Consumption Patterns over Pay Periods

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Abstract

How do individuals smooth consumption when they are paid monthly but make consumption decisions at least every week within that month? Are consumers credit constrained in the short-run with respect to non-durable consumption and can we determine the impact of this? This paper examines behaviour of individuals who receive income periodically but make consumption decisions on a more frequent basis when they are faced with an imperfect capital market with respect to the price and quantity of credit, and with price uncertainty. We present for the first time a dynamic model to characterise optimal behaviour and prove the existence of the stationary solution for each "micro" consumption period within the payment period. We simulate the numerical solution to the model and there is a clear *u*-shape over the payment period. At a given level of wealth, consumption is highest in the first week after income receipt, decreases in the second and third weeks and increases again in the fourth. A pseudo maximum likelihood estimator is applied to data from the Family Expenditure Survey to estimate the coefficient of relative risk aversion. However there is evidence of substantial measurement error in the data and we estimate that this accounts for about 50% of the total variation in the data.

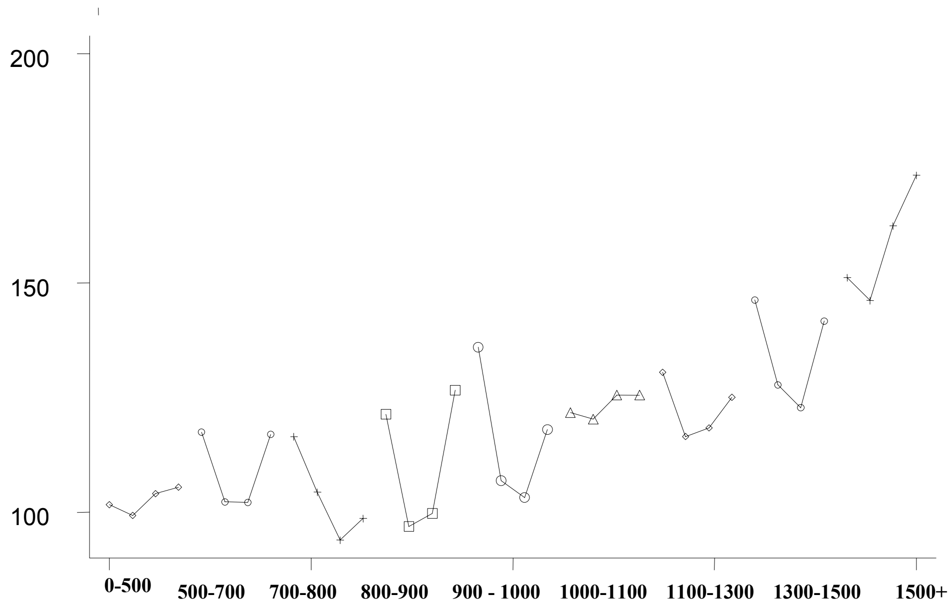
1. Introduction

In the recent past a new problem has arisen for consumers with respect to consumption decisions. Currently (1998/99), close to 58% of working persons in the UK receive their income on a monthly basis but they make consumption decisions more frequently. How does such an individual choose to allocate their spending over the month? Do individuals spend the same proportion of income in each week or are expenditures significantly higher on receipt of their income and then reduced until the next payday? An additional issue is how this short-run consumption decision fits into a life-cycle consumption plan i.e. combining food and other non durable consumption with mortgage and other credit repayments and pension plan contributions. Prima facia evidence in Figure 1 shows the pattern of non-durable expenditure over the payment cycle for nine different income groups and there is a clear pattern. For most groups consumption is relatively high in the first week when income is received, then decreases for the second and third and is then relatively high in the fourth week, in anticipation of payment being received at the end of that week.

However, does such a pattern reflect optimal behaviour? Traditional consumption theory suggests that consumption should be smooth given that income is known with certainty each month. However, there is still uncertainty with regard to expenditure levels which arises from many sources: repairs to household durables, school trips for children, spontaneous purchases, and variation in prices. Intuition suggests that individuals may postpone "unnecessary" consumption until all information is known so that they have resources to deal with any unexpected negative shocks during the month: this in effect is the precautionary motive for saving widely discussed in the literature (see Carroll and Kimball 2001). This implies positive consumption growth during the month as uncertainty is gradually resolved, particularly between the second last and last week of the payment cycle, when all information is known and income is received in the next week. At a given level of wealth, the optimal consumption is likely to be different in the second week of the payment cycle than in the fourth week because in the fourth week all uncertainty is resolved and income is received in the next period whereas in the second there are still two weeks of uncertain expenditure remaining.

The pattern shown in Figure 1 is consistent with consumers showing some degree of caution during the earlier weeks: although expenditure is high when monthly income is received, there is a substantial drop in weeks two and three, but a sharp increase in the final week when uncertainty is resolved and income is expected.

Figure 1 Average Non-durable expenditure by income group over payment cycle 98/99



However, an additional factor in the consumption decision may be a limit on unsecured borrowing for non-durable consumption for a short period of time. In this case the outcome will differ from when we assume individuals hold liquid assets or have unlimited credit to use as a buffer against negative shocks and who therefore have an expenditure pattern which in independent of income. In that case the liquidity constraints will reinforce the cautious behaviour of individuals as a response to uncertainty as pointed out in Deaton 1991, page 1222. Therefore the decisions on consumption earlier in the payment cycle will take account of the fact that there is limited credit with which to finance consumption until income is received again when positive wealth is exhausted. This also implies that an individual will not be at their credit limit before week four because consumption would then be zero until the start of the next cycle. In addition, the Marginal Propensity to Consume (MPC) out of wealth and borrowings will differ during the month; early in the month the MPC out of debt should be low, while both should increase as the next receipt of income gets closer. If there is an interest rate differential between borrowing and saving, then borrowing will not occur until all positive wealth is depleted.

Section 2 of this paper presents some illustrative empirical evidence showing the existence of a pattern in expenditure decisions over a payment cycle. Section 3 presents a simple model of simultaneous consumption and income decisions incorporating a limit on borrowing and an interest rate differential. Section 4 describes the extension and solution to this model which allows for periodic receipt of income relative to consumption decisions. Section 5 describes the estimation procedure and presents some Monte Carlo results. Section 6 concludes.

2. Empirical description of Short-Run Consumption decisions

With a given a level of uncertainty and the possibility of liquidity constraints, the consumption level consistent with any level of wealth will depend on the point at which the individual is in the payment cycle. However, as already noted above, this is not due to preferences or discount rates that vary over time: only the variation in the amount of uncertainty which makes the marginal utility of any given level of expenditure a function on the point in the payment cycle at which the expenditure occurs. Thus the point in the payment cycle acts as a shift factor in the utility function which is similar to incorporating demographic factors in the long-run consumption models. In the weeks in which there is more uncertainty, the marginal utility of consumption for given expenditure is shifted downwards relative to weeks when there is less uncertainty and by implication consumption must be lower in the weeks with more uncertainty.

The framework here closely follows Zeldes (1989), using a CRRA utility in order to derive an approximation to the Euler equation.

