The economic foundations of demographic transition

Rodrigo Cerda*

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Abstract

The paper develops a general equilibrium model where population sources, such as fertility and mortality rates, are chosen variables. It is shown that the evolution of population over time depends on income and relative prices of mortality and fertility rates. Initially as a country develops, countries should face a period with increasing fertility and higher population growth rates but later fertility and population growth rate should decrease as their relative prices increase. It is also shown that multiple equilibria may arise. An equilibrium with low levels of asset will have lower per capita income, but larger fertility, mortality and population growth rates.

*The University of Chicago and Pontificia Universidad Catolica de Chile. Correspondence address: 1215 E. Hyde Park Blvd, Apt. 204, Chicago, IL, 60615; e-mail: r-cerda@uchicago.edu.

1. Introduction

During large part of history, population and per capita income were stable. However, since the mid of nineteenth century on Europe, population and per capita income growth rates considerably accelerated. Approximately since 1920, fertility and mortality rates started to decrease while per capita income continued on its trend. This is a characteristic shared with some lags by the rest of the world.

The analysis of Malthus explained considerably well the evolution of population and per capita income before 1850 but it fails to explain the later development. His analysis (and later interpretations) had two main assumptions, namely fertility rate was a normal good with constant relative price and there was some fixed resource on the production function, usually land, that produces decreasing returns to scale on labor (or capital stock). Those assumptions will be fairly modify in this paper. First, instead of fertility as main source of population growth rate, it will be studied the evolution of fertility and mortality rates. Those two variable will not be restricted to have constant relative prices. Second, it will be allowed that at least locally, the marginal product of capital may not be decreasing. Those two modifications will have direct effects over the conclusions of the model. The former effect will be one of the determinants of the demographic transition. In fact, it will be shown that as a country develops the relative price

of children increases with respect to the price of longevity. This effect will have an important impact on fertility and mortality rates over time. The later effect will allow the possibility of multiple equilibria that can be ranked as a function of their per capita income. An equilibrium with large per capita income will be associated with lower mortality and fertility rates.

The paper is developed in the following way. Section 2 discusses the economic environment while section 3 shows the characteristics of the model when the net marginal product of capital is monotonically decreasing. Section 4 allows the net marginal product of capital being non decreasing at least locally. In that case, multiple equilibria may arise. Section 5 includes fiscal policy and it shows how fiscal policy can bring the economy from a "Malthusian trap" to a developed equilibrium. Finally, section 6 concludes.

2. The economic environment

In this economy, there are Z identical households and time is continuous. In the representative household, $n_t N_t$ individuals are born at time t -where N_t is the size of the household and n_t is the fertility rate at time t- while a fraction λ of the current members of the households are dying at every period of time. The fraction of people dying depends on household's decisions. In fact, health goods can be purchased to increase the expected lifetime of every member of the

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household. Let $\lambda(h_t)$ be the fraction of the household members that die at the end of period t, where h_t indicates the level of health goods purchased at time t. It is that assumed $\lambda(h_t)$ is a decreasing function of the health level at time t.

The household preferences will now be stated. Let U be the welfare function of the household. The overall welfare function will be a weighted sum of all future utility flows that are determined by the instantaneous utility function, $u(c_t)$, where c_t is the per capita consumption at time t. As usual the weights of the utility flows depends on a constant discount factor ρ, but also on the size of the household on the future. The size of the household will depend on fertility and health decisions. In fact, normalizing the initial size of the household equal to one, the size of the household at time t is $\exp(\int_0^t (n_s - \lambda(h_s))ds)$. Hence, we can define U as:

$$U = \int_0^\infty u(c_t)e^{-\rho t - \int_0^t (\lambda(h_s) - n_s)ds} dt \qquad (1)$$

Where $e^{-\int_0^t \lambda(h_s)ds}$ is the survival density function, for individuals born on the initial generation until time t[1] and it depends on the set of health expenditure made by individuals between time zero and t. The utility function u($\bullet$) is increasing while the function $\lambda(\bullet)$ is decreasing on its arguments. Also, both function

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[1] Or the hazard rate of being alive at period t given that the individuals were born a time t=0.

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satisfy Inada conditions.

Let consider the budget constraint faced by the household at time t. The household at each period of time is endowed with a per capita unit of time and some initial level of assets, k_t. The household is also endowed with a per capita production function that depends positively on labor and physical capital. The production function provides some units of a physical good as output. Initial assets are used as capital stock at the beginning of the period. The labor supply decision is the following. Individuals on the household do not use all their time on the production of the physical good, as they also allocate part of their time to childbearing because childbearing is time intensive. Let $\phi(n_t)$ the amount of time spent on raising children and 1-$\phi(n_t)$ be the amount of labor supplied, where $\phi(n_t) : R_+ \to R_+$ and $\phi'(n_t) > 0$. The production function is f(k_t, l_t) : $R_+^2 \to R_+$, where f$_k$,f$_l$, $f_{kl} > 0$ and k indicates capital stock or level of assets while l$_t$ indicates labor supply. From above, l$_t$ =1-$\phi(n_t)$.

The physical goods obtained through the production process may be used as consumption goods that provide current utility, capital goods that are used as assets and carried over to next period of time and health goods that increase the survival probability. Assets are left for members of the household alive next period of time. The family will be composed by the current members that survive and

newborns. At the beginning of next period of time, period "t+1"[2], the uncertainty is resolved and it is known the fraction of dead members of the household .

Hence the per capita budget constraint faced by the household at time t is the following:

$$\overset{\bullet}{k} = f(k_t, 1 - \phi(n_t)) - c_t - h_t - k_t[n_t - \lambda(h_t)] \tag{2}$$

Finally, there is also a borrowing condition imposed on households. This is the usual transversality condition that imposes the value of household's asset to approach zero[3] as time approaches to infinity. Basically, at the end of the planning horizon there would not be valuable assets left. To state the condition, let μ_t be the shadow price of assets at time t. The condition is:

$$\lim_{t \to \infty} \{k_t \mu_t\} \geq 0 \tag{3}$$

Summing up the household's problem is to choose the flow of per capita consumption, per capita health good, and fertility rate $\{c_t, h_t, n_t\}_{t=0,\dots\infty}$, to maximize (1) subject to (2)-(3), and the condition that c_t, h_t and n_t should be non negatives.

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[2]This is a continuous time economy, hence the discretization of period is not rigourosly correct. However, in this case it is used to enfasis the recursivity on the economy.
[3]Meaning that no debt is left at the end of the planning horizon.

