}}{R_{t+1}-1}\right)^\chi = \frac{R_t}{R^*} \left(\frac{R^*-1}{R_t-1}\right)^{\frac{R^*-1}{A}} \left(\frac{R_t}{R_t-1}\right)^\chi \quad (31)$$

where χ was defined in (30).

Using definitions (5) and (7), the interest rate rule (21) and equation (29), or equation (31), it is straightforward to notice that in steady-state and regardless of the type of rule under analysis (contemporaneous or forward-looking) we have

$$\bar{\pi}^N = \bar{\epsilon} = \bar{\pi}$$

$$\bar{R} = \rho(\bar{\pi}) = \bar{\pi}/\beta \quad (32)$$

Figure 1 depicts the left- and the right-hands side of equation $\rho(\pi) = \pi/\beta$ using the particular functional form (21). From this figure it is clear that there are two steady states. Under the first one the steady-state CPI-inflation rate corresponds to $\bar{\pi} = \pi^*$ and the slope of the interest rate rule is $\rho'(\pi^*) = \frac{A}{\pi^*}$, which is greater than $\frac{1}{\beta}$. Therefore, at this steady state the interest rate rule satisfies $\frac{\rho'(\pi^*)\pi^*}{\rho(\pi^*)} = \frac{\beta A}{\pi^*} > 1$ and following Leeper (1991) we say that the monetary policy rule is “active”. That is, in response to a one percent increase in the CPI-inflation, the government raises the nominal interest rate by more than one percent leading to an increase in the real interest rate. On the other hand, under the second steady state, the CPI-inflation rate corresponds to $\bar{\pi} = \pi^L$ and the slope of the interest rate rule is $\rho'(\pi^L) = \frac{A}{\pi^L}$, which is less than $\frac{1}{\beta}$. Hence at this steady state the interest rate rule satisfies $\frac{\rho'(\pi^L)\pi^L}{\rho(\pi^L)} = \frac{A}{\pi^L} < \frac{1}{\beta}$ and following Leeper (1991) we say that the monetary policy rule is “passive”. In this case in response to a one percent increase in the CPI-inflation, the government raises the nominal interest rate by less than one percent leading to a decrease in the real interest rate.

The existence of two steady states is crucial for our dynamics results.⁸ In addition it is important to observe that the steady state equation (32) in this small open economy model corresponds to the same steady state equation that arises in closed economy models such as in Benhabib, Schmitt-Grohé and Uribe (2002). However this does not imply that the dynamics of the small economy model under interest rate rules

⁸For a more formal proof of the steady state multiplicity see the Appendix.

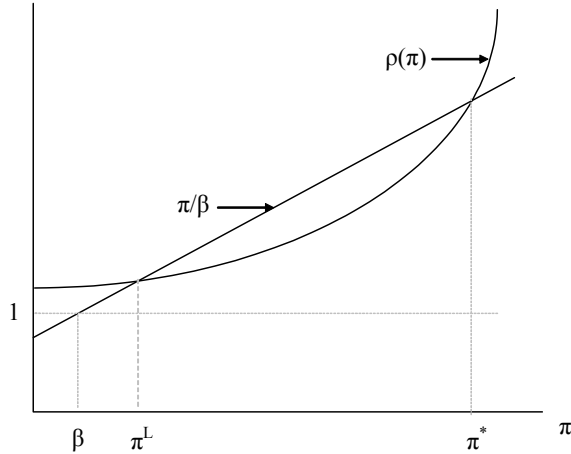


Figure 1: Multiplicity of steady states. π^* represents the target inflation rate or the inflation rate in the active steady-state whereas π^L represents the inflation rate in the passive steady state.

must be equal to the dynamics of the closed economy models. For instance the equilibrium dynamics of the contemporaneous rule model are driven by equation (29) that differs from the equation derived in Benhabib et al. in the exponent χ (defined in (30)). In their model this exponent is always positive. In our model this exponent can be positive or negative and depends on some additional structural parameters of the small open economy such as the degree of openness of the economy, α . This fact leads to much richer dynamics as we will discuss below.

For the above reasons, it is helpful to summarize how the “driving” parameter χ is affected by the relative risk aversion coefficient, σ , and by the degree of openness, α , since we will be specifically considering environments differing with respect to those coefficients.⁹

To simplify the exposition, let $\chi(\alpha, \sigma) = \frac{(1-\theta_N)(1-\gamma)(1-\alpha)(\sigma-1)}{1+(\sigma-1)\varpi(\alpha)}$ where $\varpi(\alpha) = [\gamma(\alpha + (1-\alpha)\theta_N) + (1-\gamma)] \in (0, 1)$. We study the function $\chi(\alpha, \sigma)$. It is straightforward to prove the following facts:

Fact 1: $\chi(1, \sigma) = \chi(\alpha, 1) = 0$.

Fact 2: if $\sigma \in (0, 1)$ then $\chi(\alpha, \sigma) < 0$ for any $\alpha \in (0, 1)$.

Fact 3: if $\sigma > 1$ then $\chi(\alpha, \sigma) > 0$ for any $\alpha \in (0, 1)$.

Fact 4: $\frac{\partial \chi(\alpha, \sigma)}{\partial \sigma} = \frac{(1-\gamma)(1-\theta_N)(1-\alpha)}{[1+(\sigma-1)\varpi(\alpha)]^2} > 0$ for any $\alpha \in (0, 1)$.

⁹The parameter χ also depends on the relative importance of real money balances in transactions, γ , and on the non-traded sector production parameter θ_N . However we will not pursue any “bifurcation of equilibria” analysis with respect to them. The message we want to convey in this paper is that the degree of openness in economies pursuing CPI-inflation targeting can play a big role. As it turns out, such effect is present or not depending on the relative risk aversion coefficient. Of the remaining parameters the share of real money balances in the utility function affects the nature of equilibria (more on this below). We will pursue a sensitivity analysis with respect to this parameter.

Fact 5: if $\sigma > 1$, then $\frac{\partial \chi(\alpha, \sigma)}{\partial \alpha} = -\frac{\sigma(1-\gamma)(1-\theta_N)(\sigma-1)}{[1+(\sigma-1)\varpi(\alpha)]^2} < 0$.

Fact 6: if $\sigma \in (0, 1)$, then $\frac{\partial \chi(\alpha, \sigma)}{\partial \alpha} = -\frac{\sigma(1-\gamma)(1-\theta_N)(\sigma-1)}{[1+(\sigma-1)\varpi(\alpha)]^2} > 0$.

Facts 2 and 3 make the point that the sign of χ depends on the relative risk-aversion coefficient. The remaining facts describe how χ is affected by this coefficient and by the degree of openness of the economy.

4 The Equilibrium Analysis Under a Contemporaneous Taylor Rule

In this paper, we focus on rules that are active at the intended target steady state inflation rate, π^* . For the time being passive rules will not be at the center of our discussion. For both the contemporaneous and the forward looking policy set-ups, the equilibrium analysis is made of two parts. First we will check if active interest rate rules are sufficient for the local determinacy of equilibrium of the target steady state. Active rules have been strongly advocated by monetary economics academics as being simple, transparent and above all stabilizing, in particular for what concerns closed economies. Then, we will question the robustness of local results by studying the full global dynamics of the model.

We will start by analyzing active contemporaneous interest rate rules with respect to the CPI-inflation. In order to motivate them we remember the estimations by Lubik and Schorfheide (2003) of contemporaneous rules for United kingdom, Canada, Australia and New Zealand.

4.1 Local Analysis

From log-linearizing equation (29) around the target inflation rate we obtain

$$\hat{R}_{t+1} = \left(\frac{1 - \frac{\chi}{R^* - 1}}{\frac{R^*}{A} - \frac{\chi}{R^* - 1}} \right) \hat{R}_t \quad (33)$$

The following proposition summarizes the local determinacy of equilibrium analysis for contemporaneous rules.

Proposition 1 *Suppose the government follows an active contemporaneous interest rate rule given by $R_t = \rho(\pi_t)$ with $\frac{\rho'(\pi^*)\pi^*}{\rho(\pi^*)} = \frac{A}{R^*} > 1$, with $R^* = \frac{\pi^*}{\beta}$, and let χ be defined as in (30).*

1. *If $\sigma \in (0, 1)$ then the model displays a unique local equilibrium.*
2. *Assume that $\sigma > 1$. If $\chi < \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A}\right)$ then the model displays a unique equilibrium. On the other hand, if $\chi > \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A}\right)$ then the model displays multiple equilibria.*

Proof. To prove this proposition we use (33). For 1 note that if $\sigma < 1$ then from **Fact 2**, we conclude that $\chi < 0$. This result and the zero bound on the nominal interest rate imply that $\frac{\chi}{R^* - 1} < 0$. This inequality and the assumption of an active rule, that is $\frac{A}{R^*} > 1$, help us to see that $\left(\frac{1 - \frac{\chi}{R^* - 1}}{\frac{R^*}{A} - \frac{\chi}{R^* - 1}}\right) > 1$. But this means that the mapping (33) becomes explosive. This feature of the mapping in conjunction with the fact that R_t

is a non-predetermined variable imply that there exists a unique equilibrium that corresponds to the active steady state.

For 2 note that if $\sigma > 1$ then from **Fact 3** we derive that $\chi > 0$. This result and the zero bound on the nominal interest rate imply that $\frac{\chi}{R^*-1} > 0$. Since the rule is active, $\frac{R^*}{A} < 1$, we have to consider three exclusive cases for the possible values of $\frac{\chi}{R^*-1}$. **Case a:** $\frac{\chi}{R^*-1} > 1$; **Case b:** $\frac{\chi}{R^*-1} < \frac{R^*}{A}$ and **Case c:** $\frac{R^*}{A} < \frac{\chi}{R^*-1} < 1$. We proceed by analyzing each case.

For **case a**, since $\frac{\chi}{R^*-1} > 1$ and $\frac{R^*}{A} < 1$, then $\frac{R^*}{A} < 1 < \frac{\chi}{R^*-1}$. These inequalities in turn imply that $0 < \left(\frac{1 - \frac{\chi}{R^*-1}}{\frac{R^*}{A} - \frac{\chi}{R^*-1}} \right) < 1$ which means that the mapping (33) becomes non-explosive. This feature of the mapping in conjunction with the fact that R_t is a non-predetermined variable imply that there exist multiple equilibria in which R_t converges asymptotically to its steady state.

For **case b**, since $\frac{\chi}{R^*-1} < \frac{R^*}{A}$ and $\frac{R^*}{A} < 1$, then we can derive that $\frac{\chi}{R^*-1} < \frac{R^*}{A} < 1$. These inequalities in turn imply that $\left(\frac{1 - \frac{\chi}{R^*-1}}{\frac{R^*}{A} - \frac{\chi}{R^*-1}} \right) > 1$, which means that the mapping (33) becomes explosive and by a similar analysis to the one before we conclude that the model displays a unique equilibrium.

Finally for **case c**, since $\frac{R^*}{A} < \frac{\chi}{R^*-1} < 1$ then we can conclude that $\left(\frac{1 - \frac{\chi}{R^*-1}}{\frac{R^*}{A} - \frac{\chi}{R^*-1}} \right) < 0$. Moreover if $\chi > \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A} \right)$ then $0 < 1 - \frac{\chi}{R^*-1} < -\frac{R^*}{A} + \frac{\chi}{R^*-1}$, which in turn implies that $-1 < \left(\frac{1 - \frac{\chi}{R^*-1}}{\frac{R^*}{A} - \frac{\chi}{R^*-1}} \right) < 0$. Therefore the mapping (33) becomes non-explosive and the model displays multiple equilibria. On the other hand, if $\chi < \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A} \right)$ then it is straightforward to prove that this inequality implies that $\left(\frac{1 - \frac{\chi}{R^*-1}}{\frac{R^*}{A} - \frac{\chi}{R^*-1}} \right) < -1$. Hence the aforementioned mapping becomes explosive and the model displays a unique equilibrium.

Summarizing, we have that for $\sigma > 1$, if $\frac{R^*}{A} < 1 < \frac{\chi}{R^*-1}$ or $\frac{R^*}{A} < \frac{\chi}{R^*-1} < 1$ with $\chi > \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A} \right)$, then there exists multiple equilibria. On the other hand, if $\frac{\chi}{R^*-1} < \frac{R^*}{A} < 1$ or $\frac{R^*}{A} < \frac{\chi}{R^*-1} < 1$ with $\chi < \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A} \right)$ then there exist a unique equilibrium. Hence part 2 of Proposition 1 follows. ■

The results stated in Proposition 1 point out that conditions under which contemporaneous interest rate rules induce aggregate instability in the small open economy by generating local multiple equilibria depend not only on the interest rate response coefficient to the CPI-inflation. They also depend on some structural parameters such as the relative risk aversion coefficient, σ , and other parameters that affect χ such as the degree of openness, α .

In particular, for a very low relative risk aversion coefficient ($\sigma < 1$) an active interest rate rule will lead to a unique equilibrium regardless of the values of the other structural parameters of the model. On the other hand, for a very high relative risk aversion coefficient ($\sigma > 1$), an active interest rate rule may destabilize the economy depending on the values of some other structural parameters and how they affect χ . As was mentioned above we are interested in understanding how the equilibrium dynamics of the small open economy varies with respect to the relative risk aversion coefficient σ , and the degree of openness, α . In this sense we defined the function $\chi(\alpha, \sigma)$. To grasp the role that α may play in the determinacy of equilibrium analysis consider the following extreme cases as a first approximation. Assume that $\sigma > 1$ and that there is a value for the degree of openness of the economy $\hat{\alpha} \in (0, 1)$ such that given the other structural parameters we have that $\chi(\hat{\alpha}, \sigma) = \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A} \right)$. First, if the economy is extremely open, that is $\alpha \rightarrow 1$, then $\chi \rightarrow 0$ by **Fact 1**. Hence by part 2 of Proposition 1 the model displays a unique equilibrium. Second, if the

economy is very closed namely $\alpha \rightarrow 0$ then by **Fact 5** we know that $\chi(0, \sigma) > \chi(\hat{\alpha}, \sigma) = \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A}\right)$. But by part 2 of Proposition 1 this means that the model displays multiple equilibria.

Focusing on the plane α vs σ we can derive formally the local equilibrium frontier, $\alpha^I(\sigma)$. This frontier divides the aforementioned plane into regions of values of the degree of openness, α , and the relative risk aversion coefficient, σ , under which the model displays local multiple equilibria or a local unique equilibrium for active contemporaneous interest rate rules. A sufficient and necessary condition for the existence of such a frontier is the following.

Assumption 1: $\frac{(1-\gamma)(1-\theta_N)}{1-\gamma(1-\theta_N)} > \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A}\right)$.

Moreover define

$$\chi(0, \sigma) \equiv \frac{(1-\sigma)(1-\gamma)(1-\theta_N)}{[1-\gamma(1-\theta_N)](1-\sigma)-1} \quad (34)$$

We are going to make this Assumption 1 and carry it throughout the paper, both for local and global analysis. As can be seen the validity of this assumption depends on the parameter γ , among others, that measures the importance of money for transaction purposes (more on this below).

The frontier $\alpha^I(\sigma)$ is implicitly defined by $\chi(\alpha, \sigma) = \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A}\right)$ and using Assumption 1 we can characterize it explicitly in the following proposition.

Proposition 2 Consider the plane α vs σ , define $\Upsilon^I \equiv \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A}\right)$ and let $\chi(0, \sigma)$ be defined as in (34). Under Assumption 1 and $\sigma > 1$ the local equilibrium frontier

$$\alpha^I(\sigma) = \left[1 + \Upsilon^I \left(\frac{\gamma}{1-\gamma}\right)\right]^{-1} \left[1 - \frac{\Upsilon^I}{\chi(0, \sigma)}\right]$$

is a well-defined function, strictly increasing and concave for $\sigma \geq \sigma^{I*} > 1$, where $\sigma^{I*} \equiv 1 + \frac{\left[\frac{\Upsilon^I}{1-\gamma(1-\theta_N)}\right]}{\left[\frac{(1-\gamma)(1-\theta_N)}{1-\gamma(1-\theta_N)} - \Upsilon^I\right]}$. Moreover it satisfies $\lim_{\sigma \rightarrow \sigma^{I*}} \alpha^I(\sigma) = 0$ and $\lim_{\sigma \rightarrow \infty} \alpha^I(\sigma) = \alpha^{I*}$, where $\alpha^{I*} \equiv 1 - \frac{\Upsilon^I}{[1-\Upsilon^I(1-\gamma)](1-\theta_N)} \in (0, 1)$.

Proof. See Appendix. ■

A graphical description of the local equilibrium frontier is given by Figure 2, with respect to the driving parameters α and σ . From Figure 2 one might conclude that given $\sigma < \sigma^{I*}$ the rule always guarantees a local unique equilibrium as long as the government implements an active monetary policy. In addition one might derive the same conclusion for very open economies (high α) and regardless of the relative risk aversion coefficient.¹⁰ In other words, given the relative risk aversion coefficient, the more open the economy is the more likely is that an active rule leads to a unique equilibrium. It is in this sense that an active interest rate rule might be viewed as stabilizing¹¹.

¹⁰In fact this result is more general since a quick inspection of Proposition 1 suggests that if the economy is very open, an active rule leads to a unique equilibrium regardless of the values of the other structural parameters.

¹¹This result might seem controversial given some result from the closed economy literature on Taylor rules. In our model, as the economy gets very closed ($\alpha \rightarrow 0$), for sufficiently high risk aversion, active rules deliver indeterminacy (which contrasts with the general optimistic view on aggressive targeting). However, as Benhabib et al. (2001a) have pointed out, the ability of active rules to deliver stability depends on whether consumption and real money balances are complements or substitutes. In our model, real money balances and consumption may be Edgeworth substitutes $U_{cm} < 0$ or Edgeworth complements $U_{cm} > 0$. Our results for the extreme case of a closed economy are fairly consistent with the Benhabib et al. conclusions.