We assume a CRRA utility function of the form

$$u(c_t, z_t) = \frac{c_t^{1-\rho}}{1-\rho}\exp(z_t),$$

where ρ is the coefficient of risk aversion, c_t is consumption and z represents the effect of different points in the payment cycle. If there is no limit on the amount of borrowing, then in each period the individual maximises the sum of utility in all future periods in his lifetime i.e.

$$\max E_t \sum_{t=0}^{T} \beta^t \frac{c_t^{1-\rho}}{1-\rho}\exp(z_t)$$

subject only to a budget constraint governing the evolution of wealth

$$w_{t+1} = (1+r)(w_t + y - c_t),$$

where β is the discount rate, y is regular income, r is the interest rate. The first order conditions for this problem leads to an Euler equation which implies that discounted marginal utility should be constant between time periods

$$\frac{c_t^{-\rho}\exp(z_t)\beta(1+r)}{c_{t+1}^{-\rho}\exp(z_{t+1})} = 1 + e_{t+1},$$

where both r and β are assumed to be constant over time and across households. The left hand side of this equation should differ from one only by the expectation error e_{t+1}, which has mean zero and is independent of information known at period t including monthly income. Taking logs of both sides and rearranging gives the equation for consumption growth as

$$\Delta \ln(c_{t+1}) = \frac{1}{\rho}\left[(z_{t+1} - z_t) - \ln(1+r) - \ln(\beta) + \ln(1+e_{t+1})\right] \qquad (2.1)$$

Given that changes in regular monthly are likely to be known in advance, we assume that the level of income is known with certainty. However significance of income in explaining consumption growth would indicate the importance of liquidity constraints because the individual cannot simply borrow as necessary to achieve their optimal path and so that path cannot be independent of the timing of receipt of income. The econometric specification of (2.1) therefore also includes the level of monthly income and is given by

$$\Delta \ln e_{iw} = \alpha \ln y_{it} + \gamma_{w,w-1} d_{w,w-1} + \varepsilon_{iw} \tag{2.2}$$

where $\Delta \ln e_{iw}$ = growth in expenditure in week w, for individual *i*,

y_{it} = level of pay received by individual *i* at the start of month *t,*

$d_{w,w-1}$ = dummies for the transition between weeks *w, w-1,*

ε_{iw} = stochastic error term which has mean zero and is iid.

If *d* represents a dummy variable which takes the value equal to one in *w*, (d_w - d_{w-1}) becomes a zero sum dummy over the payment cycle for each individual. The dummy variables capture the effect of moving between weeks and thus their significance indicates the importance of the changing level of uncertainty. We estimate this Euler equation using data from the Family Expenditure Survey (FES) in the UK which records expenditure for two consecutive weeks for each individual in the survey, as well as the dates over which expenditure is recorded, and the date, amount, and period covered by the last received income. Thus for all monthly paid individuals, the weeks of the payment cycle over which they are observed is clear. The FES for 1996/97, 1997/98 and 1998/99 are used but the sample is restricted to those individuals who are in full time employment, are usually paid monthly and whose last income receipt is their usual income[2]. We separate the sample into individuals who may be constrained with respect to short-term credit based on whether they are recorded as holding a credit card in the dataset. However this depends on the individual paying an annual charge for their card and therefore we supplement this by also recording whether individuals made any purchases on credit during their diary period. While this is not a perfect indicator of liquidity, close to 55% of our sample

[2] The estimations exclude "outliers" which are defined as persons whose non-durable consumption in any one week is greater than their monthly income. These observations (less than 3%) biased the results towards the insignificance of income. Monthly effects are also controlled for but the coefficients are not reported.

is classified as unconstrained which seems realistic although conservative.[3] This may be explained by the fact that we record individuals who did not use their credit card during the fortnight or pay an annual charge during the previous year as constrained.

The results of the estimation of equation 2 are presented in Table 1: monthly dummies were included although the coefficients are not reported. The effect of moving between different weeks in the payment cycle is measured relative to the effect of going from week four when uncertainty is least to week one when it is greatest. The significant coefficient on d_{43} for all years indicates individuals respond in a positive way to the resolution of all uncertainty. The negative sign on d_{32} is consistent with the pattern in Figure 1 for the income groups where consumption falls throughout the month and increases at the end. Because uncertainty is less in week three than week two, we would expect this coefficient to be positive however, although the negative values are insignificant in two – thirds of the results. The results are also quite clear when the sample is split into groups which we consider to be liquidity constrained (without a credit card) and those who are not. Again for all years, the effect of payment cycle is significant for the constrained group, while there is no effect on the unconstrained group lending weight to the argument that those with access to credit have less need to act in a cautious manner. However these individuals also have higher income we cannot determine whether these individuals are dissaving or borrowing.

In 97/98 and 98/99 the level of monthly income is significant for the whole sample even when the position in the payment cycle is controlled for. This points towards the possibility of liquidity constraints given that the level of income is certain when consumption decisions are made although the coefficient on income is positive which is at odds with the results of virtually all previous tests for excess sensitivity (see Table 5.1, Browning and Luscardi, 1996).[4]

[3] However, following the findings of Japelli et al (1998), this may still provide a more accurate classification that wealth to income ratio.

[4] In this short-run framework, higher income relieves short-run liquidity constraints and leads to a higher level of consumption and hence increased consumption growth. In the long-run, lagged movement of

Table 1

Estimation of Euler Equation

	All observations			Unconstrained Sample			Constrained Sample		
	98/99	**97/98**	**96/97**	**98/99**	**97/98**	**96/97**	**98/99**	**97/98**	**96/97**
Dependent variable: dlndexp									
N	3225	3169	2688	1750	1762	1342	1475	1407	1346
llastpay	0.060*	0.060*	0.044	0.082*	0.099*	0.024	0.180	0.026	0.073
	0.024	*0.024*	*0.029*	*0.036*	*0.034*	*0.046*	*0.034*	*0.035*	*0.041*
d21	-0.076*	-0.010	-0.069*	-0.068	-0.038	-0.081	-0.084*	0.023	-0.055
	0.027	*0.028*	*0.029*	*0.040*	*0.039*	0.043	*0.035*	*0.038*	*0.041*
d32	-0.043	-0.049*	-0.032	-0.088*	-0.029	-0.0003	0.007	-0.075*	-0.063
	0.028	*0.025*	*0.030*	*0.042*	*0.035*	*0.044*	*0.035*	*0.038*	*0.041*
d43	0.067*	0.081*	0.125*	0.050	0.053	0.052	0.084*	0.112*	0.191*
	0.026	*0.027*	*0.031*	*0.037*	*0.036*	0.045	*0.037*	*0.040*	*0.043*
constant	-0.518*	-0.416*	-0.331	-0.708*	-0.678*	*-0.168*	-0.202	-0.186	-0.529*
	0.173	0.*170*	*0.201*	0.263	*0.250*	*0.331*	*0.234*	*0.238*	*0.270*

When the two groups are considered separately however, income is always insignificant for the group we consider to be constrained, but this may still be due to misclassification of those unconstrained individuals. More difficult to explain is the significance of income for the unconstrained group in 97/98 and 98/99 which is directly at odds with the presence of liquidity constraints. However, income is generally the greatest determinant of access to credit and therefore it is possible that we are capturing this relationship here. In results not reported here, we correct for selectivity in credit card ownership by using a Heckman selection estimator and in that case, the level of income and the payment cycle are insignificant for the unconstrained group. Also including squared terms of log income, which simply relaxes the assumption that expenditure growth is a linear function of log income, also results in insignificant income terms for the unconstrained group.

income towards its permanent income level leads to a lower long-run consumption growth rate as the

3. A Model of Short-Run Consumption

Before looking at the behaviour of agents with periodic income relative to their consumption decisions, a simpler model is presented to analyse the case of simultaneous payment and consumption decisions, when account is taken of the interest rate differential between borrowing and saving evident in capital markets. We also incorporate an upper bound on the level of borrowing given that the amount of unsecured short-term borrowing is very likely to be limited for at least some of the population, particularly those with low income. Uncertainty is assumed to arise from variation in prices which also captures the choices individuals can make over expenditure when similar products are differentiated with respect to price. Although no assumptions regarding preferences are made at this stage, once the marginal utility function is not linear, the optimal path of the individual will be changed because expectations are affected by uncertainty.