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To satisfy second order conditions, we assume that the following conditions holds: $u'' \leq 0, \lambda'' \geq 0, -\frac{\phi''}{\phi'} + \frac{f_{ll}\phi'}{f_l} < 0$ and $\frac{\lambda''}{\lambda'} + \lambda'(\frac{\phi''}{\phi'} - \frac{f_{ll}\phi'}{f_l}) < 0$ - where the primes indicates derivatives. The first assumption states that as usual the current utility function must be concave on consumption, implying that individuals like to smooth consumption over time. The second condition deals with the convexity of $\lambda(\bullet)$. As $\lambda(\bullet)$ is convex, the instantaneous fraction of people surviving, 1-$\lambda(\bullet)$, is concave on health, hence a similar argument to the one of consumption follows. The third condition deals with fertility rate. The condition states that the problem must be concave on fertility as a whole. Two effects appear. First, the concavity on the opportunity cost of time and later the concavity of the production function on labor supply. Finally, the last condition deals with the overall concavity of the utility function on population.

3. The dynamics of the economy

This section will characterize the dynamics of the economy. First, it will be shown that the economy can be represented completely on the space (μ, k). Using this property and characterizing the evolution of (μ, k), we describe the evolution of the economy.

The problem determined by equations (1) to (3) can be characterized by the usual first order conditions and the costate equations. Those conditions can be

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summarized by the following equations:

$$\frac{u^{'}(c_t)}{u(c_t)\left[-\lambda^{'}(h_t)\right]} = \frac{1}{[1-\lambda^{'}(h_t)k_t]} \tag{4}$$

$$\frac{u(c_t)\left[-\lambda^{'}(h_t)\right]}{u(c_t)} = \frac{[1-\lambda^{'}(h_t)k_t]}{[f_l\phi^{'}(n_t)+k_t]} \tag{5}$$

$$\frac{\dot{\mu_t}}{\mu_t}+f_k = n-\lambda(h_t) \tag{6}$$

Condition (4) has a very intuitive interpretation. It is the equality between marginal rate of substitution of consumption goods and health goods with their ratio of prices. Holding relative prices constant and increasing marginally health consumption, has as benefit the marginal gain in expected utility and its cost is given by the marginal loss in utility, as we decrease consumption. Notice that the relative price of health goods measured in units of consumption includes the unit price of goods plus the marginal increase on assets that must be left. Equation (5) has a similar interpretation. The left hand side is the marginal rate of substitution between health goods and children while the right hand side is the ratio of prices between health goods and fertility rate.

Notice that equations (4) and (5) show that the ratio of prices depends on the level of assets and further on fertility and mortality rates. Those two characteristics will provide us some interesting results. First an increase on the level of asset

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will produce as usual an income effect, but it will also modify the ratio of prices. Second, as the relative prices are function of fertility and mortality rates, initial changes on those variables will be stretched out by changes on relative prices.

Equation (6) is the costate equation and it describes the evolution of the shadow price over time through an arbitrage condition on assets. The left hand side indicates the benefit of holding a unit of asset per capita. This benefit is given by the capital gain $\frac{\overset{\bullet}{\mu_t}}{\mu_t}$, plus the rental rate obtained by supplying the assets on the production function during period t. The right hand side shows that the total cost of holding asset is the population growth rate. In fact, the total level of assets must be equally shared among the family members during next period, but if population growth rate increases, larger assets must be accumulate to hold constant the per capita level.

This information will allow us to define some implicit demand functions that characterize the optimal allocations of households. Those demand functions are determined from the first order conditions and the budget constraint faced by the household at per capita level. Let $E(k_t)$ be current expenditure on health, consumption and children, as a function of the state variable. The demand functions at per capita level are defined by $h(E(k_t),\mu_t)$, $c(E(k_t),\mu_t)$, $n(E(k_t),\mu_t)$ -health, consumption goods and children respectively. As usual, the demand functions depend on prices given by μ_t and current income -expenditure. As current ex-

penditure depends on k_t, the demand functions can be written as $h(k_t, \mu_t)$, $c(k_t, \mu_t)$ and $n(k_t, \mu_t)$. The properties of the demand function are presented in the following lemma.

Lemma 3.1. : *The problem is characterized by the first order conditions and the budget constraint that determine the following demand functions:*

$$c_t = c(\underset{+}{k_t}, \underset{-}{\mu_t})$$

$$h_t = h(\underset{?}{k_t}, \underset{-}{\mu_t})$$

$$n_t = n(\underset{?}{k_t}, \underset{+}{\mu_t})$$

Moreover, $n_k \geq (<)0, h_k \leq (>)0$ when $k_t \leq (>)k^*$, where k^* is some threshold level of assets per capita.

Proof: See Mathematical appendix

This lemma built on the functional separability of preferences over time that allows us to solve the problem as a two stage problem. The first stage, as illustrated on the lemma, solves the problem given current expenditure, which is function of the state variable. Hence, we obtain the implicit demand as function of current expenditure. On the second stage, the distribution of assets over time is chosen. Given the concavity on the utility functions the current expenditure is an increasing function of assets on this second stage. Thus, it follows that the implicit demand functions are directly related to the level of assets.

The intuition for those results is the following. Consider first, an increase on the shadow price of assets. In that case, there are larger incentives to accumulate assets. Hence alternative uses of output, as per capita consumption and per capita health expenditure, decrease to accumulate larger assets. Also, larger incentives to accumulate assets are linked with larger future consumption and larger population, as they are future oriented goods. Thus fertility rate increases.

Consider now an increase on the initial level of assets per capita. Consumption goods are positively affected as they are normal goods. Fertility and mortality rate presents ambiguous effects though. In fact, they face two opposites effects. First, there is an income effect, and second there is a substitution effect when the per capita level of assets vary, as relative prices of fertility and mortality rates are distorted -see equations (4) and (5).

In fact, the income affects positively population growth rate because population is a normal good as a whole. The way to link this effect with fertility and mortality rates is the following. Given an increase on the level of expenditure on population, the efficient way to increase population growth rate is throughout larger fertility rate and larger mortality rate. This result follows from the linearity of population growth rate on fertility and the concavity on health expenditure. Hence, we could marginally increase population growth rate by reallocating health resources to fertility, given the initial level of population expenditure. The income

effect exploits this fact.

The way the substitution effect works is explained next. Equation (4) shows that as assets increase, the relative price of consumption goods in terms of health goods decrease[4]. Hence, we face incentives to substitute away resources from population to consumption goods. Consider the effects on the planes (c,h) and (c,n) when the level of assets increases while the level of the shadow price, μ, is held constant. The budget constraint should move outwards first, due to the income effect, but second the relative prices change favoring consumption and offsetting partly the initial income effect over population expenditures. The income effect dominates initially but later the substitution effect does.

