Fiscal Policy Rules in an Overlapping Generations Model with Endogenous Labour Supply

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Abstract

A ¯scal policy rule in which taxation is a function of existing government debt (a "wealth-tax") is usually believed to be e®ective in providing stability. Using a discrete-time version of Blanchard's overlapping generations model, extended to include money and an endogenous labour supply we show that, contrary to the intuition, a wealth tax might not be enough to ensure the existence of a unique, well de¯ned, saddle-path equilibrium. We suggest that a government willing to run a positive and sustainable level of debt could use an alternative ¯nancing rule, imposing an additional tax component, that is a function of the di®erence between the real interest rate and the tax rate on wealth.

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1 Introduction

Public debts in several European countries have reached very high levels in the 1980s. This phenomenon has been the cause of great concern among policy makers and public opinion. Expressions like unsustainability, instability and default risk have entered the day-to-day debate on economic policies. The fears related to growing debts have been translated in to the strict constraint imposed on EU economies by the Maastricht Treaty and the Stability and Growth Pact.

Among academic economists, similar issues have always been regarded as important. Interest in sustainability of ¯scal policy can be found, for instance, in Keynes (1923) and Domar (1944). Christ (1979) studies the implications of di®erent rules for stability, in an ad hoc Keynesian framework.

The events of the 1980s have caused a revival of the interest in ¯scal policy stability. Attempts have been made to de¯ne, on empirical grounds, indicators of sustainability. Blanchard et al. (1990), for example, construct short, medium and long-term indicators of sustainability. Their exercise, based on the idea that a sustainable policy is one that does not violate the intertemporal budget constraint, has the merit to acknowledge the importance of forward looking behaviour in a®ecting policy outcomes. This approach, however, can be criticized on the grounds of being mostly an accounting exercise, that heavily depends on how good the forecasts about future variables are.

In our opinion, there is a need to use modern, fully microfounded models to investigate which ¯scal policy rules are "stable", in the sense that they are consistent with the existence of a well de¯ned equilibrium and of a unique convergent path. The most natural candidates for this kind of analysis are models of overlapping generations, in which Ricardian equivalence is broken and the debt is allowed to have real e®ects. A contribution in this direction is the work by Rankin and Ro±a (1999), that uses a Diamond (1965) type, two-period-lives model to investigate the existence of a maximum sustainable level of debt. The main question they want to address is whether there can occur "catastrophes", de¯ned as situations in which a well-de¯ned debt steady-state suddenly ceases to exist while other variables, like consumption and the capital stock, still lay in an economically feasible range of values. Because of their interest in catastrophes, they mostly concentrate on comparing various steady-states for constant levels of debt, and they do not conduct a comparative analysis of different fiscal policy rules.

In this paper, we aim at comparing the dynamic effects of different fiscal policy rules, including ones in which the level of debt is endogenous, rather than being fixed at some constant, exogenous level. To do this, we use a modified version of the perpetual youth model provided by Blanchard (1985), in which agents face in every period a positive probability of death. Our point of departure is the discrete time treatment of Blanchard's model provided by Frenkel and Razin (1996). In the original Blanchard framework, disposable income is either given[1] or it follows an exogenously imposed declining path. In our model, by endogenising the labour supply, we aim to give a better account of the impact of the labour-leisure trade off decisions of agents. On the other hand, we assume that labour is the only factor of production. This is equivalent to holding the level of capital fixed. Since in Blanchard (1985) capital is endogenous, our contribution is orthogonal to his in this respect.

The model that we present is similar to others that have been recently developed in the literature, especially Leith and Wren-Lewis (2000), who use a perpetual youth model to study the interaction between monetary and fiscal policy. While they introduce nominal rigidities in the analysis, they retain the original assumption of an exogenous labour supply. Another similar model is developed by Heijdra and Ligthart (2000), whose focus is not on debt, but on comparing the macroeconomic effects of three different tax regimes (capital, labour income and consumption tax).

We conduct three policy experiments. For comparative purposes, we start by looking at the case in which the government is not allowed to use debt at all. We find that the introduction of a positive probability of death is not enough, by itself, to cause an effect of balanced budget expansions on the real interest rate. This policy, on the other hand, reduces both consumption and leisure. The overall welfare effect is therefore negative.

We then study a policy similar to the one considered by Blanchard (1985), in which a government is initially holding its debt constant, and subsequently decides to increase the level of debt to a new, higher, steady state. As government expenditure is constant, we are assuming that taxes adjust endoge-

[1] Disposable income is endogenous in Blanchard's model (since the real wage is endogenous), but it is given to the agent himself.

nously to meet the increased payments of interest. We show that this policy is likely to raise the real interest rate.

We finally consider the case of a "wealth tax", in which taxation is an increasing function of government debt (that enters positively agents' wealth). Contrary to what we could expect, making taxes an increasing function of existing debt does not authomatically guarantees stability. In other words, the presence of a wealth tax might not be enough to ensure the existence of a unique, well-defined saddle path leading to the equilibrium. In this situation, it could be the case that a huge increase in the tax coefficient on debt is needed in order to have a saddle path solution. Such an increase, however, could be not easy to implement for the government, because of political pressures. We suggest an alternative rule that can yield the same outcome, in which the government drastically reduces the tax rate on debt but adds another tax component, that is a function of the difference between the real interest rate and the tax coefficient on debt wealth. Our intention here is not to suggest that such a rule would be optimal, but only to give some insights in to what policy could be followed by a government that is in a position of having to control is debt, but that is prevented from implementing more stringent policies because of some political reasons. We believe that this situation reflects the dilemma faced by some European governments in the late 1980s and early 1990s, that were in a situation of having to reduce drastically their debt, but could not rely on very large parliamentary majorities to undertake more structural policies, like heavy taxation of wealth or permanent cuts in government expenditure.

The paper is organized as follows. Section 2 introduces the model, section 3 analyzes some steady-state and dynamic properties in the case in which there is no public sector, that is nested in our more general specification. Sections 4 and 5 look at the effects of different policy rules, while Section 6 draws some conclusions.

2 The Model

2.1 Private Agents

We consider a closed economy. In every period each agent faces a constant probability of death $(1 - q)$: We also assume no population growth. The size of the cohorts of agents born in every period is constant across time and can be normalized to 1. The size of the world population is therefore constant as well and equal to $\sum_{a=0}^{\infty} q^a = \frac{1}{1-q}$. Only one good is produced in the economy. Agents gain utility from consumption, money balances and leisure. In what follows, we introduce the optimization problem of a representative agent of age a at time t. Before proceeding with the illustration of the model, it is useful to clarify our terminology, that becomes more complicated because of the introduction of overlapping generations. We will call variables relating to an individual of age a individual variables, while aggregate variables will be the one obtained aggregating across individuals of all the different ages, and per-capita variables will be aggregate variables divided by the size of the population.

The representative agent maximizes the expected utility function[2]:

$$E(U_t) = \sum_{s=t}^{\infty} (\beta q)^{s-t}[\log(C_{a+s-t;s}) + \chi \log \frac{M_{a+s-t;s}}{P_s} + \psi \log(1 - L_{a+s-t;s})] \quad (1)$$

Preferences are homothetic and separable in consumption, real balances and leisure. The endowment of time in each period is normalized to 1: $L_{a+s-t;s}$ is the quantity of labour supplied in every period, $(1 - L_{a+s-t;s})$ is leisure. A standard assumption in this framework is the existence of insurance companies. We assume that insurance companies pay a net premium of $(\frac{1-q}{q})$ on the agent's financial wealth for each period of his life, while they encash the agent's financial wealth if the agent dies[3]. Agents can hold financial wealth as real balances or as government debt. In addition, they supply labour and pay lumps-sum taxes. The representative agent's period t budget constraint in real terms is, therefore:

$$D_{a;t+1} + \frac{M_{a;t}}{P_t} + C_{a;t} = \frac{1}{q}[\frac{M_{a-1;t-1}}{P_t} + (1 + r_t)D_{a-1;t}] + \frac{W_{a;t}}{P_t} L_{a;t} - \tau_t \quad (2)$$

Where $D; M; W; r; \tau$ and P denote respectively government debt, nominal money, nominal wage, real interest rate, real lump-sum taxes and prices. It

[2] $C_{a+s-t;s}$ denotes consumption of an agent of age $a + s - t$ at time s: An analogous notation holds for the other variables.

[3] As agents die in each period with probability q; these arrangements ensure a safe return of 1 on money and of $(1 + r_t)$ on debt.

can be shown (see Appendix) that the maximization of (1) subject to (2) and to a standard No-Ponzi Game condition (eq. ?? in the Appendix) is equivalent to the unconstrained maximization of:

$$\sum_{s=t} (\beta q)^{s-t}[\log(C_{a+s-t,s}) + \chi \log \frac{M_{a+s-t,s}}{P_s} + \psi(1 - L_{a+s-t,s})] + \tag{3}$$

$$+ \lambda \{ \frac{1}{q}[\frac{1}{1+i_t}\frac{M_{a-1,t-1}}{P_{t-1}} + D_{a-1,t}](1+r_t) - \sum_{s=t} \alpha_{s,t} q^{s-t}(C_{a+s-t,s} + \frac{i_{s+1}}{1+i_{s+1}}\frac{M_{a+s-t,s}}{P_s} +$$

$$- \frac{W_{a+s-t,s}}{P_s} L_{a+s-t,s} + \tau_t)\}$$

Where the expression in the curly brackets is the agent's intertemporal budget constraint, i is the nominal interest rate, and $\alpha_{s,t}$ is the present value factor, defined as:

$\alpha_{s,t} = 1$ when $s = t$; and

$\alpha_{s,t} = \frac{1}{(1+r_{t+1})\cdot\ldots\cdot(1+r_s)}$ when $s > t$.

The first-order conditions with respect to $C_{a+s-t,s}$, $M_{a+s-t,s}$ and $L_{a+s-t,s}$ are given by:

$$C_{a+s-t,s} = \frac{1}{\lambda}\frac{\beta^{s-t}}{\alpha_{s,t}} \tag{4}$$

$$L_{a+s-t,s} = 1 - \frac{1}{\lambda}\frac{\beta^{s-t}}{\alpha_{s,t}}\frac{P_s}{W_{a+s-t,s}}\psi \tag{5}$$

$$\frac{M_{a+s-t,s}}{P_s} = \frac{1}{\lambda}\chi\frac{\beta^{s-t}}{\alpha_{s,t}}\frac{1+i_{s+1}}{i_{s+1}} \tag{6}$$

Equation (4) implies the following Euler equation for individual consumption:

$$\frac{C_{a+s-t+1,s+1}}{C_{a+s-t,s}} = \beta\frac{\alpha_{s,t}}{\alpha_{s+1,t}} = \beta(1+r_{s+1}) \tag{7}$$

Equation (7) is the discrete-time equivalent of Blanchard's expression, it states the fact that individual consumption rises if the real interest rate is bigger than the subjective discount rate $\frac{1-\beta}{\beta}$. The rate of growth of individual consumption does not depend on wealth. As we are going to see in what follows, however, the level of individual consumption is a function of total wealth, and the rate of growth of per-capita consumption is a function of human wealth.

