Multilevel Modeling of Ordinal Responses

Sophia Rabe-Hesketh
Graduate School of Education & Graduate Group in Biostatistics
University of California, Berkeley
Institute of Education, University of London

Joint work with Anders Skrondal
Division of Epidemiology
Norwegian Institute of Public Health

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Outline

- Latent response and generalized linear model formulations
- Two-level random intercept models
- Conditional versus marginal effects
- Example: Recovery after surgery
- Different kinds of predicted probabilities
- Relaxing proportional odds assumption
- Modeling heteroscedasticity at the unit and cluster levels

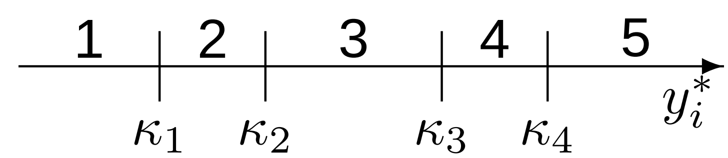
Ordered categorical responses

- Small number of mutually exclusive ‘categories’, $y_i = 1, \ldots, S$
- Categories are ordered, so y_i is **ordinal**
- Examples:
  - Severity of symptom (e.g. pain): none, moderate, severe
  - Frequency of behavior: never, occasionally, daily
  - Agreement (Likert scale):
    disagree strongly, disagree, agree, agree strongly

Latent response models

- Latent (unobserved) continuous response y_i^* underlies observed ordinal response y_i
- Threshold model determines observed response:

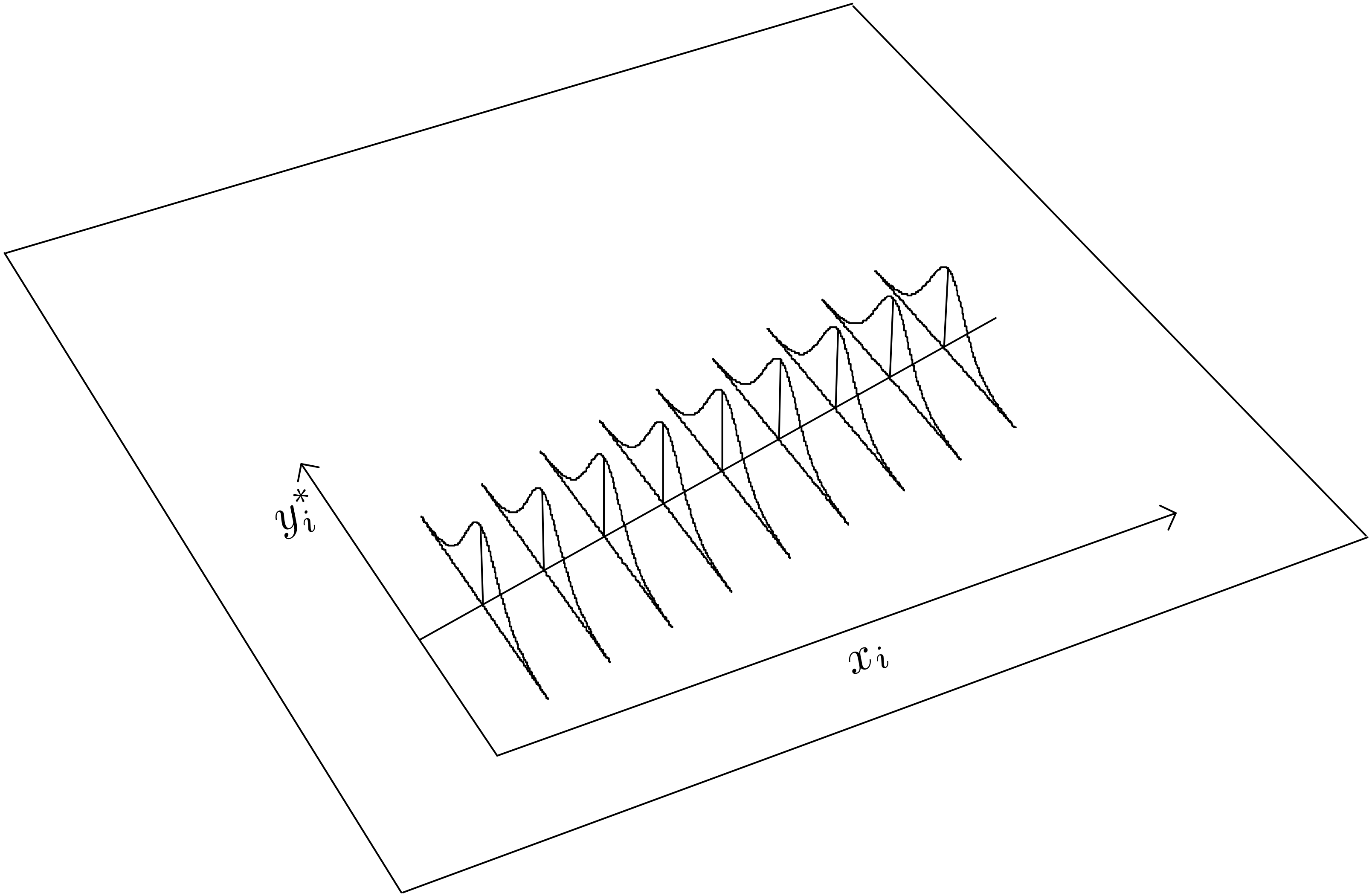
$$y_i = \begin{cases} 1 & \text{if} \quad y_i^* \leq \kappa_1 \\ 2 & \text{if} \quad \kappa_1 < y_i^* \leq \kappa_2 \\ \vdots & \vdots \qquad \vdots \\ S & \text{if} \quad \kappa_{S-1} < y_i^* \end{cases}$$



- Latent response modeled as linear regression without intercept (for identification)

$$\begin{aligned} y_i^* &= \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \epsilon_i \\ &= \mathbf{x}_i'\boldsymbol{\beta} + \epsilon_i \end{aligned}$$

Illustration: Latent response formulation ($S = 3$)