Notice that within population choice variables the substitution effect also appears, as shown by the ratio of prices on equation (5). In fact when the level of assets increases, the relative price of children in term of health goods raises. In other words, children become expensive compared to mortality investments. This substitution effect biases population sources to larger expenditures on mortality rather than fertility allocations.

As indicated on the lemma, there is a cut-off level of assets k* that determines when the substitution effect offsets the income effect. Hence initially fertility and

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[4]An analogous property can be show the relative price of consumption goods in terms of fertility rate.

mortality rates increases due to the income effect and later they decrease due to the substitution effect.

Lemma 3.1 stated above indicates properties over fertility and health expenditure, but not over population growth rate. The main properties over population growth rate of the shadow price of assets and the level of assets will be characterized in the next lemma.

Lemma 3.2. *Population growth rate is positively affected by the shadow price of per capita asset. Also population growth rate is positively affected by an increase on the level of assets when $k_t \leq k^*$ and negatively affected by an increase on the level of per capita assets when $k_t \geq k^*$.*

Proof: See mathematical appendix

Population is a normal good, but for large enough level of assets, relative prices are distorted and population is substituted away to increase per capita consumption. Also larger shadow prices of assets are associated with larger population growth rate. Notice that the household's welfare function has as property the complementarity between current population and current per capita consumption. Further, larger shadow price of assets are associated with larger physical capital accumulation which encourages future consumption. Therefore, larger μ_t is associated with larger population growth rate.

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Lemma 3.1 and lemma 3.2 will be use later when the economy's behavior is characterized. Those two lemmas will be key in showing the evolution of the demographic transition as a country develops.

4. The characterization of the overall economy

The next step is to characterize the behavior of the aggregate economy over time. It turns out that the economy can be fully characterized on the space (k, μ). In fact using the household's problem plus the implicit demand functions, we can define the maximized Hamiltonian of the household problem, H^0:

Definition 4.1. *The maximized Hamiltonian* $H^0{:}\Re_+^2 \to \Re$ *is defined as*

$$H^0(k,\mu) = \max_{c,n,h} H^0(k,\mu,c,n,h)$$

To characterize the model, we will determine the evolution of assets per capita and its shadow price over time. The evolution of assets is obtained by using the household budget constraint, plus the implicit demand functions:

$$H_\mu^0 = \overset{\bullet}{k} = f(k_t, 1 - \phi(n_t(k_t, \mu_t))) - c_t(k_t, \mu_t) - h_t(k_t, \mu_t) - k_t[n(k_t, \mu_t) - \lambda(h_t(k_t, \mu_t))]$$

$$(7)$$

Also, we may characterize the evolution of the shadow price over time. Using the arbitrage condition on the asset market, equation (6) and the implicit demand

function, we obtain:

$$-H_k^0 = \overset{\bullet}{\mu}_t = -[f_k(k_t, 1 - \phi(n_t(k_t, \mu_t))) - n(k_t, \mu_t) + \lambda(h_t(k_t, \mu_t))]\mu_t \qquad (8)$$

Notice that equations (7) and (8) are analogous to the first order conditions of the maximized Hamiltonian with respect to μ and capital stock respectively, using the envelope theorem.

The information on the dynamics obtained from equations (7) and (8)[5] determines the evolution of consumption goods, health goods and fertility rate over time, throughout the implicit demand functions derived above. Hence, we may easily determine the behavior of the economy.

Before analyzing the path of the variable over time, an important caveat that determine the results will be stated. In the traditional neoclassical theory, a marginal increase in capital stock decreases its rental rate unambiguously. Over time and as the stock of capital increases, there are less incentives to accumulate capital and the economic prosperity vanishes. In our case, the story may be different. As the economy develops, population growth rate varies and it may affect the rental rate obtained from capital stock. To clarify the effect, it will be

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[5]Plus the initial condition on capital stock.

useful to define the net rental rate of capital stock as:

$$NR_t = f_k - (n_t - \lambda(h_t))$$

The intuition to use this net rental rate is that a marginal increase on capital stock produce a benefit determined by its marginal product. This benefit must be shared with others members of the household though. Thus, a larger population growth rate decreases the per capita return from assets. In general, the literature does not exploit this effect as it assumes population as exogenous. Hence, the rental rate is decreasing on assets, following f_k. In our case, at least locally, the net rental rate could be non-decreasing on assets per capita if population growth rate offsets the effect of the decreasing marginal product of capital stock[6]. Two cases will be analyzed next: (1) the monotonically decreasing net rental rate and (2) the non-monotonically decreasing rental rate of the economy.

4.1. Monotonically decreasing net rental rate of assets

In this subsection, we show that there is a unique and stable equilibrium. During the transition to the equilibrium a demographic transition occurs. The following proposition states the result for this case.

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[6] Also f_k may be disturbed if fertility rates varies, everything else equal.

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Proposition 4.2. : *When $\frac{\partial NR_t}{\partial k_t} \leq 0$, there exists a unique equilibrium $(k \mid_{ss}$*

, $\mu \mid_{ss})$. If $k_0 < k^* < k\mid_{ss}$, there exists a demographic transition. Further, when:

1. $k_0 < k_t < k^*$, fertility, mortality and population growth rate present ambigu-

ous effects

2. $k^* < k_t < k\mid_{ss}$ fertility, mortality and population growth rate decrease

unambiguously until $k_t = k\mid_{ss}$.

Thereafter they remain constant.

Proof: See mathematical appendix

The phase diagram of the model illustrates the proposition. On any equilib-

rium the locus of points corresponding to $\overset{\bullet}{k} = 0$ must be increasing on the plane

(μ, k) while the locus of points $\overset{\bullet}{\mu} = 0$ must be decreasing. There is a unique saddle

path and a unique equilibrium. The slope of the saddle path is decreasing, as

shown in graph 1.

[Insert graph 1]

Suppose the initial level of assets is smaller than its long run level. In that

case, the shadow price decreases while the level of assets increases during the

transition. From proposition 3.2, we know that when the level of assets is smaller

than k^* and assets increases, population growth rate must be negatively affected

by the evolution of the shadow price but positively affected by the evolution of

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assets. Hence, on this range the population growth rate faces two opposite forces and the total effect is ambiguous. Also, on this range fertility and mortality rates are ambiguously affected by the same argument.