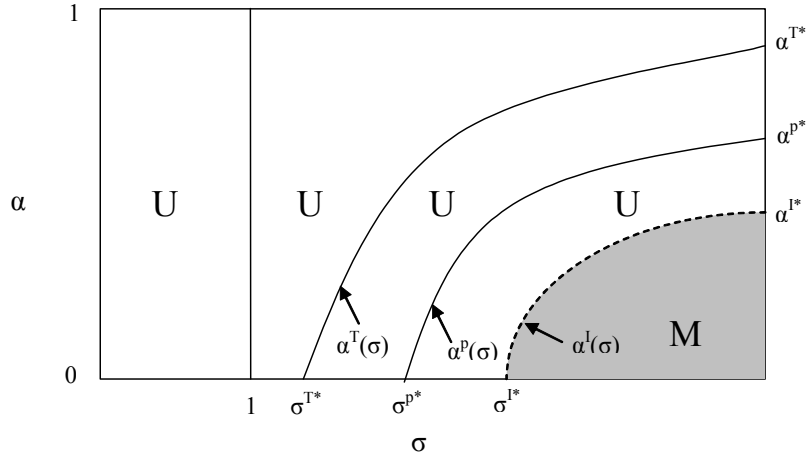


Figure 2: Local equilibrium analysis for an active contemporaneous interest rate rule. This figure shows the local equilibrium regions. M stands for local multiple equilibria and U stands for a local unique equilibrium. α corresponds to the degree of openness of the economy and σ denotes the relative risk aversion coefficient.

By sticking to local dynamics, one might conclude that active rules are likely to deliver unique equilibria (and therefore real stability) for most parametrizations. In particular, central banks seem to have an “easy” task in quite open economies. The point of this paper is to show that this is not necessarily the case under a global equilibrium analysis. The economy might be likely to end up on a variety of other dynamic paths, all consistent with rational expectation equilibria. The next section pursues the analysis.

4.2 Global Dynamics

The global equilibrium dynamics for a contemporaneous rule are completely described by the following non-linear difference equation

$$\left[\frac{R_{t+1} - 1}{R_{t+1}} \right]^\chi \left[\frac{R^* - 1}{R_{t+1} - 1} \right]^{\frac{R^* - 1}{A}} = \left[\frac{R_t - 1}{R_t} \right]^\chi \frac{R^*}{R_t} \quad (35)$$

that characterizes the equilibrium path of the gross nominal interest rate R_t . The object of this section is to study the equilibrium paths consistent with this difference equation. We will show how the degree of openness (α) and the relative risk aversion coefficient (σ) are fundamental to understand the stability of the steady state equilibria and the dynamic behavior of the non-stationary paths converging to them. To start with, we are going to study both sides of (35). Let

$$K(R_{t+1}) = \left[\frac{R_{t+1} - 1}{R_{t+1}} \right]^\chi \left[\frac{R^* - 1}{R_{t+1} - 1} \right]^{\frac{R^* - 1}{A}} \quad (36)$$

$$J(R_t) = \left[\frac{R_t - 1}{R_t} \right]^\chi \frac{R^*}{R_t} \quad (37)$$

The dynamic equation is therefore summarized by

$$K(R_{t+1}) = J(R_t)$$

The multiple steady states result motivates us to look for saddle path connections between the low and the high inflation stationary points. Clearly if K were monotonic we would have well-defined forward dynamics. Well-defined backward dynamics would result if J were monotonic instead. Studying the behavior of those two functions is therefore a first necessary step for global results.

Lemma 3 Consider the function $J(R)$ defined in (37), with $R > 1$ (the zero-bound condition). The followings hold:

1. if $\sigma \in (0, 1)$, $J(R)$ is strictly decreasing for $R > 1$ and for any $\alpha \in (0, 1)$, with $\lim_{R \rightarrow 1} J(R) = \infty$ and $\lim_{R \rightarrow \infty} J(R) = 0$;
2. if $\sigma > 1$, $J(R)$ hump-shaped with a peak at $R^J = 1 + \chi > 1$ for any $\alpha \in (0, 1)$, with $\lim_{R \rightarrow 1} J(R) = \lim_{R \rightarrow \infty} J(R) = 0$;

Proof. See Appendix. ■

In order to study the behavior of function K it is useful to define the properties of another function defined as $\alpha^p(\sigma)$. The reason is that this function that divides the plane α vs σ into two regions will define the values of α and σ for which the function K is either strictly decreasing or hump-shaped. The function $\alpha^p(\sigma)$ is implicitly defined by $\chi(\alpha, \sigma) = \frac{R^* - 1}{A}$ and using Assumption 1 we can characterize it explicitly in the following Lemma.¹²

Lemma 4 Consider the plane α vs σ , define $\Upsilon^p \equiv \frac{R^* - 1}{A}$ and let $\chi(0, \sigma)$ be defined as in (34). Under Assumption 1 and $\sigma > 1$ the frontier

$$\alpha^p(\sigma) = \left[1 + \Upsilon^p \left(\frac{\gamma}{1 - \gamma} \right) \right]^{-1} \left[1 - \frac{\Upsilon^p}{\chi(0, \sigma)} \right]$$

is a well-defined function, strictly increasing and concave for $\sigma \geq \sigma^{p*} > 1$ where $\sigma^{p*} \equiv 1 + \frac{\left[\frac{\Upsilon^p}{1 - \gamma(1 - \theta_N)} \right]}{\left[\frac{(1 - \gamma)(1 - \theta_N)}{1 - \gamma(1 - \theta_N)} - \Upsilon^p \right]}$. Moreover it satisfies $\lim_{\sigma \rightarrow \sigma^{p*}} \alpha^p(\sigma) = 0$ and $\lim_{\sigma \rightarrow \infty} \alpha^p(\sigma) = \alpha^{p*}$ where $\alpha^{p*} \equiv 1 - \frac{\Upsilon^p}{[1 - \Upsilon^p(1 - \gamma)](1 - \theta_N)} \in (0, 1)$. In addition $\alpha^{p*} > \alpha^{I*}$ and $\sigma^{p*} < \sigma^{I*}$, where α^{I*} and σ^{I*} were defined in Proposition 2.

Proof. See Appendix. ■

A graphical representation of the frontier $\alpha^p(\sigma)$ can be seen in Figure 2. Using the definition of the frontier $\alpha^p(\sigma)$ in the plane α vs σ , it is possible to pursue the characterization of the function K in the following manner.

Lemma 5 Consider the function $K(R)$ defined in (36), with $R > 1$, and recall the definitions of $\alpha^p(\sigma)$, σ^{p*} , and Υ^p in Lemma 4. The followings hold:

¹²In the Lemma we use Assumption 1. This assumption is more than we need to have a well-defined function $\alpha^p(\sigma)$. In fact we only need that $\frac{(1 - \gamma)(1 - \theta_N)}{1 - \gamma(1 - \theta_N)} > \frac{R^* - 1}{A}$. However we keep Assumption 1 to be able to compare the results from the local equilibrium analysis with the results from the global equilibrium analysis.

1. If $\sigma \in (0, 1)$, $K(R)$ is strictly decreasing for $R > 1$ and for any $\alpha \in (0, 1)$. Moreover $\lim_{R \rightarrow 1} K(R) = \infty$ and $\lim_{R \rightarrow \infty} K(R) = 0$;

2. Assume $\sigma > 1$ then

(a) for any $\sigma \in (1, \sigma^{p*})$ and $\alpha \in (0, 1)$ or for any $\sigma \in [\sigma^{p*}, \infty)$ such that $\alpha \geq \alpha^p(\sigma)$, $K(R)$ is strictly decreasing;

(b) for any $\sigma \in (\sigma^{p*}, \infty)$ such that $\alpha < \alpha^p(\sigma)$, $K(R)$ is hump-shaped with a peak at $R^K = \frac{\chi A}{R^* - 1} > 1$.

Proof. See Appendix. ■

We can now define some parametric zones with respect to α and σ within which the equilibrium dynamics will be extensively studied. In order to accomplish this task it is important to notice the following. For the values of the parameters α and σ defined in part 2(a) of Lemma 5, $J(R)$ peaks at $R^J = 1 + \chi$ and $K(R)$ is monotonically decreasing with respect to R . From the steady state analysis we know that they meet twice, at the target interest rate R^* and at $R^L < R^*$. The higher steady state has to occur on the decreasing portion of the function $J(R)$. The lower steady state intersection can instead occur both above, below or at R^J (namely, both on the increasing, decreasing or peaking portion of J). In other words we can have $R^L \gtrless R^J$. The equilibrium dynamics will be substantially different according to which case we will be considering. Since $R^J = 1 + \chi(\alpha, \sigma)$ we need to study the parametric ranges of α and σ over which $1 + \chi(\alpha, \sigma) \gtrless R^L$ or equivalently $\chi(\alpha, \sigma) \gtrless R^L - 1$. This implies we have to define a new frontier or curve $\alpha^T(\sigma)$ that describes the values of α and σ such that $\chi(\alpha, \sigma) = R^L - 1$.

We will also make another assumption that seems empirically reasonable¹³.

Assumption 2: $R^* - 1 > A(R^L - 1)$.

Using Assumptions 1 and 2 we can characterize the frontier $\alpha^T(\sigma)$ in the following Lemma.

Lemma 6 Consider the plane α vs σ , define $\Upsilon^T \equiv R^L - 1$ and let $\chi(0, \sigma)$ be defined as in (34). Under Assumptions 1, 2 and $\sigma > 1$ the frontier

$$\alpha^T(\sigma) = \left[1 + \Upsilon^T \left(\frac{\gamma}{1 - \gamma} \right) \right]^{-1} \left[1 - \frac{\Upsilon^T}{\chi(0, \sigma)} \right]$$

is a well-defined function, strictly increasing and concave for $\sigma \geq \sigma^{T*} > 1$ where $\sigma^{T*} \equiv 1 + \frac{\left[\frac{\Upsilon^T}{1 - \gamma(1 - \theta_N)} \right]}{\left[\frac{(1 - \gamma)(1 - \theta_N)}{1 - \gamma(1 - \theta_N)} - \Upsilon^T \right]}$. Moreover it satisfies $\lim_{\sigma \rightarrow \sigma^{T*}} \alpha^T(\sigma) = 0$ and $\lim_{\sigma \rightarrow \infty} \alpha^T(\sigma) = \alpha^{T*}$, where $\alpha^{T*} \equiv 1 - \frac{\Upsilon^T}{[1 - \Upsilon^T(1 - \gamma)](1 - \theta_N)} \in (0, 1)$. In addition $\alpha^{T*} > \alpha^{p*}$ and $\sigma^{T*} < \sigma^{p*}$, where α^{p*} and σ^{p*} were defined in Lemma 4.

Proof. See Appendix. ■

Figure 2 shows the $\alpha^T(\sigma)$ frontier. With this figure we can study the regions for which $R^J = 1 + \chi(\alpha, \sigma) \gtrless R^L$. Note that since $\alpha^T(\sigma)$ describes all the feasible combinations of σ and α such that $\chi(\alpha, \sigma) = R^L - 1$ then we can pursue the following analysis. Take a pair (σ^T, α^T) such that $\chi(\alpha^T, \sigma^T) = R^L - 1$. Given **Fact 5** any $\alpha \geq \alpha^T$ implies that $\chi(\alpha, \sigma^T) \leq \chi(\alpha^T, \sigma^T) = R^L - 1$. But this implies that $R^J = 1 + \chi(\alpha, \sigma^T) \leq R^L$. In other words for any $\sigma \geq \sigma^{T*}$ and any $\alpha \geq \alpha^T(\sigma)$, we have that the function $K(R)$ meets twice the function

¹³It can be shown that this requires $R^* > A \frac{R^* - 1}{A} \left[1 + \frac{R^* - 1}{A} \right]$.

$J(R)$ in its decreasing part. This analysis, Lemmas 4 and 6 and Propositions 3 and 5 allow us to divide the parametric space α vs σ into 6 zones:¹⁴

1. Zone 1: $\sigma \in (0, 1)$ and $\alpha \in (0, 1)$. Both $J(R)$ and $K(R)$ are strictly decreasing and meet twice.
2. Zone 2: $\sigma \in (1, \sigma^{T*})$ and $\alpha \in (0, 1)$. $K(R)$ is strictly decreasing and $J(R)$ is hump-shaped but $K(R)$ cuts $J(R)$ at R^L and R^* , both on the decreasing side of $J(R)$.
3. Zone 3: $\sigma \geq \sigma^{T*}$ and $\alpha \in [\alpha^T(\sigma), 1)$. The properties of $K(R)$ and $J(R)$ are the same as in Zone 2, with $R^L = R^J$ over $\alpha^T(\sigma)$.
4. Zone 4: $\sigma \in (\sigma^{T*}, \sigma^{p*})$ and $\alpha \in (0, \alpha^T(\sigma))$. $K(R)$ is strictly decreasing and $J(R)$ is hump-shaped. But $K(R)$ cuts $J(R)$ at $R^L < R^J$ and $R^* > R^J$.
5. Zone 5: $\sigma \geq \sigma^{p*}$ and $\alpha \in [\alpha^p(\sigma), \alpha^T(\sigma))$. The properties of $K(R)$ and $J(R)$ are the same as in Zone 4.
6. Zone 6: $\sigma > \sigma^{p*}$ and $\alpha \in (0, \alpha^p(\sigma))$. Both $J(K)$ and $K(R)$ are hump-shaped.

The next step is to show that within each of those zones we can face very different dynamics. We will be spelling out clearly what additional assumptions (mostly sufficient ones) will be needed to get endogenous cycles and chaotic dynamics. The point of this work is in fact to show that rich dynamics are likely to occur not that they occur for sure. We will be focusing on equilibrium dynamics for initial conditions between the two steady states

From simple inspection of the parametric zones it looks like we have left the case $\sigma = 1$ out of the picture. This is the standard log-utility case, where utility is separable with respect to all its arguments. It can be easily shown that in that case no interesting dynamics will occur other than the standard closed economy liquidity trap of Benhabib, Schmitt-Grohé and Uribe (2002). Technically it is important to clarify that in our model a standard liquidity trap corresponds to a case in which given an initial nominal interest rate between the two steady-states, the nominal interest rate converges monotonically to the lower steady-state. We refer to this case as a liquidity trap following Benhabib et al (2001b, 2002). They argue that the dynamical features of the aforementioned converging path resembles the dynamical properties of a standard liquidity trap.

We start by showing the possibility of liquidity traps.

Proposition 7 *Standard liquidity traps or deflationary paths, i.e. interest rate equilibrium paths converging asymptotically to the lower steady state for any $R_0 \in (R^L, R^*)$, occur if the pair (σ, α) belongs either to Zone 1, 2, or 3 (see Figure 3 and use the aforementioned definitions of the zones).*

Proof. The result follows straight from showing that the mapping $R_{t+1} = f(R_t)$ for $R_t \in (R^L, R^*)$ has the following features: 1) $f'(R_t) > 0$ for any $R_t > 1$; 2) $f'(R^*) > 1$; 3) $f'(R^L) < 1$; 4) $f(R_t) < R_t$ (a simple graph makes the arguments clear). Point 1) follows from applying the Implicit Function Theorem: $f'(R_t) = \frac{J'(R_t)}{K(R_{t+1})} < 0$. Point 2) and 3) follow from computing $\frac{J'(R_t)}{K(R_{t+1})}$ at $R_{t+1} = R_t = R^*$ and $R_{t+1} = R_t = R^L$. To prove point 4) take $R_0 \in (R^L, R^*)$. $R' = f(R_0)$ is defined by $K(R') = J(R_0)$. But also $K(R_0) < J(R_0)$.

¹⁴Although these zones are not marked in Figure 2, this figure is still useful to understand the definition of the zones.

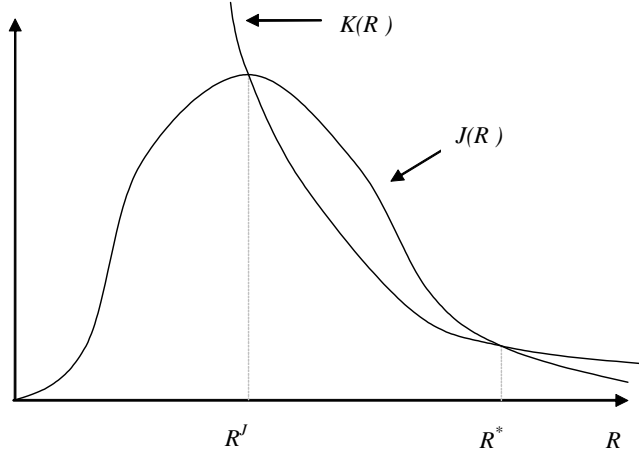


Figure 3: Graphs of the functions J and K to study the dynamics of the model in Zones 1, 2, and 3. See Proposition 7. This is the standard liquidity traps case. Dynamics are similar if the leftmost crossing point of J and K was to the right of function J 's peak

Since K is monotonically decreasing $R' = f(R_0) < R_0$. Since R_0 was taken arbitrarily, the result follows. The sequence R_t is then monotonically decreasing within the closed and compact set $[R^L, R^*]$. Therefore it should converge to a point within the set. The unique stationary point below R_0 is R^L . A standard liquidity trap occurs: the economy is driven asymptotically to the passive (low inflation) steady state. ■

The next few propositions will focus on Zone 4 and 5. Within these zones K is monotonically decreasing. Forward dynamics are then well defined. However, the hump-shaped behavior of J creates opportunities for complex erratic paths. For degrees of openness and risk aversion within those ranges defined in the aforementioned zones we will show that both two-period cycle equilibria and chaotic dynamics (cyclical orbits of any periodicity) are possible.