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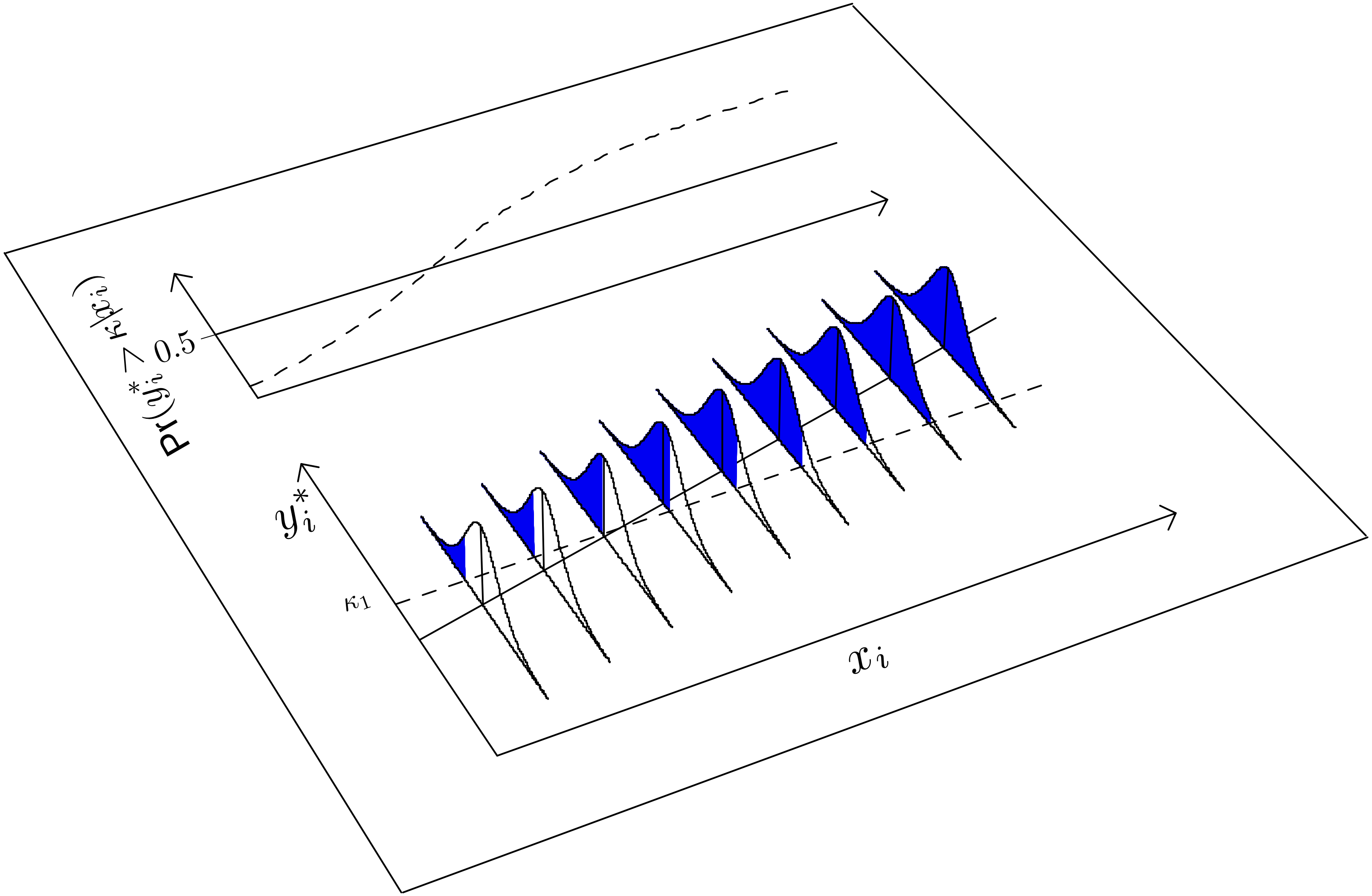