When $k_t > k^*$ and assets per capita increases towards its long run level, population growth rate decreases unambiguously. On this range, there is a large substitution effect, as explained above, and hence assets negatively population growth rate, reinforcing the effect of the shadow price of assets. By the same argument, fertility and mortality rates decrease unambiguously.

As shown in graph 1, the dynamics in this case indicates that any country starting with a low level of assets per capita with respect to the steady state will accumulate assets and increase its level of per capita income until the steady state is reached. Once the long run equilibrium is reached per capita income, mortality and fertility rates remain constant. Thus as a country develops, a demographic transition plus an increase on per capita income should be observed.

Such experience could be illustrated by European countries. They show initially a period of increasing population growth rate until 1875, due to large fertility rates and advances on mortality rates. The population growth rate remains high until 1920 when it starts to decrease reaching a positive but small value at the end of the 20th century and some times even near to zero. Fertility and mortality rate increase or remain stable until the last quarter of the 19th century where

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they start to fall. Figure 1 is the England case.

[Insert figure 1]

4.2. Non-monotonically decreasing net rate of return of capital

In this section, the assumption about the net rental rate of return from capital stock be strictly decreasing on assets will be removed. This is an important assumption for the model in the above subsection as it assures the existence of a unique equilibrium. In fact, it will be allowed that the net rental rate of capital be non-decreasing locally. Meaning that for some range of values of assets, the net rental rate of capital stock would be increasing. This variation on the model allows the slope of the locus of stationary points corresponding to $\dot{\mu}= 0$ be positive in some ranges. In this case, multiple equilibria can arise as shown in Graph 2.

[Insert Graph 2]

Graph 2 illustrates the existence of three equilibrium points denote as A,B and C. In fact, the number of equilibrium points depends on the number of times both locus cross. In this case, the equilibrium points are three mainly because the slope of the locus corresponding to the shadow price's dynamic behavior has just one range of values where its slope is positive. If there are more than one range of points with this property, more equilibrium points could exist. On the

graph, points A and C are similar to the equilibrium point obtained in the case of monotonically decreasing rental rate of capital. On those points, the equilibrium is stable and the net rental rate of the economy is locally decreasing. On point B, the contrary holds. The net rental rate of capital is increasing locally and the equilibrium is unstable. The following proposition states the main results on this case:

Proposition 4.3. *When the net rental rate of capital in the economy is not globally non-decreasing, there could exist multiple equilibria. Only the equilibria where the net rental rate of the economy is decreasing are stables. On the range $k_t > k^*$, equilibia with larger assets per capita should have lower fertility rates, lower mortality rates, lower population growth rate and larger per capita incomes.*

Proof: See Mathematical appendix.

Proposition 4.3, as shown in its proof, focuses on stable equilibria that correspond to stationary points where $\partial NR_t / \partial k_t \leq 0$ and hence the locus $\dot{\mu} = 0$ is upward sloping. The former property is an important characteristic of those equilibriums. In fact, even when it is allowed the possibility of non-decreasing rental rate of assets, in a stable equilibrium this case does not hold. This follows from maximization, as households hold assets until their marginal return of those assets equals its marginal cost, given by the shadow price. If we allow that the

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equilibrium has a rental rate non-decreasing on assets, the households should be better-off accumulating more assets. In mathematical language, we should be in a minimum, instead of a maximum.

In graph 2, point B is the threshold. Any country with initial level of assets per capita to the right of the point B converges to C. In same way, any country with initials levels of assets per capita to the left of B converges to A.

Finally, notice that as the equilibrium level of assets per capita is larger on the range $k_t > k^*$, the smaller should be population growth rate but larger per capita income. As we know that on that range fertility and mortality rate are decreasing functions of assets and increasing functions of the shadow price, equilibria with larger assets per capita must be associated with lower fertility and mortality rates unambiguously.

The existence of multiple equilibria is interesting because it matches the European history before and after the mid of the nineteenth century. In fact, Europe before 1850 resembles an equilibrium with low per capita level of assets while the world after 1850 resembles an equilibrium (or a transition to it) with higher per capita level of assets, as this equilibrium has a larger per capita income, but lower fertility,mortality and population growth rates.

A question that remains is how Europe (or other countries) could jump from one equilibrium to the other. The next section argues that fiscal policy may allow

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to a country to go from the first equilibrium to the second type of equilibrium. Obviously some other alternative explanations may exist, however this paper addresses fiscal policy as a starting point.

5. Government and the "Malthusian trap"

At the end of last section it was shown that there could exist multiple equilibria on the economy. In this section it will be argued that government through fiscal policy may bring the economy from the equilibrium with low level of assets per capita to the another with a high level of assets per capita. In this case, the transition will be characterized by a demographic transition joined with increases on per capita income.

Consider the equilibrium with the lowest level of assets on graph 2. It has larger fertility and mortality rates and lower income per capita. This case will be denoted as the "Malthusian trap". It will be next shown that the economy could go from this equilibrium, point A , to a developed equilibrium, point C of the graph, through fiscal policy. This last equilibrium is denoted as "developed equilibrium" since it has a larger per capita income, lower mortality rate and lower fertility rate which are characteristics of the developed countries on the 20th century.

5.1. Escaping from the poverty trap

In this subsection, the government is included on the theoretical model developed through the paper. The way to include the government will be to assume that the government levies some per capita lump sum taxes at each period of time. The government collects the taxes and spends them completely on health goods. The way the government spends on health is different than the way private spend on health. In fact the government would provide goods like water supply and sewage systems or public immunizations. Those goods have a different impact over the survival function, hence the death hazard function at time t will be $\lambda(h_t, g_t)$, where g_t is government spending per capita and $\lambda_h, \lambda_g \leq 0$. The amount of taxes collected per individual is assumed exogenously determined by the government.

As a matter of notation, let τ_t be the lump sum taxes paid by the family in per capita terms, and let g_t be the amount of health good provided by the government and received by the family at time t. The problem faced by the households is similar to the original problem. It can be written now as:

$$\max_{\{c_t, h_t, n_t\}_{t=0,..,\infty}} \int_0^\infty u(c_t) e^{-\rho t - \int_0^t (\lambda(h_s, g_s) - n_s) ds} \, dt$$

$$\dot{k} = f(k_t, 1 - \phi(n_t)) - c_t - h_t - \tau_t - k_t[n_t - \lambda(h_t, g_t)]$$

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$$0 \leq \lim_{t \to \infty} \{k_t \mu_t\}$$

This problem have similar properties on the households' side as the original problem analyzed in the last section. The government budget constraint will be assumed to hold with equality at any period t, $\tau_t = g_t$.