Substituting (4), (5) and (6) in to the intertemporal budget constraint and solving for $\frac{1}{\lambda}$ we obtain:

$$\frac{1}{\lambda} = \left(\frac{1-q\beta}{1+\chi+\psi}\right)\left\{(1+r_t)\frac{1}{q}\left[\frac{1}{1+i_t}\frac{M_{a-1,t-1}}{P_{t-1}} + D_{a-1,t}\right] + H_{a,t}\right\} \qquad (8)$$

Where $H_{a,t}$ is human wealth, defined as:

$$H_{a,t} = \sum_{s=t}^{\infty} \alpha_{s,t} q^{s-t}\left(\frac{W_{a+s-t,s}}{P_s} - \tau_s\right)$$

Human wealth is defined as the present discounted value of potential gross earnings (that would be earned if the agent chose to consume no leisure), minus taxes. Of course, as leisure provides utility, agents will not choose to supply a quantity 1 of work in each period. This can be seen by substituting (8) back in to the first order conditions, and deriving the individual consumption, leisure and real balance demand functions for period t:

$$C_{a,t} = \left(\frac{1-q\beta}{1+\chi+\psi}\right)\left\{(1+r_t)\frac{1}{q}\left[\frac{1}{1+i_t}\frac{M_{a-1,t-1}}{P_{t-1}} + D_{a-1,t}\right] + H_{a,t}\right\} \qquad (9)$$

$$L_{a,t} = 1 - \psi\frac{P_t}{W_{a,t}}C_{a,t} \qquad (10)$$

$$\frac{M_{a,t}}{P_t} = \chi\frac{(1+i_{t+1})}{i_{t+1}}C_{a,t} \qquad (11)$$

The expression $\frac{1-q\beta}{1+\chi+\psi}$ in (9) is the propensity to consume out of total (financial plus human) wealth. The fact that this parameter is constant over time is a consequence of our logarithmic specification, that implies a unit intertemporal elasticity of substitution. The constant propensity to consume is an inverse function of the weights on real balances and leisure in the utility function (χ and ψ) and of the agent's temporal horizon (it decreases as the effective discount factor $q\beta$ increases).

To gain some intuition on the meaning of equation (10), it is useful to rearrange it as:

$$(1 - L_{a;t})W_{a;t} = \psi P_t C_{a;t}$$

The above expression tells us that there is an inverse proportionality between expenditure on consumption and expenditure on leisure (defined in terms of the opportunity cost of not working). Finally, equation (11) is a standard money demand equation in microfounded models.

Before proceeding to define the aggregate variables, it is useful to specify the behavior of firms and of the government.

2.2 The Behavior of Firms

To make aggregation possible, we assume that the agents supply their labour in a perfectly competitive market. For the same reason, we assume that the marginal productivity of labour is invariant across ages. Anotherr simplifying assumption is that labour is the only factor of production, with constant returns. The technology used by firms is therefore:

$$Y_t = L_t \tag{12}$$

Where L_t is the quantity of labour used in the production process. Under these assumptions, from the profit maximization condition we obtain $\frac{W_t}{P_t} = 1$ in every period t.

2.3 The Government

In this paper we abstract from useful government spending. The government therefore spends on public expenditure that does not affect private utility. Government expenditure can be financed by seigniorage, lump-sum taxes and debt, according to the single-period budget constraint:

$$G_t + (1 + r_t)D_t = \tau_t + \frac{(M_t - M_{t-1})}{P_t} + D_{t+1} \tag{13}$$

In addition to this, the government must also respect a No-Ponzi game condition. It is important to notice that, since the government has an infinite life horizon, the real interest rate applied to D_t in (13) is $(1 + r_t)$; as opposed to $\frac{(1+r_t)}{q}$ in the private agents' budget constraint.

2.4 Per-Capita Variables

We are now ready to start the aggregation process. Summing across ages we get the aggregate variables that, once divided by the size of the population $\frac{1}{1-q}$, give the per-capita variables. All per-capita variables will be indexed by the superscript PC. It is also useful to define formally total wealth as the sum of financial and human wealth:

$$TW_t = (1 + r_t)[\frac{1}{1 + i_t} \frac{M_{a-1;t-1}}{P_{t-1}} + D_{a-1;t}] + H_{a;t}$$

Accordingly, per-capita consumption is given by:

$$C_t^{PC} = \sum_{a=0} (1 - q)q^a C_{a;t} = (\frac{1 - q\beta}{1 + \chi + \psi})TW_t^{PC} \quad (14)$$

Where $(1 - q)q^a$ is the proportion of agents of age a in the world population[4], and:

$$TW_t^{PC} = \sum_{a=0} (1 - q)q^a TW_t = H_t^{PC} + (1 + r_t)[\frac{1}{1 + i_t} \frac{M_{t-1}^{PC}}{P_{t-1}} + D_t^{PC}] \quad (15)$$

$$H_t^{PC} = \sum_{a=0} (1 - q)q^a \{ \sum_{s=t} \alpha_{s;t} q^{s-t} (\frac{W_s}{P_s} - \tau_s) \} = \sum_{s=t} \alpha_{s;t} q^{s-t} (1 - \tau_s) \quad (16)$$

$$M_{t-1}^{PC} = \sum_{a=0} (1 - q)q^{a-1} M_{a-1;t-1}$$

$$D_t^{PC} = \sum_{a=0} (1 - q)q^{a-1} D_{a-1;t}$$

[4] The size of each cohort of agents is normalized to 1, and each agents has a probability of surviving in every period equal to q. For the law of large numbers, therefore, q^a is the number of agents of each cohort that survive till the age a.

Notice that, in the aggregation of wealth, we have used the fact that both taxes and real wages are invariant across ages, and that real wages can be set to 1 with the special production function (12). As a consequence, per-capita human wealth is equal to individual wealth for each agent.

Similarly, as prices and interest rates are independen of age, per-capita money demand is given by:

$$\frac{M_t^{PC}}{P_t} = \chi \frac{(1 + i_{t+1})}{i_{t+1}} C_t^{PC} \tag{17}$$

Finally, aggregating the labor-leisure equation(10), and using again the result that $\frac{W_s}{P_s} = 1$ independently of age, we get:

$$L_t^{PC} = \sum_{a=0}^{\infty} (1 - q) q^a L_{a;t} = 1 - \psi(1 - q) \sum_{a=0}^{\infty} q^a C_{a;t} = 1 - \psi C_t^{PC} \tag{18}$$

The latter relationship is useful to illustrate an important characteristic of the model, namely the fact that private consumption and output (equal to the quantity of labour supplied) are determined by government expenditure. To show this, notice that in this simple closed economy, equilibrium in the goods market, in per-capita terms, boils down to:

$$Y_t^{PC} = L_t^{PC} = C_t^{PC} + G_t \tag{19}$$

Solving for L_t^{PC} and C_t^{PC} from (18) and (19) we obtain:

$$Y_t^{PC} = L_t^{PC} = \frac{1}{1 + \psi} + \frac{\psi}{1 + \psi} G_t \tag{20}$$

$$C_t^{PC} = \frac{1}{1 + \psi} - \frac{1}{1 + \psi} G_t \tag{21}$$

In the case in which leisure does not provide utility ($\psi = 0$); equations (20) and (21) reproduce the neo-classical result of no effect on output and complete crowding-out of consumption following a fiscal expansion ($dY{=}dG = 0$, $dC{=}dG = -1$). In this case agents supply inelastically all their endowment of time. The balanced-budget multiplier derived in the IS/LM literature, ($dY{=}dG = 1$; $dC{=}dG = 0$); on the other hand, emerges in the limiting case in which $\psi \to 1$.

While Y_t^{PC} and C_t^{PC} can be expressed as functions of an exogenous, predetermined variables like government expenditure, the real interest rate is a function of its future levels, behaving like a "jump" variable. In order to see this, it is useful to go through the intermediate step of characterizing the dynamic behavior of consumption.

2.5 Per-Capita Consumption Dynamics

The dynamic of per-capita consumption is given by (see Appendix for the derivation):

$$C_t^{PC} = \left(\frac{1 - q\beta}{1 + \chi + \psi}\right)(1 - q)H_t^{PC} + (1 + r_t)q\beta C_{t-1}^{PC} \tag{22}$$

In the case of infinite life ($q = 1$) equation (22) reduces to a standard Euler equation. In that case human wealth is not important for predicting future consumption. The above expression also nests the logarithmic case in the Frenkel and Razin (1996) model, in which money and leisure do not provide utility ($\chi = \psi = 0$).

3 Steady State and Dynamics without Government

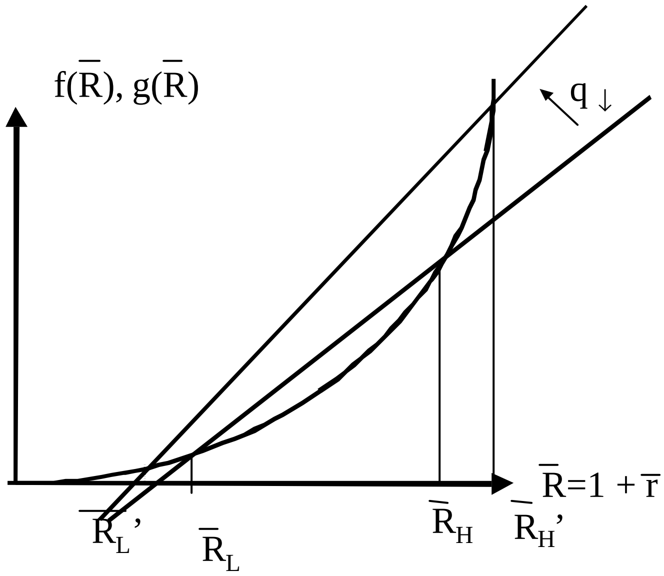
We will now characterize the steady-state and the dynamics of the model. It is convenient to consider first the case in which government expenditure, taxes and debt are permanently fixed to zero. As the model displays multiple equilibria, one problem is how to discriminate between them. The preliminary analysis of this section gives some insights about this, that will also turn out to be useful once we reintroduce the government in the model.