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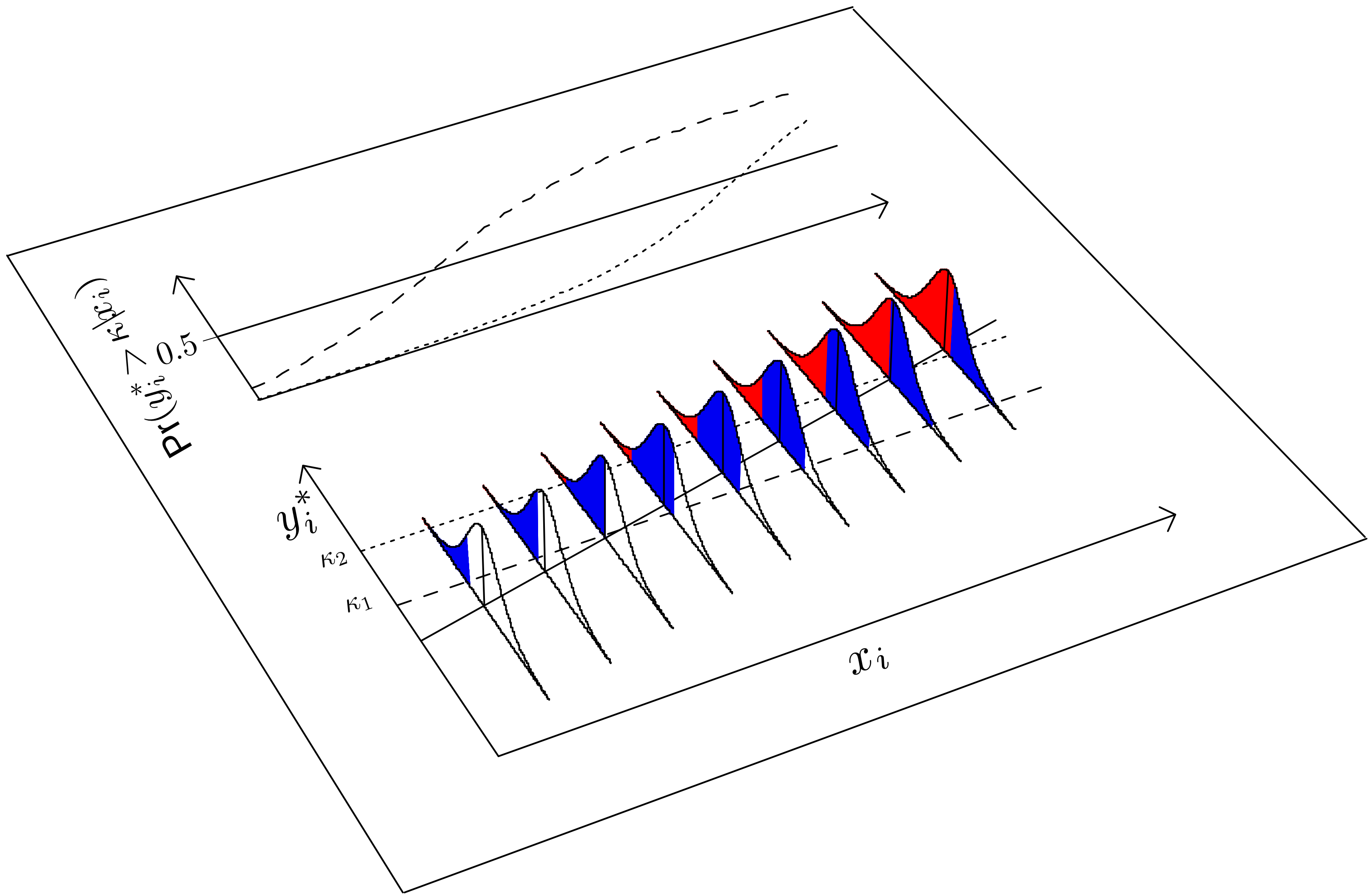
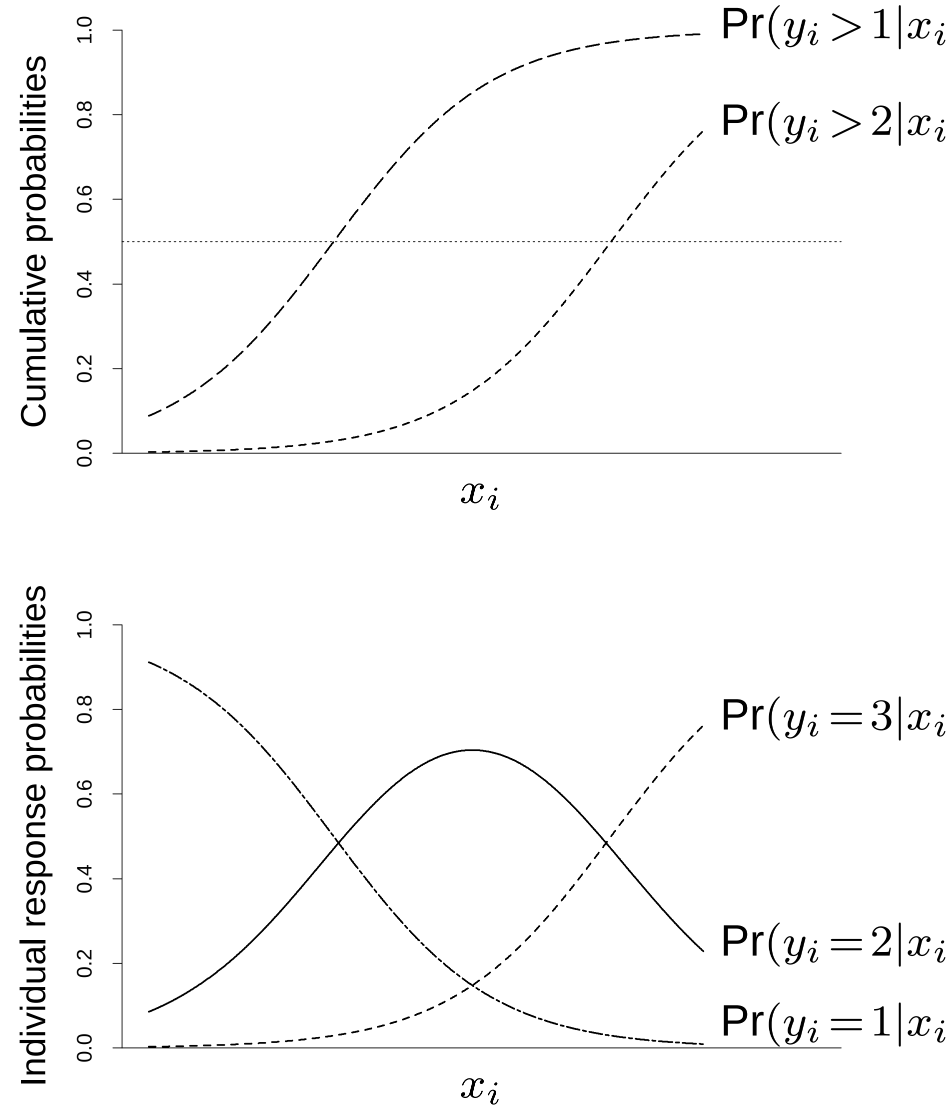


Cumulative versus individual response probabilities

$$
\begin{aligned}
\text{Pr}(y_i > 1) &= \text{Pr}(y_i = 2) + \text{Pr}(y_i = 3) \\
\text{Pr}(y_i > 2) &= \text{Pr}(y_i = 3)
\end{aligned}
$$

$$
\begin{aligned}
\text{Pr}(y_i = 3) &= \text{Pr}(y_i > 2) \\
\text{Pr}(y_i = 2) &= \text{Pr}(y_i > 1) - \text{Pr}(y_i > 2) \\
\text{Pr}(y_i = 1) &= 1 - \text{Pr}(y_i > 1)
\end{aligned}
$$



From latent response to generalized linear model formulations

- Cumulative probabilities:

$$\begin{aligned}\Pr(y_i > s|\mathbf{x}_i) &= \Pr(y_i^* > \kappa_s|\mathbf{x}_i) = \Pr(\mathbf{x}_i'\boldsymbol{\beta} + \epsilon_i > \kappa_s|\mathbf{x}_i)\\ &= \Pr(-\epsilon_i \le \mathbf{x}_i'\boldsymbol{\beta} - \kappa_s|\mathbf{x}_i) = F(\mathbf{x}_i'\boldsymbol{\beta} - \kappa_s)\end{aligned}$$

  - $F(\cdot)$ is the cumulative density function of $-\epsilon_i$
- This is a generalized linear model where
  - $F(\cdot)$ is inverse **link function** and $\mathbf{x}_i'\boldsymbol{\beta} - \kappa_s$ is **linear predictor**

Distribution of $-\epsilon_i$	Var(ϵ_i)	Link $g = F^{-1}$	Model	Stata command
Logistic	$\pi^2/3$	Cumulative Logit	Prop. odds	`ologit`, `gllamm`
Standard normal	1	Probit	Ordinal probit	`oprobit`, `gllamm`
*Gumbel	$\pi^2/6$	Compl. log-log	Compl. log-log	`gllamm`

*Asymmetric (non-equivalent model if scale reversed)

Cumulative logit model and odds ratios

- With a logistic distribution for ϵ_i, a **logit link** (one covariate x_i):

$$\underbrace{F^{-1}}_{g}[\mathrm{Pr}(y_i > s|x_i)] \;=\; \log\underbrace{\left[\frac{\mathrm{Pr}(y_i > s|x_i)}{\mathrm{Pr}(y_i \le s|x_i)}\right]}_{\mathrm{odds}_{(y_i>s|x_i)}} \;=\; \beta x_i - \kappa_s$$

$$\begin{aligned}\beta &= \log[\mathrm{odds}(y_i > s|x_i = a+1)] - \log[\mathrm{odds}(y_i > s|x_i = a)] \\ &= \log\left[\frac{\mathrm{odds}(y_i > s|x_i = a+1)}{\mathrm{odds}(y_i > s|x_i = a)}\right]\end{aligned}$$