The problem of utility maximisation in this framework is one of dynamic programming with the consumer in each period choosing a consumption level which maximises current utility plus their preference for future wealth, given constraints and expectations. Consumption will be determined by the real value of wealth which depends on the realisation of current price p_t and therefore the value function has two state variables, nominal wealth and price. The realisation of future price is denoted by π and we assume both p and π are drawn from the distribution **F** with support over which expectations are taken. We also assume that price does not exhibit any persistence: if this was not the case, the current price level would contain information about the price distribution in the next period over which expectations are taken. The budget

income level stabilizes. It is this long-run effect that previous studies using panel data have captured.

constraint for each time period *t* is $p_t c_t \le w_t + y + d_t$, and the evolution of wealth follows the process[5]: $w_{t+1} = \left(w_t + y - p_t c_t\right)(1+r) + d_t(r-\delta)$,

where $p_t c_t$ = expenditure on consumption,

w_t = wealth,

y = regular income,

d_t = debt in period t, subject to an upper limit of $\bar{d}$,

r = return on savings ,

δ = cost of borrowing.

Wealth at the start of each period before income is received is denoted by w_t. We assume that y, $\bar{d}$, r and δ are exogenous and known with certainty and that the differential between borrowing and saving rates is constant.

The Bellman equation for this problem is

$$V\left(w,p\right) = \max_{\substack{pc \le w+d+y \\ 0 \le d \le \bar{d}}} \left\{ u\left(c\right) + \beta E_{\mathbf{P}} \left[V\left((1+r)(w+y-pc) + d(r-\delta), \pi \right) \right] \right\} \tag{3.1}$$

where the value function *V(.),* and the felicity function $u(c)$ are assumed to be increasing, monotonic and concave. It is assumed that the discount factor β is such that $\beta(1+\delta) < 1$ i.e. the consumer is "impatient" and thus has a preference for consumption in the present. For a given wealth and price realisation, the control variable consumption is chosen to maximise the right hand side, subject to a lower limit on negative wealth. The problem fulfils the conditions for continuity and differentiability of the value function as laid down by Benveniste and Scheinkman (1979) and so the value function in 3.1 will hold for all levels of wealth and price.

The Lagrangean equation for the problem gives

[5] It would not be unreasonable to assume that over a sufficiently short period of time, the return on savings equals zero.

$$L(c,d;\gamma,\mu_1,\mu_2) = u(c) + \beta E\left[V\left((1+r)(w+y-pc)+d(r-\delta),\pi\right)\right]$$
$$-\gamma\left(pc-(w+y+d)\right)+\mu_1(d)+\mu_2\left(\bar{d}-d\right)$$

giving rise to the first order conditions

$$\begin{aligned}
\frac{\partial L(c,d;\gamma,\mu_1,\mu_2)}{\partial c} &= u'\left(c^*\right) - \beta(1+r)p\frac{\partial EV}{\partial w}\left(.^*\right) - \gamma p = 0,\\
\frac{\partial L(c,d;\gamma,\mu_1,\mu_2)}{\partial d} &= \beta(r-\delta)\frac{\partial EV}{\partial w}\left(.^*\right) + \gamma + \mu_1 - \mu_2 = 0,\\
\frac{\partial L(c,d;\gamma,\mu_1,\mu_2)}{\partial \gamma} &= pc^* - (w-y+d^*) \le 0,\\
\frac{\partial L(c,d;\gamma,\mu_1,\mu_2)}{\partial \mu_1} &= d^* \ge 0,\\
\frac{\partial L(c,d;\gamma,\mu_1,\mu_2)}{\partial \mu_2} &= \bar{d} - d^* \ge 0.
\end{aligned} \tag{3.2}$$

The solution to this problem can be characterised by examining different values of the multipliers. Firstly if γ=0, this implies that the agent does not consume all cash in hand, which can only occur if the consumer is a "saver". In this situation borrowing must be zero[6], μ_1>0, μ_2=0, and there is positive wealth carried over to the next period. In all other cases γ>0, i.e. consumption is greater than or equal to the sum of wealth and income, and assets carried between periods and less than or equal to zero. Obviously μ_1>0 implies μ_2=0, and vice versa, but $\mu_1 = \mu_2 = 0$ can occur if the level of optimal borrowing is positive but below the limit. Thus, when the individual consumes their cash in hand, γ>0, but has zero borrowing, μ_1>0, consumption exactly equals wealth in that period, and there are no assets or liabilities carried between periods. Then wealth equals income in the next period. Alternatively, if γ>0 and there is some level of positive borrowing, $\mu_1 = \mu_2$=0, consumption is equal to the optimal level of debt $d*$, plus cash on hand and a negative wealth of $-(1+\delta)d*$ is carried into the next period. The final possibility is when γ>0, μ_2>0 and μ_1=0, at which point the individual is at his credit limit, consuming the maximum amount available and has maximum debt at the start of next period $-(1+\delta)\bar{d}$ but consumption is still below the optimal. Therefore we have only four possible "regimes" for the individual

$$
\begin{aligned}
&\text{Regime 1: } pc^* < w + y, && d^* = 0, && \gamma=0,\ \mu_1>0,\ \mu_2=0,\\
&\text{Regime 2: } pc^* = w + y, && d^* = 0, && \gamma>0,\ \mu_1>0,\ \mu_2=0,\\
&\text{Regime 3: } pc^* = w + y + d^*, && 0<d^*<\bar{d}, && \gamma>0,\ \mu_1=\mu_2=0,\\
&\text{Regime 4: } \overline{pc} = w + y + \bar{d}, && d^* = \bar{d}, && \gamma>0,\ \mu_1=0,\ \mu_2>0.
\end{aligned}
\tag{3.3}
$$

3.2 Regime boundaries

The regime within which (3.1) is maximised depends on the level of optimal consumption relative to the real value of wealth and will determine how much borrowing or saving occurs. The boundaries between regimes can be identified from the first order conditions, (3.2). Following Deaton (1991) denote cash on hand as the sum of wealth and income. Considering first when all cash on hand is consumed, $\gamma>0$, but optimal debt is zero so $\mu_1>0$, $\mu_2=0$, i.e. there is zero wealth carried between periods, so the first order conditions from equation 3.2 are

$$\frac{\partial u(c)}{\partial c} - \beta(1+r)p\frac{\partial EV(0,\pi)}{\partial w} - \gamma p = 0 \text{ and}$$

$$\beta(r-\delta)\frac{\partial EV(0,\pi)}{\partial w} + \gamma + \mu_1 = 0 .$$

Solving for γ gives

$$\frac{1}{p}\frac{\partial u(w+y/p)}{\partial c} > \beta(1+r)\frac{\partial EV(0,\pi)}{\partial w} , \tag{3.4}$$

because $\gamma>0$ and then substituting this into the expression for μ_1, leads to

[6] A consumer will never borrow and not consume that extra cash when the cost of borrowing is greater than the return on savings which rules out $\gamma=0$, $\mu_1=\mu_2=0$ and $\gamma=0$, $\mu_1>0$ $\mu_2=0$.

$$\frac{1}{p}\frac{\partial u\left(w+y/p\right)}{\partial c}<\beta(1+\delta)\frac{\partial EV\left(0,\pi\right)}{\partial w}. \tag{3.5}$$