We characterize the dynamic behavior of the economy as in last section, but now we introduce the government budget's constraint:

$$\dot{k} = f(k_t, 1 - \phi(n_t)) - c_t - h_t - g_t - k_t[n_t - \lambda(h_t, g_t)] \qquad (7')$$

$$\dot{\mu}_t = -[f_k(k_t, 1 - \phi(n_t)) - n_t + \lambda(h_t, g_t)]\mu_t \qquad (8')$$

where $c_t = c_t(k_t, \mu_t), h_t = h_t(k_t, \mu_t)$ and $n_t = n_t(k_t, \mu_t)$.

The system is essentially the same as in the last section, but now the asset equation includes the negative effect of the lump sum tax, and the hazard rate also includes the positive effect of government expenditure. Conditional on some level of government expenditure per capita, the phase diagram can be described by graph 2.

Suppose now, that the economy has as long run equilibrium a "Malthusian trap" -point A in graph 4. The next proposition indicates in this case, a perma-

nent increase in government expenditure capita could produce a take-off, bringing the economy to point C. The result depends on the fact the government spends resources in a different way, than the households. For each dollar taken away from households and spent completely on health, households used to spend some part on consumption and some other part on health. Intuitively, mortality rate is decreased by the reallocation of resources made by the government. Individuals as they have larger chances of survival, accumulate more assets. This effect pushes the economy out of the old equilibrium to the equilibrium with higher assets per capita. The next proposition states this result.

Proposition 5.1. *An increase in health government expenditure per capita might produce a take-off from a "Malthusian trap" to the new equilibrium with higher assets per capita.*

Proof: See Mathematical Appendix.

Graph 3 shows the dynamics to the new equilibrium. Dotted lines are the disturbed locus, and point D is the new equilibrium. The transition shows some interesting patterns. First the shadow price jumps to the new saddle path from point A to F and then, it starts to move to the new equilibrium through the new saddle path. This transition produces a demographic transition and an increase on per capita income. Instantaneously, fertility, mortality and population growth

rate increase, following the jump on the shadow price of assets. The initial jump on the shadow price of assets follows from the fact that government is taken away resources from the households and the relatively large scarcity of capital raises μ_t. Hence, there are large incentives to accumulate assets and by complementarity on the utility function to increase population. After the initial jump, we move through the saddle path decreasing fertility, mortality and population growth rate until reaching the new equilibrium. The same effect as above apply.

[Insert Graph 3]

Why do we need a government? Why do not the households reallocate resources by themselves? In fact, the policy is not ex-post Pareto improving. There is some positive measure of individuals that reduce their per capita consumption by paying the lump sum tax and die during the transition. Those individuals would have been better-off by remaining on their initial condition as they would have enjoyed large per capita consumption.

6. Summary

The paper develops a general equilibrium model where population growth rate and its components are chosen variables. It is shown that as a country develops a demographic transition occurs. The demographic transition occurs mainly be-

cause relative prices are distorted. First, the relative price between fertility and mortality rate increases and hence fertility rates tend to decrease while a larger amount of health goods are allocated lowering mortality rates given the level of expenditure on population. Also a second effect occurs as the relative price of population in terms of consumption goods increase. In that case, individuals substitute away from population towards larger per capita consumption. Those two effects argue for lower fertility and mortality rates over time.

Further, the endogeneity of population growth rate may produce multiple equilibria. Those equilibria can be characterized by their level of assets per capita. In fact poorer countries, measured by lower level of asset per capita, will have a lower per capita income but larger mortality, fertility and population growth rates. As multiple equilibria coexist on the model, it is possible for a poor country to reach an equilibrium with larger assets per capita through health fiscal policy. It should be noticed that even though fiscal policy is emphasized in this paper, positive shocks over the marginal product of capital or exogenous mortality reductions may also have the same effect of bringing the economy from the "Malthusian" trap to the developed equilibrium. Hence, alternative explanations as industrial revolution or better health status due to nutrition improvements may also play a role.

7. Mathematical Appendix

7.1. Proof of lemma 3.1 and lemma 3.2

Using the individual first order conditions and budget constraint we have:

(9) $\qquad u'(c_t)e^{-\rho t - \int_0^t (\lambda(h_s) - n_s)ds} = \mu_t$

(10) $\qquad u(c_t)e^{-\rho t - \int_0^t (\lambda(h_s) - n_s)ds}(-\lambda'(h_t)) = \mu_t[1 - \lambda'(h_t)k_t]$

(11) $\qquad u(c_t)e^{-\rho t - \int_0^t (\lambda(h_s) - n_s)ds} = \mu_t[f_l\phi'(n_t) + k_t]$

(12) $c_t + h_t + k_t[n_t - \lambda(h_t)] = \Omega(k_t)$

Where $\Omega(k_t)$ is total current expenditure. Equations (9)-(12) determine the existence of solutions of the form $c(\mu_t; k_t)$, $h(\mu_t; k_t)$ and $n(\mu_t; k_t)$. Using comparative statics, we will characterize the sign of the implicit derived functions.

First to characterize the responses on the functions when the shadow price changes, replace the implicit functions on (9), (10) and (11) and differentiate with respect to the shadow price. The system obtained is the following:

$$\begin{bmatrix} \frac{U_{cc}}{U_c} & 1 & -\lambda' \\[2mm] \frac{U_c}{U} & 1 + \frac{-f_l\phi'' + f_{ll}(\phi')^2}{f_l\phi' + a} & -\lambda' \\[2mm] \frac{U_c}{U} & 1 & \frac{\lambda''}{\lambda'(1 - \lambda' a)} - \lambda' \end{bmatrix} \begin{bmatrix} \frac{\partial c}{\partial \mu} \\[2mm] \frac{\partial h}{\partial \mu} \\[2mm] \frac{\partial n}{\partial \mu} \end{bmatrix} = \begin{bmatrix} \frac{1}{\mu} \\[2mm] \frac{1}{\mu} \\[2mm] \frac{1}{\mu} \end{bmatrix}$$

$$\Leftrightarrow \Theta \begin{bmatrix} \frac{\partial c}{\partial \mu} \\[2mm] \frac{\partial h}{\partial \mu} \\[2mm] \frac{\partial n}{\partial \mu} \end{bmatrix} = \begin{bmatrix} \frac{1}{\mu} \\[2mm] \frac{1}{\mu} \\[2mm] \frac{1}{\mu} \end{bmatrix}$$