3.1 Steady State

In an economy without public sector, consumption and output are permanently fixed at the "natural" level $\frac{1}{1+\psi}$: It follows that a steady-state version of equation (22) is[5]:

[5] Barred variables denote the steady-state.

Figure 1: Effect of a reduction in q on steady-state $\bar R$



$$1+\bar r = 1 - (1+\psi)\left(\frac{1-q\beta}{1+\chi+\psi}\right)(1-q)\bar H^{PC} \tag{23}$$

Where $\bar H^{PC}$ is the steady-state level of human wealth with $\tau = 0$; i.e.:

$$\bar H^{PC} = \sum_{s=t} \left(\frac{q}{1+\bar r}\right)^{s-t} = \frac{1+\bar r}{1+\bar r - q} \tag{24}$$

Substituting (24) in (23) and denoting with $\bar R = 1+\bar r$ the gross real interest rate we can derive the following quadratic equation in $\bar R$:

$$\bar R^2 - \left\{q + \frac{1}{q}\left[1 - \left(\frac{1-q\beta}{1+\chi+\psi}\right)(1-q)(1+\psi)\right]\right\}\bar R + \frac{1}{=} = 0 \tag{25}$$

To solve explicitly for $\bar R$ from this equation would be possible, but not very illuminating. The implications of (25) are more easily understood looking at Figure 1.

The two solutions of (25) are the points in which the parabola $f(\bar{R}) = \bar{R}^2$ meets the line $g(\bar{R}) = \{q + \frac{1}{q\beta}[1 - (\frac{1-q\beta}{1+\chi+\psi})(1-q)(1+\psi)]\}\bar{R} - \frac{1}{\beta}$. Given the ranges of values of the parameters, the slope of this line is obviously positive.

From Figure 1 it is clear that we are faced with 2 possible equilibria. As the real interest rate is expected to behave like a jump variable, one way of discriminating between them is to select the unstable one. This can only be done after characterizing the dynamics of R. However, we can say something about the two different steady-states before looking at the dynamics, if we assume that deviations from the Ricardian equivalence case ($q = 1$), are not too large. The first argument is a type of Samuelson correspondence principle: the equilibrium that has desirable stability properties must also yield desirable comparative static properties. In this case, it is easy to check that, as the derivative of $\{q + \frac{1}{q\beta}[1 - (\frac{1-q\beta}{1+\chi+\psi})(1-q)(1+\psi)]\}$ computed at $q = 1$, being equal to $\frac{\beta-1}{\beta}(\frac{\chi}{1+\chi+\psi})$, is negative, in the neighborhoods of this value a fall in q will imply an increase of the "higher" equilibrium (that is a movement from $\bar{R}_H$ to $\bar{R}_H'$ in Figure 1) and a fall of the "lower" equilibrium (from $\bar{R}_L$ to $\bar{R}_L'$ in Figure 1). It follows that, if the deviation from the infinite life case is not too big (if q is not too much smaller than 1), $\bar{R}_H$ displays the more sensible result in terms of comparative static: when the probability of surviving to the next period become smaller, agents become more short sighted and therefore the real interest rate increases (present consumption becomes more costly in terms of future consumption). This would suggest to restrict our attention to the higher equilibrium $\bar{R}_H$. Another argument that leads to the same conclusion can be developed considering that, when agents have infinite lives, our model collapses to a discrete time version of the Ramsey (1928) model. When $q = 1$, equation (25) yields the two solutions $\bar{R}_L = 1$ and $\bar{R}_H = \frac{1}{\beta}$. Since $\bar{R}_H = \frac{1}{\beta}$ is the solution of the Ramsey model, while $R_L = 1$ is not, the higher equilibrium is more satisfactory. Another reason to select the higher equilibrium in the $q = 1$ case comes from the observation that, since there is no inflation in the steady state, $\bar{R} = 1$ implies $\bar{r} = \bar{i} = 0$, i.e. an infinite money demand. In addition, it is clear from equation (24) that $\bar{R} = 1$ also implies an infinite level of steady-state human wealth. It is also possible to argue that the higher steady state is the one consistent with the individual dynamics of wealth and consumption. The first thing to notice is that in a steady-state, although per-capita variables are constant, there is still some dynamics at the individual level. Since each agent is born with zero non-human wealth, in order to have a steady-state with positive per-capita financial (non-human wealth) wealth, individuals must be accumulating financial wealth, as long as they stay alive. Since in the steady-state individual human wealth, equal to per-capita human wealth, is constant, this implies that in the steady state individual total wealth is growing. Remembering that, with logarithmic preferences, individual consumption is proportional to total wealth (eq. 9), it is clear that, in the steady-state, individual consumption must be growing. From equation (7) we can see that, in order to have growing individual consumption, it must be $\overline{R} > \frac{1}{\beta}$. If we start from the Ricardian equivalence case ($q = 1$); and then we marginally reduce q, only the higher equilibrium satisfies the condition for growing consumption.

As we are going to see in what follows, the analysis of the dynamics of the model leads to the same conclusion. In particular, it will allow us to prove that, even for large deviations from Ricardian equivalence, reducing q rises (lowers) the higher (lower) equilibrium. This conclusion will give more generality to the arguments developed above.

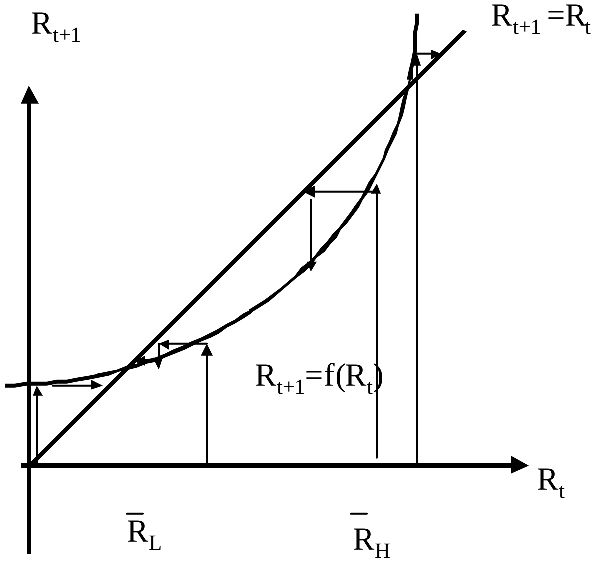
3.2 Dynamics

In this simplified version, the dynamics of the model can be summarized by a first-order non-linear difference equation for the gross real interest rate, given by (see Appendix for the derivation):

$$R_{t+1} = \frac{1}{\frac{1}{q}\left[1 - (1+\psi)\frac{(1-q\beta)}{(1+\chi+\psi)}(1-q)\right] + q\beta - \beta R_t} \tag{26}$$

Equation (26) reduces to the quadratic expression that characterizes the steady state if we impose $R_t = R_{t+1} = \overline{R}$. The dynamics out of the steady state can be investigated with the help of Figure 2, where we plot equation (26) together with the $R_t = R_{t+1}$ line. Equation (26) is a hyperbola, that cuts the $R_t = 0$ axis at $\dfrac{1}{\frac{1}{q}\left[1 - (1+\psi)\frac{(1-q\beta)}{(1+\chi+\psi)}(1-q)\right]}$. We can restrict our attention to the positive arm, that tends to the vertical asymptote $R_t = \dfrac{\frac{1}{q}\left[1 - (1+\psi)\frac{(1-q\beta)}{(1+\chi+\psi)}(1-q)\right]}{\beta}$ as $R_{t+1} \to \infty$.

Figure 2: Dynamics in the no-Government case



From equation (26) it is evident that, starting form every point on the left or on the right of $\overline{R}_L$; the economy will converge back to $\overline{R}_L$; while the opposite happens around $\overline{R}_H$: Therefore, $\overline{R}_H$ is the unstable equilibrium. As this is a forward-looking, rational-expectations model, this property should not be regarded as problematic. On the contrary, it is a very desirable feature. The logic of the rational expectations method is that, if there is a unique possibility that ensures boundedness, this is the one that will be selected. In other words, the real interest rate acts as a jump variable, making the model "well behaved" in terms of dynamics. This con¯rms that, in what follows, we can restrict our attention on $\overline{R}_H$:

From Figure 2 is also possible to derive an analysis of the e®ects of the probability of death on the real interest rate that is not limited to the case in which the value of q lies in the vicinity of 1. It is possible, infact, to show that the derivative of $\frac{1}{q}[1 ¡ (1+Ã)\frac{(1 ¡ q¯)}{(1+Â+Ã)}(1 ¡ q)] + q¯$ with respect to q is equal to $\frac{(q^2¯ ¡ 1)}{q^2}\frac{Â}{(1+Â+Ã)}$, that is unambiguously negative. This implies that a decrease in q; by raising the denominator of (26), will shift the hyperbola downward, thus raising $\overline{R}_H$ and lowering $\overline{R}_L$: In the unstable steady-state, an increase in the temporal horizon of agents decreases the real interest rate. This confirms that the unstable steady-state is the one that yields the more sensible result in terms of comparative static and of consistency with the dynamics of individual variables, even if deviations from Ricardian equivalence are large (if q is considerably smaller than 1).

As we are going to see in what follows, when we reintroduce the public sector in the model we can use similar arguments to discriminate between equilibria. The conclusions drawn from this simple version of the model are consistent with the more general case.

3.3 The Role of Money

Before reintroducing the government in the analysis, it is interesting to make a digression about the role of money in this simple version of the model. It is straightforward to show that, in a non-monetary version of the model ($\hat{A} = 0$), the two solutions are $\overline{R}_L = 1$ and $\overline{R}_H = \frac{1}{}$ even when $q < 1$: In other words, it is the presence of money that allows the real interest rate to deviate from the Ramsey solution. In a non monetary economy, agents would choose a flat pattern of consumption even when Ricardian equivalence does not hold. The intuition behind this is that, with no money and no government debt, the supply of assets is zero. In order for the assets market to clear, therefore, even the demand for assets must be zero. This is exactly what happens at $\overline{R}_H = \frac{1}{}$: For levels of the interest rate lower than this (including $\overline{R}_L = 1$), consumption must be decreasing (eq. 7). Since human wealth is constant in the steady-state, this can only happen if agents are accumulating negative financial wealth. The demand for assets, therefore, should be negative, whereas we know that the supply is zero. It follows that solutions different from $\overline{R}_H = \frac{1}{}$ are not acceptable, because they imply that the assets market fails to clear.