It is important to observe that the parametric zones defined above do not divide the parametric space α vs σ between “zones with standard liquidity traps occurring with probability one” and “zones with cycles and chaos occurring with probability one”. Zones 1, 2, and 3 simply can not have cyclical or chaotic equilibria. A necessary (but not sufficient) condition for oscillating equilibria to occur is that the implicitly forward looking mapping defined in (35) be not monotonic. That is, it must have at least one critical point: a peak or a trough. Clearly those zones 1, 2 and 3 do not display these feature over the range (R^L, R^*) . More formally, within zones 4 and 5 we will be looking for some kind of “flip bifurcations”, namely for parameters defining thresholds above/below which the steady states loose/gain stability.¹⁵

In order to proceed with the analysis, define \underline{R} and \tilde{R} implicitly as follows:

¹⁵Another kind of bifurcation would be to check for parametric thresholds such that we observe the appearance and disappearance of steady states. Clearly this can not be the case here over the “risk aversion-openness space” since steady states have been shown to depend solely on monetary policy parameters.

$$\begin{aligned} K(\underline{R}) &= J(R^J) \Leftrightarrow \underline{R} = f(R^J) \\ K(\tilde{R}) &= J(R^*) \Leftrightarrow R^* = f(\tilde{R}) \end{aligned}$$

Given the fact that for zones 4 and 5 $K(R)$ is monotonically decreasing and that within these zones $R^L < R^J$, we have that $\underline{R} < R^J$. The hump-shaped $J(R)$ guarantees that $\tilde{R} < R^*$ (see Figure 4). We have to consider the three cases $\underline{R} \gtrless \tilde{R}$ separately. The idea here is to show that an attractive set exists under different parametrizations and that within such a set cycles and chaos are likely to occur.

Assumption 3¹⁶: $f_{\min} = f(R^J) \geq \tilde{R}$

Proposition 8 *If Assumption 3 is satisfied, the followings hold:*

1. The mapping f is such that $f : [\underline{R}, R^*] \rightarrow [\underline{R}, R^*]$. Moreover for any $R_t \in (\tilde{R}, \underline{R})$, $R_t \in (\underline{R}, R^*)$.
2. Period 2 cycles exists within such set.
3. Topological chaotic dynamics, in a Li-Yorke sense are possible.

Proof.

1. The proof is trivial and therefore omitted.
2. Define a function $g(R) = R - f^2(R)$. We need to distinguish between the two cases in Assumption 3:
 - a) $f(R^J) = \tilde{R}$; b) $f(R^J) > \tilde{R}$.

(a) In this case \tilde{R}, \underline{R} and the set invariant under mapping f is therefore $[\tilde{R}, R^*]$. It should be clear that $g(R^L) = g(R^*) = 0$, $g(R^J) < 0$ and $g(\tilde{R}) < 0$. Furthermore $g'(\bar{R}) = 1 - [f'(\bar{R})]^2$, \bar{R} being either one of the two steady states. This implies that $g'(R^*) < 0$ since $f'(R^*) = \frac{J(R^*)}{K(R^*)} = \frac{A}{R^*} > 1$. Since the mapping f is continuous, there exists a point $R_u \in (R^J, R^*)$ such that $g(R_u) = 0$. The period-2 cycle is then $\{R_l, R_u\}$, with $R_l = f(R_u) \in (\underline{R}, R^L)$ (this is left to the reader).

(b) The set invariant under mapping f is now $[\underline{R}, R^*]$. As in case a), $g(R^L) = g(R^*) = 0$, and $g'(R^*) < 0$. But now $g(\underline{R}) \leq 0$ and $g(R^J) \gtrless 0$. If $g(R^J) = 0$, the period-2 cycle is $\{R^J, \underline{R}\}$. If it is $g(R^J) < 0$, then as before there exists a point $R_u \in (R^J, R^*)$ such that $g(R_u) = 0$. However, if $g(R^J) > 0$, there is no guarantee that the function g has an additional zero. A sufficient (but not necessary) condition is that $g'(R^L) < 0$. But this is simply the requirement that $1 - [f'(R^L)]^2 < 0$, which occurs if $f'(R^L) = \frac{J(R^L)}{K(R^L)} < -1$. It can be shown that such sufficient assumption is satisfied for $\chi > 0.5 \left[R^L \left(1 + \frac{R^* - 1}{A} \right) - 1 \right]$. This implies the existence of a point $R_u \in (R^L, R^J)$ such that $g(R_u) = 0$.

¹⁶If $f_{\min} = f(R^J) < \tilde{R}$ there is not non-trivial mapping-invariant set. This case displays a different type of multiplicity. It can be shown that there exist set of point within the mapping-invariant set that leave such a set after a finite number of iterations, and settle to an exploding path diverging from the active steady state. More detailed results are available from the authors upon request.

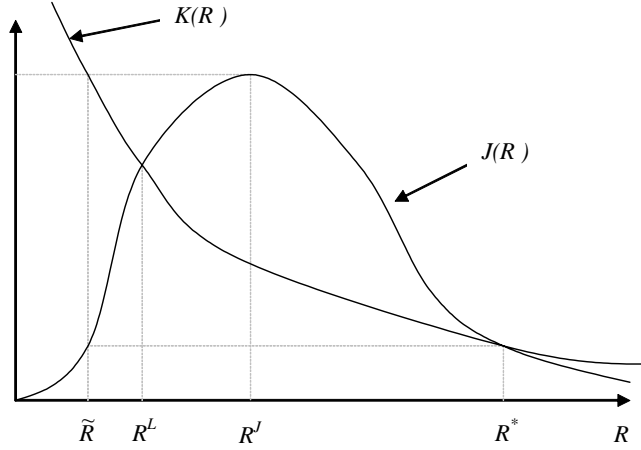


Figure 4: Graphs of the functions J and K to study the dynamics of the model in Zones 4 and 5, with $\tilde{R} = \underline{R}$. See Proposition 8.

3. Define a function $h(R) = R - f^3(R)$. Again we need to distinguish between the two cases in Assumption 3: **a)** $f(R^J) = \tilde{R}$; **b)** $f(R^J) > \tilde{R}$.

(a) It should be clear that $h(R^L) = h(R^*) = 0$, $h(R^J) < 0$, $h(\tilde{R}) < 0$. and $h'(R^*) < 0$. We show that the Li-Yorke sufficient condition for the existence of topological chaos is satisfied. Let $R^c \in (R^J, R^*)$ be such that $K(R^J) = J(R^c)$, in other words R^c is the pre-image of R^J . We will have that $R^J = f(R^c)$, $\underline{R} = f^2(R^c) < R^J < R^c$ and $R^* = f^3(R^c) > R^c$. The Li-Yorke sufficient condition is satisfied. Furthermore, since the function h is continuous, $h(R^J) < 0$ and $h'(R^*) < 0$, it follows that $h(\hat{R}) = 0$ for some $\hat{R} \in (R^J, R^*)$. A period-3 cycles exists and by Sarkovskii's Theorem cycles of any orbit exist.

(b) As above $h(R^L) = h(R^*) = 0$ and $h'(R^*) < 0$. But now $h(R^J) \geq 0$. Obviously, if $h(R^J) = 0$ the period-3 cycle is $\{R^J, \underline{R}, f(\underline{R})\}$. If $h(R^J) < 0$ by continuity of the function there exists a zero of h between R^J and R^* , and since $h'(R^L) > 0$ there is also a zero between R^L and R^J . In any case a period-3 cycles occurs. In both cases because of Sarkovskii's Theorem cycles of any periodicity exist. Furthermore by Li-Yorke there exist chaotic dynamics: period 3 implies chaos. However, if $h(R^J) > 0$ there is no guarantee that cycles of period 3 and/or of any higher order exist. Therefore $h(R^J) \leq 0$ is sufficient to have chaotic dynamics.

■

To conclude, in this section we showed that apart from standard liquidity traps, active contemporaneous interest rate rules in small open economies can generate very complex dynamics for plausible parametric ranges. We showed that contemporaneous inflation targeting can produce both monotonically and cyclically deflationary paths (the latter not converging to any stationary point).

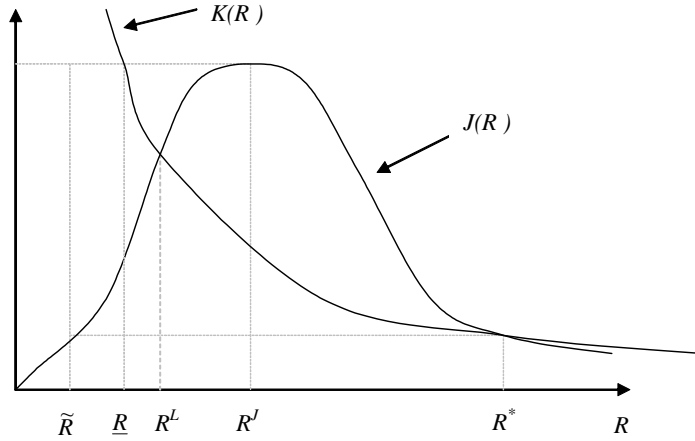


Figure 5: Graphs of the functions J and K to study the dynamics of the model in Zones 4 and 5, with $\tilde{R} < \underline{R}$. See Proposition 8.

The reader has probably noticed that we did not cover zone 6 in details. Within this zone both the K and the J functions are hump-shaped. This case is isomorphic to the case studied by Benhabib, Schmitt-Grohé and Uribe (2002) in the closed economy set-up. It can be easily shown that under some assumptions chaotic dynamics and cycles can arise around the active steady state. The proofs of this results are omitted in this version of the paper but are available upon request from the authors.

4.3 Local Uniqueness vs. Global Multiplicity

To what extent local and global dynamics analyses can lead to conflicting results? Figure 6 puts together our local and global analysis results.

By sticking to local dynamics (around the target steady state), we would conclude that the “Taylor principle” of active monetary policy leads to stability for any degree of openness, α , as long as the relative risk aversion coefficient, σ , stays below a threshold σ^{I^*} with $\sigma^{I^*} > 1$; and for any degree of openness above some function $\alpha^I(\sigma)$ with domain $\sigma \in [\sigma^{I^*}, \infty)$.

On the other hand, by studying the whole dynamics we found that within those local uniqueness ranges, monotonic deflations, cyclical and chaotic equilibria can occur. It is interesting to see how openness affects the dynamics for $\sigma > \sigma^{p^*}$. For very closed economies, cycles/chaos around the active steady state can occur. In other words, the economy is highly unstable but without risks of falling into deflationary paths. As the economy opens more, specifically above some threshold $\alpha^p(\sigma)$, the probability of entering vicious deflationary spirals become positive. The economy is not only highly unstable but might end up in a liquidity trap (cyclical or monotonic). In other words, given the coefficient of relative risk aversion, we have shown that the more open the economy is, the more likely is that a contemporaneous active rule will drive the economy into a liquidity trap. On the other hand, the more closed the economy is, the more likely is that the same rule will

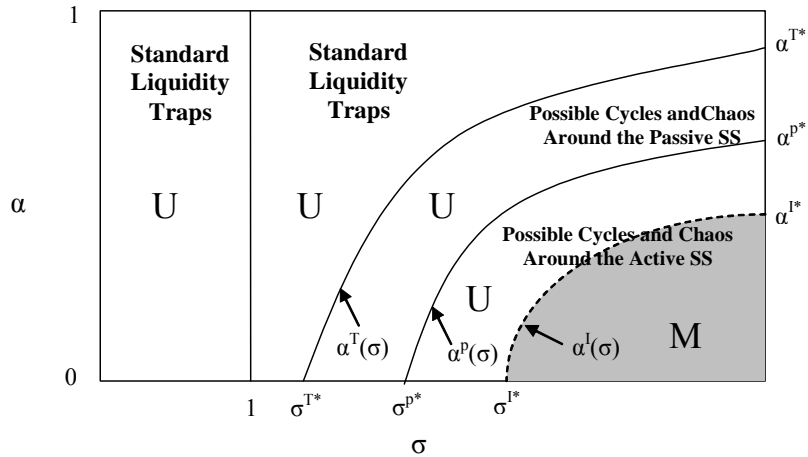


Figure 6: Equilibrium analysis for an active contemporaneous interest rate rule. This figure shows a comparison between the local equilibrium analysis and the global equilibrium analysis. M stands for local multiple equilibria and U stands for a local unique equilibrium. α corresponds to the degree of openness of the economy and σ denotes the relative risk aversion coefficient.

lead to cycles and chaotic dynamics around the inflation target.

We can confirm these theoretical results pursuing a calibration-simulation exercise. We set the time unit to be a quarter and use Canada as the representative economy. From Mendoza (1995) we borrow the labor income shares for the non-traded sector θ_N . The steady-state inflation, π^* , and the steady state nominal interest rate, R^* , are calculated as the average of the CPI-inflation and the Central Bank discount rate between 1983-2002. Then the subjective discount rate is calculated as $\beta = \pi^*/R^*$. We use the estimate of Lubik and Schorfheide (2003) for the Canadian interest rate response coefficient to inflation, $\frac{A}{R^*}$. Estimates for the share of expenditures on real money balances, $1 - \gamma$, for Canada are not available.¹⁷ For the United States, estimates of this parameter vary from 0.0146 to 0.039 depending on the specification of the utility function and method of estimation.¹⁸ We set $1 - \gamma$ equal to 0.03 and will pursue a sensitivity analysis with respect to this parameter. Table 2 presents the values of the parameters.

Table 2: Parametrization

θ_N	β	π^*	R^*	$1 - \gamma$	$\frac{A}{R^*}$
0.56	0.99	$1.031^{\frac{1}{4}}$	$1.072^{\frac{1}{4}}$	0.032	2.24

In our analysis we will vary the degree of openness of the economy, α , and the relative risk aversion

¹⁷Imrohorglu (1994) presents some estimates of currency substitution between the Canadian dollar and the U.S. dollar, but in our model only domestic money enters into the utility function.

¹⁸See Poterba and Rotemberg (1987), Finn, Hoffman and Schlagenhauf (1990) and Holman (1998).

coefficient, σ . However an estimate for Canada of the former parameter can be obtained using the average imports to GDP share during 1983-2002. This yields $\alpha = 0.31$. In contrast an estimate of the relative risk aversion coefficient is more difficult to obtain. The RBC literature usually sets this parameter to 2.¹⁹ Since setting this parameter immediately implies to assume that consumption and real money balances are either Edgeworth substitutes or complements we will use different values. That is $\sigma \in \{0.8, 2, 2.5\}$.

Using this parametrization we can show quantitatively how misleading the local equilibrium analysis for active contemporaneous rules is. Setting $\sigma = 2.5$ and using the parameters in Table 2 we can calculate $\sigma^{I^*} = 3.14$. Since $\sigma = 2.5 < \sigma^{I^*} = 3.14$ then doing a local equilibrium analysis and using Figure 6 we would conclude that the active contemporaneous rule is not destabilizing since it leads to a unique local equilibrium. However the global equilibrium analysis conveys a different message. In Figure 7 we present the global dynamics of the model for different degrees of openness $\alpha \in \{0.01, 0.37, 0.90\}$. It basically shows the first three iterates of the difference equation (29), which describes the equilibrium dynamics for the nominal interest rate. In all the panels the straight line corresponds to the 45° degree line. In particular notice that for $\alpha = \{0.01, 0.37\}$, almost closed and open economies respectively, the second and third iterates, $R_{t+2} = f^2(R_t)$ and $R_{t+3} = f^3(R_t)$, have fixed points different from the steady state values R^* and R^L . This implies that there exist two and three period cycles. By Sarkovskii's (1964) theorem, the existence of three-period cycles implies that the map F has cycles of any periodicity. Furthermore by Li and Yorke (1975), the existence of three-period cycles implies chaos. However it is important to observe that there is a difference in terms of the dynamics of the almost closed economy ($\alpha = 0.01$) and the open economy ($\alpha = 0.37$). Whereas in the former cycles and chaos arise around the active steady state. In the latter these types of dynamics are present around the passive steady state. On the other hand for very open economies ($\alpha = 0.90$), no cycles and chaotic dynamics appear. Only deflationary paths (liquidity traps) converging to the passive steady state are possible.

Finally, it is important to mention the relevance of Assumption 2 in the global analysis. This assumption assures the existence of cycles and chaos around the passive steady state. If we relax it, that is if we assume that $R^* - 1 \leq A(R^L - 1)$, then the aforementioned dynamics are not present in the global analysis. In other words, the model only displays standard liquidity traps and cycles and chaos around the active steady state.

5 The Equilibrium Analysis Under a Forward- Looking Taylor Rule

We proceed to study forward-looking interest rate rules with respect to the CPI-inflation. In order to motivate them we remember the estimations by Clarida Gali and Gertler (1998) of forward-looking rules for United kingdom, Germany, France, Italy and Japan; and the estimations by Corbo(2000) for Chile, Colombia, Peru, Costa Rica and El Salvador.

¹⁹See Mendoza (1991) among others.

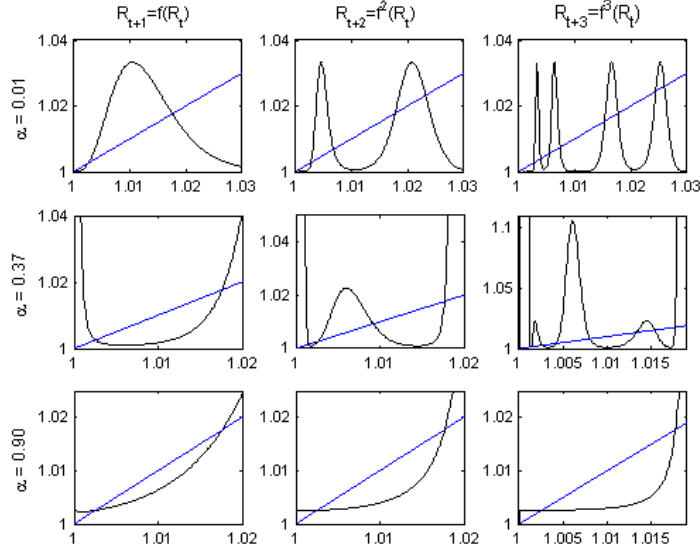


Figure 7: Active Contemporaneous Interest Rate Rules. This graph shows that depending on the degree of openness of the economy (α) two-period cycles and three-period cycles around the active and the passive steady states are possible.

5.1 Local Analysis

In order to pursue a local determinacy of equilibrium analysis for forward-looking interest rate rules we log-linearize equation (31) around the target steady states R^* . This yields

$$\hat{R}_{t+1} = \left[1 + \frac{\frac{R^*}{A} - 1}{\frac{\chi}{R^* - 1}} \right] \hat{R}_t \quad (38)$$

The following proposition summarizes the local determinacy of equilibrium analysis for forward-looking rules.