Notice that $\text{Det}(\Theta) < 0$ by SOC. Applying Cramer's rule, we have:

$$\frac{\partial c}{\partial \mu}\mu = \frac{(\frac{\lambda''}{\lambda'(1-\lambda'k)})(\frac{-f_l\phi''+f_{ll}(\phi')^2}{f_l\phi'+k})}{Det(\Theta)} < 0$$

$$\frac{\partial n}{\partial \mu}\mu = \frac{(\frac{\lambda''}{\lambda'(1-\lambda'k)})(\frac{U_c}{U}-\frac{U_{cc}}{U_c})}{Det(\Theta)} > 0$$

$$\frac{\partial h}{\partial \mu}\mu = \frac{(\frac{-f_l\phi''+f_{ll}(\phi')^2}{f_l\phi'+k})(\frac{U_{cc}}{U_c}-\frac{U_c}{U})}{Det(\Theta)} < 0$$

Next, the effect of the shadow price over the population growth rate can be

easily determined:

$$\frac{\partial(gpop)}{\partial \mu} = \frac{\partial n}{\partial \mu} - \lambda'\frac{\partial h}{\partial \mu} = \frac{[\frac{U_c}{U}-\frac{U_{cc}}{U_c}][\frac{\lambda''}{\lambda'(1-\lambda'k)}+\lambda'(\frac{-f_l\phi''+f_{ll}(\phi')^2}{f_l\phi'+k})]}{\mu Det(\Theta)} > 0$$

Since $\frac{\lambda''}{\lambda'(1-\lambda'k)} + \lambda'(\frac{-f_l\phi''+f_{ll}(\phi')^2}{f_l\phi'+k}) < 0$ by SOC.

The same method allow us to determine the effects of k. In fact, using equa-

tions (9) to (12), and differentiating now with respect to k_t, we get the following

system:

$$\begin{bmatrix} \frac{U_{cc}}{U_c}-\frac{U_c}{U} & 0 & \frac{f_l\phi''-f_{ll}(\phi')^2}{f_l\phi'+k} \\ 0 & \frac{\lambda''}{\lambda'} & \frac{\phi''}{\phi'}-\frac{f_{ll}}{f_l}\phi' \\ 1 & 1-\lambda'k & f_l\phi'+k \end{bmatrix} \begin{bmatrix} \frac{\partial c}{\partial k} \\ \frac{\partial h}{\partial k} \\ \frac{\partial n}{\partial k} \end{bmatrix} = \begin{bmatrix} -\frac{1}{f_l\phi'+k} \\ 0 \\ \Omega'(k_t)+\lambda-n \end{bmatrix}$$

$$\Leftrightarrow \Lambda \begin{bmatrix} \frac{\partial c}{\partial k} \\ \frac{\partial h}{\partial k} \\ \frac{\partial n}{\partial k} \end{bmatrix} = \begin{bmatrix} -\frac{1}{f_l\phi'+k} \\ 0 \\ \Omega'(k_t)+\lambda-n \end{bmatrix}$$

Notice that $\text{Det}(\Lambda) > 0$ by SOC. Also notice that as the utility functions are

concave on consumption, $\Omega'(k_t)+\lambda-n$ is positive. Hence the distribution of ex-

penditure over time is such that the current expenditure is an increasing function

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of the state variable.

Applying Cramer's rule, we have:

$$\frac{\partial c}{\partial a} = [\frac{-1}{Det(\Lambda)}][\frac{\lambda''}{\lambda'} - (\frac{\phi''}{\phi'} - \frac{f_{ll}}{f_l}\phi')\frac{(1-\lambda'k)}{(f_l\phi'+k)}] + \frac{(\Omega'(k_t)+\lambda-n)}{Det(\Lambda)}[\frac{-\lambda''}{\lambda'}(\frac{f_l\phi''-f_{ll}(\phi')^2}{f_l\phi'+k})] > 0$$

$$\frac{\partial h}{\partial a} = \frac{(\frac{\phi''}{\phi'} - \frac{f_{ll}}{f_l}\phi')}{Det(\Lambda)}[(\Omega'(k_t)+\lambda-n)(\frac{U_{cc}}{U_c} - \frac{U_c}{U}) + \frac{1}{(f_l\phi'+k)}]$$

$$\frac{\partial n}{\partial a} = \frac{\frac{\lambda''}{\lambda'}}{Det(\Lambda)}[(\Omega'(k_t)+\lambda-n)(\frac{U_{cc}}{U_c} - \frac{U_c}{U}) + \frac{1}{(f_l\phi'+k)}]$$

Notice that $[(\Omega'(k_t)+\lambda-n)(\frac{U_{cc}}{U_c} - \frac{U_c}{U}) + \frac{1}{(f_l\phi'+k)}]$ has an ambiguous sign. The

first term is an income effect while the second a substitution effect. When k is

small, as Inada conditions apply on the utility function, the income effect offsets

the substitution effect, while when k is larger the substitution effect offsets the

income effect. By continuity of preferences it exists some wealth level k* where

both effects exactly offset each other. Hence, when k is smaller(larger) than k*

the income(substitution) effect is the one that matters. In that case it follows

that when k<k*, $\frac{\partial h}{\partial k} < 0, \frac{\partial n}{\partial k} > 0$, while when k>k*, the contrary holds.

The effect over total population growth rate is:

$$\frac{\partial(gpop)}{\partial k} = \frac{\partial n}{\partial k} - \lambda'\frac{\partial h}{\partial k} = \frac{\frac{\lambda''}{\lambda'}+\lambda'(\frac{f_{ll}}{f_l}\phi'-\frac{\phi''}{\phi'})}{Det(\Lambda)}[(\Omega'(k_t)+\lambda-n)(\frac{U_{cc}}{U_c} - \frac{U_c}{U}) + \frac{1}{(f_l\phi'+k)}]$$

but $\frac{\lambda''}{\lambda'} + \lambda'(\frac{f_{ll}}{f_l}\phi' - \frac{\phi''}{\phi'}) < 0$ to assure SOCs are satisfied. Hence $\frac{\partial(gpop)}{\partial k} > (<)0$

when k is smaller (larger) than k*. Those results establish the lemmas. Q.E.D.