Thus, when the marginal utility of real cash on hand falls between the discounted value of having no assets or debt in the future and the agent consumes exactly cash on hand. If the current real marginal utility of real cash, $\frac{1}{p}\frac{\partial u\left(w+y/p\right)}{\partial c}$, is less than the discounted future value of zero savings, i.e. the opposite of 3.4, optimal consumption is below the level of real cash on hand and saving occurs, and the consumer is in regime 1. Similarly, if it is greater than the discounted future value of zero borrowing, consumption is financed out of debt in order to reach optimal consumption.[7]

When the consumer is constrained i.e. consumption is equal to $w+y+\bar{d}$, debt is at the upper limit, $\gamma>0$, μ_2>0, μ_1=0, so the first order conditions in this case are (from equation 3.2)

$$\frac{\partial u\left(w+y+\bar{d}/p\right)}{\partial c}-\beta p(1+r)\frac{\partial EV\left(-(1+\delta)\bar{d},\pi\right)}{\partial w}-\gamma p=0 \text{ and}$$

$$\beta(\delta-r)\frac{\partial EV\left(-(1+\delta)\bar{d},\pi\right)}{\partial w}+\gamma-\mu_2=0.$$

Again solving for γ and substituting into μ_2 gives

$$\frac{1}{p}\frac{\partial u\left(w+y+\bar{d}/p\right)}{\partial c}>\beta(1+\delta)\frac{\partial EV\left(-(1+\delta)\bar{d},\pi\right)}{\partial w}. \tag{3.6}$$

When this condition holds the real marginal utility of maximum possible consumption is greater than the discounted future value of having maximum debt, the consumer uses all available credit up to the limit. In this case, wealth combined with total available credit is below the agents optimal consumption, so the credit constraint is binding and if it were possible, marginal utility would be further decreased. However, this is not possible and so consumption remains below the optimal. If the opposite of 3.6 is true but the individual still has a preference for consumption greater than cash on hand, then positive borrowing will lower current marginal utility of consumption and increase the future value of wealth until the two terms equate at $\mu_1 = \mu_2 = 0$ when there is some negative wealth carried between periods. These conditions allow the identification of the regime that an individual will fall into, given a price realisation, p_t a wealth level w, income y, fixed interest rates, discount rate and preferences. However, the effect of price realisation is ambiguous: a low price will increase the value of real wealth, decreasing marginal utility but will increase the real value of marginal utility.

These first order conditions and corresponding regimes can be written as an augmented Euler equation where $\lambda(c)$ denotes the marginal utility of consumption, and the conditions relating consumption in any two periods, *t* and *t+1* are

$$\frac{1}{p_t}\lambda\left(c_t^*\right) = \max\left\{\begin{array}{l} \beta(1+r)E_t\lambda\left(c_{t+1}^*\right), \\ \min\left\{\frac{1}{p_t}\lambda\left(\frac{w_t+y}{p_t}\right), \beta(1+\delta)\mathrm{E}\lambda\left(c_{t+1}{}^*\right)\right\}, \\ \frac{1}{p_t}\lambda\left(\frac{w_t+y+\bar{d}_t}{p_t}\right) \end{array}\right\} \tag{3.7}$$

with expectations taken with respect to price. Consider first if the consumer did not have access to any credit, then the maximum value that consumption could take would be $(w_t + y)/p_t$ and once c_t^* is below that, marginal utility of consumption is above $\frac{1}{p_t}\lambda\left(\frac{w_t+y}{p_t}\right)$ and the Euler equation is fulfilled in an unconstrained way. When the credit constraint binds, total cash on hand $(w_t + y)/p_t$ is the highest possible consumption and this gives a lower bound on marginal utility. Therefore

$$\frac{1}{p_t}\lambda\left(c_t^*\right) = \max\left\{\beta(1+r)E_t\lambda\left(c_{t+1}^*\right), \frac{1}{p_t}\lambda\left(\frac{w_t+y}{p_t}\right)\right\}$$

[7] Obviously if $r = \delta$, there are only three regimes and every consumer is either a saver or a borrower.

as shown in Deaton (1991). With access to credit markets limited to $\overline{d}_t$, maximum consumption is now $(w_t + y + \overline{d}_t)/p_t$. If c_t^* is below the constrained level, $(w_t + y + \overline{d}_t)/p_t$, marginal utility will be higher than at the maximum possible consumption level. Hence, the individual chooses the maximum available marginal utility either $\beta(1+r)E_t\lambda\left(c_{t+1}^*\right)$ when not constrained or $\frac{1}{p_t}\lambda\left(\frac{w_t + y + \overline{d}_t}{p_t}\right)$ when constrained. With the addition of an interest rate differential, there may be consumers who have a wealth level and preferences such that optimal consumption is exactly cash in hand because they are not "impatient" enough to borrow, or "patient" enough to save. i.e. both equation 3.4 and 3.5 hold. However for some individuals in this regime, δ may be sufficient reward for saving which would raise the discounted value of savings and lower current consumption so that $\frac{1}{p_t}\lambda\left(c_t^*\right) > \frac{1}{p_t}\lambda\left(\frac{w_t + y}{p_t}\right)$. However this is not possible given capital market imperfections and so consumption would instead equal cash in hand and the consumer must be satisfied with a lower marginal utility – hence the need for the minimum condition in 3.7.

Assuming that a stationary solution exists to equation 3.7, this will define the optimal relationship between expenditure and wealth and prices, and can be written as $pc = f\left(w, p\right)$, for all wealth and price. In that case, the marginal utility of real wealth $V_w\left(w, p\right)$ can be denoted as $q\left[w, p\right]$ and assume the following properties of the functions hold

$$\frac{\partial f\left(w,p\right)}{\partial w} \geq 0; \quad \frac{\partial f\left(w,p\right)}{\partial p} \leq 0, \text{ and}$$

$$\frac{\partial q\left[w,p\right]}{\partial w} \leq 0; \quad \frac{\partial q\left[w,p\right]}{\partial p} \propto \frac{\partial\left(\frac{1}{p}\lambda\left(c\right)\right)}{\partial p}; \tag{3.8}$$

i.e. the consumption policy function is increasing in wealth but decreasing in prices while the marginal utility of money is unambiguously decreasing in wealth but the effect of price depends on the slope of real marginal utility of consumption with respect to price. Substituting $c = f(w,p)/p$ for consumption gives the marginal utility of money as $q[w,p]=\frac{1}{p}\lambda\left(\frac{1}{p}f(w,p)\right)$, and the policy function becomes $f(w,p)=p.\lambda^{-1}\left(p.q[w,p]\right)$.

