Financial Engineering with Reverse Cliquet Options

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Abstract

Index-linked securities are offered by banks, financial institutions and building societies to investors looking for downside risk protection whilst still providing upside equity index participation. This article explores how reverse cliquet options can be integrated into the structure of a guaranteed principal bond.

Pricing problems are discussed under the standard Black-Scholes model and under the constant-elasticity-of-variance model. Forward start options are the main element of this structure and new closed formulae are obtained for these options under the latter model. Risk management issues are also discussed. An example is described showing how this structure can be implemented and how the financial engineer may forecast the coupon payment that will be made to investors buying this product without exposing the issuing institution to risk of loss.

Key words: cliquet options, structured products, constant-elasticity-of-variance model, forward-start options,

JEL codes: G13, G24.

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From an investor's point of view traditional equity-linked instruments provide an opportunity to participate indirectly in the performance of a single share. For the last two decades increasingly complex, customised structures have been created in a way that enables, in many cases, regulatory constraints on the use of derivative securities, such as forwards, futures and options, to be by-passed. Convertible bonds provide a good example of an instrument that customarily has a pay out profile of a call option and that have been available to investors for many years. Liquid Yield Option Notes™ ® (LYONs™ ®) evolved as a variation on the convertible bond theme. These securities were structured to provide investors with equity performance with a strong element of built-in price stability and are described and analysed in McConnell and Schwarz [1986,1992]. The evolution of single stock LYONs™ ® led to the development of many variations in single stock linked notes and in the late 1980s equity index-linked instruments began to appear, for example, equity linked certificates of deposits explained in Gastineau and Purcell [1993].

The growth of derivative markets globally, coupled with more informed investor understanding of the risk and return characteristics of structured investment opportunities, has led to an enormous growth in the number and variety of equity index-linked securities being offered by banks, mortgage banks, and building societies. The recent decline in the level of the major international equity indexes worldwide has further stimulated investor demand for financial products that limit downside risk whilst still offering upside equity index participation. Recent guaranteed bond and note issues, for example, can be found which draw on the performance of the EuroSTOXX50 index and offer investors a callable certificate issued at a price above par, which guarantees a minimum return of par plus the full positive return on the underlying benchmark index. In the case of the bond not being called by the issuer the maturity redemption value of the bond can be expressed as:

$$B_{mat} = P \cdot Max\left[1, 1 + \left(\frac{I_T - I_0}{I_0}\right)\right] \tag{1}$$

where B_{mat} is the bond's redemption value, P the guaranteed amount (par), I_T the index level at the bond's maturity date, I_0 the initial index level or strike price.

A second example issues a bond at par and offers a minimum redemption value above par over a specified time period but with a reduced participation level in the underlying equity index. At maturity the bond's redemption value can be expressed as:

$$B_{mat} = P \cdot Max\left[1 + y, 1 + x\left(\frac{I_T - I_0}{I_0}\right)\right] \tag{2}$$

where: y represents guaranteed return above par expressed as a proportion, and x represents the benchmark index participation level as a proportion.

The pricing and hedging of these types of structures is well-known (Eales [2000]; Das [2001]). The financial institution offering the instrument will, ideally, invest in a zero coupon bond for a price less than the sum invested and use the residual to purchase the appropriate quantity of call options on the index. This approach to structuring a hedged investment instrument is most effective in a low volatility high interest rate economic climate.

A variation on this can be found in equity index-linked cliquet participation notes. These instruments make use of cliquet which are well-established instruments. They were first introduced in France using the CAC 40 equity index as the underlying security. Cliquets are also called ratchet options in the literature because they are based on resetting the strike of a derivative structure to the last fixing of the reference underlying. Ratchets can be regular as described by Howard [1995] or compound as discussed by Buetow [1999]. For the latter type there are no intermediary payments, all gains being used to increase the volume of the derivative that is used as a vehicle for the ratchet. A wide range of ratchet caps and floors in an interest rate context described in Martellini et al. [2003].

In an equity context a similar example of the use of ratchets can be found in a note which offers a minimum redemption value set above par and whose redemption yield is related to the monthly percentage changes in a specified index over a defined period of time. To manage the risk of large index movements the monthly percentage returns are collared in a tight band around the periodically reset index strike price.

$$B_{mat} = P \cdot Max\left[1 + y, 1 + \sum_{t=0}^{T} \max\left[-z\%, \min\left(z\%, \left(\frac{I_{t+1} - I_t}{I_t}\right)\right)\right]\right] \quad (3)$$

A similar approach can be adopted when seeking to price and hedge this structure as that described in the guaranteed instruments introduced earlier. Following the purchase of a zero coupon bond residual funds can be used to buy a set of cliquet call and put options with monthly expirations extending to the bond's maturity date. The portfolio of options required to create this position will be long ATM calls combined with short OTM calls and Short ATM puts combined with long OTM puts. Clearly the availability of any residual funds derived from the portfolio of options will help determine the feasibility, the attractiveness and the competitiveness of the instrument. A mirror image instrument could be constructed which links coupon to the percentage changes in an index to falls rather than rises index.

The pricing of a cliquet option typically proceeds by regarding it as a portfolio of at-the-money (ATM) forward start options. A cliquet bestows on the holder the right to buy a regular at-the-money call with time to maturity T at some future specified date T_1. Thus,

$\tau_1 = T_1 - t$ is the length of time that elapses before the forward start option comes into existence and $\tau = T - t$ is the length of time to maturity. An early approach used in the pricing of a forward start option is presented by Rubinstein [1991]. This method bases the risk-neutral value of an ATM forward start call option on the expected value of the underlying security at time t_1 and results in the option value reducing to that of a regular ATM call *where the time to maturity is the effective time* $\tau - \tau_1$, Zhang [1998]. This implies that the Black and Scholes pricing formula can be used to obtain the cliquet option's price (call or put). If the tenors are defined by the partition $t_1 < t_2 < ... < t_{n+1} \equiv T$ then:

$$Cliquet_{put}(t) = \sum_{i=1}^{n} S(t)\left[e^{-r(t_{i+1}-t_i)-\delta(t_i-t)} N\left(-d_2^{ATM}(i)\right) - e^{-\delta(t_{i+1}-t)} N\left(-d_1^{ATM}(i)\right)\right] \tag{4}$$

where $S(t)$ represents the underlying asset at time t, r the risk free rate of interest, δ is the dividend yield, σ represents volatility, and:

$$\begin{aligned} d_1^{ATM}(i) &= \frac{\left(r - \delta + 0.5\sigma^2\right)}{\sigma}\sqrt{t_{i+1} - t_i} \\ d_2^{ATM}(i) &= d_1 - \sigma\sqrt{t_{i+1} - t_i} \end{aligned} \tag{5}$$

Pricing forward start options is the key to pricing cliquets. A forward start option is a particular case of multi-stage options, which are derivatives allowing decisions to be made via conditions evaluated at intermediate time points during the life of the contingent claim (see Etheridge [2002]). Multistage options can be priced similarly to options on stocks paying discrete dividends at intermediate points over the life of the option. Under general common assumptions, the pricing equation of multistage options in a risk-less world is the well-known Black-Scholes PDE:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V}{\partial S^2}+rS(t)\frac{\partial V}{\partial S}-rV=0 \tag{6}$$

with some final condition such as $V(T,S) = G(S)$.