In other words, with no money and no bonds in the economy, there is nothing that can be used as a "reserve of value", allowing agents to diversify their consumption intertemporally. Even with finite lives, therefore, the pattern of individual consumption can not deviate from that of disposable income. Introducing money allows us to depart from a situation in which individual consumption is flat.

4 Effects of Fiscal Policy

We will now reintroduce the government in the model. In the policy rules studied in this section government debt is either not allowed or exogenous. The analysis of a case in which debt is endogenous is carried out in next section.

In the first policy experiment we look at a balanced budget expansion ($\overline{G} = \overline{\tau}$; $\overline{D} = 0$). In the second one we consider the steady-state effects of increasing debt from a constant level to another constant level. In the latter case, government expenditure is kept constant and taxes are assumed to adjust endogenously. As the focus is on fiscal policy, in both cases we hold the money supply permanently fixed at a constant level $\overline{M}$; ruling out seigniorage. For the reasons explained above we restrict our attention to the unstable equilibrium $\overline{R}_H$:

4.1 Balanced-Budget Expansions

We now turn to the case in which the government is allowed to spend but not to use debt. Assuming a constant level of expenditure perfectly matched by lump-sum taxes in every period, the steady state of human wealth becomes: $\overline{H}^{PC} = \sum_{s=t}^{\infty}(\frac{q}{1+\overline{r}})^{s-t}(1-\overline{\tau}) = (1-\overline{\tau})\frac{1+\overline{r}}{1+\overline{r}-q}$. We can therefore derive a modified version of equation (25):

$$\overline{R}^2 = \{q - \frac{1}{q\beta}[1 - \frac{(1-\overline{\tau})}{(1-\overline{G})}(\frac{1-q\beta}{1+\chi+\psi})(1-q)(1+\psi)]\}\overline{R} - \frac{1}{\beta} \tag{27}$$

As $\overline{G} = \overline{\tau}$; the above equation reduces to (25). This implies a quite unexpected result. Although our model is based on some "non neo-classical" assumptions, like the deviation from Ricardian Equivalence, balanced-budget fiscal expansions turn out not to affect the real interest rate. In Blanchard (1985), the real interest rate is equated to the marginal productivity of capital. As a balanced-budget expansion decreases capital in his model, the real interest rate increases. The assumption that labour is the only factor of production implies that, even with finite horizons, following a balanced-budget expansion our model behaves like the Ramsey one, in which the real interest rate is independent of movement in G, rather than like Blanchard's.

As in the steady state inflation is zero, nominal and real interest rates coincide. This implies no direct effect via the interest rate on real balances. Remembering equations (20) and (21), it is clear that, following a once and for all fiscal expansion, there will be a step increase in the quantity of labour supplied and a step decrease in consumption. Both consumption[6] and leisure fall. Money demand, being a function of consumption, falls as well. The overall welfare effect of a balanced-budget fiscal expansion is therefore negative.

The welfare results of our model are qualitatively the same that can be derived, for the long run[7], in a closed-economy version of the Redux model presented by Obstfeld and Rogoff (hereafter OR, 1995, p.703) The latter is nested in the model presented Ganelli (2000), when government expenditure does not provide utility. In the present model and in the OR model the output multiplier is positive and the consumption multiplier is negative, and both are less than one in absolute value. If we assume $\psi < 1$; the negative welfare effect is mitigated in our case, compared to the OR model. When $\psi = 1$ our model and the closed economy version of the OR model coincide.

A paper that looks at the consequences of a balanced-budget expansion in a sticky-price, continuous-time, perpetual-youth model with capital accumulation is Rankin and Scalera (1995). Their results are quite different from ours. In their model the long-run consumption multiplier is positive and the output multiplier is above unity. The authors explain this as a consequence of the fact that they have investment and capital accumulation. With no capital accumulation, their model would give the usual Keynesian balanced-budget multiplier ($dY/dG = 1$; $dC/dG = 0$). As we have already stressed in section 2.4, this result only emerges here in the not very realistic case of an infinite weight of leisure in agents' preferences. In other words, the presence of an endogenous labour supply in our model is sufficient to deviate from the neo-classical result of a zero output multiplier, even in a flexible price world, but is not enough to generate the polar result of Rankin and Scalera (1995).

4.2 The Case of Constant Debt

We now turn our attention to another policy, in which the debt is fixed exogenously. We will therefore look at the steady-state effect of an increase

[6] The reduction in steady-state consumption in our model is consistent with both the Ramsey and Blanchard's models.

[7] As the present model is a flexible-prices one, it would not be appropriate to compare our results with the short-run ones in Obstfeld and Rogoff, where prices are sticky.

from one constant level of $\overline{D}$ to a new constant level. From the government budget constraint with constant $\overline{G}$, $\overline{D}$ and $\overline{M}$; we have:

$$\tau_t = \overline{G} + r_t\overline{D}$$

When the government decides to raise the level of steady-state debt, $\overline{G}$ is kept constant, and taxes adjust endogenously to meet the increased interest payments.

The steady-state value of human wealth is now $\overline{H}^{PC} = \sum_{s=t}^{\infty}(\frac{q}{1+\overline{r}})^{s-t}(1 - \overline{G} - \overline{r}\overline{D}) = \frac{1+\overline{r}}{1+\overline{r}-q}(1 - \overline{G} - \overline{r}\overline{D})$; and the steady-state equation for R can be expressed as:

$$\overline{R}^2 = \frac{[\frac{(1+\psi)(1-q)(1-q\beta)}{(1+\chi+\psi)(1-G)}\overline{D} + \frac{(1+\psi)(1-q)(1-q\beta)}{(1+\chi+\psi)} - q^2\beta - 1]}{\frac{(1+\psi)(1-q)(1-q\beta)}{(1+\chi+\psi)(1-\overline{G})}\overline{D} - q\beta}R + \tag{28}$$
$$+\frac{q}{\frac{(1+\psi)(1-q)(1-q\beta)}{(1+\chi+\psi)(1-\overline{G})}\overline{D} - q\beta}$$

In this case, it is no longer possible to investigate our policy experiment without using some numerical examples. Before introducing simulations, however, we present what we can conclude using only analytical methods. The steady-state solutions for the real interest rate are still given by the points where the parabola $f(\overline{R}) = \overline{R}^2$ meets a straight line, that we now denote $h(\overline{R})$. Equation (28), however, shows that the signs of the slope and of the intercept of the line are no longer unambiguous.

As $\overline{G}$ denotes per-capita government expenditure, a sensible assumption is that the maximum amount of work available in each period, 1, cannot be all used to produce public goods. The quantity $(1 - \overline{G})$ is therefore positive. This implies that we can derive two threshold values such that, if $\overline{D}$ is bigger than these, the magnitudes

$$\frac{(1+\psi)(1-q)(1-q\beta)}{(1+\chi+\psi)(1-\overline{G})}\overline{D} - q\beta$$

and

$$\frac{(1+\psi)(1-q)(1-q\beta)}{(1+\chi+\psi)(1-\overline{G})}\overline{D} + \frac{(1+\psi)(1-q)(1-q\beta)}{(1+\chi+\psi)} - q^2\beta - 1$$

are positive. The values are:

$$\overline{D} > \frac{(1 + \text{Â} + \text{Ã})(1 \text{ ¡ } \overline{G})}{(1 + \text{Ã})(1 \text{ ¡ } q)(1 \text{ ¡ } q^{-})} q^{-} = S_{\overline{D}}$$

and

$$\overline{D} > [(1 + q^{2-})\frac{(1 + \text{Â} + \text{Ã})}{(1 + \text{Ã})(1 \text{ ¡ } q)(1 \text{ ¡ } q^{-})} \text{ ¡ } 1](1 \text{ ¡ } \overline{G}) = S_{\overline{N}}$$

Whether $\overline{D}$ is bigger or not than $S_{\overline{D}}$ determines the sign of the intercept, while the sign of the slope, being determined by a ratio, depends on both conditions. Since we can prove (see below) that $S_{\overline{N}} > S_{\overline{D}}$ always, there are only three possible regimes:

(i). $\overline{D} < S_{\overline{D}} < S_{\overline{N}}$: In this case the intercept is negative, but the slope is still positive (because both numerator and denominator in the ratio are negative). This case is qualitatively similar to the one with no government or balanced budgets (See Fig. 3).

(ii). $S_{\overline{D}} < \overline{D} < S_{\overline{N}}$: In this case the intercept is positive, and the slope is negative (as we have a negative numerator and a positive denominator in the ratio). See Fig. 4.

(iii). $\overline{D} > S_{\overline{N}} > S_{\overline{D}}$: In this case the intercept is positive, and the slope is positive (the numerator and the denominator of the ratio that gives the sign of the slope are both positive). See Fig 5. Notice that in both cases (ii) and (iii) only one solution with a positive $\overline{R}$ is possible

Before proceeding to the comparative static analysis of the di®erent cases, notice that $S_{\overline{D}} > S_{\overline{N}}$ can be rewritten, after algebraic passages, as:

$$1 > \frac{(1 + \text{Â} + \text{Ã})}{(1 + \text{Ã})}[1 + \frac{q}{(1 \text{ ¡ } q)(1 \text{ ¡ } q^{-})}]$$

the above inequality implies that 1 should be bigger than a number bigger than 1, and is therefore a contradiction. This means that we must have $S_{\overline{N}} > S_{\overline{D}}$ always, and the case in which $S_{\overline{N}} < \overline{D} < S_{\overline{D}}$ is therefore ruled out.