Rewriting 3.7 in terms of $q[w,p]$

$$q[w,p]=\max\left\{\begin{array}{l}\beta(1+r)\int_P q\left[(1+r)\left(w+y-p\lambda^{-1}\left(pq[w,p]\right)\right),\pi\right]dF(\pi),\\ \min\left\{\frac{1}{p}\lambda\left(\frac{w+y}{p}\right),\beta(1+\delta)\int_P q\left[(1+\delta)\left(w+y-p\lambda^{-1}\left(pq[w,p]\right)\right),\pi\right]dF(\pi)\right\},\\ \frac{1}{p}\lambda\left(\frac{w+y+\bar{d}}{p}\right)\end{array}\right\} \quad (3.9)$$

the solution to which will describe the marginal utility of money and therefore define the policy function. For ease of notation denote the following functions

$$H_1\left(q[w,p],w,p\right)=\beta(1+r)\int_P q\left[(1+r)(w+y-p\lambda^{-1}\left(pq[w,p]\right),\pi\right]dF(\pi)$$

$$\underline{\lambda}=\frac{1}{p}\lambda\left(\frac{w+y}{p}\right)$$

$$H_2\left(q[w,p],w,p\right)=\beta(1+\delta)\int_P q\left[(1+\delta)(w+y-p\lambda^{-1}\left(pq[w,p]\right),\pi\right]dF(\pi)$$

$$\bar{\lambda}=\frac{1}{p}\lambda\left(\frac{w+y+\bar{d}}{p}\right),$$

where $H_1\left(q[w,p],w,p\right)$ and $H_2\left(q[w,p],w,p\right)$ are the discounted future value of savings and borrowings, respectively, for current wealth and price, w, p, and marginal utility of money $q[w,p]$. Extending the work of Deaton and Laroque (1992) and under a similar set of assumptions it is possible to show the following

Theorem 1

There is a unique stationary solution $q\left[w,p\right]$, the solution to the functional equation 3.9, which is non-increasing and continuous in wealth. In addition the solution can be characterised as follows

$$q\left[w,p\right]=H_1\left(q\left[w,p\right],w,p\right), \text{ when } \beta(1+r)\int_P q\left[0,\pi\right]dF(\pi)\geq\underline{\lambda};$$

$$q\left[w,p\right]=\underline{\lambda}, \qquad \text{when } \begin{cases}\beta(1+r)\int_P q\left[0,\pi\right]dF(\pi)\leq\underline{\lambda}, \text{ and}\\ \beta(1+\delta)\int_P q\left[0,\pi\right]dF(\pi)\geq\underline{\lambda};\end{cases}$$

$$q\left[w,p\right]=H_2\left(q\left[w,p\right],w,p\right), \text{ when } \begin{cases}\beta(1+\delta)\int_P q\left[-(1+\delta)\overline{d},\pi\right]dF(\pi)\geq\overline{\lambda}, \text{ and}\\ \beta(1+\delta)\int_P q\left[0,\pi\right]dF(\pi)\leq\underline{\lambda};\end{cases}$$

$$q\left[w,p\right]=\overline{\lambda}, \qquad \text{when } \beta(1+\delta)\int_P q\left[-(1+\delta)\overline{d},\pi\right]dF(\pi)\leq\overline{\lambda};$$

Proof available upon request.

This theorem formalises the intuition given above: behaviour is determined by the value of wealth and optimal consumption today relative to valuation of future wealth, for any particular level of current wealth and prices, expectations and a constant set of preferences over consumption and wealth. When the wealth level is sufficiently high that the real value of current marginal utility from consuming all of that wealth plus income is below the future marginal value of zero wealth, it is optimal given the individuals preferences to shift wealth toward the future, increasing current marginal utility and decreasing future marginal utility of wealth until a unique level of savings equates the two terms. At a lower level of wealth, the individual will consume all cash on hand when it is optimal to carry zero assets between periods, given the cost of increasing current consumption and reducing future wealth δ, and decreasing current consumption while increasing future wealth, *r*. When the value of carrying zero debt between periods is below the value of consuming all cash on hand, the individual will incur borrowings in order to decrease current marginal utility and increase future marginal utility but these borrowings will be below the maximum amount as long as the optimal consumption is below the point of the credit limit. If wealth is sufficiently low that optimal consumption is above the sum of wealth, income and maximum credit, the individual will consume the maximum amount possible and would increase consumption if it were possible because the marginal value of future debt is below the marginal utility of consuming at the credit limit.

The stationary equilibrium function $q\left[w, p\right]$ will be non-linear given the *min* condition resulting from the interest rate differential and the *max* condition resulting from the limit on borrowing. In addition the policy function which is recoverable by using $f\left(w, p\right) = p.\lambda^{-1}\left(p.q\left[w, p\right]\right)$ is non-differentiable at the three critical values of wealth levels where the solution switches between regimes:

$w < w_1^*$, where $\beta(1+\delta)\int_P q\left[-(1+\delta)(w_1^* + y - pc), \pi\right] dF(\pi) = \overline{\lambda}$:

$$f\left(w, p\right) = w_1^* + y + \overline{d}$$

$w_1^* < w < w_2^*$ where $\beta(1+\delta)\int_P q\left[-(1+\delta)(w_2^* + y - pc), \pi\right] dF(\pi) = \underline{\lambda}$;

$$w_1^* + y + \overline{d} < f\left(w, p\right) < w_2^* + y$$

$w_2^* < w < w_3^*$, where $\beta(1+r)\int_P q\left[(1+r)(w_3^* + y - pc), \pi\right] dF(\pi) = \underline{\lambda}$;

$$w_2^* + y < f\left(w, p\right) = w + y < w_3^* + y$$

$w_3^* < w$ where $\beta(1+r)\int_P q\left[(1+r)(w_3^* + y - pc), \pi\right] dF(\pi) = \underline{\lambda}$;

$$w_3^* + y < f\left(w, p\right) < w + y$$

At low levels of wealth, therefore, the solution tracks $\overline{\lambda}$, follows an optimal path for wealth levels between w_1^* and w_2^* where some positive borrowing incurs, follows $\underline{\lambda}$ between w_2^* and

w_3^*, and above w_3^* again follows an optimal path with consumption less than cash on hand. Given $\overline{\lambda} < \underline{\lambda}$, the solution function $q\left[w,p\right]$ will be flatter than λ while above $\underline{\lambda}$ and between $\overline{\lambda}$ and $\underline{\lambda}$.

4. Multi consumption periods within each payment period

4.1 Model

In this section the focus shifts to a model with a number of consumption periods, but where payment is received periodically. There are three possible situations for the consumer: payment occurs in this period and not in the next period, payment does not occur this period or next period, and payment does not occur this period but does occur next period. Thus, a three period model is sufficient to capture the characteristics of the problem but any number of interim periods can be included. It seems natural to consider individual periods as weeks and each three period cycle as a month. The value function for each week will reflect the proximity to receipt of income i.e. in week 1 it will reflect that income is received but that any debt will incur an interest cost in each week until the next payment, while behaviour in week 3 accounts for the receipt of income in following period and the immediate repayment of debt. Combining this with the cost differential between savings and borrowings ensures that debt will never be incurred until all positive wealth is depleted. We assume that the positive return on savings and the cost of borrowing are incurred each week, so the source of borrowing is more like an overdraft than a credit card. The evolution of wealth between weeks follows the process

$$w_{i+1} = \left(w_i + y - p_i c_i\right)(1+r) + d_i(r-\delta), \text{ where } y=0 \text{ for } i\neq 1.$$

The maximisation problem therefore written as a Bellman equation for each week is:

$$V_1\left(w,p\right) = \max_{\substack{pc \le w+y+d \\ 0 \le d \le \frac{\underline{d}}{(1+\delta)^2}}} \left\{u\left(c\right) + \beta \mathrm{E}_P\left[V_2\left((1+r)(w+y-pc)+(r-\delta)d,\pi\right)\right]\right\}$$

$$V_2\left(w,p\right) = \max_{\substack{pc \le w+d \\ 0 \le d \le \frac{\underline{d}}{(1+\delta)}}} \left\{u\left(c\right) + \beta \mathrm{E}_P\left[V_3\left((1+r)(w-pc)+(r-\delta)d,\pi\right)\right]\right\}$$

$$V_3(w,p) = \max_{\substack{pc \le w+d \\ 0 \le d \le \bar{d}}} \left\{ u(c) + \beta \mathrm{E}_P \left[V_1 \left((1+r)(w-pc) + (r-\delta)d, \pi \right) \right] \right\}$$