The Feynman-Kac solution of the above equation is:

$$V(S(t),t)=e^{-r(T-t)}\widetilde{E}[G(S(T))|S(t)] \tag{7}$$

where the expectation operator is taken under the risk-neutral measure.

The forward start option is an option that comes into existence at time T_1 and has maturity T. The following backward procedure can be used to calculate the price of this option:

(a) Calculate the final payoff of the option at time T.

(b) Calculate the value of the payoff from (a) at time T_1; this is given as the solution of the Black-Scholes PDE with $t = T_1$.

(c) Check the conditions and calculate the terminal value of the option at T_1 and for $t < T_1$ use the Black-Scholes PDE to get the solution

$$V(S(t),t)=e^{-r(T_1-t)}\widetilde{E}[V(S(T_1),T_1)|S(t)] \tag{8}$$

Out-of-the-money (OTM),in-the-money cliquets (ITM), and more exotic structures can also be handled in the same partial differential equation (PDE) pricing framework.

In the same vein Monte Carlo simulation (MCS) and quasi-MCS can be used to price cliquets taking into account the element of path dependency ignored by the standard Black and Scholes formula. Buetow [1999] suggests that pricing this type of instrument accurately is best undertaken using different methods and comparing the results obtained.

These pricing methods, however, all suffer from the assumption of constant volatility. Wilmott [2002] highlights the problems associated with this assumption and illustrates the dangers faced by writers of cliquets when ignoring volatility risk. It can be shown that the gamma of a cliquet option is the sum of gamma values for regular options because the gamma of a forward start option is zero before the starting time. This may create the impression that risk management is easy in this case. However, for this type of option, hedging can be quite complex because the delta, vega and theta have discontinuities around reset times.

This article explores how reverse cliquet options can be integrated into the structure of a guaranteed principal bond. Pricing problems are discussed under the standard Black-Scholes model and under the constant-elasticity-of-variance model. Forward start options are the main element of this structure and under the latter model the pricing of these important options is not easy. This problem is solved and *en passant* new closed formulae are derived for forward start options under the CEV model.

I. FINANCIAL ENGINEERING WITH REVERSE CLIQUETS

Unlike the structures discussed so far, reverse cliquet options are best employed either when volatility levels are substantially higher than historically observed volatilities and are expected to revert back to normal or when investors hold the view that the markets are likely to become more bullish (puts) or bearish (calls).

A reverse cliquet can be integrated into the structure of a guaranteed principle bond. This is achieved by creating a pool of funds derived from, for example, investors augmenting their investment sum by writing forward start options. The fund starts with a value of greater than 100%[1] and is drawn on over time if and when the written options expire in-the-money (ITM). Under this construction the bond may guarantee full return of principal invested and offer a higher than market coupon that declines as the underlying asset, to which the bond is linked, declines in value (put) or rises in value (calls) as measured on pre-specified future dates. Coupons could be paid on defined intermediate dates or as a single payment at the instrument's maturity.

Reverse Cliquet with Put Options

If it is assumed that investor's views are bullish concerning equity market performance and that volatilies are high, a bond could be offered which pays out an amount determined by the total initial option net income fund less the sum of the declines in the benchmark index either at maturity or on intermediate coupon dates t_1 t_2 , ...$t_{n+1.}$ = T.

From the issuing institution's perspective one way in which the structure could be engineered would be to combine a zero coupon bond, purchased using the investor's deposit, together with a portfolio of income generating forward start written put options. The put option premia represents an additional pool of funds that will need to be drawn on should the underlying asset decline in value in any period.

There is clearly a real risk in the structure that needs to be addressed. Large falls or a series of falls in the asset's value may result in the additional funds being exhausted and the investor's investment principal being used to meet settlement obligations. In such situations, to ensure that the principal return guarantee is met, the institution offering the product will need to meet the cost from their own funds. To avoid this potentially expensive problem each cliquet in the portfolio will need to have insurance in place to ensure that potential losses are capped.

[1] The figure of 100% being the investor's initial cash investment (P).

Exhibit 1 illustrates this for the case of a single period whilst Exhibit 2 suggests the instrument's construction.

Insert Exhibit 1 Here

Insert Exhibit 2 Here

A possible course of action that would create a series of appropriate loss limits would be for the institution to purchase offsetting OTM forward start put options for each of the short forward start put options held in the portfolio. This introduces a conflict. The long OTM options will act as a drain on the funds which are being used to enable the offering of a higher than market coupon as an incentive to the investor. On the one hand the product requires a coupon high enough to attract investors on the other the risk of severe market index falls must be capped, achieving this by buying OTM cliquet options will exert a downward pull on the coupon.

II. PRICING UNDER CEV MODEL

The pricing mechanism for reverse cliquets falls under the Black-Scholes umbrella. The essential step is pricing forward start options and as described by Zhang [1998] or

Etheridge [2002]. The key point is the factorization of the value of the option, at the time point where the option comes into existence, as the product of the underlying stock and a multiplicative factor that does not depend on the underlying.

Fundamentally the Black-Scholes-Merton model is based on the modelling of the underlying assuming geometric Brownian motion. While this has been a great theoretical development empirical observations have questioned some of the assumptions or implications of this celebrated model. The main criticism stems from the assumption of constant variance which is contradicted by the empirical evidence showing that volatility changes with stock price[2]. Since geometric Brownian motion cannot account for the empirical observation that the variation of stock returns is declining, most of the time, as the stock price levels rises we are led into considering a more complex Ito process than a standard geometric Brownian motion.

In this section we develop this idea and model the underlying with a constant-elasticity-of-variance (CEV) process and derive the price of the forward start options that are the building block for the reverse cliquets. Once this is achieved everything else regarding financial engineering with reverse cliquets follows more or less the same methodology as described above.

The CEV model for an asset S is described by the following SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)^{\alpha} dZ(t) \tag{9}$$

where μ is the drift parameter, $\alpha > 0$ is a constant parameter and all other variables and parameters are exactly as for a geometric Brownian motion. This alternative stochastic process for pricing options has been proposed by Cox & Ross [1976] and they provided closed-

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[2] Schmalensee & Trippi (1978) found evidence of a negative relationship between stock price changes and changes in implied volatility while Black (1976) discovered using ten years of data of six stocks that a proportional increase, respectively decrease, in the stock price is associated with a larger proportional increase, respectively decrease in the standard deviation of the stock.

formulae for pricing European vanilla options when $\alpha < 1$. Empirical evidence shows that the CEV model in general outperforms the Black-Scholes model. MacBeth and Merville [1980] and Emanuel and MacBeth [1982] found empirical evidence supporting this conclusion on stock options markets while Hauser and Bagley [1986] showed similar results on the currency options markets. For the particular case of square-root process, that is for $\alpha = 0.5$, Beckers [1980] revealed that Black-Scholes ITM call and OTM put prices evaluated at implicit volatilities of at-the-money options are lower than those counterparts calculated with the CEV model. The CEV model implies a smile pattern that is frequently encountered on equity, index and currency options markets. However, the CEV model still leaves some Black-Scholes smile effects unexplained such as underpricing of ITM puts and OTM calls. Fortunately, for the structured product presented here the OTM puts are important.

