Recent Developments in Multilevel Modeling

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2. One-level models
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- New commands `xtmelogit` and `xtmepoisson`
- Mixed effects for binary and count responses
- They work just like `xtmixed` does
- Random intercepts and random coefficients
- You can have multiple levels of nested random effects
- Various predictions, including random effects and their standard errors
- We'll be discussing binary responses and `xtmelogit`

For a series of $i = 1, \ldots, M$ independent panels, let

$$P(y_{ij} = 1|\mathbf{u}_i) = H(\mathbf{x}_{ij}\beta + \mathbf{z}_{ij}\mathbf{u}_i)$$

where

- there are $j = 1, \ldots, n_{ij}$ observations in panel i
- $\mathbf{x}_{ij}$ are the p covariates for the fixed effects
- β are the fixed effects
- $\mathbf{z}_{ij}$ are the q covariates for the random effects
- $\mathbf{u}_i$ are the random effects, specific to panel i
- $\mathbf{u}_i$ are normal with mean $\mathbf{0}$ and variance matrix $\mathbf{\Sigma}$
- $H()$ is the logistic cdf

- You can also think of this model in terms of a latent response $y_{ij} = I(y^*_{ij} > 0)$ where

$$y^*_{ij} = \mathbf{x}_{ij}\boldsymbol{\beta} + \mathbf{z}_{ij}\mathbf{u}_i + \epsilon_{ij}$$

- The errors ϵ_{ij} are logistic-distributed with mean zero and variance $\pi^2/3$, independent of $\mathbf{u}_i$

- Random effects are not directly estimated, but instead characterized by the elements of $\boldsymbol{\Sigma}$, known as *variance components*

- You can, however, "predict" random effects

- As such, you fit this model by estimating β and the variance components in $\boldsymbol{\Sigma}$

- A maximum-likelihood solution requires integrating out the distribution of $\mathbf{u}_i$.

- A tricky proposition in nonlinear models such as logit

Example

- 1989 Bangladesh fertility survey (Huq and Cleland 1990)
- Ng et al. (2006) analyze data on 1,934 women, who were polled on their use of contraception
- Data were collected from 60 districts containing urban and rural areas
- Covariates include age, urban/rural area, and indicators for number of children
- Among other things, we wish to assess a district effect on contraception use

- For woman j in district i, consider this model for $\pi_{ij} = P(\texttt{c\_use}_{ij} = 1)$

$$\begin{aligned}\text{logit}(\pi_{ij}) &= \beta_0 + \beta_1\texttt{urban}_{ij} + \beta_2\texttt{age}_{ij} + \\ &\qquad \beta_3\texttt{child1}_{ij} + \beta_4\texttt{child2}_{ij} + \beta_5\texttt{child3}_{ij} + u_i\end{aligned}$$

- The u_i represent 60 district-specific random effects
- You can use `xtlogit` (option `re`) to fit this model and estimate σ_u^2, the variance of the u_i
- `xtlogit` will also give an LR test for $H_o : \sigma_u^2 = 0$, by comparing log likelihoods with `logit`
- You could also use `xtmelogit` on this model

- Introducing a random coefficient, we now consider

$$\text{logit}(\pi_{ij}) = \beta_0 + \beta_1 \texttt{urban}_{ij} + \mathcal{F}_{ij} + u_i + v_i \texttt{urban}_{ij}$$

- $\mathcal{F}_{ij}$ is shorthand for the fixed-effects specification on age and children
- This model allows for distinct random effects for urban and rural areas within each district
- For rural areas in district i, the effect is u_i
- For urban areas, $u_i + v_i$
- You need `xtmelogit` to fit this model

```
. xtmelogit c_use urban age child* || district: urban
Refining starting values:
```

(output omitted)

```
Performing gradient-based optimization:
```

(output omitted)

```
Mixed-effects logistic regression              Number of obs      =      1934
Group variable: district                       Number of groups   =        60

                                               Obs per group: min =         2
                                                              avg =      32.2
                                                              max =       118

Integration points =   7                       Wald chi2(5)       =     97.30
Log likelihood = -1205.0025                    Prob > chi2        =    0.0000
```

c_use	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
urban	.7143927	.1513595	4.72	0.000	.4177336	1.011052
age	-.0262261	.0079656	-3.29	0.001	-.0418384	-.0106138
child1	1.128973	.1599346	7.06	0.000	.8155069	1.442439
child2	1.363165	.1761804	7.74	0.000	1.017857	1.708472
child3	1.352238	.1815608	7.45	0.000	.9963853	1.70809
_cons	-1.698137	.1505019	-11.28	0.000	-1.993115	-1.403159