Proposition 9 *Suppose the government follows an active forward-looking interest rate rule given by $R_t = \rho(\pi_{t+1})$ with $\rho'(\pi^*) = \frac{\beta A}{\pi^*} = \frac{A}{R^*} > 1$ and let χ be defined as in (30),*

1. *if $\sigma < 1$ then the model displays a unique equilibrium.*
2. *assume that $\sigma > 1$. If $\chi < \frac{1}{2}(R^* - 1) \left(1 - \frac{R^*}{A}\right)$ then the model displays a unique equilibrium. On the other hand, if $\chi > \frac{1}{2}(R^* - 1) \left(1 - \frac{R^*}{A}\right)$ then the model displays multiple equilibria.*

Proof. To prove this proposition we use (38). For 1 note that if $\sigma < 1$ then from **Fact 2**, we conclude that $\chi < 0$. This result and the zero bound on the nominal interest rate imply that $\frac{\chi}{R^* - 1} < 0$. This inequality and the assumption of an active rule, that is $\frac{A}{R^*} > 1$, help us to see that $\left(1 + \frac{\frac{R^*}{A} - 1}{\frac{\chi}{R^* - 1}}\right) > 1$. But this means that the mapping (38) becomes explosive. This feature of the mapping in conjunction with the fact that R_t

is a non-predetermined variable imply that there exists a unique equilibrium that corresponds to the active steady state.

For 2 note that if $\sigma > 1$ then from **Fact 3** we derive that $\chi > 0$. This result and the zero bound on the nominal interest rate imply that $\frac{\chi}{R^*-1} > 0$. This inequality and the assumption that the rule is active, $\frac{R^*}{A} < 1$, lead us to conclude that $\left(1 + \frac{\frac{R^*-1}{\chi}}{R^*-1}\right) < 1$. This means that in order for an active forward-looking rule to induce a unique equilibrium (for the mapping 38 to become explosive) it is necessary that $\left(1 + \frac{\frac{R^*-1}{\chi}}{R^*-1}\right) < -1$. If $\chi < \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right)$ then $\frac{R^*}{A} - 1 < -2\frac{\chi}{R^*-1}$ and therefore $\left(1 + \frac{\frac{R^*-1}{\chi}}{R^*-1}\right) < -1$. On the other hand, if $\chi > \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right)$ then it is straightforward to prove that this inequality implies that $-1 < \left(\frac{1 - \frac{\chi}{R^*-1}}{\frac{R^*}{A} - \frac{\chi}{R^*-1}}\right) < 1$. Hence the aforementioned mapping becomes non-explosive and the model displays a multiple equilibria. ■

As in the contemporaneous interest rate rule analysis, Proposition 9 points out that conditions under which active forward-looking interest rate rules lead to multiple equilibria in the small open economy depend on some structural parameters such as the relative risk aversion coefficient σ and the parameters that affect χ . In particular, for a very low relative risk aversion coefficient ($\sigma < 1$) an active interest rate rule will lead to a unique equilibrium regardless of the values of the other structural parameters of the model. On the other hand, for a very high relative risk aversion coefficient ($\sigma > 1$), an active interest rate rule may destabilize the economy depending on the values of some other structural parameters and how they affect χ . As was mentioned above we are interested in understanding how the equilibrium dynamics of the small open economy varies with respect to the relative risk aversion coefficient σ , and the degree of openness, α . In this sense we defined the function $\chi(\alpha, \sigma)$. To grasp the role that the degree of openness of the economy, α , may play in the determinacy of equilibrium analysis consider the following extreme cases as a first approximation. Assume that $\sigma > 1$ and that there is a value for the degree of openness of the economy $\hat{\alpha} \in (0, 1)$ such that given the other structural parameters we have that $\chi(\hat{\alpha}, \sigma) = \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right)$. First, if the economy is extremely open, that is $\alpha \rightarrow 1$, then $\chi \rightarrow 0$ by **Fact 1**. Hence by part 2 of Proposition 9 we conclude that the model displays a unique equilibrium. Second, if the economy is very closed namely $\alpha \rightarrow 0$, then by **Fact 5** we know that $\chi(0, \sigma) > \chi(\hat{\alpha}, \sigma) = \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right)$. But by part 2 of Proposition 9 this means that the model displays multiple equilibria.

As we did before for the local analysis of contemporaneous, it is possible to derive formally the local equilibrium frontier, $\alpha^d(\sigma)$, on the plane α vs σ . This frontier divides the aforementioned plane into values of the degree of openness, α , and the relative risk aversion coefficient, σ , under which the model displays multiple local equilibria or a unique local equilibrium for active forward-looking interest rate rules.

The frontier $\alpha^d(\sigma)$ is implicitly defined by $\chi(\alpha, \sigma) = \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right)$ and using Assumption 1 we can characterize it explicitly in the following proposition.

Proposition 10 *Consider the plane α vs σ , define $\Upsilon^d \equiv \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right)$ and let $\chi(0, \sigma)$ be defined as in (34). Under Assumption 1 and $\sigma > 1$ the local equilibrium frontier*

$$\alpha^d(\sigma) = \left[1 + \Upsilon^d \left(\frac{\gamma}{1-\gamma}\right)\right]^{-1} \left[1 - \frac{\Upsilon^d}{\chi(0, \sigma)}\right]$$

is a well-defined function, strictly increasing and concave for $\sigma \geq \sigma^{d} > 1$ where $\sigma^{d*} \equiv 1 + \frac{\left[\frac{\Upsilon^d}{1-\gamma(1-\theta_N)}\right]}{\left(\frac{(1-\gamma)(1-\theta_N)}{1-\gamma(1-\theta_N)} - \Upsilon^d\right)}$.*

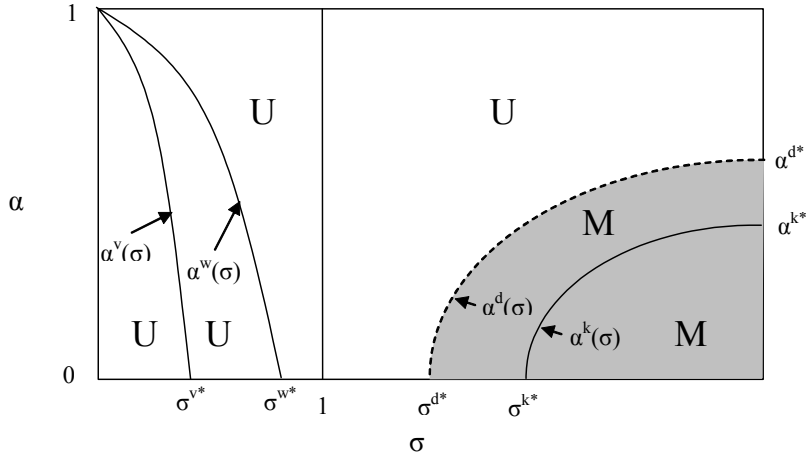


Figure 8: Local equilibrium analysis for an active forward-looking interest rate rule. This figure shows the local equilibrium regions. M stands for local multiple equilibria and U stands for a local unique equilibrium. α corresponds to the degree of openness of the economy and σ denotes the relative risk aversion coefficient.

Moreover $\lim_{\sigma \rightarrow \sigma^{d*}} \alpha^d(\sigma) = 0$ and $\lim_{\sigma \rightarrow \infty} \alpha^d(\sigma) = \alpha^{d*}$, where $\alpha^{d*} \equiv 1 - \frac{\Upsilon^d}{[1 - \Upsilon^d(1 - \gamma)](1 - \theta_N)} \in (0, 1)$.

Proof. See Appendix. ■

Figure 8 presents the frontier $\alpha^d(\sigma)$. In particular this figure shows that for $\sigma < \sigma^{d*}$ the forward-looking rule guarantees a unique equilibrium. In addition it is possible to observe that for $\sigma > \sigma^{d*}$, the more open the economy is (higher α) the more likely is that an active rule leads to a unique equilibrium. This reinforces the idea that even in the case of forward-looking rules an active rule might be viewed as stabilizing for some open economies. However as was pointed out this view may be misleading. As we will show in the global analysis of the equilibrium, active forward-looking rules may also generate deflationary paths and cyclical and chaotic dynamics.

5.2 Global Dynamics

As was said before the following difference equation summarizes the dynamics of our model under forward-looking interest rate rules:

$$\left(\frac{R_{t+1} - 1}{R_{t+1}} \right)^\chi = \frac{R^*}{(R^* - 1)^{\frac{R^* - 1}{A}}} \frac{(R_t - 1)^{\chi + \frac{R^* - 1}{A}}}{R_t^{1 + \chi}} \quad (39)$$

where χ was defined in (30). We will define the left hand-side and the right hand side of equation (39) as $K^f(R)$ and $J^f(R)$ respectively. In order to study the behavior of function $J^f(R)$ it is useful to define the properties of another function defined as $\alpha^v(\sigma)$. The reason is that this function that divides the plane α vs σ into two regions will define the values of α and σ for which the function J^f is either strictly decreasing

or hump-shaped. The function $\alpha^v(\sigma)$ is implicitly defined by $\chi(\alpha, \sigma) = \frac{1-R^*}{A}$ and can be characterized explicitly in the following Lemma.

Lemma 11 Consider the plane α vs σ , define $\Upsilon^v \equiv \frac{1-R^*}{A} < 0$ and let $\chi(0, \sigma)$ be defined as in (34),

1. if $\frac{(1-\gamma)}{\gamma} > -\Upsilon^v$ then the frontier

$$\alpha^v(\sigma) = \left[1 + \Upsilon^v \left(\frac{\gamma}{1-\gamma} \right) \right]^{-1} \left[1 - \frac{\Upsilon^v}{\chi(0, \sigma)} \right]$$

is strictly decreasing and concave for $\sigma < 1$. Moreover $\lim_{\sigma \rightarrow \sigma^{v*}} \alpha^v(\sigma) = 0$, $\lim_{\sigma \rightarrow 0} \alpha^v(\sigma) = 1$, and $\lim_{\sigma \rightarrow 1^-} \alpha^v(\sigma) =$

$-\infty$ where $\sigma^{v*} \equiv \frac{\left[1 - \frac{\Upsilon^v \gamma}{1-\gamma} \right]}{\left[1 - \frac{\Upsilon^v [1-\gamma(1-\theta_N)]}{(1-\gamma)(1-\theta_N)} \right]}$ satisfying $0 < \sigma^{v*} < 1$.

2. if $\frac{(1-\gamma)}{\gamma} < -\Upsilon^v$ then $\alpha^v(\sigma)$ never crosses the region $\alpha \in (0, 1)$ vs $\sigma \in (0, \infty)$.

Proof. See the Appendix. ■

Figure 9 shows the frontier $\alpha^v(\sigma)$ in the plane α vs σ . We can use it and the previous lemma to characterize the behavior of the function $J^f(R)$.

Lemma 12 Recall the definitions of $\alpha^v(\sigma)$, σ^{v*} and Υ^v in the Lemma 11 and assume $\frac{(1-\gamma)}{\gamma} > -\Upsilon^v$. The function $J^f(R) = \frac{R^*}{(R^*-1)^{\frac{R^*-1}{A}}} \frac{(R-1)^{\chi + \frac{R^*-1}{A}}}{R^{1+\chi}}$ has the following features:

1. It is always positive for any $R > 1$ with $\lim_{R \rightarrow 1} J^f(R) = 0$ and $\lim_{R \rightarrow \infty} J^f(R) = 0$.
2. For any $\sigma > 0$ and $\alpha > \alpha^v(\sigma)$ the function $J^f(R)$ is hump-shaped with a peak at $R^{J^f} = \frac{1+\chi}{1-\frac{R^*-1}{A}} > 1$.
3. For $\sigma \in (0, \sigma^{v*}]$ and $\alpha \leq \alpha^v(\sigma)$ the function $J^f(R)$ is strictly decreasing.

Proof. See the Appendix. ■

Lemma 13 The function $K^f(R) = \left[\frac{R-1}{R} \right]^\chi$ has the following features:

1. It is always positive for any $R > 1$;
2. For $\sigma \in (0, 1)$, the function $K^f(R)$ is strictly decreasing with $\lim_{R \rightarrow 1} K^f(R) = \infty$ and $\lim_{R \rightarrow \infty} K^f(R) = 1$.
3. For $\sigma > 1$, the function $K^f(R)$ is strictly increasing with $\lim_{R \rightarrow 1} K^f(R) = 0$ and $\lim_{R \rightarrow \infty} K^f(R) = 1$.

Proof. See the Appendix. ■

As was done for the analysis of contemporaneous rules we can now define some parametric zones with respect to α and σ within which equilibrium dynamics will be extensively studied. In order to accomplish this task it is important to notice the following. From Lemmas 12 and 13 it is clear that for $\alpha \in (0, 1)$ and $\sigma \in (0, 1)$, $J^f(R)$ may have a peak at $R^{J^f} = \frac{1+\chi}{1-\frac{R^*-1}{A}}$ and $K^f(R)$ is monotonically decreasing with respect to R . In addition, from the steady state analysis we know that $J^f(R)$ and $K^f(R)$ meet twice, at the target interest rate R^* and at $R^L < R^*$. On one hand the higher steady state has to occur on the

decreasing side of the function $J^f(R)$. On the other hand, the lower steady state intersection can occur above, below or at R^{Jf} (namely, on the increasing part, decreasing part or at the peak of $J^f(R)$). In other words we can have $R^L \gtrless R^{Jf}$. The equilibrium dynamics will be affected by the case we consider. Since $R^{Jf} = \frac{1+\chi}{1-\frac{R^*-1}{A}}$ we need to study the parametric ranges of α and σ over which $R^L \gtrless \frac{1+\chi}{1-\frac{R^*-1}{A}}$ or equivalently $\chi(\alpha, \sigma) \gtrless \left(1 - \frac{R^*-1}{A}\right) R^L - 1$. This implies we have to define a new frontier or curve $\alpha^w(\sigma)$ that describes the values of α and σ such that $\chi(\alpha, \sigma) = \left(1 - \frac{R^*-1}{A}\right) R^L - 1$. Note that since we are focusing on the case that $\sigma \in (0, 1)$ then by **Fact 2** we know that $\chi(\alpha, \sigma) < 0$ which in turn implies that we are only interested on cases for which $\left(1 - \frac{R^*-1}{A}\right) R^L - 1 < 0$ is valid.

Using Assumptions 2 we can characterize the frontier $\alpha^w(\sigma)$ in the following Lemma.

Lemma 14 Consider the plane α vs σ , define $\Upsilon^w \equiv \left(1 - \frac{R^*-1}{A}\right) R^L - 1$, $\Upsilon^v \equiv \frac{1-R^*}{A}$ and let $\chi(0, \sigma)$ be defined as in (34). Under Assumption 2 and $\frac{(1-\gamma)}{\gamma} > -\Upsilon^v$, the frontier

$$\alpha^w(\sigma) = \left[1 + \Upsilon^w \left(\frac{\gamma}{1-\gamma}\right)\right]^{-1} \left[1 - \frac{\Upsilon^w}{\chi(0, \sigma)}\right]$$

is strictly decreasing and concave for $\sigma < 1$. Moreover $\lim_{\sigma \rightarrow \sigma^{w*}} \alpha^w(\sigma) = 0$, $\lim_{\sigma \rightarrow 0} \alpha^w(\sigma) = 1$, and $\lim_{\sigma \rightarrow 1^-} \alpha^w(\sigma) = -\infty$, where $\sigma^{w*} \equiv \frac{\left[1 - \frac{\Upsilon^w \gamma}{1-\gamma}\right]}{\left[1 - \frac{\Upsilon^w [1-\gamma(1-\theta_N)]}{(1-\gamma)(1-\theta_N)}\right]}$ satisfying $0 < \sigma^{w*} < \sigma^{v*} < 1$, and σ^{v*} was defined in Lemma 11.

Figure 9 presents a graphical representation of the $\alpha^w(\sigma)$ frontier. Using this frontier we can study the regions on the α vs σ plane for which $R^L \gtrless R^{Jf} \equiv \frac{1+\chi}{1-\frac{R^*-1}{A}}$. Note that since $\alpha^w(\sigma)$ describes all the feasible combinations of σ and α such that $\chi(\alpha, \sigma) = \left(1 - \frac{R^*-1}{A}\right) R^L - 1$ then we can pursue the following analysis. Take a pair (σ^w, α^w) such that $\chi(\alpha^w, \sigma^w) = \left(1 - \frac{R^*-1}{A}\right) R^L - 1$. Given **Fact 6** any $\alpha \leq \alpha^w$ implies that $\chi(\alpha, \sigma^w) < \chi(\alpha^w, \sigma^w) = \left(1 - \frac{R^*-1}{A}\right) R^L - 1$. But this implies that $R^{Jf} = \frac{1+\chi(\alpha, \sigma^w)}{\left(1 - \frac{R^*-1}{A}\right)} \leq R^L$. In other words for any $\sigma \leq \sigma^{w*}$ and any $\alpha \leq \alpha^w(\sigma)$, we have that the function $K^f(R)$ meets twice the function $J^f(R)$ in its decreasing part. This particular feature of these functions becomes important to prove that cycles and chaotic dynamics are not possible.

Moreover from Lemmas 12 and 13 it is possible to see that for $\alpha \in (0, 1)$ and $\sigma > 1$, $J^f(R)$ has a peak at $R^{Jf} = \frac{1+\chi}{1-\frac{R^*-1}{A}}$ and $K^f(R)$ is monotonically increasing with respect to R . In addition, from the steady state analysis we know that $J^f(R)$ and $K^f(R)$ meet twice, at the target interest rate R^* and at $R^L < R^*$. The lower steady state, R^L occurs on the increasing part of the function $J^f(R)$. But the higher steady state, R^* may occur above, below or at R^{Jf} (namely, on the decreasing part, increasing part, or at the peak of $J^f(R)$). In other words we can have $R^{Jf} \gtrless R^*$. The equilibrium dynamics will be affected by the case we consider. Since $R^{Jf} = \frac{1+\chi}{1-\frac{R^*-1}{A}}$ we need to study the parametric ranges of α and σ over which $\frac{1+\chi}{1-\frac{R^*-1}{A}} \gtrless R^*$ or equivalently $\chi(\alpha, \sigma) \gtrless \left(1 - \frac{R^*-1}{A}\right) R^* - 1$. This implies we have to define a new frontier or curve $\alpha^a(\sigma)$ that describes the values of α and σ such that $\chi(\alpha, \sigma) = \left(1 - \frac{R^*-1}{A}\right) R^* - 1$ or equivalently, $\chi(\alpha, \sigma) = (R^* - 1) \left(1 - \frac{R^*}{A}\right) > 0$.