Emanuel and MacBeth [1982] determined the formulae for the case when $\alpha > 1$, which for technical mathematical reasons and different boundary behaviour is different than the formulae for $\alpha < 1$. Schroder [1989] showed how to express the CEV option pricing formulae in terms of the noncentral chi-square distribution. This is recovered here when pricing forward start options, although it is not mentioned in the text explicitly.

For the sake of clarity we focus in this section on pricing an ATM forward start call option that kicks in at time T_1 and matures at T. Similar calculations can be made for OTM or ITM forward start options. Employing risk-neutral valuation we get the value of the option at time T_1 as:

$$V(S(T_1);T-T_1) = e^{-r(T-T_1)}\widetilde{E}_t\left[(S(T)-S(T_1))^+\right] \tag{10}$$

For simplicity, and without loss of generality, we restrict to the case $\alpha = 0.5$ which is the case most investigated in the literature. Denoting by $F_t(u) = P(S(T) \le u \mid S(t))$ Cox and Ross [1976] employed the following useful result due to Feller [1951]:

$$
\begin{aligned}
&dF_t(u) = \sqrt{\theta_t \vartheta_t}\, e^{-\vartheta_t - \theta_t u} u^{-1/2} I_1(2\sqrt{\theta_t \vartheta_t u}) \cdot du, \quad \textit{for any } u > 0 \\
&F_t(0) = G(\mu;1) \\
&F_t(u) = 0, \ \textit{for any } u < 0
\end{aligned} \tag{11}
$$

$$
\text{where } \theta_t = \frac{2\mu}{\sigma_0^2[e^{\mu(T-t)} - 1]},\ \vartheta_t = S(t)\theta_t e^{\mu(T-t)},\ G(x;a) = \int_x^\infty \frac{1}{\Gamma(a)} y^{a-1} e^{-y} dy \tag{12}
$$

and $I_1(\cdot)$ is the modified Bessel function of the first kind of order one.

For risk-neutral martingale pricing one sets either $\mu = r$ or $\mu = r - \delta$ if dividends are paid continuously at rate δ. For a general strike price *X* and maturity *T* the price of a European call at time *t* is:

$$
V(S(t), T-t) = e^{-r(T-t)} \widetilde{E}_t[(S(T) - X)^+] = e^{-r(T-t)} \int_X^\infty (s - X) dF_t(s) \tag{13}
$$

and using Feller's result given above it follows that:

$$
V(S(t), T-t) = e^{-r(T-t)} \int_X^\infty (s - X) \left(\sqrt{\theta_t \vartheta_t}\, e^{-\vartheta_t - \theta_t s} s^{-1/2} \right) I_1(2\sqrt{\theta_t \vartheta_t s}) ds \tag{14}
$$

However, the modified Bessel function can be approximated with the following series:

$$
I_1(z) = \frac{z}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k!\Gamma(k+2)} \tag{15}
$$

Replacing this in equation (11) leads to:

$$
V(S(t), T-t) = e^{-r(T-t)} \int_X^\infty (s - X) \left(\sqrt{\theta_t \vartheta_t}\, e^{-\vartheta_t - \theta_t s} s^{-1/2} \right) \sum_{k=0}^{\infty} \frac{\theta_t^k \vartheta_t^k s^k}{k!(k+1)!} ds \tag{16}
$$

In the Appendix it is shown that in the end we get to:

$$V(S(T_1);T-T_1)=S(T_1)\left\{\sum_{k=1}^{\infty} g(\vartheta_{T_1};k)G(\theta_{T_1}S(T_1);k+1)-e^{-r(T-T_1)}\sum_{k=1}^{\infty} g(\vartheta_{T_1};k+1)G(\theta_{T_1}S(T_1);k)\right\} \tag{17}$$

where $g(x;m)=\frac{x^{m-1}}{\Gamma(m)}e^{-x}$ is the probability density function for a gamma distribution with mean and variance equal to *m*.

The second factor delimited by the large brackets is a function $\psi(r,\sigma,T_1,T,S(T_1))$ so that we can write:

$$V(S(T_1);T-T_1)=S(T_1)\psi(r,\sigma,T_1,T,S(T_1)) \tag{18}$$

Unfortunately, under a CEV model, we cannot continue as described above when using a Black-Scholes model because the second factor is *not* independent of the underlying. This will complicate the calculation of the value of the forward start option at time $t = 0$, however, since we can still apply risk-neutral pricing, we can write:

$$V(S(0);T)=e^{-rT_1}\widetilde{E}\left[S(T_1)\psi(r,\sigma,T_1,T,S(T_1))\right] \tag{19}$$

In the Appendix it is shown that:

$$V(S(0);T)=\Omega_1-\Omega_2 \tag{20}$$

where:

$$\Omega_1=\theta_0^2\vartheta_0^2 e^{-\vartheta_0-rT_1}\sum_{j=0}^{\infty}\frac{1}{j!(j+1)!}\sum_{k=1}^{\infty}\frac{b^{k-1}}{(k-1)!}\sum_{i=0}^{k}\frac{\theta_{T_1}^{k-i}}{(k-i)!}\frac{(2k+j-i)!}{(\theta_0+\theta_{T_1}+b)^{2k+j-i+1}} \tag{21a}$$

$$\Omega_2=\theta_0^2\vartheta_0^2 e^{-\vartheta_0-rT}\sum_{j=0}^{\infty}\frac{1}{j!(j+1)!}\sum_{k=1}^{\infty}\frac{b^{k}}{k!}\sum_{i=0}^{k-1}\frac{\theta_{T_1}^{k-i-1}}{(k-i-1)!}\frac{(2k+j-i-1)!}{(\theta_0+\theta_{T_1}+b)^{2k+j-i}} \tag{21b}$$

where $b=\theta_{T_1}e^{rT_1}$.

III. RISK CONTROL ISSUES

The way in which the guaranteed principal instrument has been created by Financial Institution A, falls in the equity index result in sums being drawn down from the fund. The fund is protected from becoming negative by the holding a portfolio of long OTM puts, which form caps. Three market scenarios can be considered (1) the equity index rises by η%, (2) the equity index remains at its current level, (3) the equity index falls by η%. On reaching maturity in cases (1) and (2) the investor's achieved coupon will be the maximum offered in the bond's indenture $C_{\max}$. Under the third scenario the achieved coupon will be determined by:

$$0 \le \left(C_{max} + \left(\sum_{t=1}^{T} \left(-1 \cdot max\left[Strike_{ATM_t} - S_{t-1}, 0 \right] + max\left[Strike_{OTM_t} - S_{t-1}, 0 \right] \right) \right) \right) \le C_{max} \qquad (22)$$

In the case of the institution providing the cliquet options the pay out will be the mirror image of those generated by the investor. Under scenarios (1) and (2) the institution will meet the coupon pay out from the funds made up of the original investment plus the net income generated by the collar. Under scenario (3) the coupon paid to the investor will be reduced by an amount reflecting the downside protected fall in the index. Exhibits 3. and 4. illustrate the effect on the coupon as a result of period-on-period declines in the index value. At around a 3.8% fall in the index the funds experiences it maximum depletion rate. Unrealistic 80% period-on-period declines result in the investor's coupon payment rising as the percentage pay out declines due to the cliquet resets being implemented at much lower index values.