--more--

Random-effects Parameters	Estimate	Std. Err.	[95% Conf.	Interval]
district: Independent				
sd(urban)	.5235464	.203566	.2443374	1.121813
sd(_cons)	.4889585	.087638	.3441182	.6947624

```
LR test vs. logistic regression:     chi2(2) =    47.05   Prob > chi2 = 0.0000
Note: LR test is conservative and provided only for reference.
```

- As with `logit`, option `or` will give odds ratios
- Use option `variance` for variances instead of standard deviations of random effects
- LR test comparing to standard logit is at the bottom, along with a note telling you the *p*-value is conservative

- Evaluating the log likelihood requires integrating out the random effects
- The default method used by `xtmelogit` is adaptive Gaussian quadrature (AGQ) with seven quadrature points per level
- AGQ is computationally intensive
- Previous methods, such as PQL and MQL, avoided the integration altogether (Breslow and Clayton 1993)
- PQL and MQL can be severely biased (Rodriguez and Goldman 1995)
- Also, being quasi-likelihood, their use prohibits LR tests

- Implicit in our previous model was the default independent covariance structure

$$\boldsymbol{\Sigma} = \mathrm{Var}\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}$$

- Assuming $\mathrm{Cov}(u_i, v_i) = 0$ means you are also assuming $\mathrm{Var}(u_i + v_i) > \mathrm{Var}(u_i)$
- Are urban areas really more variable than rural areas?
- Even worse, what if we change the coding of the random effects? Codings are not arbitrary here
- Option `covariance(unstructured)` will include this covariance in the model

```
. xtmelogit c_use urban age child* || district: urban, cov(un) var
```

(output omitted)

```
Mixed-effects logistic regression               Number of obs      =      1934
Group variable: district                        Number of groups   =        60

                                                Obs per group: min =         2
                                                               avg =      32.2
                                                               max =       118

Integration points =   7                        Wald chi2(5)       =     97.50
Log likelihood = -1199.315                      Prob > chi2        =    0.0000
```

c_use	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
urban	.8157872	.1715519	4.76	0.000	.4795516	1.152023
age	-.026415	.008023	-3.29	0.001	-.0421398	-.0106902
child1	1.13252	.1603285	7.06	0.000	.818282	1.446758
child2	1.357739	.1770522	7.67	0.000	1.010724	1.704755
child3	1.353827	.1828801	7.40	0.000	.9953882	1.712265
_cons	-1.71165	.1605617	-10.66	0.000	-2.026345	-1.396954

--more--

Random-effects Parameters	Estimate	Std. Err.	[95% Conf.	Interval]
district: Unstructured				
var(urban)	.6663222	.3224715	.2580709	1.7204
var(_cons)	.3897435	.1292459	.2034723	.7465388
cov(urban,_cons)	-.4058846	.1755418	-.7499403	-.0618289

```
LR test vs. logistic regression:     chi2(3) =    58.42   Prob > chi2 = 0.0000
Note: LR test is conservative and provided only for reference.
```

```
. estimates store corr

. lrtest no_corr corr

Likelihood-ratio test                                 LR chi2(1)  =     11.38
(Assumption: no_corr nested in corr)                  Prob > chi2 =    0.0007
```

- We can now estimate the variance of the random effects for urban areas as

$$\mathrm{Var}(u_i + v_i) = \sigma_u^2 + \sigma_v^2 + 2\sigma_{uv}$$

- If you did this, you would get $\mathrm{Var}(u_i + v_i) = 0.244$, which is actually less than $\mathrm{Var}(u_i) = 0.390$
- Better still, if you want to directly compare rural areas to urban areas, recode your random effects
- The unstructured covariance structure will ensure an equivalent model under alternate codings of random-effects variables
- Also, predictions of random effects will be what you want

```
. gen byte rural = 1 - urban

. xtmelogit c_use urban rural age child*, nocons || district: urban rural,
> nocons cov(un) var
```