- Coefficient β represents increase in log-odds per unit increase in x_i
- Exponentiated coefficient $\exp(\beta)$ represents **odds ratio** per unit increase in x_i
    - Does not depend on $s \Rightarrow$ Proportional odds

Other logit models for ordinal responses

- Adjacent category logit (see Zheng & R-H, Stata Journal 7, 313-333)

$$\log\left[\frac{\Pr(y_i = s+1|\mathbf{x}_i)}{\Pr(y_i = s|\mathbf{x}_i)}\right] \;=\; \mathbf{x}_i'\boldsymbol{\beta} + \kappa_s$$

  Data expansion followed by `clogit` or `gllamm` command

- Continuation ratio logit (see MLMUS2 - CH8)

$$\log\left[\frac{\Pr(y_i = s|\mathbf{x}_i)}{\Pr(y_i > s|\mathbf{x}_i)}\right] \;=\; \mathbf{x}_i'\boldsymbol{\beta} + \kappa_s$$

  Data expansion followed by `logit` or `gllamm` command

- Stereotype model

$$\log\left[\frac{\Pr(y_i = s|\mathbf{x}_i)}{\Pr(y_i = 1|\mathbf{x}_i)}\right] \;=\; \mathbf{x}_i'\boldsymbol{\beta}\alpha_s + \kappa_s$$

  `slogit` command

Two-level random intercept models

- Units i (e.g. subjects) nested in clusters j (e.g. schools)
- Include random intercept ζ_j for clusters

$$y_{ij}^* = \mathbf{x}_{ij}'\boldsymbol{\beta} + \zeta_j + \epsilon_{ij}, \quad \zeta_j|\mathbf{x}_{ij} \sim N(0,\psi)$$

$$\epsilon_{ij}|\mathbf{x}_{ij} \sim \begin{cases} N(0,1) & \text{for probit models} \\ \text{Logistic} & \text{for logit models} \end{cases}$$

  - ζ_j independent of ϵ_{ij}
  - ζ_j and ϵ_{ij} independent across clusters
  - ϵ_{ij} independent across units within clusters

Intraclass correlation of latent responses

- Total residual $\xi_{ij} = \zeta_j + \epsilon_{ij}$ has variance

$$\text{Var}(\xi_{ij}|\mathbf{x}_{ij}) \; = \begin{cases} \psi + 1 & \text{for probit models} \\ \psi + \pi^2/3 & \text{for logit models} \end{cases}$$

- Covariance between total residuals ξ_{ij} and $\xi_{i'j}$ of two subjects in same cluster is ψ and **intraclass correlation** is

$$\text{Cor}(\xi_{ij}, \xi_{i'j}|\mathbf{x}_{ij}, \mathbf{x}_{i'j}) \; = \begin{cases} \psi/(\psi + 1) & \text{for probit models} \\ \psi/(\psi + \pi^2/3) & \text{for logit models} \end{cases}$$

Conditional versus marginal effects

- For probit model, **conditional** or cluster-specific probabilities are

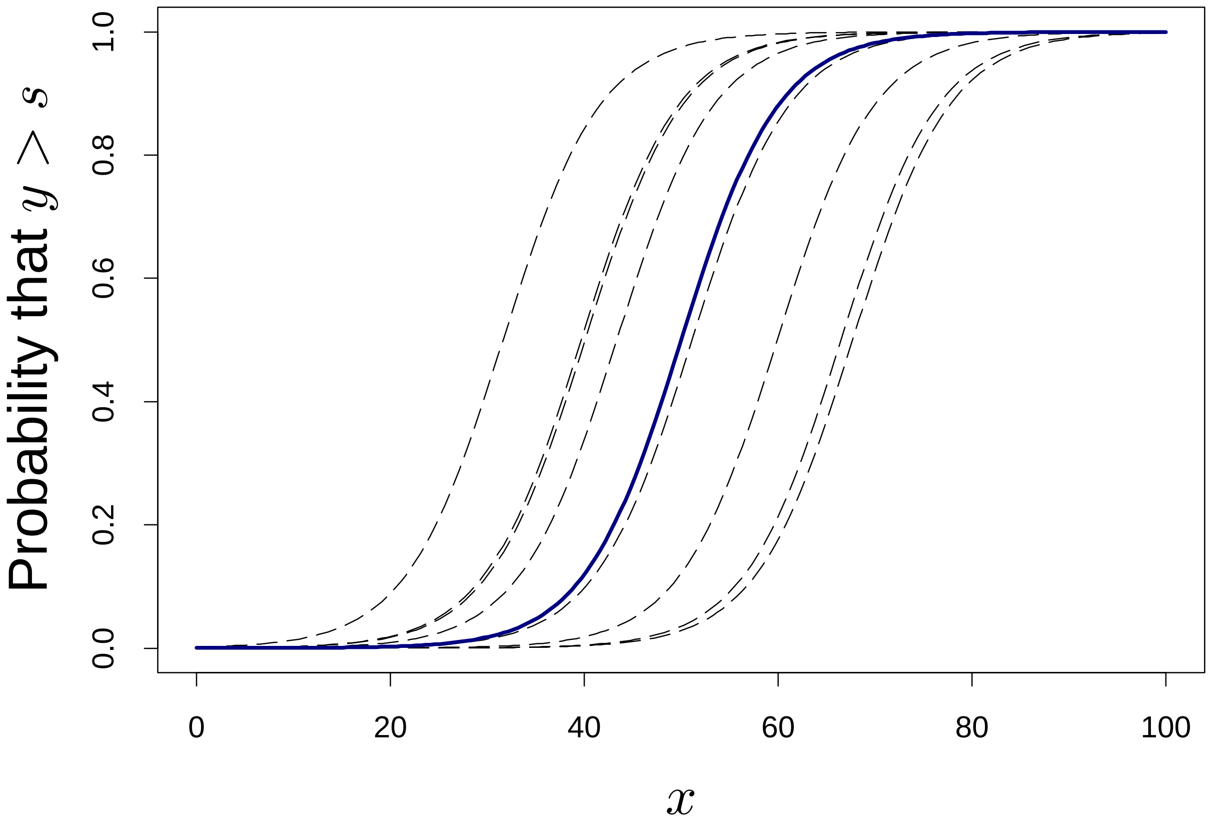
$$\text{Pr}(y_{ij} > s|\zeta_j, \mathbf{x}_{ij}) \;=\; \Phi(\mathbf{x}'_{ij}\boldsymbol{\beta} + \zeta_j - \kappa_s)$$