For simplicity we shall assume that the guaranteed amount to the investor is 100%. In other words the structured investment product guarantees the return in full of the sum invested at maturity T.

Let H denote the price, at time 0, of a zero coupon risk free bond with maturity T. Obviously $0< H < 100$ and $100-H$ is available for using in the reverse cliquet structure. Over each period of time $[t_{i-1}, t_i]$ of constant length $\Delta i = t_i - t_{i-1}$, with $i = 1,2,...,n+1$ the financial institution will sell ATM forward start put options and buy OTM forward start put options. Let $S(i)$ be the price of the index at time t_i and let $0<\eta<1$ be a factor defining the OTM strike price as $\eta S(t_{i-1})$ for the period $[t_{i-1}, t_i]$.

The payoff of the short ATM forward start put at t_i is $-\max[S(t_{i-1}) - S(t_i),0]$ and the payoff of the long OTM forward start put at the same time is $\max[\eta S(t_{i-1}) - S(t_i),0]$. This forward start spread has the combined value:

$$\max[\eta S(t_{i-1}) - S(t_i),0] - \max[S(t_{i-1}) - S(t_i),0]. \tag{23}$$

At time 0 this can be priced as a portfolio of options using risk-neutral valuation in the framework developed by Harrison and Kreps [1979]. Using the formulae[3] for forward start put options provided in Zhang [1998] the premium of the forward spread at time 0 is

$$\begin{aligned} & S(0)[e^{-r\Delta i-\delta t_{i-1}} N(-d_2^{ATM}(i)) - e^{-\delta t_i} N(-d_1^{ATM}(i))] - S(0)[\eta e^{-r\Delta i-\delta t_{i-1}} N(-d_2(i)) - e^{-\delta t_i} N(-d_1(i))] \\ & = S(0)e^{-r\Delta i-\delta t_{i-1}}[N(-d_2^{ATM}(i)) - \eta N(-d_2(i))] + S(0)e^{-\delta t_i}[\eta N(-d_1(i)) - N(-d_1^{ATM}(i))] \end{aligned} \tag{24}$$

Recall that for the OTM forward start option:

$$d_1(i) = \frac{r-\delta+\sigma^2/2}{\sigma}\sqrt{\Delta i} - \frac{\ln\eta}{\sigma\sqrt{\Delta i}} \quad and \quad d_2(i) = d_1(i) - \sigma\sqrt{\Delta i}$$

The total revenue at time 0 from the forward start engineered structure is

$$Q = \sum_{i=1}^{T} S(0)e^{-r\Delta i-\delta t_{i-1}}[N(-d_2^{ATM}(i)) - \eta N(-d_2(i))] + S(0)e^{-\delta t_i}[\eta N(-d_1(i)) - N(-d_1^{ATM}(i))] \tag{25}$$

so the seller of the reverse cliquet has $1-H+Q$ at their disposal.

Fixed Coupon

Suppose now that the investor is also rewarded with a coupon *x* (%) paid at the end of each period, where for simplicity we take $t_i \equiv i$ for all $i = 1,2,...,T$. Thus the present value of what the investor will get over the life of the product, assuming a flat interest rate *r* and continuous compounding for simplicity is:

$$x\left(\frac{1-e^{-rT}}{e^{r}-1}\right) \qquad (26)$$

At each reset time, say time *i*, the maximum payout that the reverse cliquet seller may have to pay to third parties, following the decline of the index, is:

$$S(i-1) - \eta S(i-1) \qquad (27)$$

so the total maximum payout that may be paid, at time 0, is worth:

$$L = (1-\eta)\sum_{i=1}^{T} S(i-1)e^{-ri} \qquad (28)$$

Therefore the following inequality relating the coupon rate *x* and floor level represented by the deflating factor η is:

$$1 - H + Q - L \geq x\left(\frac{1-e^{-rT}}{e^{r}-1}\right) \qquad (29)$$

The only weakness in this construction is that future index levels are uncertain. Assuming that the index follows a geometric Brownian motion[4] with drift parameter μ and volatility parameter σ it is known that:

[3] We have corrected some typos that appear in Zhang [1998].
[4] This is not exactly correct from a pure mathematical finance point of view but it seems to work well in practice and it can be therefore used as least as a very good approximation.

$$E_0[S(i)] = S(0)e^{\mu i} \tag{30}$$

Passing the expectation operator, in real world, over the above inequality constraint leads to the following *average* condition:

$$1 - H + Q \geq x\left(\frac{1-e^{-rT}}{e^r - 1}\right) + (1-\eta)\sum_{i=1}^{T} S(0)e^{\mu(i-1)}e^{-ri} = x\left(\frac{1-e^{-rT}}{e^r - 1}\right) + \frac{(1-\eta)}{e^r - e^{\mu}} S(0)\left(1 - e^{(\mu - r)T}\right)$$

(31)

Alternatively the financial engineer may try to simulate 'what-if' scenarios using the evolution equation of the index:

$$S(i) = S(0)e^{(\mu - \sigma^2/2)i + \sigma W(i)} \tag{32}$$

where W is a Wiener process.

Variable Coupon

A more common practice is to provide investors with a variable coupon that pays at each reset time or in one payment at maturity the difference between a fixed coupon rate x (%) and the level of percentage decline in the index over the ending period. For period $[t_{i-1}, t_i]$ the decline in the index is:

$$\max[S(i-1) - S(i), 0]$$

so when all payments are settled at maturity T the coupon paid is:

$$\Pi = \sum_{i=1}^{T} \max\left(x - \max[\frac{S(i-1) - S(i)}{S(i-1)}, 0], 0 \right). \tag{33}$$

As in the previous section, considering the worst case scenario that for each period the ATM put options will be exercised due to a decline of the index at or below the cap provided by the OTM options, the financial engineer must make sure that:

$$1 - H + Q \geq \Pi e^{-rT} + \sum_{i=1}^{T} \min\left(\max[S(i-1) - S(i), 0], (1-\eta)S(i-1)\right)e^{-ri} \tag{34}$$

otherwise payments may be missed or losses will be made.