The following lemma characterizes the frontier $\alpha^k(\sigma)$.

Lemma 15 Consider the plane α vs σ , define $\Upsilon^k \equiv (R^* - 1) \left(1 - \frac{R^*}{A}\right)$ and let $\chi(0, \sigma)$ be defined as in (34). If $\sigma > 1$ and $\frac{(1-\gamma)(1-\theta_N)}{1-\gamma(1-\theta_N)} > \Upsilon^k$ then the local equilibrium frontier

$$\alpha^k(\sigma) = \left[1 + \Upsilon^k \left(\frac{\gamma}{1-\gamma}\right)\right]^{-1} \left[1 - \frac{\Upsilon^k}{\chi(0, \sigma)}\right]$$

is a well-defined function, strictly increasing and concave for $\sigma \geq \sigma^{k*} > 1$ where $\sigma^{k*} \equiv 1 + \frac{\left[\frac{\Upsilon^k}{1-\gamma(1-\theta_N)}\right]}{\left[\frac{(1-\gamma)(1-\theta_N)}{1-\gamma(1-\theta_N)} - \Upsilon^k\right]}$. Moreover $\lim_{\sigma \rightarrow \sigma^{k*}} \alpha^k(\sigma) = 0$, $\lim_{\sigma \rightarrow \infty} \alpha^k(\sigma) = \alpha^{k*}$, $\sigma^{k*} > \sigma^{d*}$ and $\alpha^{k*} < \alpha^{d*}$ where $\alpha^{k*} \equiv 1 - \frac{\Upsilon^k}{[1-\Upsilon^k(1-\gamma)](1-\theta_N)} \in (0, 1)$; and σ^{d*} and α^{d*} were defined in Proposition 10.

Proof. See Appendix. ■

A graphical representation of $\alpha^k(\sigma)$ can be found in Figure 9. Using this frontier it is possible to study the regions on the α vs σ plane for which $R^* \gtrless R^{Jf} \equiv \frac{1+\chi}{1-\frac{R^*}{A}}$. To do so, it is important to observe that since $\alpha^k(\sigma)$ describes all the feasible combinations of σ and α such that $\chi(\alpha, \sigma) = (R^* - 1) \left(1 - \frac{R^*}{A}\right)$ then we can pursue the following analysis. Take a pair (σ^k, α^k) such that $\chi(\alpha^k, \sigma^k) = (R^* - 1) \left(1 - \frac{R^*}{A}\right)$. Given **Fact 5** any $\alpha \leq \alpha^k$ implies that $\chi(\alpha, \sigma^k) \geq \chi(\alpha^k, \sigma^k) = (R^* - 1) \left(1 - \frac{R^*}{A}\right)$. But this implies that $R^{Jf} \geq \frac{1+\chi(\alpha, \sigma^k)}{(1-\frac{R^*}{A})} > R^*$. In other words for any $\sigma \geq \sigma^{k*} > 1$ and any $\alpha \leq \alpha^k(\sigma)$, we have that the function $K^f(R)$ meets twice the function $J^f(R)$ in its increasing part. But as we will see, this feature leads to the no possibility of cycles and chaotic dynamics.

This analysis, Lemmas 11, 12, 13, 14, and 15 help us to divide the parametric space α vs σ into 5 zones:²⁰

1. Zone 1: $\sigma \in (0, \sigma^{w*}]$ and $\alpha \in (0, \alpha^w(\sigma)]$. $K^f(R)$ is strictly decreasing and $J^f(R)$ is either strictly decreasing or hump-shaped. They meet twice and in the case in which $J^f(R)$ is hump-shaped, they meet in the decreasing part of $J^f(R)$. This means that in this case $R^{Jf} \leq R^L$. Standard liquidity traps (or deflationary paths) can be shown to occur as we will see below.
2. Zone 2: $\sigma \in (0, 1)$ and $\alpha \in (\alpha^w(\sigma), 1)$. $K^f(R)$ is strictly decreasing and $J^f(R)$ hump-shaped. They meet twice but in this case $R^{Jf} > R^L$. As will be shown cycles and chaotic dynamics around the passive steady state may occur.
3. Zone 3: $\sigma \in (1, \sigma^{k*})$ and $\alpha \in (0, 1)$. $K^f(R)$ is always increasing and $J^f(R)$ is hump-shaped. They meet twice but in this case $R^{Jf} < R^*$. As will be shown cycle and chaotic dynamics around the active steady state may occur.
4. Zone 4: $\sigma > \sigma^{k*}$ and $\alpha \in (\alpha^k(\sigma), 1)$. The properties of $K^f(R)$ and $J^f(R)$ are the same as in Zone 3 with $R^{Jf} = R^*$ over $\alpha^k(\sigma)$.
5. Zone 5: $\sigma \geq \sigma^{k*}$ and $\alpha \in (0, \alpha^k(\sigma)]$. $K^f(R)$ is always increasing and $J^f(R)$ is hump-shaped. They meet twice but in this case $R^{Jf} \geq R^*$. In this zone monotonic inflationary paths converging to the active steady state occur and cycles and chaotic dynamics are not present.

²⁰Although these zones are not marked in Figure 9, this figure is still useful to understand the definition of the zones.

Using these parametric zones and Lemmas 11, 12, 13, 14, and 15 it is possible to prove the existence of standard liquidity traps, cycles and chaos for forward-looking rules as we did for contemporaneous rules.

We intentionally omit formal proofs on the existence of chaos and cycles for Zone 2, since this case is isomorphic to what extensively analyzed for the contemporaneous rule case (specifically what we labelled Zone 4 and 5 over there). It is hower of interest to consider more formally what occurs for relative risk aversion above one.

Proposition 16 *Let $\sigma > \sigma^{k^*}$ and $\alpha \in (0, \alpha^k(\sigma))$, namely Zone 5 For any $R_0 \in (R^L, R^*)$, $\lim_{t \rightarrow \infty} R_t = R^*$.*

Proof. The proof is trivial and therefore omitted. The reader should simply note that in this case the mapping $R_{t+1} = f(R_t)$ is monotonically increasing over the set (R^L, R^*) . ■

Therefore Zone 5 of the forward looking case displays monotonic paths converging to the target steady state for any initial value between the target and the low steady state. Now we focus on Zones 3 and 4. Following steps similar to the contemporaneous rules case, define \bar{R} and \tilde{R} as follows:

$$\begin{aligned} K(\bar{R}) &= J(R^J) \iff \bar{R} = f(R^J) \\ J(\tilde{R}) &= J(R^L) \iff R^L = f(\tilde{R}) \end{aligned}$$

It should be evident that $\bar{R} > R^J$ and that $\tilde{R} > R^J$ too. As pointed out before the forward mapping f is unimodal with a maximum at R^J . The following assumption will be used throughout the formal analysis.

Assumption 4²¹: $f_{\max} = f(R^J) \leq \tilde{R}$.

Proposition 17 *Let $\sigma \in (1, \sigma^{k^*})$ and $\alpha \in (0, 1)$, or $\sigma > \sigma^{k^*}$ and $\alpha \in (\alpha^k(\sigma), 1)$, i.e. Zone 3 and 4. If Assumption 3 is satisfied, the followings hold:*

1. *The mapping f is such that $f : [R^L, \bar{R}] \rightarrow [R^L, \bar{R}]$. Moreover for any $R_t \in (\bar{R}, \tilde{R}]$, $R_t \in [R^L, \bar{R}]$.*
2. *Period 2 cycles exists within such set.*
3. *Topological chaotic dynamics, in a Li-Yorke sense are possible.*

Proof.

1. The proof is trivial and therefore omitted.
2. Define a function $g(R) = R - f^2(R)$. We need to distinguish between the two cases in Assumption 3:
 - a) $f(R^J) = \tilde{R}$; b) $f(R^J) < \tilde{R}$.

- (a) The set invariant under mapping f is therefore $[R^L, \tilde{R}]$. It should be clear that $g(R^L) = g(R^*) = 0$, $g(R^J) > 0$, $g(\tilde{R}) > 0$. and $g'(R^L) < 0$ Since the mapping f is continous, a sufficient condition for the existence of period-2 cycles (namely of zeros of the function g) is that $g'(R^*) < 0$ as well.

²¹If $f_{\max} = f(R^J) > \tilde{R}$ there is no non-trivial mapping invariant set. This case gives rise to another type of equilibria of non cyclical nature. More specifically, we can define subset of point within the domain of f that leave such set after a finite number of iterations. They would settle on a path converging to the lower bound of the interest rate.

This is indeed holding if $f'(R^*) < -1$, i.e. over regions of the (α, σ) plane where the model displays LOCAL uniqueness (above the frontier $\alpha^d(\sigma)$). In the region between $\alpha^d(\sigma)$ and $\alpha^k(\sigma)$, we have that $f'(R^*) \in (-1, 0)$ and that sufficient condition fails. Instead $g(R^*) > 0$. We can still show existence of period-2 cycles around the active steady state R^* by construction. Let $\widehat{R} \in (R^J, \widetilde{R})$ be the pre-image of R^J , i.e. $R^J = f(\widehat{R})$. Clearly $g(\widehat{R}) = \widehat{R} - \widetilde{R} < 0$. This together with the pervious information implies the existence of at least other four zeros of g , respectively two to the right of R^* and the other two to its left.

- (b) The set invariant under mapping f is therefore $[R^L, \overline{R}]$. As in case a), $g(R^L) = g(R^*) = 0$, and $g'(R^L) < 0$. But now $g(\overline{R}) \geq 0$ and $(R^J) \geq 0$. If $g(R^J) = 0$, the period-2 cycle is $\{R^J, \overline{R}\}$. If it is an inequality, we need to make some distinction. Again if we are within a region where $f'(R^*) < -1$, by continuity there exists a point $R_l \in (R^L, R^*)$ such that $g(R_l) = 0$. Let \widetilde{R}^* be the pre-image of R^* . It has to be that $g(\widetilde{R}^*) < 0$. Then if $g(R^J) > 0$, $R_l \in (\widetilde{R}^*, R^J)$; $g(R^J) < 0$, $R_l \in (R^J, R^*)$. In both cases this implies that the second focal point of the period-2 cycle, $R_u = f(R_l) \in (R^*, \overline{R})$

If instead we lie between $\alpha^d(\sigma)$ and $\alpha^k(\sigma)$ and we have that $f'(R^*) \in (-1, 0)$, period-2 cycles occur surely if $g(R^J) > 0$. In this case since $g(R^*) > 0$, two zeros of g occur to the right of R^* , one between R^J and R^* , and one between \widetilde{R}^* and R^J . Cycles might not occur if $g(R^J) > 0$.

3. Define a function $h(R) = R - f^3(R)$. Again we need to distinguish between the two cases in Assumption 3: **a)** $f(R^J) = \widetilde{R}$; **b)** $f(R^J) < \widetilde{R}$.

- (a) It should be clear that $h(R^L) = h(R^*) = 0$, $h(R^J) > 0$, $h(\widetilde{R}) > 0$. and $h'(R^L) < 0$. We show that the Li-Yorke sufficient condition for the existence of topological chaos is satisfied. Moreover $h'(R^*) > 0$ always. But this is sufficient to show the existence of a period-3 cycle since there must be a point between R^J and R^* such that $h(R) = 0$. By Sarkovskii's Theorem cycles of any periodicity exist. The Li-Yorke condition applies too. Let $\widehat{R} \in (R^L, R^J)$ be the pre-image of R^J . We then have that $R^J = f(\widehat{R})$; $\overline{R} = f^2(\widehat{R})$ and $R^L = f^3(\widehat{R})$ with $R^L < \widehat{R} < R^L < \overline{R}$. Such condition holds.

- (b) It should be clear that $h(R^L) = h(R^*) = 0$, $h'(R^*) > 0$ and $h'(R^L) < 0$. But now $h(R^J) \geq 0$. Similarly to the analogous case for the contemporaneous rule, those conditions are enough for the existence of a period 3 cycles and therefore chaos if $h(R^J) \geq 0$.

■

The following figure summarizes the main results of the global analysis. Afterward we will use this figure to compare the results from the forward-looking rules analysis with the results from the contemporaneous rules analysis presented in Figure 6.

Figure 9 shows that forward looking rules alter the flavour of our previous conclusions for contemporaneous rules. Although as before interest rate rules that target the expected future CPI-inflation can be highly destabilizing, there are some important differences with respect to the case of contemporaneous interest rate rules. It is still valid that the degree of openness of the economy matters for the appearance of cycles and chaotic dynamics. However under forward-looking rules these types of endogenous fluctuations may appear

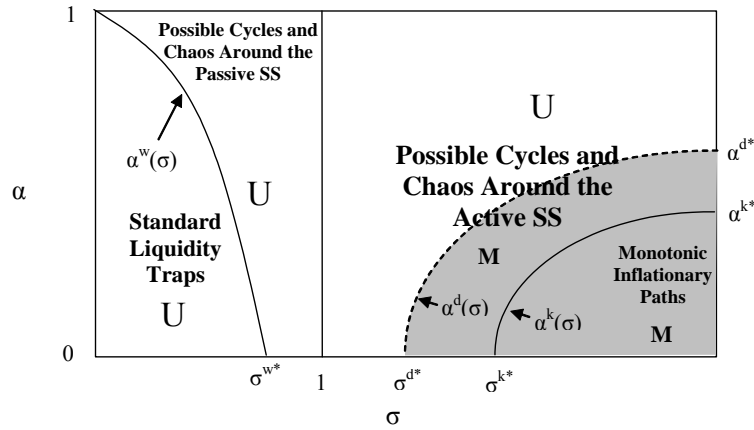


Figure 9: Equilibrium analysis for an active forward-looking interest rate rule. This figure shows the results from the global equilibrium analysis and a comparison between the local equilibrium analysis and the global equilibrium analysis. M stands for local multiple equilibria and U stands for a local unique equilibrium. α corresponds to the degree of openness of the economy and σ denotes the relative risk aversion coefficient.

even for economies with relative risk aversion coefficients smaller than one. In this case, given the relative risk aversion coefficient, the more open the economy is the more likely is that a forward-looking rule will lead the economy to cyclical and chaotic dynamics around the passive steady state.

Furthermore, in contrast to the results for contemporaneous rules, the degree of openness of the economy plays an opposite role for economies that follow a forward-looking rule and have a relative risk aversion coefficient greater than one. In other words, given a relative risk aversion coefficient bigger than one, the more open the economy is the more likely is that a forward-looking rule will drive the economy to cyclical and chaotic dynamics around the active steady state.

As was mentioned above, following the arguments and proofs developed for contemporaneous interest rate rules, it is possible to state proofs of the existence of two-period cycles and chaotic dynamics under forward-looking rules. In stead of doing this analysis we will use the aforementioned parametrization and present the following orbit diagrams for $\sigma = \{0.8, 2\}$.

Figures 10 and 11 show how depending on the degree of openness of the economy, an active forward-looking rule may drive the economy into period-2 cycles, period-4 cycles, period-8 cycles and even chaotic dynamics. Both Figures show that as the degree of openness of the economy, α , increases from zero, the rule drives the economy into a period-2 cycle, as indicated by the first split into two branches. As the economy becomes more open both branches split simultaneously yielding a period-4 cycle. A cascade of further period doublings occurs as the degree of openness of the economy increases, yielding period-8, period-16 and so on. Finally after some degree of openness the rule drives the economy into a chaotic dynamics, that is when the map (31) becomes chaotic and the attractor changes from a finite to an infinite set of point.

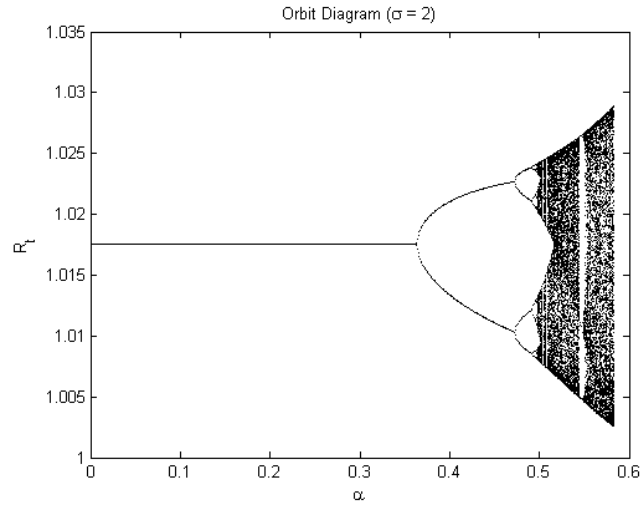


Figure 10: Orbit diagram for an active forward-looking rule and $\sigma = 2$. R_t denotes the nominal interest rate and σ stands for the relative risk aversion coefficient.

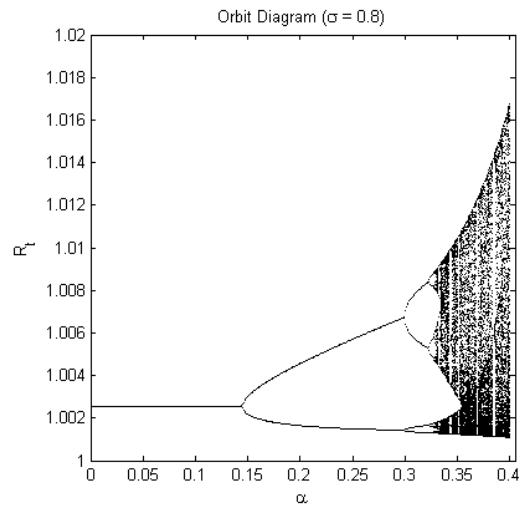


Figure 11: Orbit diagram for an active forward-looking rule and $\sigma = 0.8$. R_t denotes the nominal interest rate and σ stands for the relative risk aversion coefficient.