- **Marginal** or population-averaged response probabilities are

$$\begin{aligned}
\text{Pr}(y_{ij} > s|\mathbf{x}_{ij}) &= \text{Pr}(y^*_{ij} > \kappa_s) = \text{Pr}(\mathbf{x}'_{ij}\boldsymbol{\beta} + \xi_{ij} > \kappa_s) \\
&= \text{Pr}(-\xi_{ij} \leq \mathbf{x}'_{ij}\boldsymbol{\beta} - \kappa_s) \\
&= \text{Pr}\left(\frac{\xi_{ij}}{\sqrt{\psi+1}} \leq \frac{\mathbf{x}'_{ij}\boldsymbol{\beta} - \kappa_s}{\sqrt{\psi+1}}\right) \\
&= \Phi\left(\frac{\mathbf{x}'_{ij}\boldsymbol{\beta} - \kappa_s}{\sqrt{\psi+1}}\right)
\end{aligned}$$

- Therefore, marginal effects $\boldsymbol{\beta}/\sqrt{\psi+1}$ attenuated compared with conditional effects $\boldsymbol{\beta}$

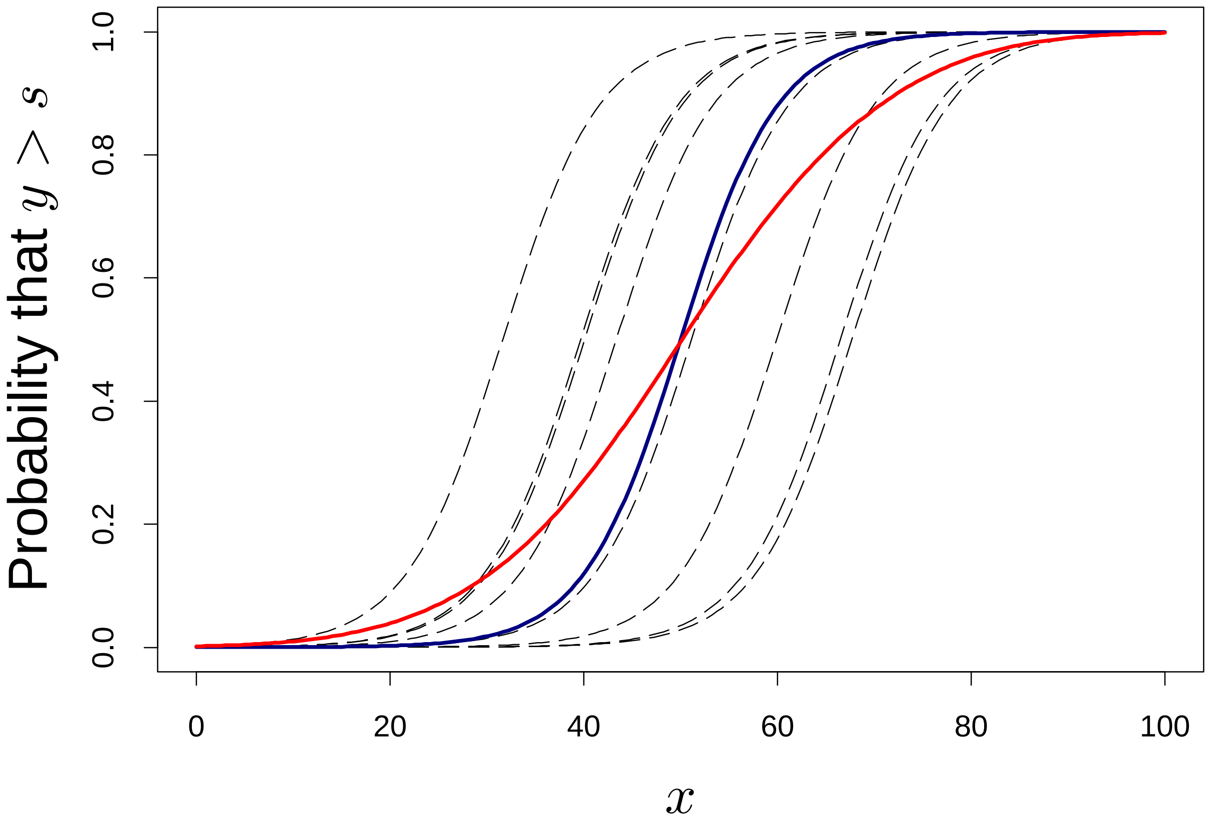
Illustration: Conditional versus marginal relationship



- - - - - conditional or cluster-specific (random sample)

—— median

Illustration: Conditional versus marginal relationship



conditional or cluster-specific (random sample)

median

marginal or population-averaged

Multilevel random coefficient models

- Consider clustered longitudinal data with occasions i (level 1) nested in student j (level 2) in schools k (level 3)
- Example of three-level random coefficient model:

$$y^*_{ijk} \;=\; [\zeta^{(2)}_{0jk} + \zeta^{(3)}_{0k}] + [\beta_1 + \zeta^{(2)}_{1jk} + \zeta^{(3)}_{1k}]x_{1ijk} + \beta_2 x_{2ijk} + \epsilon_{ijk}$$

  - $\zeta^{(2)}_{0jk}$ and $\zeta^{(3)}_{0k}$ are random intercepts at levels 2 and 3
  - $\zeta^{(2)}_{1jk}$ and $\zeta^{(3)}_{1k}$ are random coefficients of x_{1ijk} at levels 2 and 3
  - Random effects at the same level, e.g. $(\zeta^{(2)}_{0jk}, \zeta^{(2)}_{1jk})$, bivariate normal with zero means, independent across units jk
- General three-level random coefficient model

$$y^*_{ijk} \;=\; \mathbf{x}'_{ijk}\boldsymbol{\beta} + \mathbf{z}^{(2)\prime}_{ijk}\boldsymbol{\zeta}^{(2)}_{jk} + \mathbf{z}^{(3)\prime}_{ijk}\boldsymbol{\zeta}^{(3)}_{k} + \epsilon_{ijk}$$

Example: Recovery after surgery

- Study of effect of dosage of anesthetic on postsurgical recovery
- 60 children undergoing surgery were randomized into four dosage groups
- Recovery assessed 0, 5, 15, and 30 minutes after admission to recovery room
  - `score`: recovery score
    1 (no recovery), 2 ,3 ,4 (complete recovery)
  - `id2`: subject identifier
  - `dosage`: dosage of anesthetic in milligram/kilogram
  - `age`: age of child in months
  - `duration`: duration of surgery in minutes
  - `time`: time after surgery in minutes

Two-level model and estimation using gllamm

- Occasions i, child j
- Model the probability of exceeding a category s, $s = 1, 2, 3$

$$\text{logit}[\text{Pr}(y_{ij} > s|\zeta_j, \mathbf{x}_{ij})] \quad = \quad \beta_1 \texttt{dosage}_j + \beta_2 \texttt{age}_j + \beta_3 \texttt{duration}_j$$
$$+ \beta_4 \texttt{time}_i + \zeta_j - \kappa_s$$