Figure 3: Steady state with constant debt, case (i)

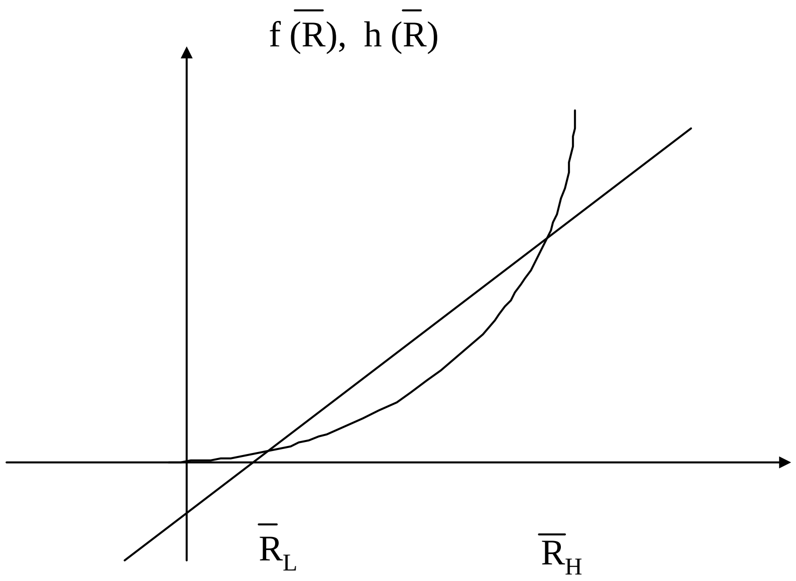


Figure 4: Steady state with constant debt, case (ii)

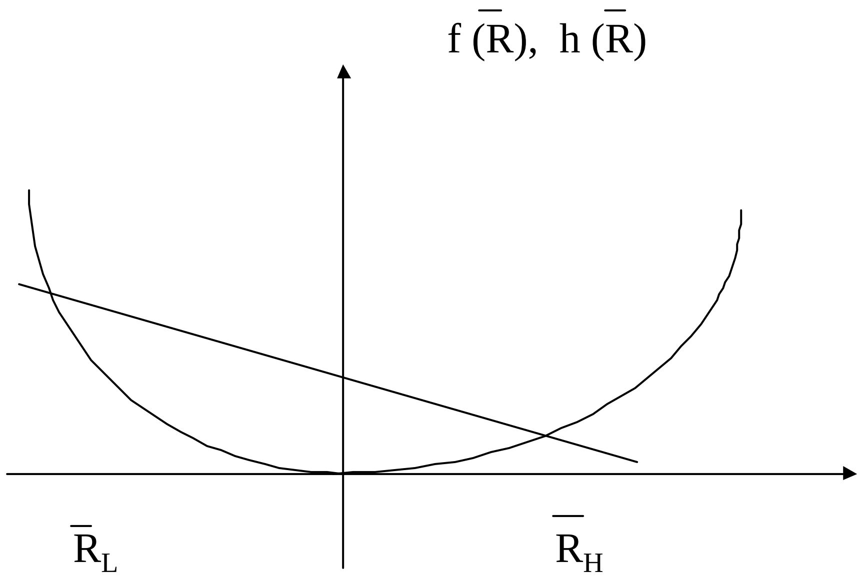
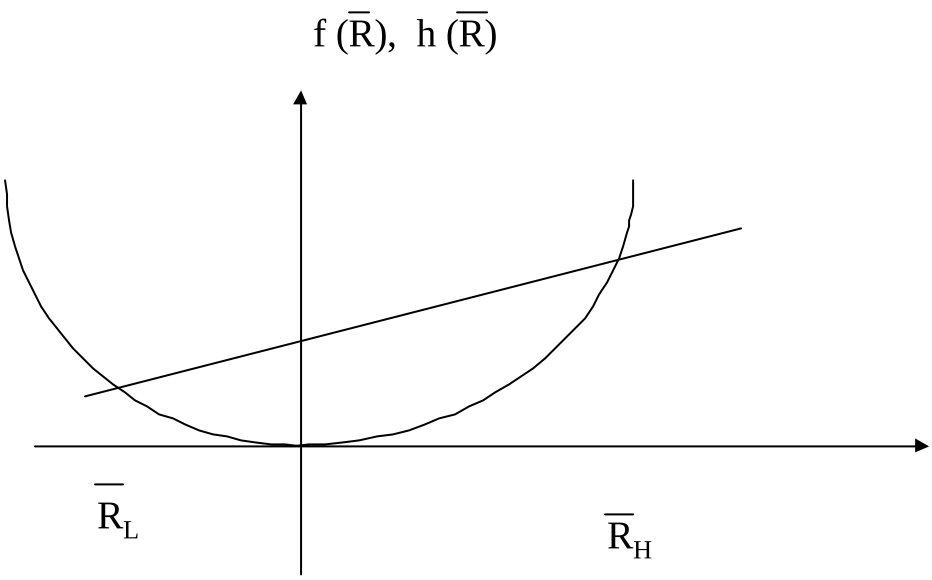


Figure 5: Steady state with constant debt, case (iii)



We now look at the effects of a steady-state increase in $\overline{D}$ in the three different regimes (i), (ii) and (iii). In doing this, we assume that the increase is $\overline{D}$ is small enough so that the effect is a change within the regime, and not a change between regimes.

(i). An increase in $\overline{D}$ shifts the intercept down. To see the effect on the slope, notice that

$$\frac{\frac{(1+\psi)(1-\theta)(1-\theta\beta)}{(1+\rho+\psi)(1-\overline{G})}\overline{D} + \frac{(1+\psi)(1-\theta)(1-\theta\beta)}{(1+\rho+\psi)} - \theta^{2\beta} - 1}{\frac{(1+\psi)(1-\theta)(1-\theta\beta)}{(1+\rho+\psi)(1-\overline{G})}\overline{D} - \theta\beta}$$

can be rewritten as

$$\frac{\frac{(1+\psi)(1-\theta)(1-\theta\beta)}{(1+\rho+\psi)(1-\overline{G})}\overline{D} + \frac{(1+\psi)(1-\theta)(1-\theta\beta)}{(1+\rho+\psi)} - \theta^{2\beta} - 1 - \theta\beta + \theta\beta}{\frac{(1+\psi)(1-\theta)(1-\theta\beta)}{(1+\rho+\psi)(1-\overline{G})}\overline{D} - \theta\beta} =$$

$$= 1 + \frac{\frac{(1+\psi)(1-\theta)(1-\theta\beta)}{(1+\rho+\psi)} + \theta\beta - \theta^{2\beta} - 1}{\frac{(1+\psi)(1-\theta)(1-\theta\beta)}{(1+\rho+\psi)(1-\overline{G})}\overline{D} - \theta\beta}$$

since the coefficient on $\overline{D}$ in the denominator of the previous expression is positive, an increase in $\overline{D}$ will increase the slope if the numerator is negative (because 1 plus something negative will become 1 plus something still negative but smaller than before in absolute value). Rearranging

$$\frac{(1+\psi)(1-q)(1-q\beta)}{(1+\chi+\psi)} + q\beta - q^2\beta - 1$$

we can see that this is the case, since this expression can be rewritten as

$$\frac{-q(1+\psi) + \chi[(q\beta(1-q) - 1]}{(1+\chi+\psi)} < 0$$

Summarizing, an increase in $\overline{D}$ has two contrasting effects on the higher steady-state $\overline{R}_H$: the fall in the intercept pushes it down, but the increase in the slope pushes it up. The final effect will be determined by which of the two prevails.

(ii). An increase in $\overline{D}$ will still imply a fall in the intercept. The slope, on the other hand, will increase, in the sense that it will still be negative, but smaller in absolute value, therefore less negatively sloped. The first effect goes in the direction of reducing R_H and the second in the direction of increasing it, with an ambiguous final result.

(iii) This case is qualitatively similar to the one presented in (i), except for the fact that now only in the higher steady state R takes a positive value. An increase in $\overline{D}$ implies a fall in the intercept, but the slope increases. Again, there are two contrasting effects on the higher interest rate: the fall in the intercept pushes it down, but the increase in the slope pushes it up. The final effect, therefore, depends on which of the two dominates.

Although the above discussion does not allow us to state unambiguously the sign of the effect of debt on the interest rate, an analytical result can be estabilished for a particuler sub-case of the situation considered in (i), that we will call "reference case". We present here this reference case before moving to numerical simulations. In our reference case steady-state debt is initially set to zero ($\overline{D} = 0$) and the utility provided by money is also zero ($\chi = 0$): A case in which initial steady-state debt is fixed to zero is quite a natural one to consider, since it is continuous with the no-debt cases previously considered. As we already know, in the reference case the two steady-state solutions for R are $\overline{R}_L = 1$ and $\overline{R}_H = \frac{1}{\beta}$. Let's now notice that equation (28) can be rewritten as:

$$(q\beta - k\overline{D})\overline{R}^2 - [1 + q^2\beta - (1 - \overline{G})k - k\overline{D}]\overline{R} + q = 0$$

where $k = \frac{(1+\psi)(1-q)(1-q\beta)}{(1+\chi+\psi)(1-\overline{G})}$. Totally differentiating and evaluating in the reference case we have:

$$[q\beta 2\overline{R} - (q + q\beta)]d\overline{R} + (-k\overline{R}^2 + k\overline{R})d\overline{D}$$

Further, evaluating at the higher steady-state $\overline{R}_H = \frac{1}{\beta}$ and rearranging we have:

$$\frac{d\overline{R}}{d\overline{D}} = \frac{k}{\beta q^2}$$

that proves the positive effect of an increase in debt on the real interest rate in the reference case.

Since from the graphical analysis previously developed we concluded that, out of the reference case, the sign of this effect is ambiguous, we have also carried out some simulations, in which we assign specific values to the parameters, and then we evaluate how the solutions to equation (28) change as the level of debt is increased. The benchmark values for the parameters are: $\beta = .9$; $q = .9$; $\psi = .1$; $\chi = .05$; $G = .3$. This reproduces a situation in which agents are not very myopic and the deviation from Ricardian equivalence is not very high. Also, the utility provided by leisure and real balances is assumed to be small compared to the one provided by consumption. Finally, we assume that one third of the maximum amount of work available in every period is used to produce public goods. The sensitivity of the results with respect to changes in the parameters has also been tested, repeating the same exercise for cases in which the deviation from Ricardian equivalence is larger ($q = .7$); or the exogenous level of government expenditure is higher ($G = .66$). Our simulations show that an increase in $\overline{D}$ is likely to increase $\overline{R}_H$. Table 1 shows the effect on $\overline{R}_H$ of an increase from $\overline{D} = 2$ to $\overline{D} = 3$, in the various cases considered. The reported values of the steady-state interest rate are, of course, highly unrealistic. It should be noticed, however, that, since our model is a quite simplified ones, it is not our intention to produce realistic quantitative estimates of the levels of the steady state variables. Our main interest here is in the qualitative result that R_H increases as the level of exogenous debt increases[8].