In general terms the Figures suggest that the more open the economy is, the more likely is that the rule will cause chaotic dynamics. However it is important to observe that the relative risk aversion coefficient also plays a role in the analysis. As we found before, for relative risk aversion coefficients greater than one, $\sigma > 1$, (that is when consumption and real money balances are substitutes), the cyclical and chaotic dynamics may occur around the active steady state as shown in Figure 10. On the other hand, for relative risk aversion coefficients smaller than one, $\sigma < 1$, (that is when consumption and real money balances are complements), the cyclical and chaotic dynamics may be present around the passive steady state as shown in Figure 11.

To conclude the global equilibrium analysis it is important to point out how the results presented in this analysis may be affected by relaxing the assumption that $\frac{(1-\gamma)}{\gamma} > -\Upsilon^v$. If it is assumed that $\frac{(1-\gamma)}{\gamma} < -\Upsilon^v \equiv \frac{R^*-1}{A}$ but $\frac{(1-\gamma)}{\gamma} > -\Upsilon^w \equiv 1 - \left(1 - \frac{R^*-1}{A}\right) R^L$ then the results are not affected. However if it is assumed that $\frac{(1-\gamma)}{\gamma} < -\Upsilon^w < -\Upsilon^v$, then the degree of openness plays no role in the analysis for $\sigma < 1$. In other words, regardless of the degree of openness, the model displays chaotic and cyclical dynamics around the passive steady state for relative risk aversion coefficients smaller than one. However if the relative risk aversion coefficient is greater than one, it is still valid that the more open the economy is the more likely is that an active forward-looking rule will drive the economy to chaotic and cyclical dynamics around the active steady state.

5.3 Local Uniqueness vs. Global Multiplicity

As we have done for the contemporaneous case, we compare the local and global results for forward-looking rules. A graphical comparison is presented in Figure 9. Once more by studying dynamics in a small neighborhood of the active steady state we would conclude that active rules deliver stability in equilibrium for any level of openness, as long as the risk aversion coefficient stays below an upper threshold σ^{d*} . For higher risk aversion coefficients a unique equilibrium still occurs as long as the economy is open enough.

The conclusions from the global analysis are radically different. In particular for very open economies with relative risk aversion coefficients greater than one active forward-looking rules may lead the economy to cyclical and chaotic dynamics around the active steady state. In addition for economies with relative risk aversion coefficients smaller than one, the aforementioned rules may drive the economy not only to liquidity traps (deflationary paths) but also to chaotic and cyclical dynamics around the passive steady state. In both cases the existence of this chaotic and cyclical dynamics are associated with the degree of openness of the economy.

6 A Sensitivity and a Quantitative Analysis

In this section, we use the parametrization of Table 2 to pursue two exercises for contemporaneous and forward-looking rules. The first one is to assess the size of the local and global uniqueness/multiplicity regions shown in Figures 6 and 9. The second one consists of studying how these regions vary accordingly to changes in parameters such as the share of expenditure of real money balances, $1 - \gamma$, the inflation target π^* , and the degree of active responsiveness to the CPI-inflation of the rule, $\frac{A}{R^*}$. As before we will do the analysis using the plane α vs σ considering feasible values for these parameters.

First we pursue the analysis for active contemporaneous rules. The results are presented in Figure 12. For

the following analysis it is useful to use this Figure and Figure 6. In Figure 12 the top-left panel represents the base case that sets $1 - \gamma = 0.03$, $\pi^* = 1.008$, and $\frac{A}{R^*} = 2.24$. From this panel it is clear that given typical values used in the RBC literature $\sigma \in (1, 3)$ and depending on the degree of openness the economy, the active contemporaneous rule may drive the economy to cycles and chaos around both the active and the passive steady states and to liquidity traps.

The top-right panel draws the frontiers that determine the local and global multiple equilibria regions, after a change in the share of expenditure of real money balances, $1 - \gamma$. It shows that a reduction in this share shifts all the frontiers down increasing the area of possible liquidity traps and reducing not only the areas of cyclical and chaotic dynamics but also the areas of local multiple equilibria. In this sense economies that are “less cash dependent” (lower $1 - \gamma$) are less likely to be driven to local and global multiple equilibria (cycles or chaos) by an active contemporaneous interest rate rule.

The effects on the frontiers from increasing the inflation target are shown in the bottom-left panel of Figure 12. From this panel we can conclude that increasing the inflation target will also shift down the frontiers. This implies that local and global multiple equilibria become less likely.

Finally the bottom-right panel represents the case of a reducing the degree of responsiveness of the rule with respect to the CPI-inflation. In this case we assume that the interest rate response coefficient to the CPI-inflation corresponds to the one in the well studied Taylor rule $\frac{A}{R^*} = 1.5$. The reduction in the level of aggressiveness of the rule with respect of inflation causes a shift down of all the frontiers meaning that the local and global multiple equilibria become less feasible. In other words, a more aggressive central bank with respect to the CPI-inflation is more likely to lead the economy to cycles and chaos than a less aggressive one.

In Figure 13 we show the results for forward-looking rules of the same two exercises we did for the contemporaneous rules. Using this Figure and Figure 10. We can pursue the following analysis. The base case is presented in the top-left panel setting $1 - \gamma = 0.03$, $\pi^* = 1.008$, and $\frac{A}{R^*} = 2.24$. It is clear that for $\sigma \in (1, 2)$ an active forward-looking rule assures a local unique equilibrium but depending on the degree of openness it also may drive the economy to cycles or chaos around the inflation target. Furthermore for the type of rules under analysis and for $\sigma < 1$, cyclical and chaotic dynamics around the passive steady state may appear depending on the degree of openness.

The top-right panel shows the effects on the frontiers that determine the local and global multiple equilibria regions, caused by a change in the share of expenditure of real money balances, $1 - \gamma$. In contrast to the results for contemporaneous rules we find that “less cash dependent economies” (lower $1 - \gamma$) are more likely to be driven to cyclical and chaotic dynamics around the active or the passive steady state by forward-looking rules.

The bottom-left panel of Figure 13 presents the results of an increase in the inflation target. This increase shifts down the frontiers implying that possible cycles, chaos and local multiple equilibria for forward-looking rules become more likely under active forward-looking rules.

Finally the bottom-right panel represents the case of a reducing the degree of responsiveness of the rule with respect to the CPI-inflation to $\frac{A}{R^*} = 1.5$ (the Taylor rule). Although in principle the reduction of the degree of aggressiveness of the rule does not affect the frontiers for $\sigma < 1$, it can be observed that it causes a shift up of all the frontiers for $\sigma > 1$. This means that local multiple equilibria become more feasible whereas

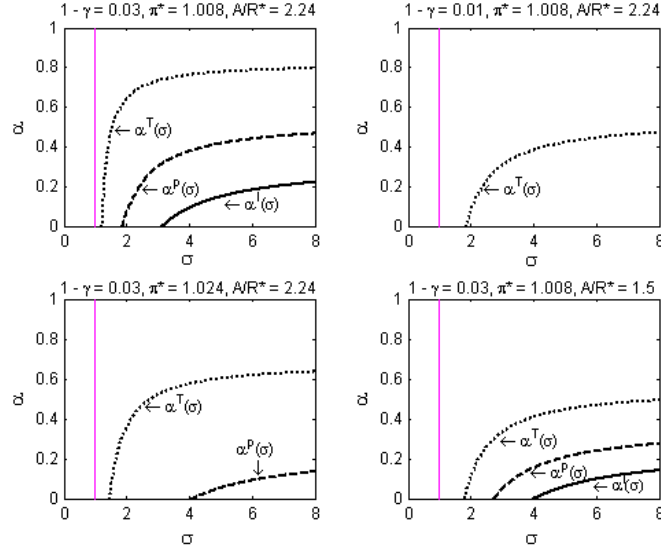


Figure 12: This figure shows how the local and global equilibrium frontiers for active contemporaneous rules vary with changes of the share of expenditures on real money balances, $1 - \gamma$, the target inflation, π^* , and the degree of responsiveness of the rule, $\frac{A}{R^*}$. The base case corresponds to the left-top panel. See also Figure 7. α corresponds to the degree of openness of the economy and σ denotes the relative risk aversion coefficient.

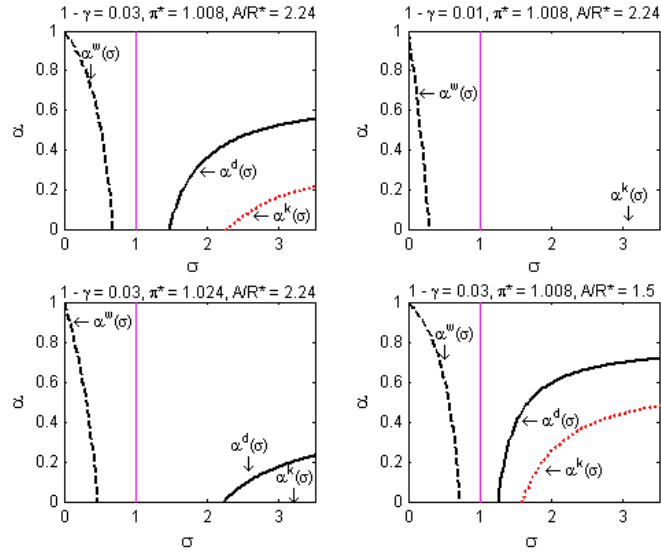


Figure 13: This figure shows how the local and global equilibrium frontiers for active forward-looking rules vary with changes of the share of expenditures on real money balances, $1 - \gamma$, the target inflation, π^* , and the degree of responsiveness of the rule, $\frac{A}{R^*}$. The base case corresponds to the left-top panel. See also Figure 10. α corresponds to the degree of openness of the economy and σ denotes the relative risk aversion coefficient.

global multiple equilibria (cycles and chaos around the inflation target) become less possible. In other words and in contrast to the contemporaneous rules results we have the following result. Although it is less likely that a more aggressive forward-looking rule with respect to the CPI-inflation will lead the economy to local multiple equilibria, it also more likely that the same rule will drive the same economy to cycles and chaos around the inflation target.

7 Avoiding Cyclical and Chaotic Equilibria:

In this section, we show how the existence of cyclical and chaotic equilibria around either the active or the passive steady state depends on the importance of real money balances for welfare. Other things being equal the equilibrium level of real balances is clearly increasing in the parameter γ . We consider separately the case of contemporaneous and forward looking rules.

7.1 Contemporaneous Inflation Rule

Throughout the paper we assumed that $\frac{\kappa}{\varpi(0)} > \frac{1}{2} (R^* - 1) \left(1 + \frac{R^*}{A}\right)$. This was necessary and sufficient for the possibility of parametric regions of, respectively, local indeterminacy and local determinacy (but global multiplicity) with respect to openness and risk aversion coefficients. Given the monetary policy related parameters (R^* and A), we study how the left hand side of the inequality above depends on γ .

Define the function $\delta(\gamma) = \frac{\kappa}{\varpi(0)}$. Clearly $\delta'(\gamma) < 0$, $\delta(0) = 1 - \theta_N$ and $\delta(1) = 0$. If $(1 - \theta_N) > \frac{1}{2} (R^* - 1) \left(1 + \frac{R^*}{A}\right)$, then $\delta(\gamma)$ crosses the lines $\frac{1}{2} (R^* - 1) \left(1 + \frac{R^*}{A}\right)$, $\frac{R^*-1}{A}$ and $R^L - 1$ at positive values of γ , that we denote, respectively by γ^c , γ^* and $\tilde{\gamma}$.

We can then observe that for $\gamma \in (0, \gamma^c)$, with $\gamma^c = \frac{1 - \theta_N - \frac{1}{2}(R^* - 1)\left(1 + \frac{R^*}{A}\right)}{1 - \theta_N - (1 - \theta_N)\frac{1}{2}(R^* - 1)\left(1 + \frac{R^*}{A}\right)}$, $\delta(\gamma) > \frac{1}{2} (R^* - 1) \left(1 + \frac{R^*}{A}\right)$ and all dynamics described in the contemporaneous rules section above are possible. However if $\gamma \in (\gamma^c, \gamma^*)$, with $\gamma^* = \frac{1 - \theta_N - \frac{R^* - 1}{A}}{1 - \theta_N - (1 - \theta_N)\frac{R^* - 1}{A}}$, $\delta(\gamma) \in \left(\frac{R^* - 1}{A}, \frac{1}{2} (R^* - 1) \left(1 + \frac{R^*}{A}\right)\right)$ and the model always delivers local uniqueness. Multiple global equilibria still occur as described. As we move to $\gamma \in (\gamma^*, \tilde{\gamma})$ with $\tilde{\gamma} = \frac{1 - \theta_N - (R^L - 1)}{1 - \theta_N - (1 - \theta_N)(R^L - 1)}$, $\delta(\gamma) \in (R^L - 1, \frac{R^* - 1}{A})$ and the model can not display cycles/chaos around the active steady state but only around the passive one. Finally for $\gamma > \tilde{\gamma}$, $\delta(\gamma) < R^L - 1$ and the forward dynamics mapping is monotonic between the two steady states. This rules out any possible cycle or chaotic behavior around both steady states. In this case multiple global equilibria are possible but take the form of monotonic deflationary paths (or standard liquidity traps) only.

From this technical observation we can conclude that erratic equilibrium dynamics of the form described extensively in the paper become less likely as the role of real money balances in transactions vanishes. Economies in which cash is greatly valued for transaction purposes (low γ) seem more vulnerable to the endogenous volatility described in this study. On the contrary, as the credit market improves and cash becomes less and less needed, such endogenous fluctuations disappear and the only risky equilibrium becomes a liquidity trap (the short interest rate falls progressively towards zero). It would be interesting then to compare the aggregate performance of developed and emerging economies following some kind of implicit CPI inflation targeting as defined here. We are currently pursuing this.

Apart from allowing us to distinguish between economies with different levels of financial innovation, these latest results on γ highlight some role for monetary policy to eliminate or at least reduce unwanted

multiplicity. In fact for a given importance of real balances (i.e. given γ), the central bank could design an active interest rate rule that could both deliver local uniqueness and eliminate cyclical/chaotic equilibria. The following proposition defines the necessary conditions.

Proposition 18 *Take a given $\gamma \in (0, 1)$. If the inflation target $\pi^* > \beta(1 + \delta(\gamma))$ - with $\delta(\gamma)$ defined above - and the interest rate reaction to current CPI inflation is bigger than one but smaller than an upper-bound $\bar{\rho}$ (defined below), then no cyclical/chaotic equilibria can occur for contemporaneous active Taylor rules.*

Proof. Since we want $R^L - 1 > \delta(\gamma)$, we need $R^* > 1 + \delta(\gamma)$, i.e. if a second steady state exists it has to be lower than the target. A second condition has to do with the steepness of the rule around the target itself. If the elasticity is exactly one, we have two coincident steady states and no dynamics occur. However the stationary equilibrium is hyperbolic and local analysis is meaningless. By continuity, as we increase the elasticity slightly above one a second steady state arises but not that far from the target. A sufficient condition for R^L to occur to the right of $1 + \delta(\gamma)$ is that $R^* (\delta(\gamma))^{R^*-1} < (1 + \delta(\gamma)) (R^* - 1)^{R^*-1}$. Taking logs of both sides, rearranging and using fact that $A = \beta\rho'(\pi^*)$, we get that this occurs if

$$\rho'(\pi^*) < \frac{R^* - 1}{\beta} \frac{\ln(R^* - 1) - \ln\delta(\gamma)}{\ln R^* - \ln(1 + \delta(\gamma))} \quad (40)$$

This poses an upper bound to the level of inflation aggressiveness consistent with local determinacy and no equilibrium cycles/chaos ■

Though the result might have been quite intuitive since the beginning, we have stated formally the importance of choosing the target inflation rate, in addition to the level of aggressiveness, in order to make endogenous fluctuations less likely. And this has can be accomplished without requiring any specific Ricardian or non-Ricardian fiscal policy rule. The latter channel might still play a role in eliminating deflationary paths too. This is the purpose of a research project we are currently dealing with.

7.2 Forward-Looking Inflation Rule

For a forward looking rule we have shown that the degree of openness plays a role on for a coefficient of relative risk aversion below one. There, we identified an openness frontier distinguishing between standard liquidity traps and cycles/chaos around the passive steady state. For risk aversion above one, the only role of openness regards the local equilibrium determinacy. At a global level cycles around the active steady state are always possible.

A necessary and sufficient condition for the existence of an α -frontier dividing liquidity traps from cycles was $\frac{(1-\gamma)}{\gamma} > -\Upsilon^w$. Given the monetary policy parameters defining Υ^w , we study for which values of γ this inequality holds.

Let $\delta(\gamma) = \frac{(1-\gamma)}{\gamma}$. The following are clearly true: $\delta'(\gamma) < 0$, $\delta(0) = \infty$ and $\delta(1) = 0$. Therefore if $-\Upsilon^w = 1 - R^L \left(1 - \frac{R^*-1}{A}\right) > 0$, there exists a $\gamma^c \in (0, 1)$ such that $\delta(\gamma^c) = -\Upsilon^w$. It follows that for $\gamma \in [\gamma^c, 1)$, $\delta(\gamma) \leq -\Upsilon^w$ the frontier α^w is not defined. This implies that standard liquidity traps, though still possible, become a "measure zero event", since the economy would display a monotonically decaying series of interest rates only for a peculiar initial condition. For values below such critical point, the frontier is well defined and liquidity traps become the unique type of self-fulfilling equilibrium for levels of openness.

The policy implication is not as clear as what obtained in the contemporaneous case. Under a forward looking rule more volatile dynamics occur when real balances are less valued for welfare. They can still occur even for more cash-dependent economies but together with deflationary paths.