(output omitted)

```
Mixed-effects logistic regression               Number of obs      =      1934
Group variable: district                        Number of groups   =        60

                                                Obs per group: min =         2
                                                               avg =      32.2
                                                               max =       118

Integration points =   7                        Wald chi2(6)       =    120.24
Log likelihood = -1199.315                      Prob > chi2        =    0.0000
```

(output omitted)

```
------------------------------------------------------------------------------
  Random-effects Parameters  |   Estimate   Std. Err.     [95% Conf. Interval]
-----------------------------+------------------------------------------------
district: Unstructured       |
                  var(urban) |   .2442916   .1450648      .0762869    .7822893
                  var(rural) |   .3897431   .1292457      .2034722    .7465379
            cov(urban,rural) |  -.0161406    .105746     -.2233989    .1911177
------------------------------------------------------------------------------
LR test vs. logistic regression:     chi2(3) =    58.42   Prob > chi2 = 0.0000

Note: LR test is conservative and provided only for reference.
```

- You've seen `Independent` and `Unstructured` in action
- Also available are `Identity` and `Exchangeable`
- You can combine these to form blocked-diagonal structures
- Such structures can reduce the number of estimable parameters
- For example, consider a random effects specification of the form

```
... || district: child1 child2, nocons cov(ex) || district: child3, nocons
```

as an alternative to a 3×3 unstructured variance matrix

Example

- The Tower of London (Rabe-Hesketh et al. 2001)
- Study of cognitive abilities of patients with schizophrenia
- Cognitive ability was measure as successful completion of the Tower of London, a computerized task (binary variable `dtlm`)
- 226 subjects, all but one tested at three difficulty levels
- Subjects were not only patients (`group==3`), but relatives (`group==2`) and nonrelated controls (`group==1`)
- We can thus propose a model having random effects shared among relatives (variable `family`) and subject-specific effects nested within families

```
. xi: xtmelogit dtlm difficulty i.group || family: || subject:, or variance
i.group          _Igroup_1-3          (naturally coded; _Igroup_1 omitted)
```

(output omitted)

```
Mixed-effects logistic regression              Number of obs      =       677
```

Group Variable	No. of Groups	Observations per Group Minimum	Average	Maximum	Integration Points
family	118	2	5.7	27	7
subject	226	2	3.0	3	7

```
                                               Wald chi2(3)       =     74.89
Log likelihood = -305.12043                    Prob > chi2        =    0.0000
```

dtlm	Odds Ratio	Std. Err.	z	P>|z|	[95% Conf.	Interval]
difficulty	.192337	.0371622	-8.53	0.000	.131704	.2808839
_Igroup_2	.7798295	.2763766	-0.70	0.483	.3893394	1.561964
_Igroup_3	.3491338	.1396499	-2.63	0.009	.1594117	.7646517

--more--

Random-effects Parameters	Estimate	Std. Err.	[95% Conf.	Interval]
family: Identity				
var(_cons)	.569182	.5216584	.0944322	3.430694
subject: Identity				
var(_cons)	1.137931	.6857497	.3492672	3.707441

```
LR test vs. logistic regression:     chi2(2) =    17.54   Prob > chi2 = 0.0002
Note: LR test is conservative and provided only for reference.
```

- `xtmelogit`, by default, uses AGQ which can be intensive with large datasets or high-dimensional models
- Computation time is roughly on the order of

$$T \sim p^2\{M + M(N_Q)^{q_t}\}$$

where

p is the number of estimable parameters
M is the number of lowest-level (smallest) panels
N_Q is the number of quadrature points
q_t is the total dimension of the random effects (all levels)