- Estimation of two-level model using adaptive quadrature in `gllamm`:

```
gllamm score dosage age duration time, ///
   i(id2) link(ologit) adapt
```

- Without a random intercept (single-level) using `ologit`:

```
ologit score dosage age duration time
```

Maximum likelihood estimates

	Single-level model			Two-level model		
Parameter	Est	(SE)	OR	Est	(SE)	OR
$10\beta_1$ [dosage]	-0.34	(0.22)	0.71	-1.28	(0.92)	0.28
$10\beta_2$ [age]	-0.22	(0.08)	0.80	-0.55	(0.33)	0.57
$10\beta_3$ [duration]	-0.09	(0.03)	0.91	-0.22	(0.14)	0.80
$10\beta_4$ [time]	0.97	(0.12)	2.64	2.35	(0.27)	10.50
$\psi = \text{Var}(\zeta_j)$		–		13.4	(4.12)	
κ_1	-2.13	(0.69)		-6.21	(2.80)	
κ_2	-0.98	(0.68)		-3.43	(2.76)	
κ_3	0.18	(0.68)		-0.73	(2.73)	
Log-likelihood		-283			-222	

Intraclass correlation: 0.80

Predicted probabilities

- Random effects distributions
  - 'Prior' distribution: $g(\zeta_j)$
  - 'Posterior' distribution, given data for person j: $\omega(\zeta_j|\mathbf{y}_j, \mathbf{X}_j)$
- Predicted **marginal** or population averaged probabilities:

$$\text{Pr}(y_{ij} > 2|\mathbf{x}_{ij}) = \int \frac{\text{exp}(\mathbf{x}'_{ij}\widehat{\boldsymbol{\beta}} + \zeta_j - \widehat{\kappa}_2)}{1 + \text{exp}(\mathbf{x}'_{ij}\widehat{\boldsymbol{\beta}} + \zeta_j - \widehat{\kappa}_2)} g(\zeta_j)\text{d}\zeta_j$$

```
gllapred p_marg, marg mu above(2)
```

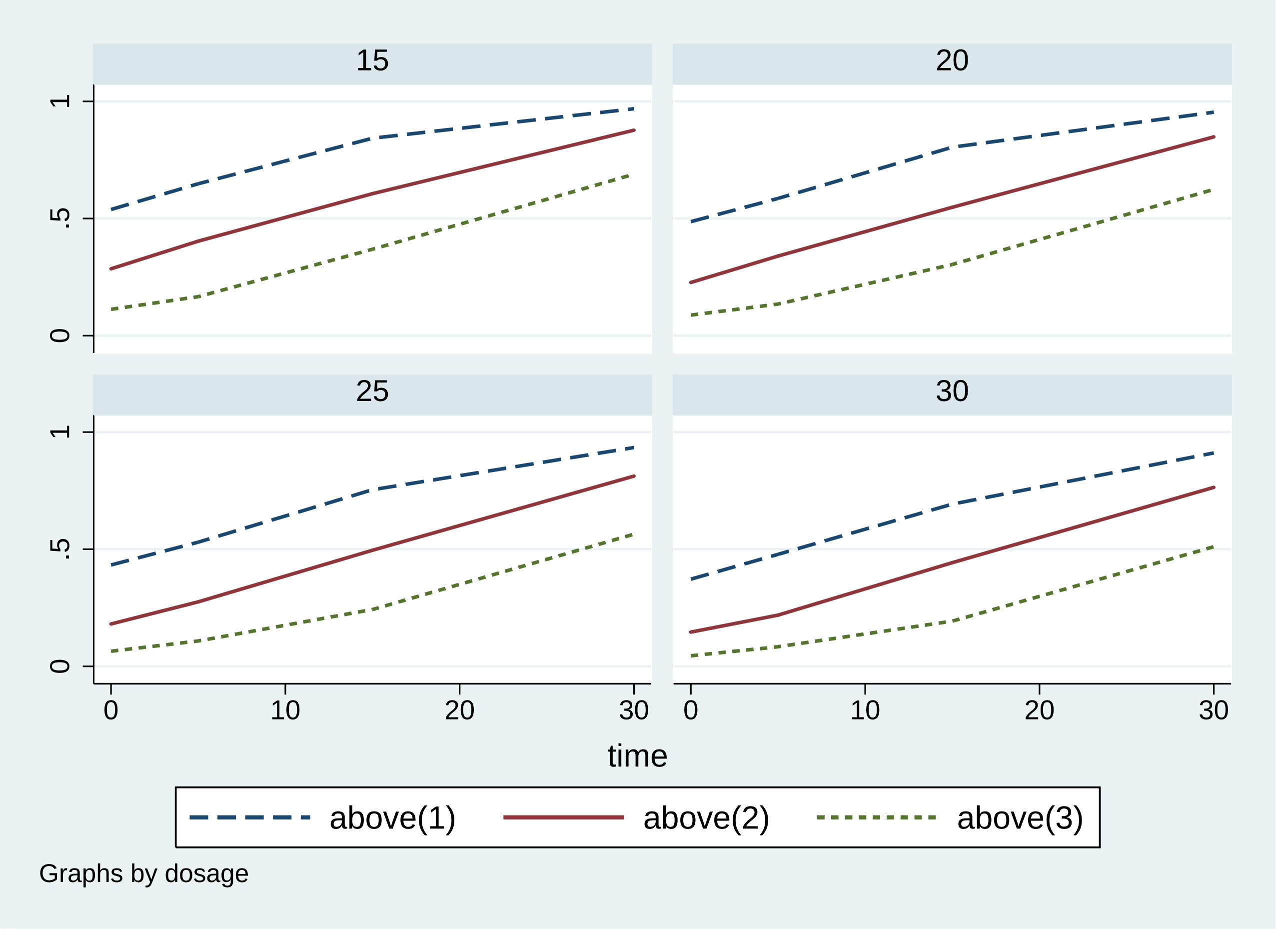
- Predicted **posterior mean** probabilities:

$$\text{E}_\zeta[\text{Pr}(y_{ij} > 2|\zeta_j, \mathbf{x}_{ij})|\mathbf{y}_j, \mathbf{X}_j] = \int \frac{\text{exp}(\mathbf{x}'_{ij}\widehat{\boldsymbol{\beta}} + \zeta_j - \widehat{\kappa}_2)}{1 + \text{exp}(\mathbf{x}'_{ij}\widehat{\boldsymbol{\beta}} + \zeta_j - \widehat{\kappa}_2)} \omega(\zeta_j|\mathbf{y}_j, \mathbf{X}_j)\text{d}\zeta_j$$

```
gllapred p_subj, mu above(2)
```