IV. APPLICATION

In order to examine how this type of product can be engineered consider the following example:

> *A non-callable bond is issued offering a minimum return of full principal invested at the end of three years or full principal plus x% - the sum of the annual declines in the defined equity index.*

Recall that the financial engineer has to establish at what level *x* can be set and this will in turn be determined by the amount available from the sale of ATM puts less the cost of the OTM puts needed to create the cap. To illustrate how the structure can be replicated we will price both long and short forward start put options that comprise the cliquet option collar initially in a Black and Scholes framework. This, of course, ignores volatility stochasticity and any volatility smile. Proceeding with this approach we assume that the discount rate is 2.35% this implies that the institution will today pay 93.27% for a zero coupon bond with a three year maturity.

$$H = \frac{FV}{(1+dr)^T} = \frac{100}{(1+0.0235)^3} = 93.27\% \tag{35}$$

Thus (1 – H) = (1 - 0.9327) = 0.0673%, implies that a 6.73% residual is immediately available to invest in the fund that will be used to make payments to the put holders if and when required. To price the forward start options we assume that the yield curve is flat and that risk free rate for all maturities is 2.52%; dividend yield is 1.58%; volatility is 25% p.a. and the life span of each option is 360-days. Using the formulae presented in equation (5) above the price of each ATM forward start option in this regime is 9.232% and since 1 regular put and 2 forward start puts are needed to cover the maturity of the bond and the number of resets the total income from ATM options will be 27.70%. In order to cover the period-by-period downside investor risk, the institution will need to buy 2 OTM forward start put options and 1 OTM regular put option for the first year of the structured product life at a total cost of 20.47%. The net contribution of the put option transactions to the fund will be 7.23%, combing this with the 6.73% residual from the zero coupon bond purchase, we have a fund of 13.96%. This fund provides an indication of the maximum coupon that the investor can expect to receive when there are no payouts from the fund at any of the reset dates. Should the index fall to a level below the relevant floor in each period the long OTM put options will be exercised ensuring that the investor receives the minimum return on the instrument, namely the original investment principal.

If we assume that one single payment will be made to the investor at the bond's maturity and that a 100 basis point transactions fee is imposed by the financial institution structuring the transaction the maximum final coupon that can be offered to the investor is 12.95%. If the issuing institution is willing to accept only 25 basis points for its services the maximum coupon that can be attached to the bond is 13.70%. In both of these cases the 100% principal guarantee can be met as illustrated in Exhibits 3 and 4.

Insert Exhibit 3 Here

Insert Exhibit 4 Here

For risk control purposes we can simulate possible paths for the index and check the amounts that will be paid under each scenario to the counterparty in the forward start options and from that derive the amount left to pay the coupons to the investor. Continuing with the same data provided above in this section and assuming in addition that r = 3% and the fixed coupon rate *x* is 3% per annum paid at maturity, our Monte Carlo simulation exercise suggests that the maximum present value of total payment made by the seller of the structured product is 16.34%. This should be compared with the 13.96% funds available. A more informative view is described in Exhibits 5. and 6. showing the total payments made under each simulated path of the index. It can be seen that there are only two scenarios where the payments made by the financial institution exceed the funds they have for the reverse cliquet.

Insert Exhibit 5 here

A possible strategy that the financial engineer may follow is to search for the coupon rate x that makes the maximum payment, over the already simulated paths of the index, equal to the total sum of funds available $1 - H + Q$. For the above example this is realised for a coupon rate $x = 2\%$ (per annum). Exhibit 6. shows that indeed for all scenarios no payment is higher than the targeted 13.96%.

Insert Exhibit 6 here

V. CONCLUSION

Structured products are establishing as a class of instruments in modern finance. Here we have investigated a product underpinned by reverse cliquet options. We provided an approach to pricing and implementing this type of structure under the standard Black-Scholes model**.** The financial engineer is able to perform calculations that determine how large the coupon offered to investors can be. The main difficulty in valuing of this structure revolves around the pricing of the required forward start options. Since a cap is also created using OTM forward start puts, we considered that models that take account of known empirical facts should be investigated in addition to the standard Black and Scholes model. For this reason we have used the CEV model as a starting point and derived a new option pricing formula for forward start options.

Future research will continue by looking at more general stochastic volatility models for the equity index and larger models that consider jointly the term structure of interest rates and a model for the equity index.

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Appendix

First we show how to calculate the following integral

$$V(S(t),T-t)=e^{-r(T-t)}\int_X^{\infty}(s-X)\left(\sqrt{\theta_t\vartheta_t}e^{-\vartheta_t-\theta_t s}s^{-1/2}\right)\sum_{k=0}^{\infty}\frac{\theta_t^k\vartheta_t^k s^k}{k!(k+1)!}ds$$

We are going to separate the integral into two integrals. Thus

$$V(S(t),T-t)=e^{-r(T-t)}\theta_t\vartheta_t e^{-\vartheta_t}\left\{\int_X^{\infty}se^{-\theta_t s}\sum_{k=0}^{\infty}\frac{\theta_t^k\vartheta_t^k s^k}{k!(k+1)!}ds-X\int_X^{\infty}e^{-\theta_t s}\sum_{k=0}^{\infty}\frac{\theta_t^k\vartheta_t^k s^k}{k!(k+1)!}ds\right\}$$

Making the change of variable $\theta_t s=y$ we get

$$\begin{aligned}V(S(t),T-t)&=e^{-r(T-t)}\vartheta_t e^{-\vartheta_t}\left\{\int_{\theta_t X}^{\infty}ye^{-y}\sum_{k=0}^{\infty}\frac{\vartheta_t^k y^k}{k!(k+1)!}\frac{1}{\theta_t}dy-X\int_{\theta_t X}^{\infty}e^{-y}\sum_{k=0}^{\infty}\frac{\vartheta_t^k y^k}{k!(k+1)!}dy\right\}\\&=S(t)\sum_{k=0}^{\infty}\frac{\vartheta_t^k}{k!}e^{-\vartheta_t}\left(\int_{\theta_t X}^{\infty}\frac{y^{k+1}}{(k+1)!}e^{-y}dy\right)-e^{-r(T-t)}X\sum_{k=0}^{\infty}\frac{\vartheta_t^{k+1}}{(k+1)!}e^{-\vartheta_t}\left(\int_{\theta_t X}^{\infty}\frac{y^k}{k!}e^{-y}dy\right)\\&=S(t)\sum_{k=1}^{\infty}\frac{\vartheta_t^{k-1}}{(k-1)!}e^{-\vartheta_t}\left(\int_{\theta_t X}^{\infty}\frac{y^k}{k!}e^{-y}dy\right)-e^{-r(T-t)}X\sum_{k=1}^{\infty}\frac{\vartheta_t^k}{k!}e^{-\vartheta_t}\left(\int_{\theta_t X}^{\infty}\frac{y^{k-1}}{(k-1)!}e^{-y}dy\right)\end{aligned}$$

If $g(x;m)=\frac{x^{m-1}}{\Gamma(m)}e^{-x}$ is the probability density function for a gamma distribution with mean and variance equal to *m* and the incomplete gamma function is defined as in the text in formula (9) it follows then that

$$V(S(t),T-t)=S(t)\sum_{k=1}^{\infty}g(\vartheta_t;k)G(\theta_t X;k+1)-e^{-r(T-t)}X\sum_{k=1}^{\infty}g(\vartheta_t;k+1)G(\theta_t X;k)$$

Hence, at time $t=T_1$, the value of the ATM forward start option is

$$V(S(T_1);T-T_1)=S(T_1)\left\{\sum_{k=1}^{\infty}g(\vartheta_{T_1};k)G(\theta_{T_1}S(T_1);k+1)-e^{-r(T-T_1)}\sum_{k=1}^{\infty}g(\vartheta_{T_1};k+1)G(\theta_{T_1}S(T_1);k)\right\}$$