Table 1. Effects of an increase in $\overline{D}$ on $\overline{R}_H$

	Benchmark	$q = .7$	$G = .66$
$\overline{D} = 2$	$R_H = 1.194$	$R_H = 2.170$	$R_H = 1.285$
$\overline{D} = 3$	$R_H = 1.235$	$R_H = 4.037$	$R_H = 1.390$

If, like the previous analysis suggests, an increase in the exogenous level of debt increases the steady-state level of the interest rate, then this policy has a two negative welfare effects. The first one arises through real balances. Since in the steady state nominal and real interest rate coincide, an increase in $\overline{R}_H$ implies that agents demand less money, and this reduces their utility.The second effect is due to the fact that an increase in the real interest rate implies an higher growth rate of consumption over an individual's lifetime (remember equation 7). Since average lifetime consumption does not change (the level of per-capita consumption does not change, see equation 21) , this effect increases the imbalance in the lifetime consumption profile. We could expect, on an intuitive basis, that this would further reduce lifetime utility.

In Blanchard (1985, p.243), a similar policy reduces the steady-state levels of both capital and consumption. In his model, therefore, the real interest rate increases unambiguously. This is what is likely to happen in our model. In our model, however, we have no effect on per-capita consumption. This difference is probably due to the fact that, keeping the level of capital constant, we have prevented movements in the real interest rate from having direct effects on consumption. In Blanchard, the decrease in capital associated with an increase in R has a direct, negative effect on consumption. This does not happen in our model, because capital does not change.

5 Introduction of a Wealth Tax

In the policy experiments considered so far, government debt was either zero or fixed at a constant level exogenously determined. We now turn to the case in which taxes are a function of the existing level of debt. In this case both debt and taxes are endogenously determined.

[8] This result is confirmed for a wide range of variation of $\overline{D}$.

Since government debt enters as an asset in the agents' portfolio, we refer to this as a "wealth tax". It is important to stress, however, that this kind of instrument, being a function of per-capita debt, should not be regarded as a wealth tax in the strictest sense, i.e. one that distorts agents' decision. From an individual's point of view, the tax is a "lump-sum" one. The size of the lump-sum depends on aggregate wealth, but a single individual has no in°uence on the latter. Examples of taxation imposed on aggregate wealth, without distortionary consequences, can be found both in the theoretical literature (for example Rankin and Scalera, 1995) and in large-scale macroeconometric models used for policy simulations (Mitchell et al. 1999).

Formally, the rule that we are considering, is expressed as:

$$¿_t = T + ¿ D_t$$

for every t, with $0 < ¿ < 1$ and $G > T$: On an intuitive basis, we would expect that if the real interest rate is smaller than the rate at which new debt feeds in to new taxes ($¿ > r_t$), this rule should grant stability, preventing the debt from exploding (see below, equation 29).

The most interesting result in this section is that, contrary to the intuition, we can not rule out cases in which this rule fails, for realistic parameters values, to ensure the existence of a well-de¯ned equilibrium with a unique convergent path. If R_t;as it appears in equation (29) below, was independent of debt (as it is the case in the Ramsey model), then stability would depend only on the sign of r_t ¡ $¿_t$: If we get instability, then, it must be because of the dynamics in R_t, which are introduced by the fact that, when $q < 1$, R_t depends on D_t: In other words, an overlapping generations economy is more likely to be unstable, under a given wealth tax rule, than a Ramsey economy. Failing to consider the implications of ¯nite horizons means that rules similar to the one that we are analysing are usually believed, especially in policy related analysis, to be e®ective in "closing" the model. Mitchell et al. (1999, pag. 171), for instance, in comparing the properties of di®erent macroeconometric models, refer to "The speci¯cation of a ¯scal policy reaction function or ¯scal closure rule that enforces the government's intertemporal budget constraint...". In their analysis, based on an in¯nite horizons theoretical framework, a tax rule that makes taxation a function of the existing stock of debt ensures convergence of debt (in the case of no real growth) if ¿ (µ in their notation) is bigger than the (exogenous) real interest rate (Mitchell et al. 1999, pag. 179).

The implications of such a rule in our model are investigated in what follows using a combination of graphical analysis and numerical simulations. The dynamics of the economy can be summarized by two non-linear difference equations in R_t and D_t:

$$D_{t+1} = (R_t - \tau)D_t + \overline{G} - T \tag{29}$$

$$R_{t+1} = \frac{1}{-(1 - \tau D_t - T)\dfrac{(1+\psi)}{(1-\overline{G})}\dfrac{(1-q\beta)}{(1+\phi+\psi)}\dfrac{(1-q)}{q} + q\beta + \dfrac{1}{q} - \beta R_t} \tag{30}$$

Equation (29) comes from substituting the tax-rule in to the period by period government budget constraint with constant $\overline{G}$, and $\overline{M}$, while (30) can be derived following the same method used for (26) and taking in to account that now $H_t^{PC} = \sum_{s=t}^{\infty} \alpha_{s;t} q^{s-t}(1-\tau_s) = \sum_{s=t}^{\infty} \alpha_{s;t} q^{s-t}(1-T-\tau D_s)$:

The locus $\Delta D_t = 0$ is a hyperbola with intercept $D_t = \dfrac{\overline{G}-T}{\tau+1}$; and $R_t = \tau + 1$ and $D_t = 0$ respectively as vertical and horizontal asymptotes. Debt converges back to the locus on the left of the vertical asymptote (where $\tau > r$), and diverges away from it on the right (where $r > \tau$). The locus $\Delta R_t = 0$ is the sum of a straight line with positive slope and of a hyperbola (see Appendix). It tends to the straight line as $R_t \to \infty$ and to the hyperbola as $R_t \to 0$. R_t decreases above the locus and increases below.

In principle, the existence of steady-state solutions could be studied analytically, imposing constant levels of D and R in equations (29) and (30) and solving the system. Doing this without assigning specific parameters to the values would not allow us to derive neat expression for the solutions. Before resorting to simulations, however, it is useful to stress that, combining equations (29) and (30), it is possible to show that the steady-state values are the solutions of a cubic equation, that therefore has either one or three real roots. All the possible cases are presented graphically in Figures 6 to 8[9]. The three possible steady states are labelled, starting with the lower, as $\overline{R}_1$; $\overline{R}_2$ and $\overline{R}_3$: If there is only one solution, this can only be in the region where $r < \tau$ (Fig. 6). The case of three solutions can happen in two different ways: all the solutions where $r < \tau$ (Fig. 7), or one where $r < \tau$ and two where $r > \tau$ (Fig. 8).

[9] The practice of drawing phase dyagrams for discrete systems is a standard one in

Figure 6: Wealth-tax rule, one solution

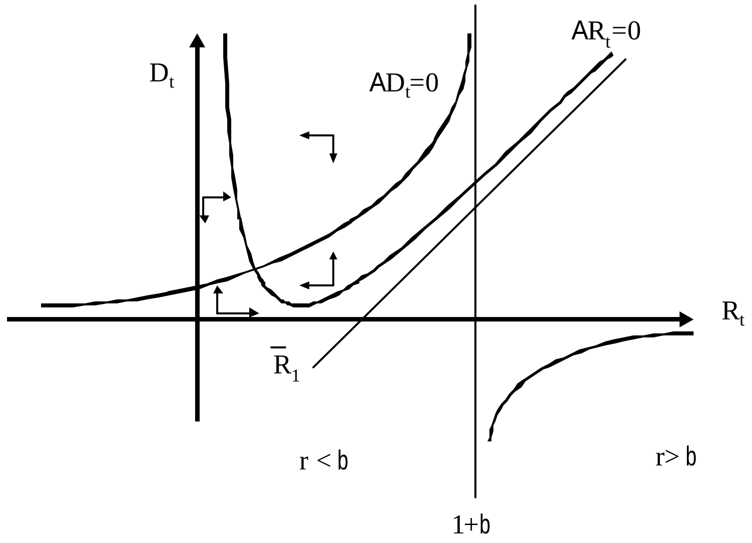


Figure 7: Wealth-tax rule: three solutions, case one

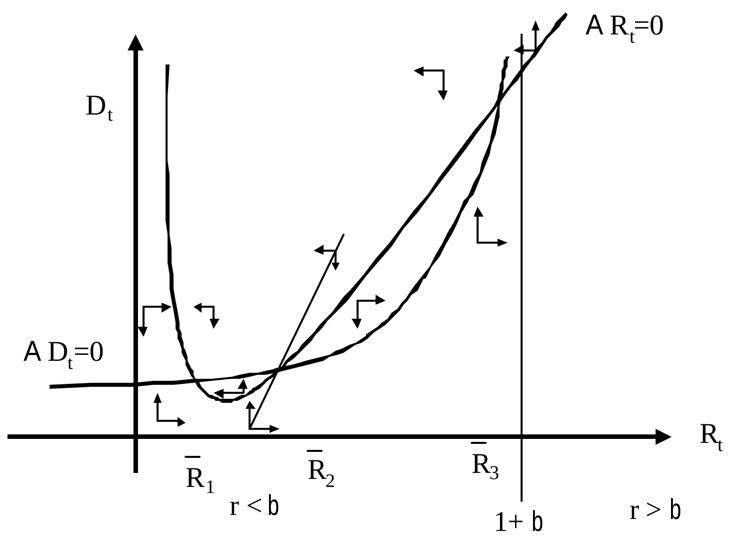


Figure 8: Wealth tax rule, three solutions: case two

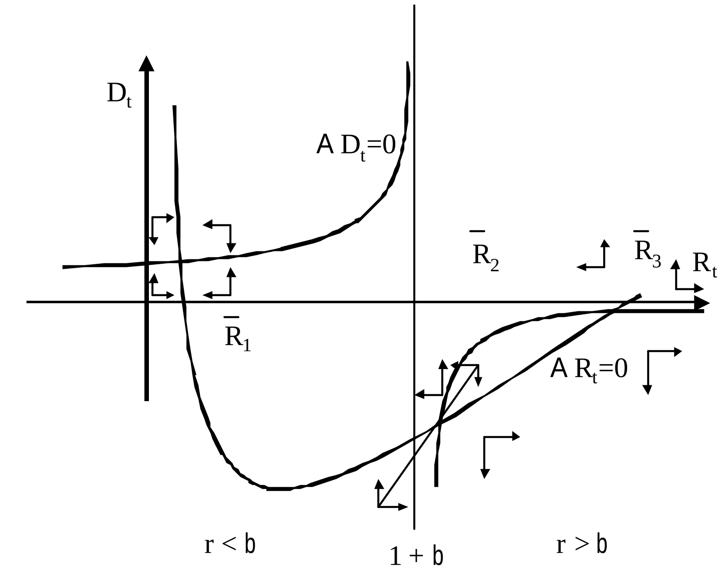
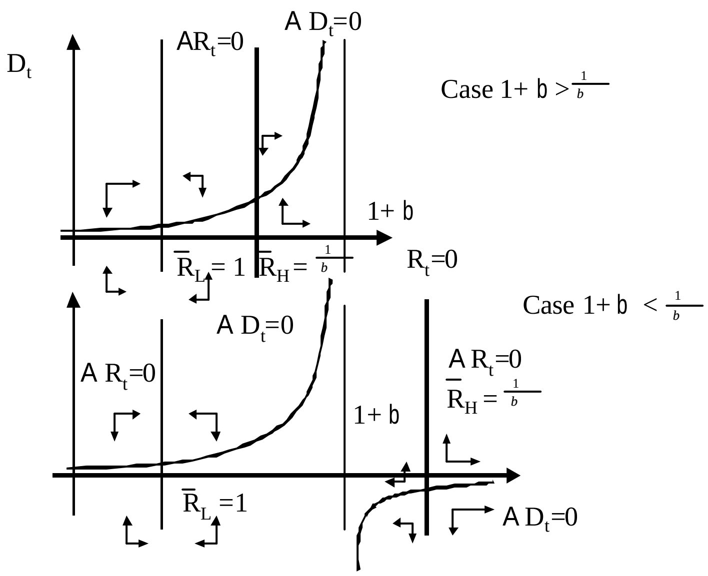