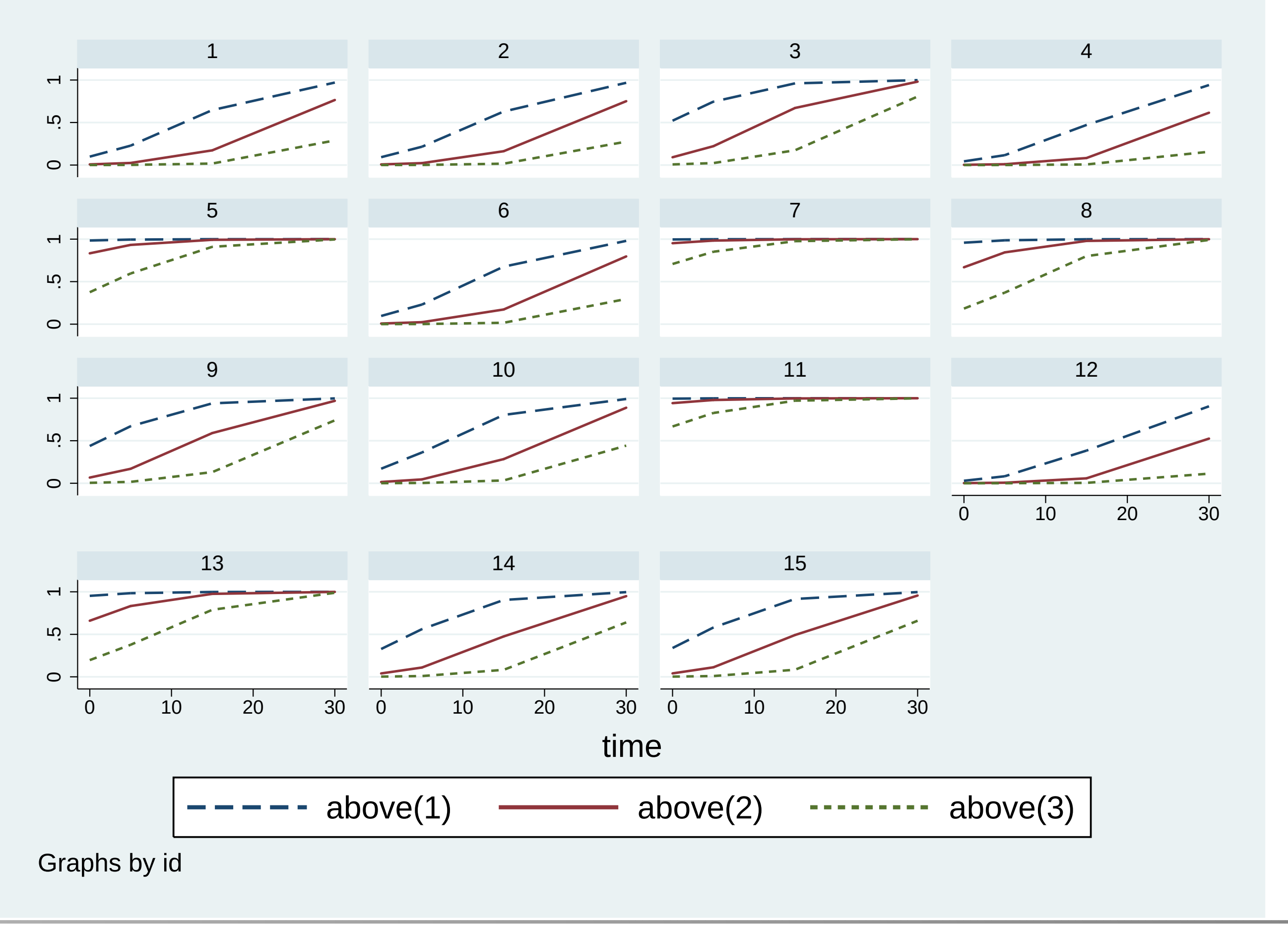
Predicted marginal probabilities

- For four dosage groups (15mg/Kg, 20mg/Kg, 25mg/Kg, 30mg/Kg) over time
- `age`=37months, `duration`=80min



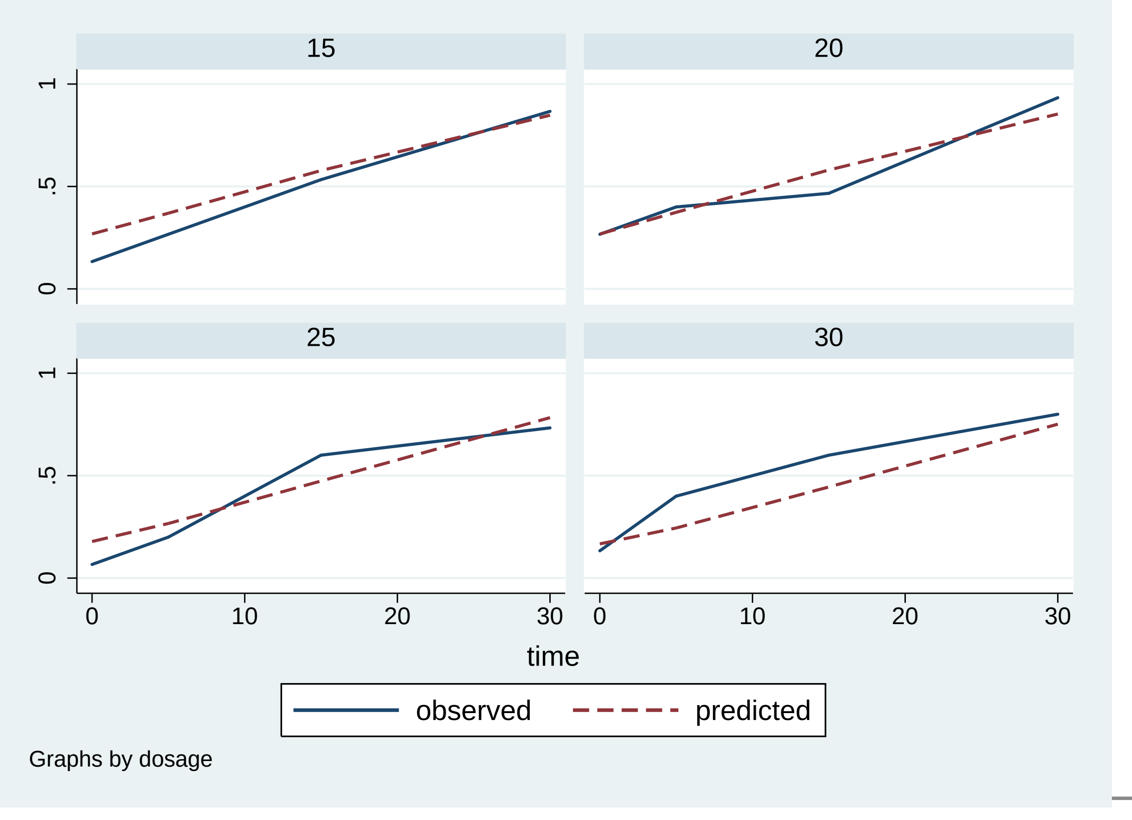
Predicted posterior mean probabilities

- For 15 subjects in dosage group 20mg/Kg over time
- `age`, `duration` as they are in the data