The second calculation detailed here is

$$V(S(0);T) = e^{-rT_1}\tilde{E}\left[S(T_1)\psi(r,\sigma,T_1,T,S(T_1))\right]$$

where $\psi(r,\sigma,T_1,T,S(T_1)) = \left\{\sum_{k=1}^{\infty} g(\vartheta_{T_1};k)G(\theta_{T_1}S(T_1);k+1) - e^{-r(T-T_1)}\sum_{k=1}^{\infty} g(\vartheta_{T_1};k+1)G(\theta_{T_1}S(T_1);k)\right\}$

Evidently $V(S(0);T) = e^{-rT_1}\int_0^{\infty} s\psi(r,\sigma,T_1,T,s)dF_0(s)$ which after replacement of ψ and the other known expressions becomes

$$e^{-rT_1}\sum_{k=1}^{\infty}\frac{b^{k-1}}{(k-1)!}\theta_0^2\vartheta_0^2 e^{-\vartheta_0}\sum_{j=0}^{\infty}\frac{1}{j!(j+1)!}\int_0^{\infty} s^{k+j}e^{-s(b+\theta_0)}G(\theta_{T_1}s;k+1)ds -$$

$$e^{-rT}\sum_{k=1}^{\infty}\frac{b^{k-1}}{k!}\theta_0^2\vartheta_0^2 e^{-\vartheta_0}\sum_{j=0}^{\infty}\frac{1}{j!(j+1)!}\int_0^{\infty} s^{k+j+1}e^{-s(b+\theta_0)}G(\theta_{T_1}s;k)ds$$

Lets denote the first term by Ω_1 and the second term by Ω_2. The key element in the subsequent calculations is the integral $I_n(A) = \int_A^{\infty} y^n e^{-y}dy$ that can be shown after some integration by parts to be equal to

$$I_n(A) = \int_A^{\infty} y^n e^{-y}dy = e^{-A}\sum_{i=0}^{n}\frac{n!}{(n-i)!}A^{n-i}$$

In order to calculate Ω_1 we need to calculate first the integral

$$\int_0^{\infty} s^{k+j}e^{-s(b+\theta_0)}G(\theta_{T_1}s;k+1)ds = \int_0^{\infty} s^{k+j}e^{-s(b+\theta_0)}\frac{1}{k!}I_k(\theta_{T_1}s)ds$$

$$= \frac{1}{k!}\sum_{i=0}^{k}\frac{k!}{(k-i)!}\theta_{T_1}^{k-i}\int_0^{\infty} s^{2k+j-i}e^{-s(\theta_0+\theta_{T_1}+b)}ds$$

$$= \sum_{i=0}^{k}\frac{\theta_{T_1}^{k-i}}{(k-i)!}\int_0^{\infty}\frac{u^{2k+j-i}}{(\theta_0+\theta_{T_1}+b)^{2k+j-i+1}}e^{-u}du$$

$$= \sum_{i=0}^{k}\frac{\theta_{T_1}^{k-i}}{(k-i)!}\frac{(2k+j-i)!}{(\theta_0+\theta_{T_1}+b)^{2k+j-i+1}}$$

Thus

$$\Omega_1 = \theta_0^2\vartheta_0^2 e^{-\vartheta_0}e^{-rT_1}\sum_{j=0}^{\infty}\frac{1}{j!(j+1)!}\sum_{k=1}^{\infty}\frac{b^{k-1}}{(k-1)!}\sum_{i=0}^{k}\frac{\theta_{T_1}^{k-i}}{(k-i)!}\frac{(2k+j-i)!}{(\theta_0+\theta_{T_1}+b)^{2k+j-i+1}}$$

Similarly

$$\Omega_2 = \theta_0^2\vartheta_0^2 e^{-\vartheta_0}e^{-rT}\sum_{j=0}^{\infty}\frac{1}{j!(j+1)!}\sum_{k=1}^{\infty}\frac{b^{k}}{k!}\sum_{i=0}^{k-1}\frac{\theta_{T_1}^{k-i-1}}{(k-i-1)!}\frac{(2k+j-i-1)!}{(\theta_0+\theta_{T_1}+b)^{2k+j-i}}$$

List of Exhibits

Exhibit 1.