Income enters the budget constraint only in the first week and the maximum borrowing in each week is defined as total limit on "monthly" borrowing $\bar{d}$, discounted by the cost of borrowing, δ, which is incurred weekly. Therefore $\bar{d}_1 \equiv \bar{d}/(1+\delta)^2$, $\bar{d}_2 \equiv -\bar{d}/(1+\delta)$, $\bar{d}_3 \equiv \bar{d}$, while the minimum wealth possible at the start of each week is given by $\underline{w}_{t+1} \equiv -(1+\delta)\bar{d}_t$. The first order conditions in 3.2 arise for each value function and so the real marginal utility of consumption for each period can be written as follows for each week

$$\frac{1}{p}\lambda\left(c_i^*\right) = \max \begin{Bmatrix} \beta(1+r)E\lambda\left(c_{i+1}^*\right), \\ \min\left\{ \frac{1}{p}\lambda\left(\frac{w+y}{p}\right), \beta(1+\delta)\mathrm{E}\lambda\left(c_{i+1}^*\right) \right\}, \\ \frac{1}{p}\lambda\left(\frac{w+y+\bar{d}_i}{p}\right) \end{Bmatrix} \tag{4.1}$$

for $\bar{d}_1 = \bar{d}/(1+\delta)^2$, $\bar{d}_2 = \bar{d}/(1+\delta)$ and $\bar{d}_3 = \bar{d}$ and where $y=0$ if $i \neq 1$.

The interpretation of the Euler equation is similar to the simpler one period case from the previous section and again encompasses the four possible positions that the individual can be in for each week. The individual will save and earn r when wealth is sufficiently high that the optimal consumption is below cash on hand, maximum utility involves substituting consumption towards future periods, and the Euler equation is fulfilled in an unconstrained way. Consumption will exactly equal cash on hand when the individual is neither sufficiently patient enough to save at r or impatient enough to borrow at δ. An optimal consumption which involves saving at δ i.e. decreasing consumption below cash in hand is unattainable, so the individual will instead accept a marginal utility level equal to $\frac{1}{p}\lambda\left(\frac{w+y}{p}\right)$ when the optimal would be above that; hence the need for the minimum condition. The Euler equation is also fulfilled without constraint in the third regime which involves an optimal level of borrowing at δ which is below the maximum level. Finally for any given level of wealth, the minimum attainable marginal utility is at the point of maximum borrowing is in the fourth regime and the solution will be at this point when all other consumption levels imply a lower unattainable marginal utility; hence the solution must be $\frac{1}{p}\lambda\left(\frac{w+y+\bar{d}_i}{p}\right)$, the marginal utility from the constrained level of consumption.

Considering the behaviour over the payment cycle, an individual who consumes in regime 2 in the first week i.e. consumes all available wealth plus income, (or repays all debt out of income and consumes the remainder) must consume out of borrowing in the remaining weeks, i.e. must be in regime 3 or 4. Similarly, an individual whose optimal consumption is very high relative to the real value of wealth and therefore chooses to be in regime 4 with maximum possible borrowing, will have zero or negative consumption in the following weeks until the next income is received. However this is unlikely to be a solution once there are restrictions placed on preferences to allow only strictly positive consumption levels.

The stationary solution to equation 4.1, will define the consumption function relationship between wealth, prices and expenditure in each week. Using the same notation as Section 3, denote $pc_i = f_i\left(w,p\right)$, for $i = 1,2,3$, for all wealth and prices. Denote the marginal utility of real wealth as $q_i\left[w,p\right]$ and we assume the optimal policy function is increasing in wealth but decreasing in prices while the marginal utility of money is decreasing in wealth but the price effects depend on the slope of real marginal utility of consumption (as in 3.8). Substituting $c = f_i\left(w,p\right)/p$ for consumption gives the marginal utility of money as $q_i\left[w,p\right]=\frac{1}{p}\lambda\left(\frac{1}{p}f_i\left(w,p\right)\right)$, and the policy function becomes $f_i\left(w,p\right) = p.\lambda^{-1}\left(p.q_i\left[w,p\right]\right)$. Writing the Euler equation 4.1 in terms of the marginal utility of money, gives the following functional equation to which there is a unique solution $q_i\left[w,p\right]$ for *i=1,2,3* for each equation given by:

$$q_i\left[w,p\right]=$$

$$\max\begin{Bmatrix}\beta(1+r)\int_P q_{i+1}\left[(1+r)(w+y-p\lambda^{-1}\left(pq_i\left[w,p\right]\right),\pi\right]dF(\pi),\\ \min\left\{\frac{1}{p}\lambda\left(\frac{w+y}{p}\right),\beta(1+\delta)\int_P q_{i+1}\left[(1+\delta)(w+y-p\lambda^{-1}\left(pq_i\left[w,p\right]\right),\pi\right]dF(\pi)\right\},\\ \frac{1}{p}\lambda\left(\frac{w+y+\bar{d}_i}{p}\right)\end{Bmatrix} \quad (4.2)$$

The functions $H_1\left(q_i\left[w,p\right],w,p\right)$, $H_2\left(q_i\left[w,p\right],w,p\right),\underline{\lambda}$, $\bar{\lambda}$, have analogous definitions to Section 3 i.e. where $H_1\left(q_i\left[w,p\right],w,p\right)$ and $H_2\left(q_i\left[w,p\right],w,p\right)$ are the discounted future value of savings and borrowings, respectively, for wealth today *w*, price today *p*, and marginal utility of money $q_i\left[w,p\right]$ in week *i*. Following the analysis in Bailey and Chambers (1996), who grouped time spells together into epochs in the analysis of season harvest prices, we can develop a similar proof in order to state the following

Theorem 2:

There is a unique set of functions $q_i\left[w,p\right]$*, the stationary solutions to the functional equations (4.2) above, which are non-increasing and continuous in wealth. In addition the solutions can be characterised as follows*

$$q_i\left[w,p\right]=H_1\left(q_{i+1}\left[w,p\right],w,p\right),\qquad \text{when } \beta(1+r)\int_P q_{i+1}\left[0,\pi\right]dF(\pi)\geq\underline{\lambda};$$

$$q_i\left[w,p\right]=\underline{\lambda},\qquad \text{when }\begin{cases}\beta(1+r)\int_P q_{i+1}\left[0,\pi\right]dF(\pi)\leq\underline{\lambda},\text{ and}\\ \beta(1+\delta)\int_P q_{i+1}\left[0,\pi\right]dF(\pi)\geq\underline{\lambda};\end{cases}$$

$$q_i\left[w,p\right]=H_2\left(q_{i+1}\left[w,p\right],w,p\right), \qquad \text{when} \begin{cases} \beta(1+\delta)\int_P q_{i+1}\left[-(1+\delta)\overline{d}_i,\pi\right]dF(\pi)\geq\overline{\lambda}, \text{ and} \\ \beta(1+\delta)\int_P q_{i+1}\left[0,\pi\right]dF(\pi)\leq\underline{\lambda}; \end{cases}$$

$$q_i\left[w,p\right]=\overline{\lambda} \qquad \text{when } \beta(1+\delta)\int_P q_{i+1}\left[-(1+\delta)\overline{d}_i,\pi\right]dF(\pi)\leq\overline{\lambda}.$$

Proof available upon request.