7.2. Proof of proposition 4.2

From equations (7) and (8), we have:

$$\text{H}^0_\mu = \overset{\bullet}{k} = f(k, 1 - \phi(n(k,\mu))) - c(k,\mu) - h(k,\mu) - k[n(k,\mu) - \lambda(h(k,\mu))]$$

$$-\text{H}^0_k = \overset{\bullet}{\mu} = -[f_k(k, 1 - \phi(n(k,\mu))) - n(k,\mu) + \lambda(h(k,\mu))]\mu$$

Where the time subscript where eliminated for simplicity. The stationary state to the system is one where both asset per capita and shadow price of assets are constant, hence we will focus on $\overset{\bullet}{\mu}_t = \overset{\bullet}{k} = 0$. The slope of the locus, in that case, are $\frac{\partial \mu}{\partial k} \mid_{\overset{\bullet}{k}=0} = -\frac{H^0_{\mu k}}{H^0_{\mu\mu}}, \frac{\partial \mu}{\partial k} \mid_{\overset{\bullet}{\mu}=0} = -\frac{H^0_{kk}}{H^0_{k\mu}}$. Since the net rental rate of the economy is decreasing, we have $H^0_{kk} < 0$. We can also sign $H^0_{\mu k}, H^0_{k\mu}$ and $H^0_{\mu\mu}$.

First, notice that around an equilibrium we have $-H^0_{k\mu} = \mu[n_\mu - \lambda' h_\mu] + \mu f_{kl}\phi' n_\mu > 0$, given the properties of the implicit demand functions obtained on lemmas 3.1 and 3.2. By symmetry of the Hamiltonian it follows that $H^0_{k\mu} = H^0_{\mu k} < 0$.Finally, Simply differentiation plus the condition $\frac{u(c_t)\left[-\lambda'(h_t)\right]}{u(c_t)} = \frac{[1-\lambda'(h_t)k_t]}{[f_l\phi'(n_t)+k_t]}$, gives us $H^0_{\mu\mu} = -c_\mu - (f_l\phi' + k)[n_\mu - \lambda' h_\mu] > 0$ by convexity of the Hamiltonian on the shadow price. Those conditions imply that on any equilibrium, we have $\frac{\partial \mu}{\partial k} \mid_{\overset{\bullet}{k}=0} > 0, \frac{\partial \mu}{\partial k} \mid_{\overset{\bullet}{\mu}=0} < 0$.

Notice that the dynamics can be determined using the phase diagram. Above the locus $\overset{\bullet}{k} = 0$, capital stock increases as the level of consumption and health decreases. Below the contrary holds. In fact, to show this property fix some level of assets (capital) and disturb the shadow price. The effect over asset accumulation is given by $\frac{\partial \overset{\bullet}{k}}{\partial \mu} = H^0_{\mu\mu} > 0$. Following the same procedure the dynamics above the locus $\overset{\bullet}{\mu} = 0$ can be stated. To determine this effect fix the level of shadow price

and distort the level of assets. In this case, we have $\frac{\partial \dot{\mu}}{\partial k} = -H^0_{kk} > 0$. These dynamics determine a saddle path to the equilibrium where shadow price of assets decreases as the level of assets increases over time.

As the slopes of the two locus have different signs, the locuses cross only one time and there is only one pair $(k \mid_{ss}, \mu \mid_{ss})$ satisfying the equilibrium. It follows that there is only a unique level of h $\mid_{ss}$ and n$\mid_{ss}$ in this economy on the long run.

During the transition we need to separate the effects for lower levels of k than k* and for larger levels. When k<k*, as we approach to the equilibrium, population growth rate is positively affected by the income effect but negatively affected by the decrease on the shadow price, hence the effect over population growth rate is ambiguous. When k>k*, relative prices are distorted and the income effect is offset. Hence, the final effect over population growth rate is negative by the properties of lemma 3.2. This last effect is reinforced by the decrease on the shadow price of assets. Hence unambiguously population growth rate decreases. Analogous arguments follow for fertility and mortality rate. Q.E.D

7.3. Proof of proposition 4.3

The proof follows the proof of proposition 4.2.The slope of the locus are the same but we allow that H^0_{kk} be positive on some range of asset, while negative as above on some other ranges of assets. Around the locuses $\dot{k}= 0$ and $\dot{\mu}= 0$ the same

dynamics hold. However, notice that the fact that H^0_{kk} may be positive plays a role. Whenever H^0_{kk} is negative, we find the same properties as in proposition 4.2, namely stable equilibrium with constant per capita income and constant fertility and mortality rates on the long run. When H^0_{kk} is positive, both locus are positively sloped. In this case, an equilibrium may exist if the locuses cross, however the equilibrium is not stable. The phase diagram shows graphically this last claim. Hence, any stable equilibrium must have a decreasing net rental rate.

Finally, any set of stable equilibria belonging to the set of assets per capita larger than k* may be ranked. Take any two equilibria on this range. The one with larger assets per capita must have lower mortality rate, fertility rate and population growth rate as indicated by lemma 3.1 and 3.2, since capital stock is larger and its shadow price is smaller. Further the one with larger assets per capita must have larger per capita income due first to larger capital stock and second to larger labor supplied to the market, as we spent less time on children. Q.E.D.

7.4. Proof of proposition 5.1

To see the effect of increasing the government expenditure we disturb both locuses by increasing g_t, while holding the level of asset per capita constant. As g_t is disturbed, both locuses will move. In fact, we will show that the locus $\overset{\bullet}{\mu}= 0$

moves downward while the locus $\overset{\bullet}{k}= 0$ moves upward for an increase on the government spending. To get this effect, fix k_t and disturb g_t, the effect over the locuses is the following:

$$\frac{\partial \mu}{\partial g} \Big|_{\overset{\bullet}{k}=0} = \frac{1-\lambda_g k}{H_{\mu\mu}} > 0$$

$$\frac{\partial \mu}{\partial g} \Big|_{\overset{\bullet}{\mu}=0} = \frac{\lambda_g}{-H_{k\mu}} < 0$$

As the locus moves in opposite directions, the current level of assets per capita on the "Malthusian" trap is no longer an equilibrium. When the push is large enough both locus will not cross anymore around the initial level of asset and we move to a higher level asset per capita through the saddle path -see graph 5. Q.E.D.