Is there any role then for monetary policy to at least get rid of cyclical patterns, as we found for the contemporaneous case? An intuitive policy prescription could be the following. Take a small open economy, characterized by a triplet (σ, α, γ) , for $\sigma < 1$. The monetary authority could design a monetary rule appropriately (by accurately choosing R^* and A , such that $\chi(\sigma, \alpha, \gamma) < \Upsilon^w$). This would bring the economy out of the cyclical pattern region, though still allowing liquidity traps.²²

Now we move to $\sigma > 1$. Here cycles disappear if $\chi(\alpha, \sigma, \gamma) > \Upsilon^k$. This would occur for any level of openness if $\Upsilon^k \equiv (R^* - 1) \left(1 - \frac{R^*}{A}\right) = 0$. But this is never true in our model since we consider specifically $R^* > 1$ and $A > R^*$. Nevertheless, Υ^k is affected by the choice of the inflation target π^* and the interest rate rule responsiveness $\rho = \frac{A}{R^*}$ since it can be written as $\Upsilon^k \equiv (\beta\pi^* - 1) \left(1 - \frac{1}{\rho}\right)$. Then, given a triplet (α, σ, γ) , the monetary authority could either a) keep ρ constant and choose the target $\pi^* < \beta^{-1} \left[1 + \frac{\chi(\alpha, \sigma, \gamma)}{1 - \frac{1}{\rho}}\right]$; or b) keep the target π^* constant and choose the responsiveness $\rho < \frac{1}{1 - \frac{\chi(\alpha, \sigma, \gamma)}{\beta\pi^* - 1}}$.

8 Backward-Looking Rules and Targeting The Non-Traded Good Inflation

8.1 Backward-Looking Rule

We have shown that contemporaneous and forward-looking rules may lead to cyclical and chaotic dynamics and more importantly that these dynamics are related to the degree of openness of the economy and to the relative risk aversion coefficient. The next step is to study the dynamics of the small open economy model under an active backward-looking rule. In this case the rule is defined as:

$$R_t \equiv 1 + (R^* - 1) \left(\frac{\pi_{t-1}}{\pi^*}\right)^{\frac{A}{R^* - 1}}; \quad \text{with } \frac{A}{R^*} > 1 \quad (41)$$

and the first order conditions of the model can be reduced to:

$$\left(\frac{R_t - 1}{R_t}\right)^x = \frac{\beta R_t}{\pi_{t+1}} \left(\frac{R_{t+1} - 1}{R_{t+1}}\right)^x \quad (42)$$

These last two equations form a system of two first-order difference equations. As is well known to derive analytic results, as before, from the non-linear study of this system is a very difficult task. Therefore we rely on simulations trying to find if for different values of α and σ the system presents cycles or chaos. The results that are available upon request show that these types of dynamics are not present under backward-looking rules. In other words, the model always converges to either the active steady state or to the passive steady-state.

²²Cycles and chaos around the passive steady state are ruled out for any level of openness if the threshold Υ^w is bigger than zero. It can be shown that this can be achieved by choosing R^* and A accordingly.

8.2 Targeting the Non-Traded Goods Inflation Rate

It is possible to pursue all the previous analysis for contemporaneous and forward-looking rules changing the target of the rule from the CPI-inflation to the Non-traded goods inflation. If the government targets the non-traded goods inflation rate, the analyses for contemporaneous and forward-looking rules correspond to study the equations (29) and (31), as before, but replacing the exponent χ by χ_0 , where χ_0 is defined as $\chi_0 \equiv \frac{\chi}{1-\alpha}$. In fact, under some assumptions, it is possible to derive similar lemmas and propositions to the ones derived before.²³ Due to space constraint, we prefer to use the parametrization of Table 2 and draw the frontiers that determine the local and global equilibria regions in the plane α vs σ for the aforementioned rules. Figure 14 presents the results. In this figure we have abused of notation using the same names $\alpha^T(\sigma)$, $\alpha^p(\sigma)$ and $\alpha^I(\sigma)$, and $\alpha^w(\sigma)$, $\alpha^d(\sigma)$ and $\alpha^k(\sigma)$ for the frontiers as we used before.²⁴ We keep the same notation to facilitate comparisons. The top panel of this figure shows the local and global multiple equilibria regions for the contemporaneous rule. To some extent it seems that given the relative risk aversion coefficient (for instance $\sigma = 2.1$), the degree of openness differentiates between the possibility of cycles and chaos around the passive steady state and the possibility of the same type of dynamics around the active steady state. It is also clear that for specific relative risk aversion coefficients, standard liquidity traps may arise for any degree of openness of the economy.

The bottom panel presents the results for the forward-looking rule. It is possible to observe that in this case the degree of openness of the economy is not as important as the coefficient of relative risk aversion coefficient in determining the possible dynamics of the model. In fact we can pursue a closer analysis of this case and construct an orbit diagram varying the relative risk aversion coefficient.

Figure 15 presents the results. The top panel corresponds to the case of an almost closed economy ($\alpha = 0.01$), while the bottom panel corresponds to an open economy ($\alpha = 0.50$). For low relative risk aversion coefficients ($\sigma < 1$) the rule drives the economy into standard liquidity traps and as the coefficient increases cycles and chaos around the passive steady state appear. For high relative risk aversion coefficients ($\sigma > 1$) the story is different. The economy is driven into chaotic dynamics around the active steady state and as the coefficient increases cycles and monotonic inflationary paths converging to the active steady-state appear.²⁵ As can be observed the role of the degree of openness of the economy in the analysis is not as important as the role played by the relative risk aversion coefficient. In fact varying the degree of openness of the economy from 0.01 to 0.50 does not affect significantly the dynamics around the passive steady state but it has an effect on the dynamics around the active steady state, as predicted by Figure 13. For instance for $\sigma = 1.5$, the forward-looking rule drives an almost closed economy ($\alpha = 0.01$) to a monotonic inflationary path converging to the active steady state, while the same rule drives the open economy ($\alpha = 0.50$) into a period-two cycle around the active steady state.

²³They are available from the authors upon request.

²⁴The functional forms of these frontiers are different from the ones in the CPI-inflation targeting analysis.

²⁵The reader may ask for the white window between the chaotic region around the passive steady state and the chaotic region around the active steady state. In that window there are paths of the nominal interest rate (R_t) going to either 1 or ∞ . Both dynamics are discarded in our analysis since we are interested exclusively in paths that satisfy $1 < R_t \leq \infty$.

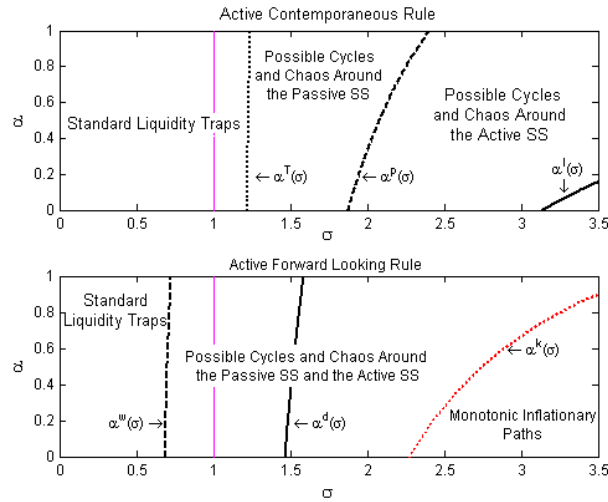


Figure 14: Targeting the non-traded goods inflation. Equilibrium analysis for an active contemporaneous rule (top panel) and an active forward-looking rule (bottom panel). This figure shows the results from the global equilibrium analysis and a comparison between the local equilibrium analysis and the global equilibrium analysis. α corresponds to the degree of openness of the economy and σ denotes the relative risk aversion coefficient.

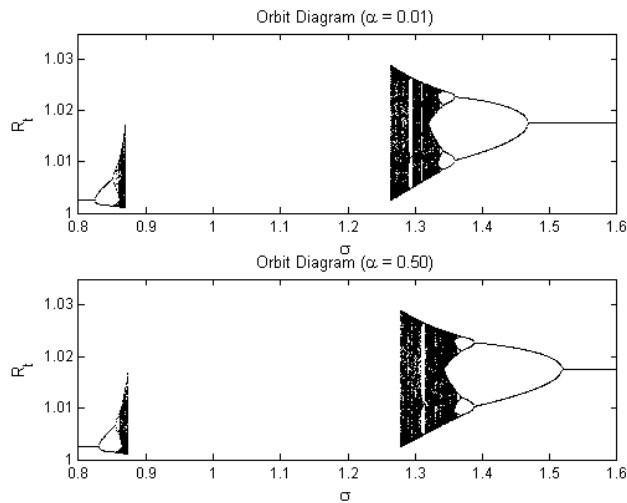


Figure 15: Targeting the non-traded goods inflation. Orbit diagrams for an active forward-looking rule. The top panel corresponds to the case of an almost closed economy ($\alpha = 0.01$), while the bottom panel corresponds to an open economy ($\alpha = 0.50$). R_t denotes the nominal interest rate and σ stands for the relative risk aversion coefficient.

9 Conclusions

In this paper we have shown that Taylor rules that are active "in the Taylor sense" around the target steady state might actually have perverse effects on a small open economy dynamics. In particular we have shown that there is an interesting interaction between the coefficient of relative risk aversion and the degree of openness (measured by the share of tradable goods in consumers' preferences) in characterizing the economic dynamics of our small open economy. To further stress the relevance of our results, we have been pursuing both a local and a global equilibrium dynamics analysis (the former being the standard approach in the monetary rules literature).

In the contemporaneous Taylor rule case, for risk aversion and openness ranges for which local analysis would conclude in favor of price stability, we highlight the possibility of standard liquidity traps, cyclical/chaotic dynamics both around a desired (targeted) and an undesired (passive) steady state. In particular, for high enough risk aversion, all these possibilities can arise according to the degree of openness of the economy. More closed economies can display high instability but still around the target. As the share of traded goods increases (the economy opens up, and so the weight of traded goods in CPI inflation increases), the likelihood of falling (monotonically or cyclically) into dangerous deflations does too. An extremely open economy seems to fall into such traps with probability one.

Forward looking Taylor rules do not seem to do a much better job. For local determinacy ranges of risk aversion and openness, we can get liquidity traps, cyclical and chaotic equilibria as for the contemporaneous case. All our analytical results are confirmed by a simple parametrization of the model.

Though from a local point of view contemporaneous or forward looking inflation targeting give basically identical results, from a global point of view there are few interesting differences. First, while for moderate risk aversions (below 1) contemporaneous rules can deliver monotonic deflationary paths only, forward looking rules could also produce cycles and chaotic dynamics around the low inflation state. Second, for risk aversion above one, liquidity traps are only possible with contemporaneous rules but not with forward looking. The latter can still create endogenous fluctuations but around the target state only. Furthermore, forward looking rules also display monotonic equilibrium paths converging to the active steady state. For what concerns local stability, they behave quite similarly. Nevertheless, the size of the local indeterminacy region is generally smaller for the contemporaneous rule case.

For the contemporaneous rule case, we found out that erratic equilibrium dynamics of the form described extensively in the paper become less likely as the role of real money balances in transactions vanishes. Economies in which cash is greatly valued for transaction purposes (low γ) seem more vulnerable to the endogenous volatility described in this study. On the contrary, as the credit market improves and cash becomes less and less needed, such endogenous fluctuations disappear and the only risky equilibrium becomes a liquidity trap (the short interest rate falls progressively towards zero). In such circumstances there is still an active role for monetary policy in eliminating at least part of the multiplicity. We showed that if the target inflation rate and the interest rate rule response to inflation are chosen appropriately (accurately monitoring the level of, exogenous, financial innovation), cycles and chaos can be completely eliminated, without requiring any specific Ricardian or non-Ricardian fiscal policy rule. The latter channel might still play a role in eliminating deflationary paths too. This is going to be part of our future research.

The relationship between cash dependency and multiplicity is more mixed in the case of forward looking

rules. A decrease in the role of money in providing transaction services does not eliminate unwanted cyclical paths and chaos. Actually it seems to be the case that for less cash dependent economies cyclical patterns are the most likely outcome.

A result of some interest is that the existence of endogenous cycles and chaos does not depend on targeting a CPI inflation rate, or, in other words, considering traded goods prices in the price index. The degree of openness still plays a role even though the interest rate reacts to non-traded goods inflation only (domestic inflation). To conclude we consider rules that react to past inflation (backward looking rules) and find out that cyclical and chaotic dynamics are not possible, although the economy can still display multiple equilibria in the form of liquidity traps or monotonic paths to the target.

The bottom line though is that price stability could be indeed a difficult task. From a technical point of view this paper points out the misleading results one would get by focusing on local techniques in judging the stabilizing properties of monetary rules. From a policy point of view it might highlight some warnings for small open economies fastly moving to inflation targeting regime through interest rate feedback rules. Central bank of developing and developed small open economies are explicitly making price stability their prime objective and therefore pursuing aggressive anti-inflationary policies.

A Appendix

A.1 Steady state multiplicity

At the steady state $R_{t+1} = R_t = \bar{R}$. Equation (29) reduces to

$$(R^* - 1)^{\frac{R^*-1}{A}} \bar{R} = R^* (\bar{R} - 1)^{\frac{R^*-1}{A}} \quad (43)$$

It is clear that $\bar{R} = R^*$ is a possible steady state. We are going to show that if the Taylor rule is active at this steady state a second lower steady state $R^L < R^*$ exists and it is unique.

Proposition 19 *If $\frac{A}{R^*} > 1$ (an active Taylor rule) then there exists a unique $R^L \in (1, R^*)$ that solves (43).*

Proof. First of all denote the left hand side and the right hand side of equation (43) as $LHS(\bar{R})$ and $RHS(\bar{R})$ respectively. Second note that

$$\begin{aligned} \lim_{\bar{R} \rightarrow 1} LHS(\bar{R}) &= (R^* - 1)^{\frac{R^*-1}{A}} > 0 \\ \lim_{\bar{R} \rightarrow 1} RHS(\bar{R}) &= 0 \end{aligned}$$

$LHS(\bar{R})$ is linear in \bar{R} with slope $(R^* - 1)^{\frac{R^*-1}{A}} > 0$. $RHS(\bar{R})$ slopes upwards as well for any $\bar{R} > 1$,

$$\frac{\partial RHS(\bar{R})}{\partial \bar{R}} = R^* \frac{R^* - 1}{A} (\bar{R} - 1)^{\frac{R^*-1}{A} - 1} > 0$$

A sufficient condition for a second solution $\bar{R} = R^L$ to exist is that the slope of the $RHS(\bar{R})$ at R^* be smaller than $(R^* - 1)^{\frac{R^*-1}{A}}$. This will guarantee that the $LHS(\bar{R})$ and $RHS(\bar{R})$ will cross at a second point $\bar{R} = R^L$

between 1 and R^* . The slope of the $RHS(\bar{R})$ at R^* is $\frac{R^*}{A} (R^* - 1)^{\frac{R^*-1}{A}}$. Hence the sufficient condition is $\frac{R^*}{A} < 1$ or equivalently $\frac{A}{R^*} > 1$.

Next we show that the second lower steady state is unique. For that it is enough to show that the $RHS(\bar{R})$ is strictly concave. Taking second derivative:

$$RHS''(\bar{R}) = R^* \frac{R^* - 1}{A} (\bar{R} - 1)^{\frac{R^*-1}{A} - 2} \left(\frac{R^* - 1}{A} - 1 \right)$$

which is strictly negative as long as $\frac{R^*-1}{A} - 2 < 0$. But this is guaranteed by the fact that $\frac{A}{R^*} > 1$. ■

A.2 Proof of Proposition 2

Proof. The explicit expression for $\alpha^I(\sigma)$ comes directly from its implicit definition $\chi(\alpha, \sigma) = \Upsilon^I$ and some algebra. $\alpha^I(\sigma)$ is well defined in the sense that for feasible values of the structural parameters and for any $\sigma \geq \sigma^{I*} > 1$ it is a continuous function with $\alpha^I(\sigma) \in (0, 1)$. To see this, note that since $\Upsilon^I > 0$, γ , $\theta_N \in (0, 1)$ and $\sigma > 1$ then it is straightforward to derive that $\alpha^I(\sigma) < 1$. Furthermore it is simple to see that the expression for $\sigma^{I*} \equiv 1 + \left[\frac{(1-\gamma)(1-\theta_N)}{1-\gamma(1-\theta_N)} - \Upsilon^I \right]^{-1} \left[\frac{\Upsilon^I}{1-\gamma(1-\theta_N)} \right]$ comes from solving $\chi(0, \sigma^{I*}) = \Upsilon^I$. Using Assumption 1 we may infer that $\sigma^{I*} > 1$. Utilizing this and **Fact 4** we can deduce that for any $\sigma \geq \sigma^{I*} > 1$, it is valid that $\chi(0, \sigma) > \chi(0, \sigma^{I*}) = \Upsilon^I$, which in turn means that $1 > \frac{\Upsilon^I}{\chi(0, \sigma)}$. But this last inequality and $\Upsilon^I > 0$, γ , $\theta_N \in (0, 1)$ and $\sigma > 1$ imply that $\alpha^I(\sigma) > 0$.

Moreover it is simple to show that $\frac{\partial \alpha^I(\sigma)}{\partial \sigma} = \left[1 + \Upsilon^I \left(\frac{\gamma}{1-\gamma} \right) \right]^{-1} \left[\frac{\Upsilon^I}{(1-\gamma)(1-\theta_N)(1-\sigma)^2} \right] > 0$ and that $\frac{\partial^2 \alpha^I(\sigma)}{\partial \sigma^2} = \left[1 + \Upsilon^I \left(\frac{\gamma}{1-\gamma} \right) \right]^{-1} \left[\frac{2\Upsilon^I}{(1-\gamma)(1-\theta_N)(1-\sigma)^3} \right] < 0$ for $\Upsilon^I > 0$, γ , $\theta_N \in (0, 1)$ and $\sigma > 1$. Therefore the function $\alpha^I(\sigma)$ is strictly increasing and concave for any $\sigma \geq \sigma^{I*} > 1$. Additionally from the definition of $\alpha^I(\sigma)$ it is straightforward to show that $\lim_{\sigma \rightarrow \sigma^{I*}} \alpha^I(\sigma) = 0$ and $\lim_{\sigma \rightarrow \infty} \alpha^I(\sigma) = 1 - \frac{\Upsilon^I}{[1-\Upsilon^I(1-\gamma)](1-\theta_N)} \in (0, 1)$ since $\alpha^I(\sigma) \in (0, 1)$ for any $\sigma \geq \sigma^{I*}$. ■

A.3 Proof of Lemma 3

Proof. First compute the derivative of the function $J(R)$ in (37) with respect to R

$$J'(R) = \frac{R^* R^\chi (R-1)^{\chi-1}}{(R^{1+\chi})^2} (1 + \chi - R)$$

For any $R > 1$, $sign[J'(R)] = sign(1 + \chi - R)$.