Assessing model fit

- Compare proportions of scores above 2 by `dosage` and `time` with $\Pr(y_{ij} > 2|\texttt{dosage}_j, \texttt{time}_i)$
  - Predict marginal probabilities with covariates as they are in the data $\Rightarrow \Pr(y_{ij} > 2|\texttt{dosage}_j, \texttt{age}_j, \texttt{duration}_j, \texttt{time}_i)$
  - Then obtain average over `age` and `duration` $\Rightarrow \Pr(y_{ij} > 2|\texttt{dosage}_j, \texttt{time}_i)$



Relaxing the proportional odds assumption for dosage

- Instead of $\beta_1\texttt{dosage}_j$:

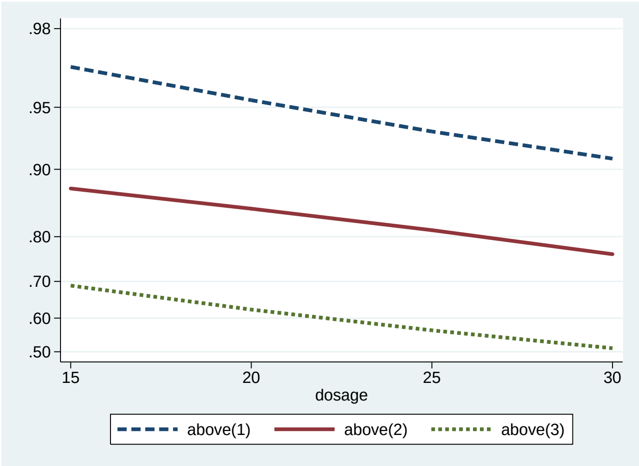
$$\begin{aligned}\text{logit}[\Pr(y_{ij} > s|\zeta_j, \mathbf{x}_{ij})] &= \beta_2\texttt{age}_j + \beta_3\texttt{duration}_j \\ &+ \beta_4\texttt{time}_i + \zeta_j - (\kappa_s + \delta_s\texttt{dosage}_j)\end{aligned}$$

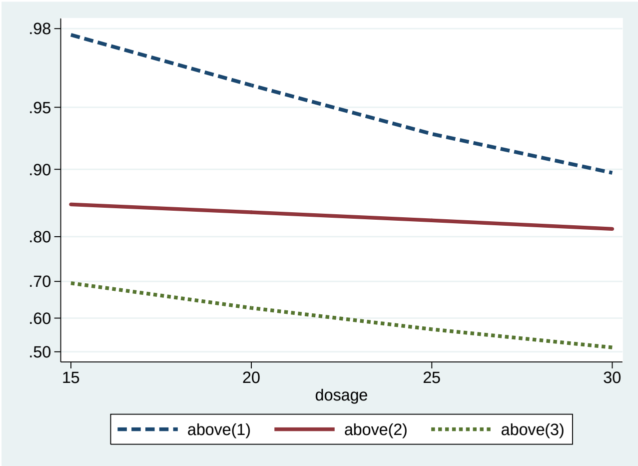
- Non-proportional odds model in `gllamm`:

```
eq thr: dosage
gllamm score age duration time, ///
   i(id2) link(ologit) thresh(thr) adapt
```

Maximum likelihood estimates

	Proportional odds		Non-proportional odds	
s	$\widehat{\beta}_1$	(SE)	$-\widehat{\delta}_s$	(SE)
1	-0.13	(0.09)	-0.20	(0.10)
2	-0.13	(0.09)	-0.05	(0.10)
3	-0.13	(0.09)	-0.13	(0.10)
	Log-lik. = -222		Log-lik. = -218, LR-test, $p = 0.02$	
	Pred. marginal prob. (time=30min)		Pred. marginal prob. (time=30min)	





Modeling heteroscedasticity at level 1

- Scaled ordinal probit link (similar to `hetprob`)

$$\Pr(y_{ij} > s|\mathbf{x}_{ij}, \zeta_j) \;=\; \Phi\left(\frac{\mathbf{x}'_{ij}\boldsymbol{\beta} + \zeta_j - \kappa_s}{\sigma_{ij}}\right)$$

$$\log(\sigma_{ij}) \;=\; \mathbf{z}'_{ij}\boldsymbol{\alpha} = \alpha_2 d_{2j} + \alpha_3 d_{3j} + \alpha_4 d_{4j}$$

```
tabulate dosage, gen(d)
eq het: d2 d3 d4  /* log sd = 0 in one group */
gllamm score dosage age duration time, i(id2) ///
   link(soprobit) s(het) from(a) adapt
```

- LR test: No evidence that level-1 variance depends on dosage group

dosage	15	20	25	30
k	1	2	3	4
$\alpha_k = \log(\sigma)$	0.00	0.06	-0.06	-0.09
σ	1.00	1.06	0.94	0.91

Modeling heteroscedasticity at level 2

- At level 2: Use factor loadings

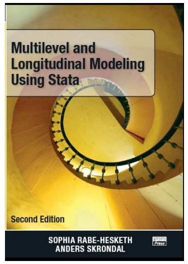
$$\begin{aligned}\text{linear predictor}_s &= \mathbf{x}'_{ij}\boldsymbol{\beta} + \zeta_j \mathbf{z}'_j \boldsymbol{\lambda} - \kappa_s, \qquad \lambda_1 = 1 \\ &= \mathbf{x}'_{ij}\boldsymbol{\beta} + \zeta_j(d_{1j} + \lambda_2 d_{2j} + \lambda_3 d_{3j} + \lambda_4 d_{4j}) - \kappa_s\end{aligned}$$

```
eq load: d1 d2 d3 d4
gllamm score dosage age duration time, i(id2) ///
   eqs(load) link(oprobit) adapt
```

- LR test: No evidence that level-2 variance depends on dosage group

dosage	15	20	25	30
k	1	2	3	4
λ_k	1.00	1.16	1.25	1.70
$\sqrt{\psi}\lambda_k$	1.65	1.91	2.06	2.80

References

- 

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- See also `http://www.gllamm.org`