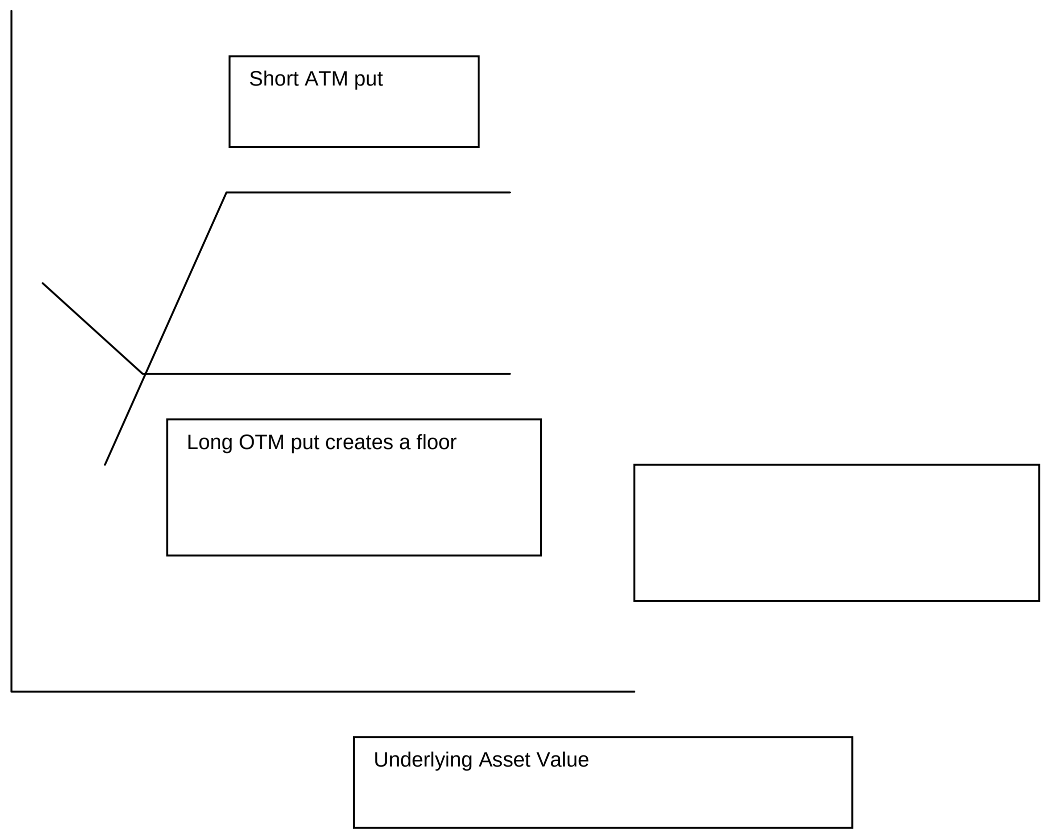
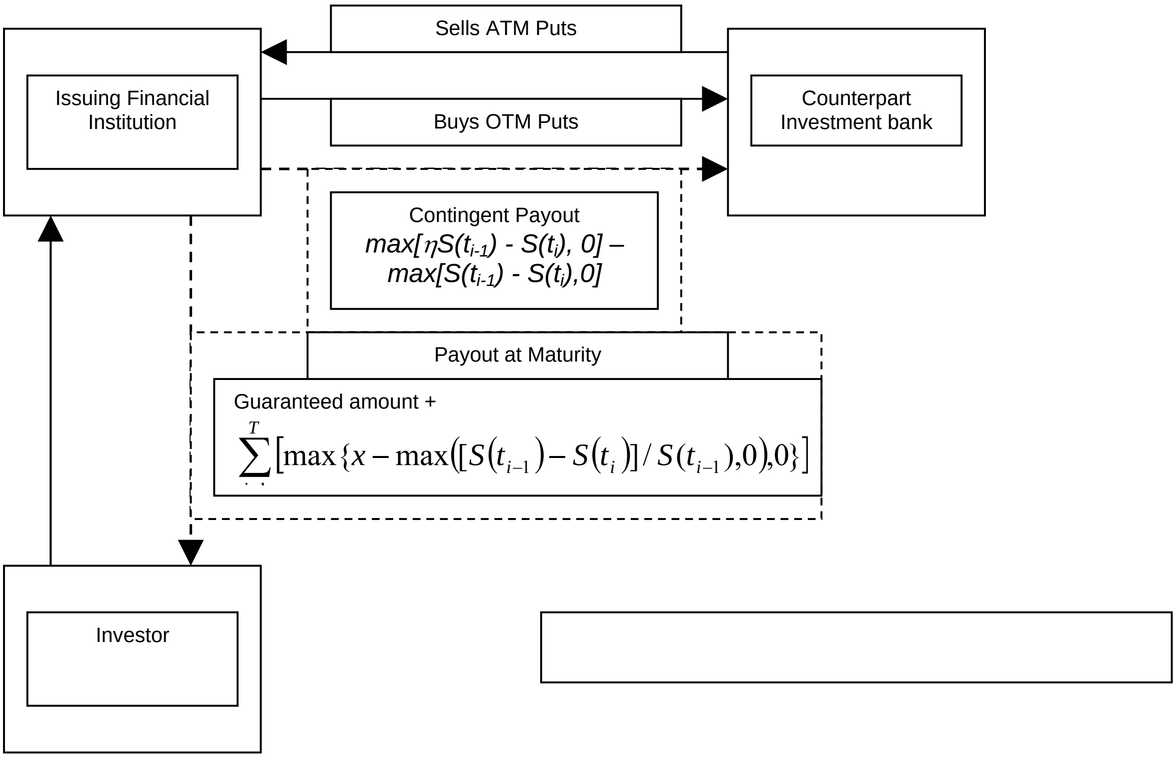
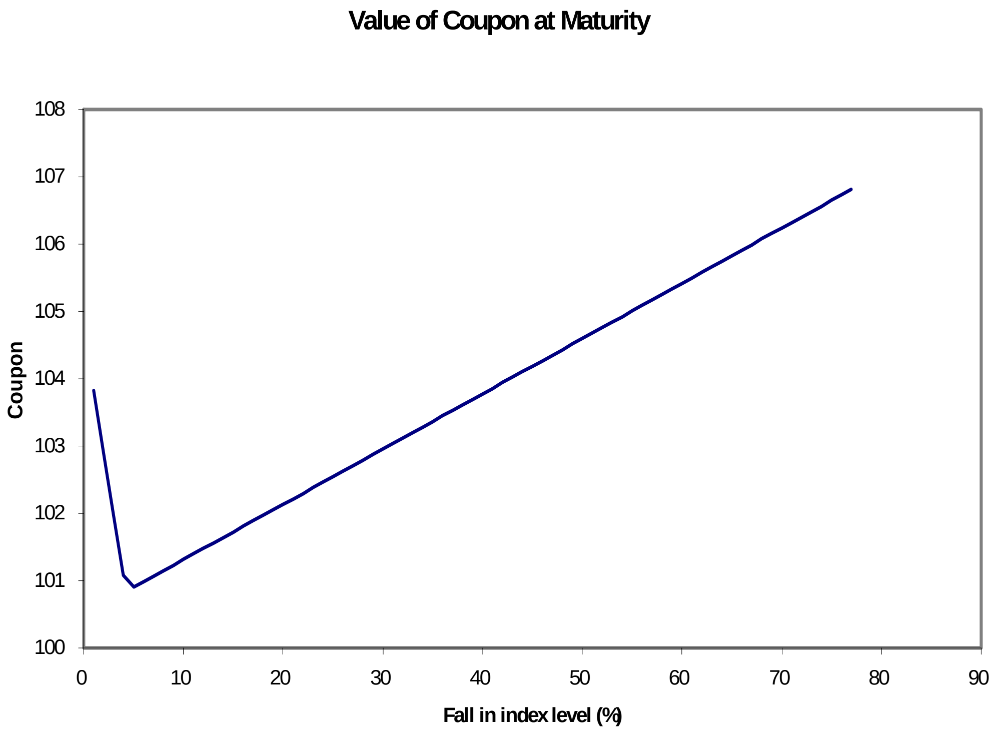


Exhibit 2.



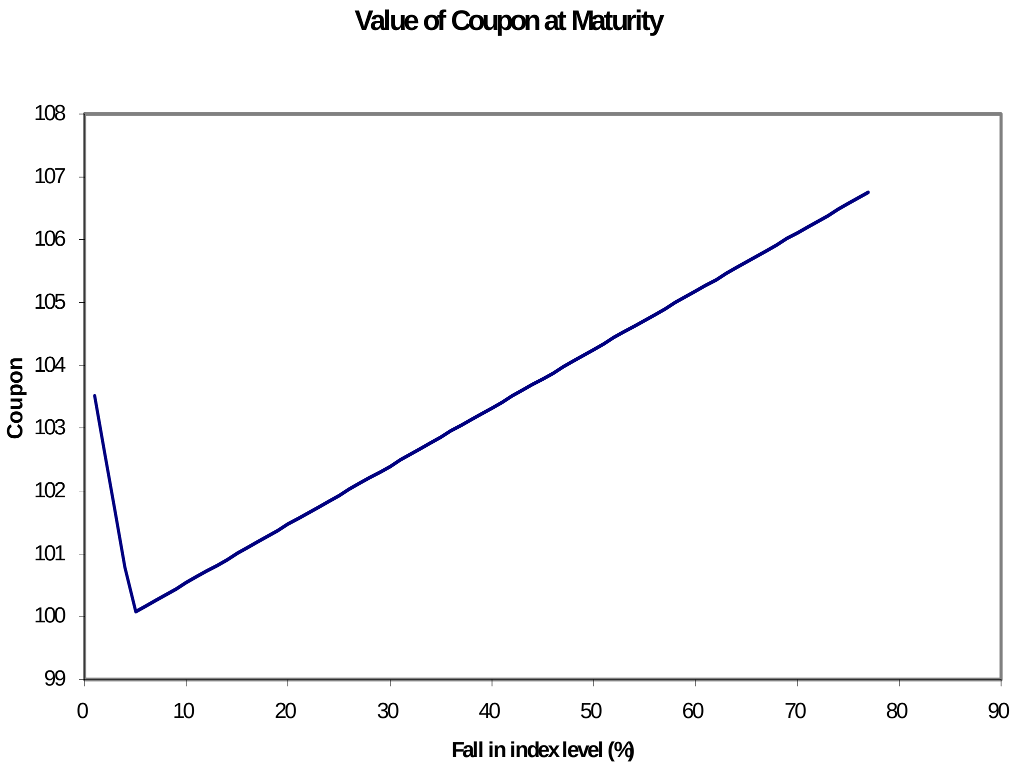
Financial Engineering structure of the indexed-linked guaranteed principal bond.