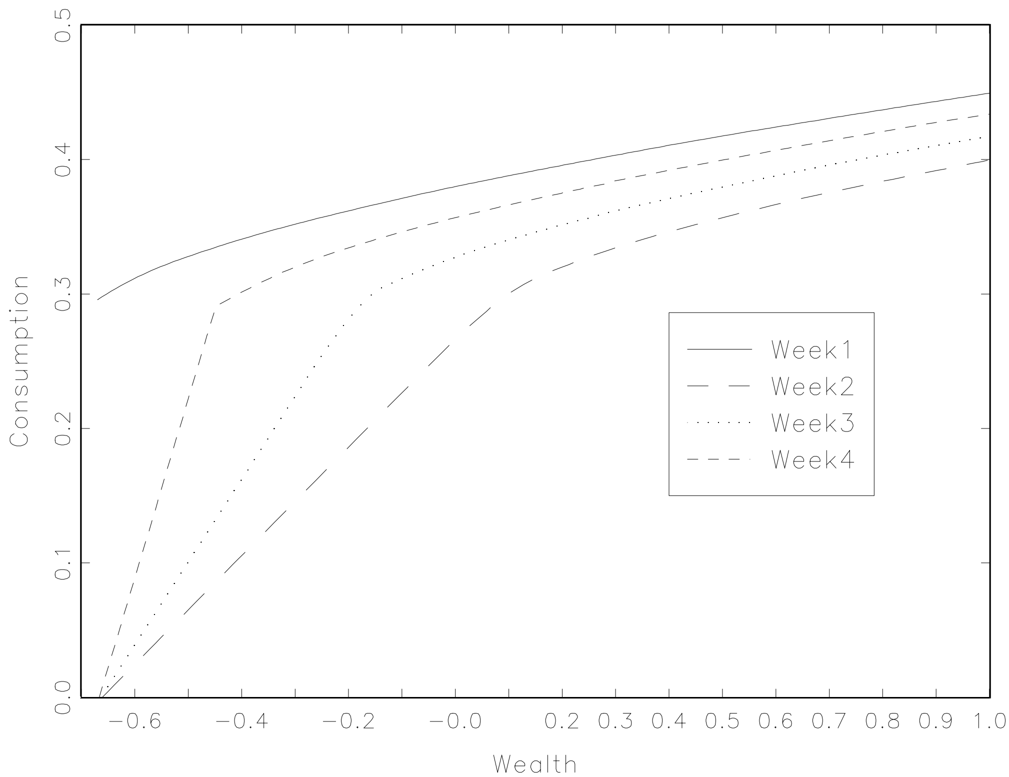
The theorem sets out the conditions for each regime: positive saving is optimal when the wealth level is sufficiently high that the real marginal utility from consuming all cash on hand, $\underline{\lambda}$, is below the discounted future real value of the marginal utility of money from zero savings. In this case, consumption is shifted towards the future so that current marginal utility increases and future marginal utility of money decreases until a unique level of saving is reached that results in equality between the two terms. When the wealth level is such that the real marginal utility from consuming cash on hand is above the discounted marginal utility of zero savings in the following week discounted at *r,* the individual has an incentive to increase consumption and carry negative wealth into the next week. However, the cost δ of borrowing may bring the discounted future marginal utility of zero borrowing above the marginal utility of cash on hand in which case the preference for higher consumption is not strong enough that the individual is willing to borrow at δ. When wealth is lower however, it is optimal for the individual to have a positive level of debt below the limit which decreases current marginal utility of consumption while increasing the future marginal utility of wealth. Borrowing is below the limit provided the marginal utility from maximum negative wealth in the following week is above the marginal utility of maximum consumption today. Finally an individual consumes at maximum debt when optimal consumption is so high relative to the real value of wealth, that even at maximum debt the individual would further decrease current marginal utility of consumption by borrowing if it were possible.

We assume CRRA preferences with utility function of the form

$$u[c]=\frac{1}{1-\rho}c^{1-\rho} \; for \;\; \rho>1,$$

where ρ is the coefficient of relative risk aversion and the simulated numerical solution is shown in Figure 2 for a CRRA utility function where we assume β = 0.95, and ρ = 2.5. The cost of borrowing δ, is set equal to 0.4% giving an annualised rate of 25% which is representative of the rates applicable on very short-term borrowing. Standard numerical methods of dynamic programming as outlined in Deaton (1991,1992) are used to reach the solution.

Figure 2: Solution for four weeks



The *u*-shape is evident at all levels of wealth: consumption is highest in the week when income is received, lowest in the next week and the increasing over the remaining weeks. The model replicates almost perfectly the pattern in the data in Figure 1. Hence the FES data is the clearly the outcome of optimal behaviour once we allow for precautionary motives and limits on borrowing for the type of expenditure considered here. This is despite the relatively simple utility function used in the simulation of the model. The kinks at the points where the solution switches between different regimes are clearly evident in week four and so for any given wealth level is this week we can predict perfectly the regime within which the individual will be and hence their borrowing/saving behaviour. The solution also shows that week four is the only week for which regime four can occur and again this is consistent with the infinite cost of zero consumption in the utility function that we have assumed.

5. Empirical Estimation

The Family Expenditure Survey (FES) in the UK provides expenditure patterns for two consecutive weeks for each individual in the survey and also records the dates over which this expenditure is recorded, and the level of and timing of the receipt of regular labour income. The sample of interest for the estimation of our model is those individuals who are in paid employment and receive income on a monthly basis, and from the data we can calculate at which point in their payment cycle each individual is observed. The model however, is one of non-durable consumption and so income should be interpreted as a measure of disposable income, i.e. after regular payments such as housing and heating costs, credit repayments etc. are made.[8] The FES gives very detailed information regarding loans and higher purchase schemes outstanding for all individuals but records housing and heating costs at a household level only. Therefore we include only single individuals who are heads of households, about three fifths of whom are homeowners and we can then deduct regular costs from monthly labour income to give a measure of disposable income available for non-durable consumption. This also allows us to use the household level indicator of credit card ownership rather than the individual variable as an indicator of possible liquidity constraints[9] and abstracts from the issue of resource pooling in households who receive income at different points in time. The data on these individuals are pooled for three years of the FES 1996-99 giving a total of 727 observations.

The first panel in Table 1 provides some summary statistics according to ownership of a credit card. There is a clear income effect in credit card possession with higher monthly income and living costs for card holders but still have higher disposable income. The second panel of Table 1 shows the mean of weekly expenditure as a proportion of monthly disposable income for each week in the cycle. The range of expenditure share is very large and it would seem difficult for any model to reconcile individuals who spend less than 0.05% of their disposable income in week one (twenty observations) with those who spend more than their whole monthly disposable income (seven observations) in week one. In addition our model predicts that expenditure in the first week should never be less than 0.20 of disposable income even at minimum wealth levels and for the worst price draw (see Figure 2) although in fact there are 208 observations in the data for which this is not true. Examining the means in each week

[8] In the model, wealth should also be interpreted as liquid cash on hand rather than (illiquid) saving or investments for long-term purposes, and income refers to disposable income.

[9] The recording of individual credit card ownership relies on the individual recalling paying an annual charge for their card but card possession at a household level is recorded irrespective of charges. In this sample only 60% hold a credit card when using the individual level question, while 71% do using the household question, which seems more accurate given the choice of sample i.e. single salaried employees.

reveals that the *u*-shape pattern over the month which is evident for the whole dataset and is predicted by the model is also clearly evident for this sub sample of individuals although for non-credit card holders, the effect is less obvious.