1. if $\sigma \in (0, 1)$, $\chi < 0$ for any α because of **Fact 2**. Therefore $1 - R + \chi < 0$ for any $R > 1$. The limits are trivial.
2. if $\sigma > 1$, $\chi > 0$ for any α from **Fact 3**. We have that $J'(R) = 0$ if and only if $R = R^J = 1 + \chi > 1$. $J(R)$ is increasing for any $R < R^J$ and decreasing for $R > R^J$. The limits are trivial.

■

A.4 Proof of Lemma 4

Proof. The proof proceeds following the same steps that we followed in the proof for Proposition 2. The only difference is that instead of using $\Upsilon^I \equiv \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A}\right)$ we use $\Upsilon^p \equiv \frac{R^* - 1}{A}$. Moreover use the fact that $\Upsilon^I > \Upsilon^p$. Finally the inequalities $\alpha^{p^*} > \alpha^{I^*}$ and $\sigma^{p^*} < \sigma^{I^*}$ follow from $\Upsilon^I > \Upsilon^p$ and the definitions of α^{p^*} , α^{I^*} , σ^{p^*} and σ^{I^*} (see Proposition 2). ■

A.5 Proof of Lemma 5

Proof. First of all the derivative of $K(R)$ in (36) with respect to R is:

$$K'(R) = \frac{(R^* - 1)^{\frac{R^* - 1}{A}} (R - 1)^{\chi + \frac{1 - R^*}{A}}}{R^\chi} \left[\frac{R^{\frac{1 - R^*}{A}} + \chi}{(R - 1)R} \right]$$

For any $R > 1$, we have that $\text{sign}[K'(R)] = \text{sign}\left[R^{\frac{1 - R^*}{A}} + \chi\right]$.

1. If $\sigma \in (0, 1)$, $\chi < 0$ for any $\alpha \in (0, 1)$ because of **Fact 2**. Therefore for $R^* > 1$ we have that $R^{\frac{1 - R^*}{A}} + \chi < 0$ for any $R > 1$. But this implies that $K'(R) < 0$. Moreover since $\frac{1 - R^*}{A} + \chi < 0$, for $\sigma \in (0, 1)$, using (36) we derive that $\lim_{R \rightarrow 1} K(R) = \infty$ and $\lim_{R \rightarrow \infty} K(R) = 0$.
2. Now consider $\sigma > 1$. We know from of **Fact 3** that this is enough to have $\chi(\alpha, \sigma) > 0$. Therefore $K'(R) = 0$ if and only if $R = R^K = \frac{\chi A}{R^* - 1}$. However we need R^K to be bigger than 1 to be a valid peak. This requires $\chi(\alpha, \sigma) > \frac{R^* - 1}{A}$. Is this true for any $\sigma > 1$ and for any $\alpha \in (0, 1)$? The answer is no and this is why there are two cases: (a) and (b).

- (a) Note that $\alpha^p(\sigma)$ is defined implicitly as the values of α and σ such that $\chi(\alpha, \sigma) = \frac{R^* - 1}{A}$ and σ^{p^*} is such that $\chi(0, \sigma^{p^*}) = \frac{R^* - 1}{A}$. For any $\sigma \in (1, \sigma^{p^*})$ and $\alpha \in (0, 1)$ we have that $\chi(\alpha, \sigma) < \frac{R^* - 1}{A}$. The reason is that from **Facts 4** and **5**, given $\alpha \in (0, 1)$ and for any $\sigma \in (1, \sigma^{p^*})$ we have that $\chi(\alpha, \sigma) < \chi(\alpha, \sigma^{p^*}) < \chi(0, \sigma^{p^*}) = \frac{R^* - 1}{A}$. Hence $\chi(\alpha, \sigma) < \frac{R^* - 1}{A}$, which in turn means that $R^K < 1$. However we have assumed that $R > 1$. Therefore it is clear that we are only interested in the decreasing part of $K(R)$ and the first part of part (a) follows.

Furthermore for any $\sigma \in [\sigma^{p^*}, \infty)$ such that $\alpha \geq \alpha^p(\sigma)$, $K(R)$ is strictly decreasing. The reason is that by definition $\alpha^p(\sigma)$ is defined implicitly as the values of α and σ such that $\chi(\alpha, \sigma) = \frac{R^* - 1}{A}$. Then using **Facts 5** it is clear that for any $\sigma \in [\sigma^{p^*}, \infty)$ and any $\alpha \geq \alpha^p(\sigma)$, we have that $\chi(\alpha, \sigma) \leq \chi(\alpha^p(\sigma), \sigma) = \frac{R^* - 1}{A}$. But this means that $R^K \leq 1$. However we have assumed that $R > 1$. Therefore it is clear that we are only interested in the decreasing part of $K(R)$ and the second part of part (a) follows.

- (b) Since $\alpha^p(\sigma)$ is defined implicitly as the values of α and σ such that $\chi(\alpha, \sigma) = \frac{R^* - 1}{A}$, then using **Facts 5** it is clear that for any $\sigma \in (\sigma^{p^*}, \infty)$ and any $\alpha < \alpha^p(\sigma)$ we have that $\chi(\alpha, \sigma) > \chi(\alpha^p(\sigma), \sigma) = \frac{R^* - 1}{A}$. But this means that $R^K = \frac{\chi A}{R^* - 1} > 1$ is a valid peak of $K(R)$. Hence $K(R)$ is hump-shaped with a peak at $R^K = \frac{\chi A}{R^* - 1} > 1$.

■

A.6 Proof of Lemma 6

Proof. The proof proceeds following the same steps that we follow in the proof for Proposition 2. The only difference is that instead of using $\Upsilon^I \equiv \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A}\right)$ we use $\Upsilon^T \equiv R^L - 1$. Moreover use the fact that $\Upsilon^I > \Upsilon^T$. For the last part of the Lemma remember that since $\Upsilon^p \equiv \frac{R^* - 1}{A}$ then Assumption 2 can be rewritten as $\Upsilon^p > \Upsilon^T$. Then utilizing this and the definitions of α^{p^*} , α^{T^*} , σ^{p^*} and σ^{T^*} the inequalities $\alpha^{T^*} > \alpha^{p^*}$ and $\sigma^{T^*} < \sigma^{p^*}$ follow (see Lemma 4). ■

A.7 Proof of Proposition 10

Proof. The proof follows the same steps that we apply in the proof for Proposition (2). The only difference is that instead of using $\Upsilon^I \equiv \frac{1}{2}(R^* - 1) \left(1 + \frac{R^*}{A}\right)$ we use $\Upsilon^d \equiv \frac{1}{2}(R^* - 1) \left(1 - \frac{R^*}{A}\right)$. Moreover use the fact that $\Upsilon^I > \Upsilon^d$. ■

A.8 Proof of Lemma 11

Proof. The explicit expression for $\alpha^v(\sigma)$ comes directly from its implicit definition $\chi(\alpha, \sigma) = \Upsilon^v < 0$ and some algebra.

1. First note that using $\Upsilon^v < 0$, $\gamma, \theta_N \in (0, 1)$, and the assumption $\frac{(1-\gamma)}{\gamma} > -\Upsilon^v$ we can infer that for any $\sigma < 1$, we have that $\frac{\partial \alpha^v(\sigma)}{\partial \sigma} = \left[1 + \Upsilon^v \left(\frac{\gamma}{1-\gamma}\right)\right]^{-1} \left[\frac{\Upsilon^v}{(1-\gamma)(1-\theta_N)(1-\sigma)^2}\right] < 0$ and that $\frac{\partial^2 \alpha^v(\sigma)}{\partial \sigma^2} = \left[1 + \Upsilon^v \left(\frac{\gamma}{1-\gamma}\right)\right]^{-1} \left[\frac{2\Upsilon^v}{(1-\gamma)(1-\theta_N)(1-\sigma)^3}\right] < 0$, which means that $\alpha^v(\sigma)$ is strictly decreasing and concave for any $\sigma < 1$. The limits are trivial using **Fact 1**, and the definition of $\alpha^v(\sigma)$ and σ^{v^*} . In particular note that the expression for σ^{v^*} comes from solving $\chi(0, \sigma^{v^*}) = \Upsilon^v$, and that $0 < \sigma^{v^*} < 1$ if $\frac{(1-\gamma)}{\gamma} > -\Upsilon^v$.

The reader may ask why we focus on values of σ such that $\sigma < 1$. The reason is that for $\sigma > 1$, the function $\alpha^v(\sigma) \notin (0, 1)$. To see this note that from the definition of $\alpha^v(\sigma)$, **Fact 3** and the assumptions $\Upsilon^v < 0$, $\gamma, \theta_N \in (0, 1)$, and $\frac{(1-\gamma)}{\gamma} > -\Upsilon^v$, we may conclude that $\alpha^v(\sigma) > 1$ for any $\sigma > 1$ with $\lim_{\sigma \rightarrow 1^+} \alpha^v(\sigma) = +\infty$.

2. First note that it is easy to check that if $\frac{(1-\gamma)}{\gamma} < -\Upsilon^v$ then $\lim_{\sigma \rightarrow 1^-} \alpha^v(\sigma) = +\infty$ and $\lim_{\sigma \rightarrow 1^+} \alpha^v(\sigma) = -\infty$. Second using the expressions derived in part 1 of this proof for $\frac{\partial \alpha^v(\sigma)}{\partial \sigma}$ and $\frac{\partial^2 \alpha^v(\sigma)}{\partial \sigma^2}$, **Fact 2**, $\Upsilon^v < 0$, $\gamma, \theta_N \in (0, 1)$ and $\frac{(1-\gamma)}{\gamma} < -\Upsilon^v$ we can derive that $\frac{\partial \alpha^v(\sigma)}{\partial \sigma} > 0$ and $\frac{\partial^2 \alpha^v(\sigma)}{\partial \sigma^2} > 0$ for any $\sigma < 1$. Which means that $\alpha^v(\sigma)$ is strictly increasing and convex for any $\sigma < 1$. However using this and since $\lim_{\sigma \rightarrow 0} \alpha^v(\sigma) = 1$ and $\lim_{\sigma \rightarrow 1^-} \alpha^v(\sigma) = +\infty$, then we can conclude that for any $0 \leq \sigma \leq 1$ we have that $\alpha^v(\sigma) \geq 1$.

Moreover from the definition of $\alpha^v(\sigma)$, **Fact 3**, $\Upsilon^v < 0$, $\gamma, \theta_N \in (0, 1)$ and $\frac{(1-\gamma)}{\gamma} > -\Upsilon^v$ we can observe that for any $\sigma > 1$ we have that $\alpha^v(\sigma) < 0$. Hence if $\frac{(1-\gamma)}{\gamma} < -\Upsilon^v$ then $\alpha^v(\sigma)$ never crosses the region $\alpha \in (0, 1)$ vs $\sigma \in (0, \infty)$.

■

A.9 Proof of Lemma 12

Proof.

1. Trivial.
2. By differentiating once and after some algebra we obtain

$$J^{f'}(R_t) = \frac{R^* \left(1 - \frac{R^*-1}{A}\right)}{(R^*-1)^{\frac{R^*-1}{A}}} \frac{(R-1)^{\chi-1 + \frac{R^*-1}{A}}}{R^{2+\chi}} \left[\frac{1+\chi}{1 - \frac{R^*-1}{A}} - R \right]$$

Therefore for any $R > 1$, $\text{sign}[J^{f'}(R)] = \text{sign}\left[\frac{1+\chi}{1 - \frac{R^*-1}{A}} - R\right]$.

Note that if $1 > -\chi$ then $J^f(R)$ is a hump-shaped function with a peak at $R^{J^f} = \frac{1+\chi}{1 - \frac{R^*-1}{A}}$ since $J^{f'}(R_t) > 0$ for $\frac{1+\chi}{1 - \frac{R^*-1}{A}} > R$ and $J^{f'}(R_t) < 0$ for $\frac{1+\chi}{1 - \frac{R^*-1}{A}} < R$. We will assume $1 > -\chi$ and relax this assumption in part 3 of this proof. If there is a peak at $R = R^{J^f}$, in order for it to be a valid peak we need that $R^{J^f} > 1$. In other words we need that $\chi > \frac{1-R^*}{A}$. It is important to remember that $1 < R^*$ and therefore $\frac{1-R^*}{A} < 0$. If $\sigma > 1$ then **Fact 3** guarantees that $1 > -\chi$ and since $\chi > 0$ then we have a valid peak. This in turn means that $J^f(R)$ is hump-shaped for any $\alpha \in (0, 1)$ and $\sigma > 1$. On the other hand if $\sigma \in (0, 1)$ we know from Lemma 11 that $\alpha^v(\sigma)$ define all the combinations of α and σ for which $\chi(\alpha, \sigma) = \frac{1-R^*}{A} = \Upsilon^v$. Let's take the particular pair (α^v, σ^v) such $\chi(\alpha^v, \sigma^v) = \frac{1-R^*}{A} = \Upsilon^v$. Then using the assumption $\frac{(1-\gamma)}{\gamma} > -\Upsilon^v$, Lemma 11 and **Fact 6** we know that for any $\alpha > \alpha^v$ we have that $\chi(\alpha, \sigma^v) > \chi(\alpha^v, \sigma^v) = \frac{1-R^*}{A} = \Upsilon^v$ which means that $R^{J^f} > 1$. But this result together with the result for $\sigma > 1$ imply that for any $\sigma > 0$ and $\alpha > \alpha^v(\sigma)$ the function $J^f(R)$ is hump-shaped with a peak at $R^{J^f} = \frac{1+\chi}{1 - \frac{R^*-1}{A}} > 1$.

3. Assume that $1 > -\chi$. If $\sigma \in (0, \sigma^{v*}]$ we know from Lemma 11 that $\alpha^v(\sigma)$ define all the combinations of α and σ for which $\chi(\alpha, \sigma) = \frac{1-R^*}{A} = \Upsilon^v$. Let's take the particular pair (α^v, σ^v) such $\chi(\alpha^v, \sigma^v) = \frac{1-R^*}{A} = \Upsilon^v$. Then using the assumption $\frac{(1-\gamma)}{\gamma} > -\Upsilon^v$, Lemma 11 and **Fact 6** we know that for any $\alpha \leq \alpha^v$ we have that $\chi(\alpha, \sigma^v) \leq \chi(\alpha^v, \sigma^v) = \frac{1-R^*}{A} = \Upsilon^v$ which means that $R^{J^f} \leq 1$. That is we do not have a valid peak since we assumed that $R > 1$. In this case we are only interested in the strictly decreasing part of $J^f(R)$. What if $1 < -\chi$? In this case $\frac{1+\chi}{1 - \frac{R^*-1}{A}} - R$ is always negative which implies from $\text{sign}[J^{f'}(R)] = \text{sign}\left[\frac{1+\chi}{1 - \frac{R^*-1}{A}} - R\right]$ that $J^{f'}(R) < 0$. Therefore $J^f(R)$ is always strictly decreasing.

■

A.10 Proof of Lemma 13

Proof.

1. Trivial.
2. By differentiating $K^f(R)$ once we obtain

$$K^{f'}(R) = \chi \frac{(R-1)^{\chi-1}}{R^{1+\chi}}$$

which means that $\text{sign}[K^{f'}(R)] = \text{sign}[\chi]$ for any $R > 1$. Using this and **Fact 2** we conclude that for any $\sigma \in (0, 1)$, the function $K^f(R)$ is strictly decreasing. The limits are trivial.

3. Use $\text{sign}[K^{f'}(R)] = \text{sign}[\chi]$ for any $R > 1$, and **Fact 3** to conclude that for any $\sigma > 1$, the function $K^f(R)$ is strictly increasing. The limits are trivial.

■

A.11 Proof of Lemma 14

Proof. The explicit expression for $\alpha^w(\sigma)$ comes directly from its implicit definition $\chi(\alpha, \sigma) = \Upsilon^w$ and some algebra. The proof follows the same steps as in the proof for Lemma 11 taking into account the following. First notice that using Assumption 2 we may conclude that $\Upsilon^w \equiv \left(1 - \frac{R^*-1}{A}\right) R^L - 1 < 0$ since $\frac{R^L-1}{R^L} < R^L - 1 < \frac{R^*-1}{A}$. Second it is simple to show that since the rule is active $\frac{R^*}{A} < 1$ then $\left(1 - \frac{R^*-1}{A}\right) R^L - 1 < -\frac{R^*-1}{A}$, or equivalently that $-\Upsilon^w < -\Upsilon^v$. With these results proceed following the same steps as the ones in the proof for Lemma 11. The result $\sigma^{w*} < \sigma^{v*} < 1$ follows from the definitions of σ^{w*} and σ^{v*} and from $-\Upsilon^w < -\Upsilon^v$. ■

A.12 Proof of Proposition 15

Proof. The proof follows the same steps as the ones to prove Proposition 10. The only difference is that instead of using $\Upsilon^d \equiv \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right)$ we use $\Upsilon^k \equiv (R^* - 1)\left(1 - \frac{R^*}{A}\right)$. Moreover instead of using Assumption 1 we use $\frac{(1-\gamma)(1-\theta_N)}{1-\gamma(1-\theta_N)} > \Upsilon^k$. Finally $\sigma^{k*} > \sigma^{d*}$ and $\alpha^{k*} < \alpha^{d*}$ follow from the definitions of σ^{k*} , σ^{d*} , α^{k*} , and α^{d*} , and applying the fact that $\Upsilon^k > \Upsilon^d$. ■

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