Figure 9: Wealth tax rule in the q=1 case



It is evident that steady-states in the region where $r > \tau$ can only occur for negative values of debt. The intuition behind this is that when the real interest rate is bigger than the taxation coefficient, existing debt generates new debt at a faster rate than it increases taxation. As a result, it is only possible to have a stable level of debt if this is negative, i.e. if agents are borrowing from the government. In this case an increase in r is good news for the government's finance. We are induced to pay little attention to the possibility of such an outcome, however, on the basis of the observation that equilibria with negative values of government debt are not very likely in reality. On the normative side, we are interested in sustainability, so we do not want to suggest a rule that would eliminate government debt completely[10].

What about the region in which $r < \tau$? From Figures 6, 7 and 8 it is clear that the steady-state $\bar{R}_1$ in this region is globally stable. However, we are quite doubtful about the practical relevance of this equilibrium, for several reasons. The first one is that it implies a multiplicity problem, i.e. there is no unique path along which the system converges. It is also of some utility to draw some analogy with the particular case in which Ricardian equivalence holds ($q = 1$): This yields a discrete-time version of the Ramsey model with a wealth tax. In this situation, debt financing becomes irrelevant for the real interest rate, and the solutions for $\bar{R}$ are only two, and equivalent to the ones that we obtain when there is no government in the model: $\bar{R}_L = 1$ and $\bar{R}_H = \frac{1}{\beta}$. The $\Delta R = 0$ locus collapses to two vertical lines in correspondence of the two solutions for R. Figure 9 describes the dynamics in this case. It is clear that the $\bar{R}_L$ is always a sink, whereas $\bar{R}_H$ implies a positive (negative) debt and is a saddle (source) if $1 + \tau > \frac{1}{\beta}$ ($1 + \tau < \frac{1}{\beta}$). The equilibrium $\bar{R}_H$ with $1 + \tau > \frac{1}{\beta}$ is therefore the one that yields the case that we consider more satisfactory in terms of the stability properties, i.e. the saddle path.

In this case the steady-state solution for debt is given by:

$$\bar{D} = \frac{\bar{G} - T}{1 + \tau - \frac{1}{\beta}}$$

modern macroeconomics (see, for example, Blanchard and Fischer, 1989, pp. 230-31).

[10] A positive level of debt can be a desirable property of an economy, as long as it is sustainable. In our model, for instance, debt facilitates consumption smoothing over time.

that implies sensible comparative static properties ($\frac{d\overline{D}}{d\overline{G}} > 0$; $\frac{d\overline{D}}{d\tau} < 0$, $\frac{d\overline{D}}{d\beta} < 0$).

As already discussed in Section 3, when $q = 1$ the solution $\overline{R}_L$ can be ruled out because it would imply infinite real balances. This is consistent with our previous discussion in the simpler version of the model, in which we have ruled out the lower real interest rate equilibrium because it did not display the right dynamics and comparative static characteristics. Even though we must be careful in mechanically transferring these conclusion to the case of no Ricardian Equivalence ($q < 1$), we believe that they corroborate, to some extent, our decision to deem the $\overline{R}_1$ steady state, that is the one corresponding to $\overline{R}_L$ in the non-Ricardian case, as unacceptable. In addition, comparing Figure 9 with Figure 6, 7 and 8 we can develop a further argument, perhaps the strongest one, to rule out $\overline{R}_1$. In Figure 9, where $q = 1$, the lower equilibrium yields a value of 1 for the gross real interest rate. Figures 6, 7 and 8, however, suggest that when we marginally reduce q from 1 to a value less than 1, the $\Delta R_t = 0$ locus goes from a straight to an hyperbolic shape. This implies that the $\overline{R}_L$ steady-state goes from 1 to a value smaller than 1, becoming what we have denoted as $\overline{R}_1$ in figures 6, 7, and 8. A gross real interest rate smaller than 1, however, means $\overline{r} < 0$. Since in the steady-state, with constant prices, real and nominal interest rate coincide, the lower steady state implies not only a negative real interest rate, but also a negative nominal interest rate, that is obviously economically meaningless. The numerical simulations that we provide below (see Table 4) support this reasoning, yielding always a value less than 1 for $\overline{R}_1$. We are quite confident, therefore, that we can rule out this equilibrium.

From the above analysis it follows that the only case in which the economy converges along a uniquely well defined path to a steady-state with positive debt is when we have three steady states in the region where $r < \tau$ (Fig 8).

In this case the second steady-state $\overline{R}_2$ is a saddle-path, while the third one $\overline{R}_3$ is a source. It is useful, at this point, to see which cases are likely to emerge for given parameter values, and how the government's choices can affect the outcome. In our simulation exercises on this case, we start considering the same benchmark values as in Section 4 ($\beta = .9$; $q = .9$; $\psi = .1$; $\chi = .05$; $G = .3$), complemented by $T = .2$ and $\tau = 0.25$. This yields a solution in which the three steady state are in the region $r < \tau$, and a well defined, convergent steady-state exists. The result of the simulations are summarized in Table 4[11]. Figure ## was generated in Maple setting the parameters at the benchmark levels, and it confirms the shapes of the loci already illustrated in the theoretical analysis of Figure ##.

Table 4. Steady-state values of R and D for different numerical examples (In column 2,3,4 only the parameters reported have been altered with respect to the benchmark case)

Benchmark	$q = .85$	$\tau = .2$	$\tau = .01$
$\bar{R}_1 = .992$	$\bar{R}_1 = .987$	$\bar{R}_1 = .992$	$\bar{R}_1 = .997$
$\bar{R}_2 = 1.16$	$\bar{R}_2$ complex root	$\bar{R}_2$ complex root	$\bar{R}_2 = 1.053$
$\bar{R}_3 = 1.207$	$\bar{R}_3$ complex root	$\bar{R}_3$ complex root	$\bar{R}_3 = 1.068$
$\bar{D}_1 = .338$	$\bar{D}_1 = .380$	$\bar{D}_1 = .482$	$\bar{D}_1 = 7.815$
$\bar{D}_2 = 1.107$	$\bar{D}_2$ complex root	$\bar{D}_2$ complex root	$\bar{D}_2 = -2.309$
$\bar{D}_3 = 2.324$	$\bar{D}_3$ complex root	$\bar{D}_3$ complex root	$\bar{D}_3 = -1.713$

How do changes in the parameters of the model affect the solutions? Keeping everything else constant, a reduction in q from .9 to .85 gives complex roots for the second and third steady state. Therefore, we are left only with the $\bar{R}_1$ solution[12] (See Table 4, column 2). A moderate reduction in τ (for example, from .25 to .2) has the same effect (Table 4, column 3), while if the reduction is drastic (for example, from .25 to .01), we have the case of one steady-state with positive debt and two with negative debt (Table 4, column 4). Notice that in this numerical example we have a saddle-path corresponding to positive debt when $1 + \tau = 1.25 > \frac{1}{\beta} = 1.11$; and a saddle path with negative debt when $\frac{1}{\beta} = 1.11 > 1 + \tau = 1.01$. The eigenvalues for the relevant (positive debt) steady-state have been calculated for this numerical example and are reported in Table 5, confirming that it is a saddle-path.

Figure ##:Simulations, the benchmark case

[11] In the benchmark case, the selected steady state implies a value of $r \approx 16\%$. Although somewhat more acceptable than the ones obtained in Section 5, the latter is still a quite unrealistic equilibrium value. Again, it is worth stressing that is not our purpose to produce realistic estimates of the magnitudes of the variables.

[12] A similar result emerges if we reduce the discount rate to $\beta = .85$ keeping $q = .9$.

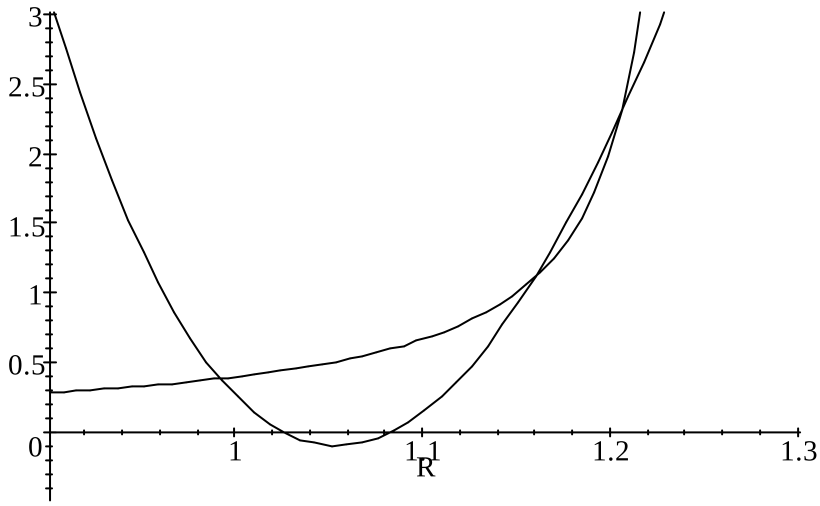


Table 5. Eigenvalues for the $(\overline{R}_2; \overline{D}_2)$ steady-state in the benchmark case

Steady-state	Eigenvalues
$\overline{R}_2 = 1.16; \overline{D}_2 = 1.107$	$\lambda_1 = .951; \lambda_2 = 1.169$

Although the properties of the solutions depend on the vector of all the parameters, the previous analysis suggests that a well-defined saddle-path is more likely to emerge for higher values of τ. In some cases, the tax coefficient on debt needed to ensure a saddle-path with positive debt could be quite high. We therefore suggest that a government that is faced with the dilemma of designing a policy rule that is sustainable and at the same time allows a positive debt steady state could choose an alternative policy rule. In this new rule taxes are a function not only of debt, but also of the divergence between the real interest rate and the taxation coefficient on debt. To clarify our motivations in proposing such a rule, let's consider the following parametrization, in which the deviation from Ricardian equivalence is larger than in the benchmark case: $\beta = .9$; $q = .685$; $\psi = .1$; $\chi = .05$; $G = .3$; $T = .2$ and $\tau = 0.25$. This is a case that produces only one real solution for the steady-state (Table 6, column 1); that we disregard for the usual reasons. If the government increases τ to 0.6; however, we can have a saddle-path with positive debt (Table 6, column 2).