- The real killer is $(N_Q)^{q_t}$

- Ideally, you want enough quadrature points such that adding more points doesn't change much
- In complex models, this can very time consuming, especially during the exploratory phase of the analysis
- Sometimes you just want quicker results, and you may be willing to give up a bit of accuracy
- Use option `laplace`, equivalent to $N_Q = 1$
- The computational benefit is clear – one raised to any power equals one

```
. xi: xtmelogit dtlm level i.group || family: || subject:, or variance laplace
i.group           _Igroup_1-3         (naturally coded; _Igroup_1 omitted)
  (output omitted )
Mixed-effects logistic regression               Number of obs      =       677

--------------------------------------------------------------------------
                |   No. of       Observations per Group       Integration
 Group Variable |   Groups    Minimum    Average    Maximum      Points
----------------+---------------------------------------------------------
         family |      118          2        5.7         27           1
        subject |      226          2        3.0          3           1
--------------------------------------------------------------------------

                                                Wald chi2(3)       =     76.09
Log likelihood = -306.51035                     Prob > chi2        =    0.0000

------------------------------------------------------------------------------
        dtlm | Odds Ratio   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       level |   .2044132   .0377578    -8.60   0.000     .1423248    .2935872
   _Igroup_2 |   .7860452   .2625197    -0.72   0.471     .4084766    1.512613
   _Igroup_3 |   .3575718   .1354592    -2.71   0.007     .1701774    .7513194
------------------------------------------------------------------------------

--more--
```

Random-effects Parameters	Estimate	Std. Err.	[95% Conf.	Interval]
family: Identity				
var(_cons)	.522942	.4704255	.0896879	3.04911
subject: Identity				
var(_cons)	.7909329	.5699273	.1926568	3.247095

```
LR test vs. logistic regression:     chi2(2) =    14.76   Prob > chi2 = 0.0006
Note: LR test is conservative and provided only for reference.
Note: log-likelihood calculations are based on the Laplacian approximation.
```

- Odds ratios and their standard errors are well approximated by Laplace
- Variance components exhibit bias, particularly at the lower (subject) level
- Model log-likelihoods and comparison LR test are in fair agreement
- These behaviors are fairly typical
- If anything, it shows that you can at least use laplace while building your model

One further advantage of `laplace` is that it permits you to fit crossed-effects models, which will have high-dimension

Example

- School data from Fife, Scotland (Rabe-Hesketh and Skrondal 2005)
- Attainment scores at age 16 for 3,435 students who attended any of 148 primary schools and 19 secondary schools
- We are interested in whether the attainment score is greater than 6
- We want random effects due to primary school and secondary school, but these effects are not nested

- Consider the model

$$\text{logit}\{(\Pr(\texttt{attain}_{ijk} > 6)\} = \beta_0 + \beta_1 \texttt{sex}_{ijk} + u_i + v_j$$

  for student k who attended primary school i and secondary school j

- Since there is no nesting, you can use the level designation `_all:` to treat the entire data as one big panel
- Use factor notation `R.`*varname* to mimic the creation of indicator variables identifying schools
- However, notice that we can treat one set of effects as nested within the entire data

```
. xtmelogit attain_gt_6 sex || _all:R.sid || pid:, or variance
Note: factor variables specified; option laplace assumed
```

(output omitted)

```
Mixed-effects logistic regression               Number of obs      =      3435

--------------------------------------------------------------------------------
                |     No. of       Observations per Group         Integration
 Group Variable |     Groups    Minimum    Average    Maximum        Points
----------------+---------------------------------------------------------------
           _all |          1       3435     3435.0       3435             1
            pid |        148          1       23.2         72             1
--------------------------------------------------------------------------------

                                                Wald chi2(1)       =     14.28
Log likelihood = -2220.0035                     Prob > chi2        =    0.0002

------------------------------------------------------------------------------
 attain_gt_6 | Odds Ratio   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         sex |    1.32512   .0986968     3.78   0.000     1.145135    1.533395
------------------------------------------------------------------------------

--more--
```

Random-effects Parameters	Estimate	Std. Err.	[95% Conf.	Interval]
_all: Identity				
var(R.sid)	.1239739	.0694743	.0413354	.3718252
pid: Identity				
var(_cons)	.4520502	.0953867	.298934	.6835937

```
LR test vs. logistic regression:     chi2(2) =   195.80   Prob > chi2 = 0.0000
Note: LR test is conservative and provided only for reference.
Note: log-likelihood calculations are based on the Laplacian approximation.
```

- `xtmelogit` and `xtmepoisson` are new to Stata 10
- We discussed `xtmelogit` – the same holds true for `xtmepoisson`
- Computations can get intensive
- The Laplacian approximation is a quicker alternative
- You can fit crossed-effects models, and large ones with creative nesting
- Work in this area is ongoing

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