Exhibit 3



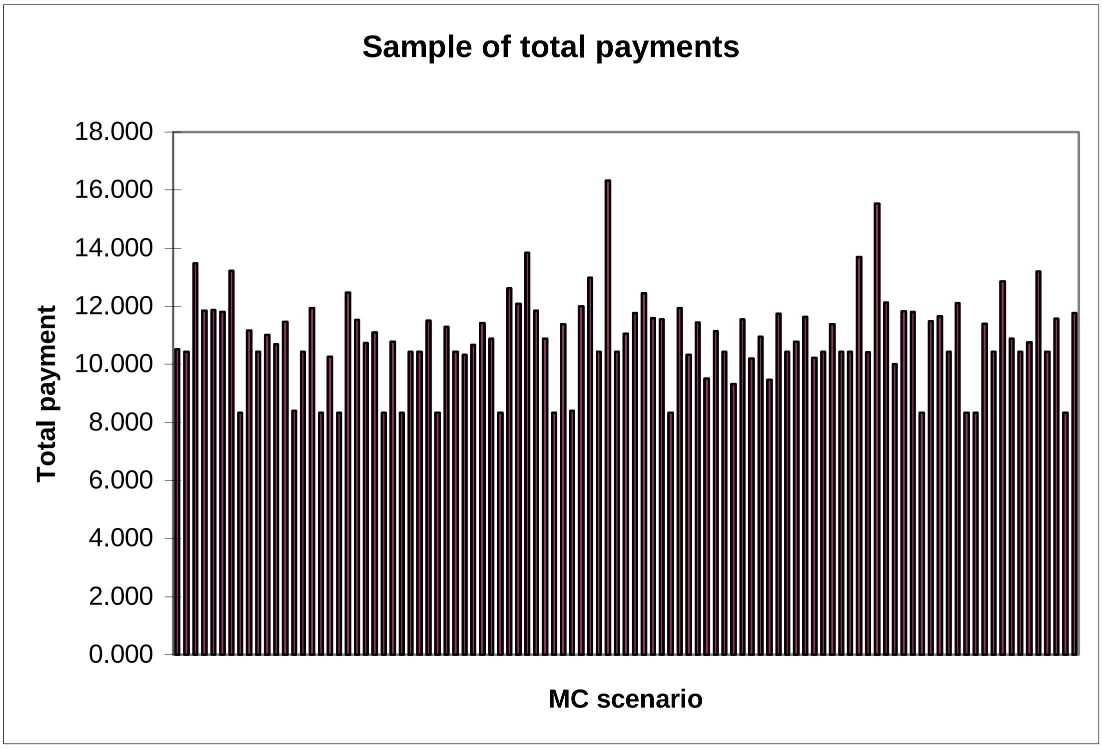
The value of coupon at maturity with respect to the fall in the underlying index on a simulated scenario where the issuer is charging 100 basis points; the principal is always guaranteed.

Exhibit 4



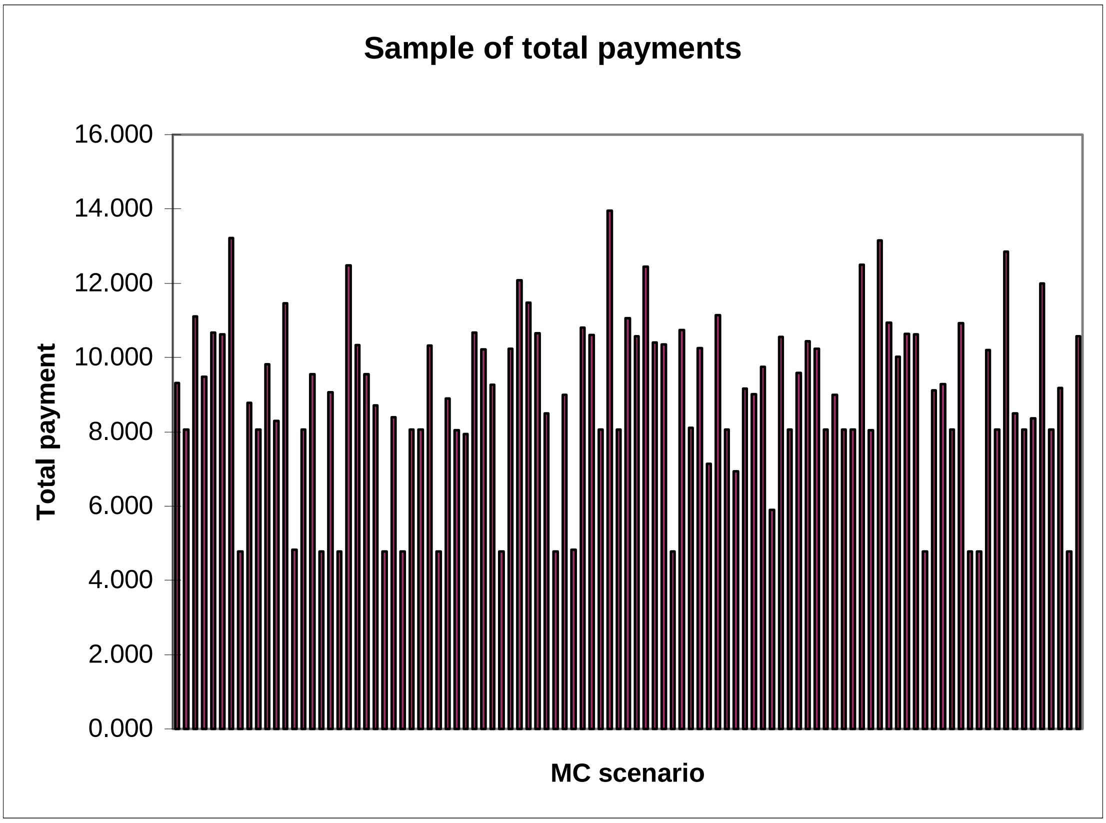
The value of coupon at maturity with respect to the fall in the underlying index on a simulated scenario where the issuer is charging 25 basis points; the principal is always guaranteed.

Exhibit 5



Monte Carlo simulations when the coupon rate is x=3% p.a. There are only two cases where the total payment is higher than 13.956.

Exhibit 6



Monte Carlo simulations when the coupon rate is x=2% p.a. There are no cases where the total payment is higher than 13.956.