A saddle-path with positive debt, however, can also be obtained if the government introduces the alternative rule:

$$\tau_t = T + \tau_1 D_t + \tau_2 (r_t - \tau_1) \tag{31}$$

setting the following tax rates: $T = .25$; $\tau_1 = .07$; $\tau_2 = .5$ (Table 6, column 3). The eigenvalues corresponding to this steady-state are reported in Table 7

Table 6. Introduction of a interest-tax rule

$q = .685$	$q = .685; \tau = 0.6$	$q = .685; \tau_1 = 0.7; T = .25$
$\overline{R}_1 = .965$	$\overline{R}_1 = .958$	$\overline{R}_1 = .98$
$\overline{R}_2$ complex root	$\overline{R}_2 = 1.25$	$\overline{R}_2 = 1.17$
$\overline{R}_3$ complex root	$\overline{R}_3 = 1.49$	$\overline{R}_3 = 1.2$
$\overline{D}_1 = .351$	$\overline{D}_1 = .158$	$\overline{D}_1 = 1.053$
$\overline{D}_2$ complex root	$\overline{D}_2 = .285$	$\overline{D}_2 = .0048600$
$\overline{D}_3$ complex root	$\overline{D}_3 = .877$	$\overline{D}_3 = .1054469$

Table 7. Eigenvalues for the $(\overline{R}_2; \overline{D}_2)$ steady-state in interest-tax rule case

Steady-state	Eigenvalues
$\overline{R}_2 = 1.17; \overline{D}_2 = .0048$	$\lambda_1 = -.059$; $\lambda_2 = 1.104$

With this rule we can have a saddle-path with positive level of debt in the region where $r > \tau$. The intuition behind this is quite straightforward. As we said previously, when the real interest rate is bigger than the taxation coefficient on debt, taxes are not growing enough to close the debt if this is positive, so stability can only be achieved for negative levels of debt. In the case of the new rule, we can have a positive debt steady state because, even though the real interest rate is bigger than the taxation coefficient on debt, the additional tax component increases with the real interest rate, preventing the debt from exploding.

From equation (31) it is clear that the new tax rule is taking into account not only the level of the debt (the stock), but also the stream of payments for the government that the existing level of debt is generating. The new taxation component is proportional to the net gains, for the agents, from holding a unit of debt, that is a flow variable. If the $\tau_1 D_t$ component can be assimilated to a wealth-tax (that hits a stock), the $\tau_2(r_t - \tau_1)$ component can be considered a tax on income from financial capital.

Formally, with the new tax rule the difference equations governing the system are:

$$D_{t+1} = (R_t - \tau_1)D_t - \tau_2(R_t - 1 - \tau_1) + \overline{G} - T \tag{32}$$

and

$$R_{t+1} = \frac{1}{f - [1 - \tau_1 D_t - T - \tau_2(R_t - 1 - \tau_1)]\frac{(1+\psi)}{(1-\overline{G})}\frac{(1-q\beta)}{(1+\phi+\psi)}\frac{(1-q)}{q}g + q\beta + \frac{1}{q} - \beta R_t} \tag{33}$$

The $\Delta D_t = 0$ locus is still the sum of an hyperbola and of a straight line (see Appendix). The line is still upward sloping if the coefficient τ_2 is not too big (formally if $\tau_2 < \frac{\beta}{(1+\psi)}\frac{(1+\phi+\psi)}{(1-q\beta)}\frac{q}{(1-q)}$)

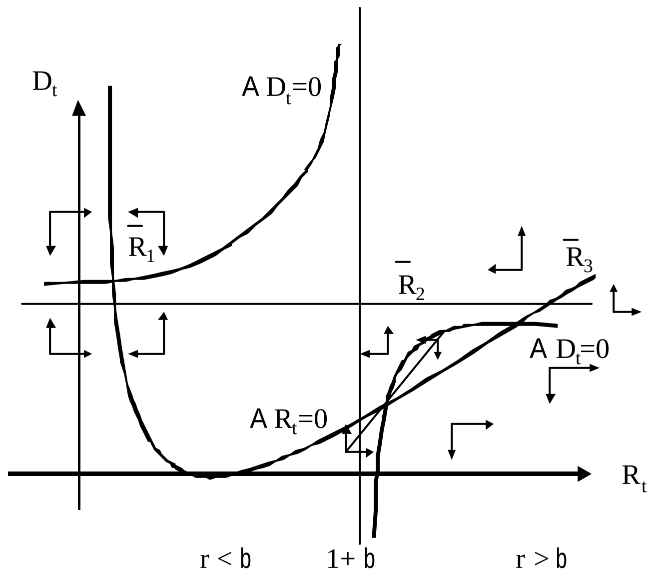
In graphical terms, the introduction of the new component in taxation shifts the horizontal asymptote of the $\Delta D_t = 0$ locus above zero (to $D = \tau_2$); and makes possible a saddle-path equilibrium for a positive level of debt in the region where $r > \tau$ (Fig. 10).

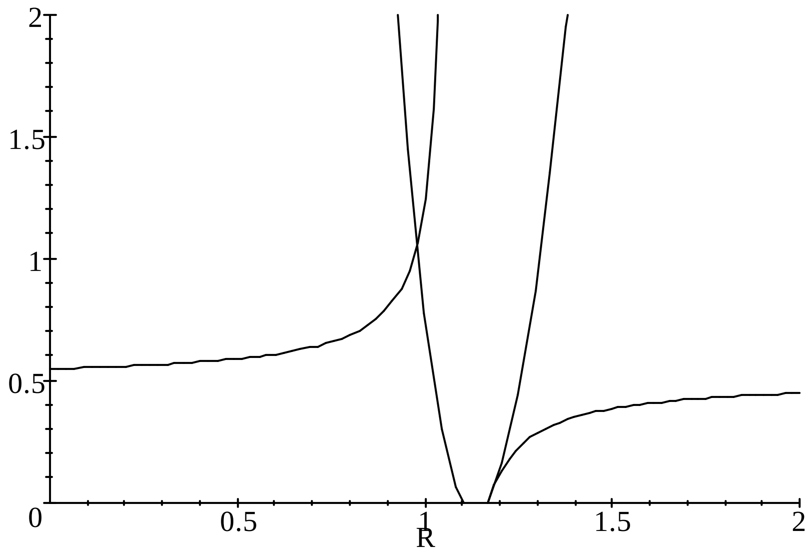
The plots based on the simulations for the case of the interest-rate rule are showed in Figure ##[13]. they confirm the theoretical shapes envisaged in Figure 10.

Figure ##:Simulations: the interest rate rule

[13] Although R2 and R3 look almos coincident here for scale reasons, the plot of Figure ## is based on the simulations reported in Table ##.

Figure 10: Interest-rate tax rule





In the case of the numerical example that we have provided, the main advantage of using this alternative rule lays in the fact that a fiscal package that implies a much lower tax rate on debt ($\tau_1 = 0.7$), a small increase in the lump-sum component in taxes (T increases from .2 to .25) and the introduction of the new tax component (at the rate $\tau_2 = .5$); could be more feasible, from a political point of view, than the alternative of a huge increase of the tax coefficient on debt to .6. A rule like the one that we are proposing would approximate the effects of a rule in which debt inclusive of interest (for example $\tau_t = \tau R_t D_t$) but would have the advantage of being more feasible from a political point of view. As we already stressed in the introduction, we do not intend to argue that such a rule would be optimal. Nevertheless, we believe that our analysis could give some useful indication to policy makers about a possible way to follow in a situation in which political constraint prevent more drastic measures, like an increase in the taxation of debt (or a reduction in government expenditure). Although we are not explicitly considering in the model the possibility that agents could refuse to subscribe new public debt when this is being taxed (or when the tax coefficient on this increase drastrically, how it would be necessary to achieve stability in the example that we have summarixed in Table 6, column 2 . The latter is an example of what we mean by political constraints. Our analysis also gives some warnings about the excessive faith put in the literature and in policy analysis in fiscal "closure" rules since, as we have seen, these can fail to generate a well-defined steady-state for sensible parameters values.

Our analysis is, of course, subject to several caveats. One is the practical working of a rule that makes taxation a function of the real interest rate. It could be problematic, for example, to decide which exact measure of the nominal rates and prices to choose to build the real rate that taxes should target. Furthermore, even if in our model output is fixed, in real economies increases in the real interest rate are likely to be associated with periods of recession. This means that also the alternative rule that we are proposing could be the object of political criticism, since it would be problematic to introduce a rule that automatically increases taxes during a recession. In addition to this, such a rule would give to the monetary authorities a certain degree of (indirect) power on fiscal policy.

6 Conclusions

This paper uses a modified version of the Blanchard (1985) model of perpetual youth to investigate the dynamic effects of different fiscal policy rules.

The main finding of the paper is that a simple fiscal closure rule, based on a wealth tax, could be insufficient to ensure the existence of a well defined saddle-path equilibrium even when the tax rate exceeds the real interest rate. We suggest that an alternative way of solving this problem could be to add another taxation component, that takes in to account the level of the real interest rate.

Our model has many limitations, in particular a very simple production structure and fully flexible prices. The latter is reflected in the fact that, even with an endogenous labour supply, aggregate per-capita consumption and output are completely determined by the level of government expenditure, and therefore are fixed at some sort of "natural level" if government expenditure is constant[14]. The present version of the model can, therefore, be seen as a first step for the construction of models in which departures from pure neo-classical assumptions will be larger. In particular, adding nominal rigidities and imperfect competition, and adopting a two-country framework, will allow an analysis of the effects of debt policies in a modified version of the OR model used in paper 1. Given that OR themselves stress the importance of breaking down Ricardian equivalence in their framework, this seems worth pursuing[15].

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