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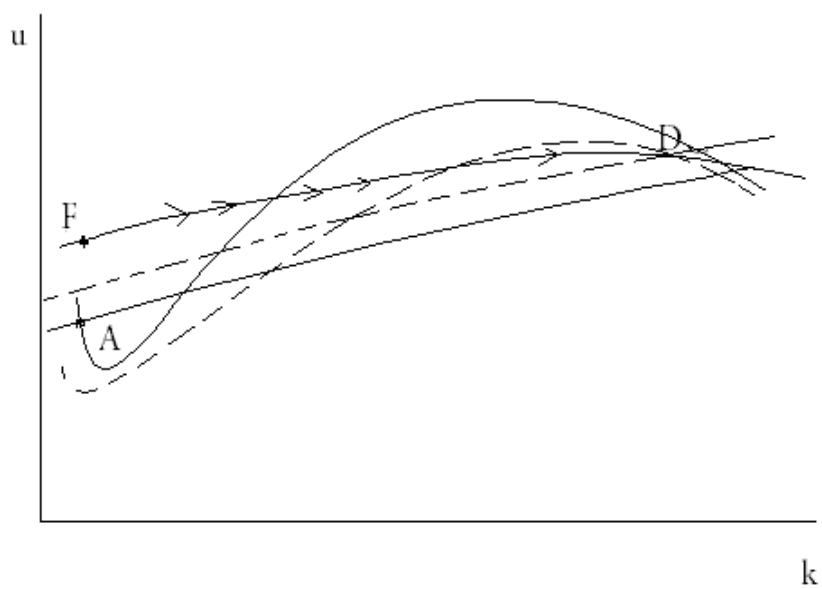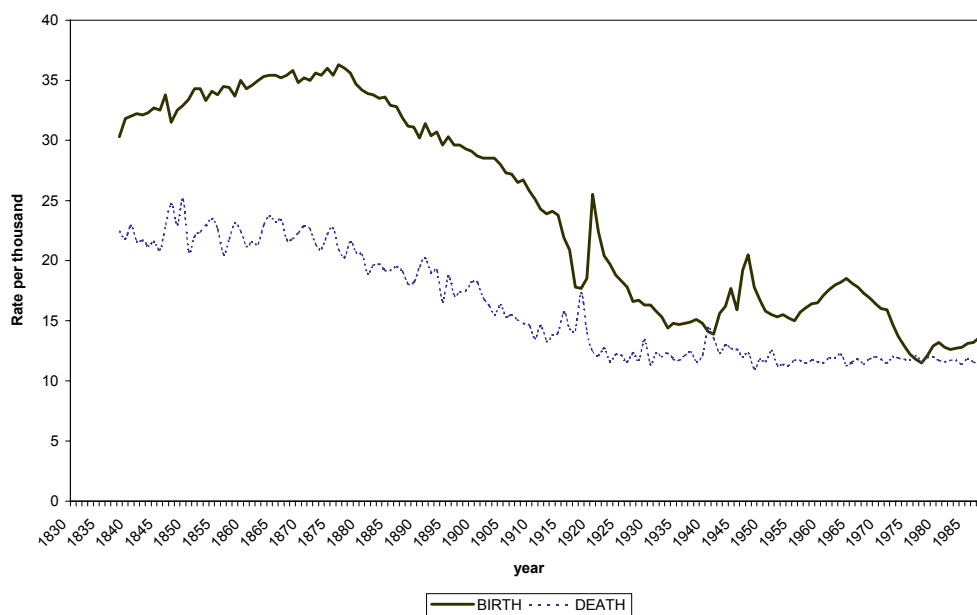
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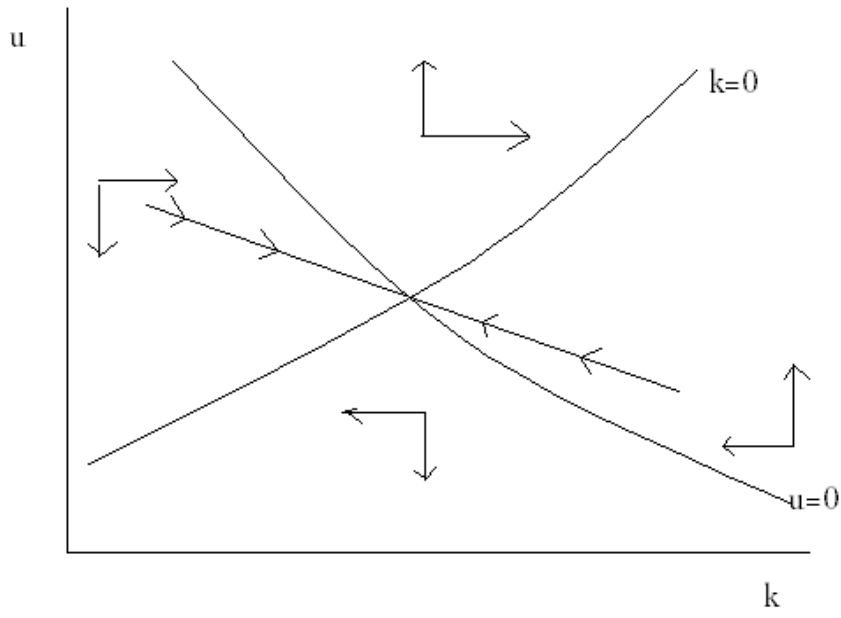
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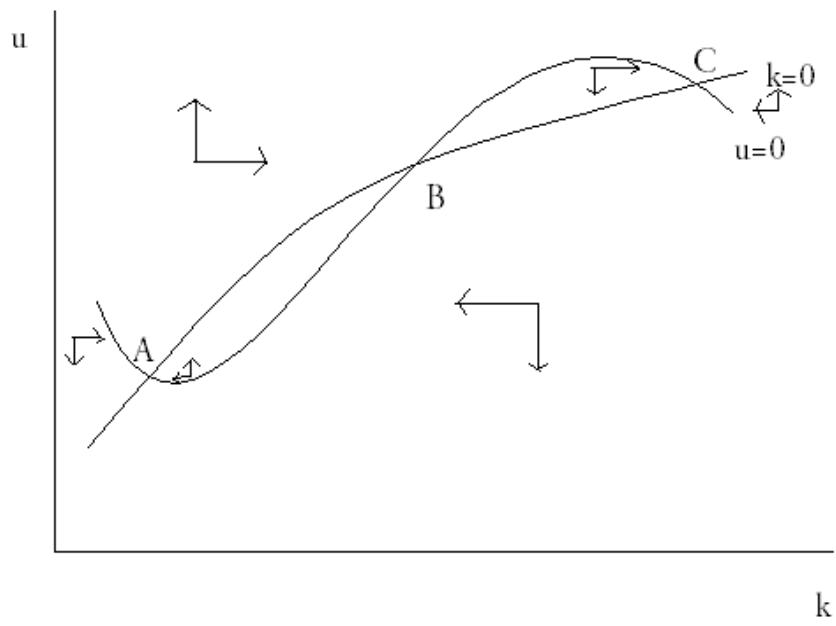
Graph 3

Figure 1

England (1838-1988)

Source: Mitchell, B.R., "International historical statistics, Europe 1750-1988".