Table 3

	Credit Card holders: N=516		***Non Credit Card holders: N=211***	
	Mean	*Std Dev*	*Mean*	*Std Dev*
Age	34.7	*9.4*	33.4	*9.7*
Monthly Income	1277.27	*659.58*	893.76	*408.57*
Housing Cost	378.04	*261.58*	282.98	*170.79*
Electricity	39.39	*28.62*	29.93	*26.85*
Credit Repayments	62.44	*94.88*	43.35	*77.49*
Disposable Income	764.44	*535.05*	553.21	*367.81*

Weekly expenditure as a share of monthly disposable income

	Credit Card holders					***Non Credit Card holders***				
	N	*Mean*	**Std Dev**	*Min*	*Max*	***N***	*Mean*	**Std Dev**	*Min*	*Max*
Week 1[10]	**249**	0.23	*0.21*	0.02	2.72	**94**	0.23	*0.33*	0.02	2.71
Week 2	**263**	0.21	*0.21*	0.02	2.63	**111**	0.30	*0.38*	0.01	2.06
Week 3	**266**	0.24	*0.24*	0.01	1.70	**117**	0.25	*0.33*	0.01	2.38
Week 4	**253**	0.28	*0.40*	0.01	3.87	**178**	0.27	*0.30*	0.004	2.44

5.1 Estimation procedure

The simulated policy function show in Figure 2 is calculated over an equally spaced grid of points of W possible values for wealth and H possible values for price. Therefore the numerical solution to the policy function is a W x H table of values each corresponding to a particular combination of wealth and price. This solution allows us to relate any level of wealth and price to the optimal consumption by linear interpolation between those grid points. Thus for any given wealth, price, and parameter vector θ, we can calculate the optimal outcome as

$p_t c_t = f_t\left(w_t, p_t; \theta\right)$. We can then estimate the parameters which most closely match the observed expenditure level to that implied by this non-linear policy function. However, in the data neither wealth nor price data are available and so the estimation instead relies on the relationship between expenditure in adjacent weeks. In short, we take expenditure in week *t* and by inverting the policy function for week *t* we can find for any given price, the level of wealth combination consistent with that expenditure observation, i.e. observed expenditure will give an implied level of cash on hand $w_t = f_t^{-1}\left(p_t c_t; p_t, \theta\right)$. The success of this relies upon the monotonicity of the function if there is to be a unique association between an expenditure observation and wealth level. In addition, the concavity of the consumption function means that the solution points for expenditure for the wealth and price points on the grid are not equally spaced. However, the use of linear interpolation ensures that neither of these issues is a problem although it does require relatively large values for *W* and *P*. Once the implied wealth level w_t is calculated, the cash on hand carried into the *next* period *t+1* can be calculated using the equation for the evolution of wealth: $w_{t+1} = (1+\zeta)(f_t^{-1}\left(p_t c_t; p_t, \theta\right) + y - p_t c_t)$ where *y=0* if *t≠1*, $\zeta = \delta$ if $(w_t - p_t c_t) < 0$, and $\zeta = r$ otherwise. Conditional on expenditure in *t* and price outcomes in *t* and *t+1*, p_t, π_{t+1} respectively, expenditure in *t+1* can therefore be calculated as

$$\left(p_{t+1}c_{t+1}; p_t c_t, p_t, \pi_{t+1}, \theta\right) = f_{t+1}\left(\left(1+\zeta\right)\left(f_t^{-1}\left(p_t c_t; p_t, \theta\right) + y - p_t c_t\right), \pi_{t+1}; \pi_{t+1}, \theta\right),$$

and by taking an expectation over the price distribution in the second period, we can arrive at an expectation of expenditure for the second week of observation conditional upon the observed level of expenditure in the first week, the price outcome in the first week and the parameter vector which is given by

$$E_{\pi_{t+1}}\left(p_{t+1}c_{t+1}; p_t c_t, p_t, \theta\right) = E_{\pi_{t+1}}\left(f_{t+1}\left(\left(1+\zeta\right)\left(f_t^{-1}\left(p_t c_t; p_t, \theta\right) + y - p_t c_t\right), \pi_{t+1}; \theta\right); p_t c_t, p_t\right)$$

However the implied level of cash on hand $w_t = f_t^{-1}\left(p_t c_t; p_t, \theta\right)$ must be calculated for all possible first period prices p_t. Therefore taking an expectation again over the price distribution gives the first conditional moment of the second period expenditure as

$$\begin{aligned} m\left(p_t c_t; \theta\right) &= E_{p_t}\left[E_{\pi_{t+1}}\left(p_{t+1}c_{t+1}; p_t c_t, \theta\right)\right] \\ &= E_{p_t}\left[E_{\pi_{t+1}}\left(f_{t+1}\left(\left(1+\zeta\right)\left(f_t^{-1}\left(p_t c_t; p_t, \theta\right) + y - p_t c_t\right), \pi_{t+1}; \theta\right); p_t c_t\right)\right]. \end{aligned}$$

The second moment consists of the expectation of variance of expenditure $p_{t+1}c_{t+1}; p_t c_t, p_t, \pi_{t+1}, \theta$ around each conditional mean $E_{\pi_{t+1}}\left(p_{t+1}c_{t+1}; p_t c_t, p_t, \theta\right)$, this being

[10] The calculation for Week 1 omits an observation for which the reported expenditure is 26 times income: this observation distorted the mean and standard deviation

due to the uncertainty of prices in $t+1$, plus the expected variance around each mean conditional only on observed expenditure i.e.

$$v(p_t c_t;\theta) = E_{\pi_{\tau+1}}\left[\left(p_{t+1}c_{t+1};p_t c_t,p_t,\pi_{t+1},\theta\right) - E_{\pi_{t+1}}\left[p_{t+1}c_{t+1};p_t c_t,p_t,\theta\right]\right]^2$$

$$+E_{p_t}\left[E_{\pi_{t+1}}\left[p_{t+1}c_{t+1};p_t c_t,\theta\right] - E_{\pi}\left[E_{\pi_{t+1}}\left(p_{t+1}c_{t+1};\theta\right)\right]\right]^2.$$

The second extra terms is due to the fact that first period price is not observed to the econometrician.

Estimation is carried out using the pseudo maximum likelihood estimator (PMLE) of Gourieroux et al, (1984). The general methodology used is therefore very similar to that used by Deaton and Laroque (1995,1996) which estimated a model of commodity prices where the policy function is also dependant on two unobservable variables. The log of the PMLE written as a function of the K-vector of parameters θ and $j=1.....N$ observations in the dataset

$$\ln L = \sum_{j=1}^{N} \ln l_j(\theta) = -\frac{1}{N}\sum_{j=1}^{N}\frac{\left(p_{jt+1}c_{jt+1} - m\left(p_{jt}c_{jt};\theta\right)\right)^2}{v\left(p_{jt}c_{jt};\theta\right)} - \frac{1}{N}\sum_{j=1}^{N}\ln\left(v\left(p_{jt}c_{jt};\theta\right)\right) \quad (5.1)$$

The variance covariance matrix for the PMLE is calculated as

$$V = J^{-1}\left(G'G\right)J^{-1} \quad (5.2)$$

where G is a *(N-1)* x *K* matrix where element $G_{jk} = \frac{\partial \ln l_j}{\partial \theta_k}$ and J is a *K* x *K* matrix with element $J_{ik} = \frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_k}$.

The relationship between wealth and expenditure has been shown to be non-linear with kinks existing at the points where the solution switches between the four regimes, Therefore the likelihood will be non-differentiable with respect to the parameters at these points, but as shown in Laroque and Salanie (1994), a PMLE will result in consistent estimates of the parameters despite the non-differentiability. However, the non-differentiability leads to substantial difficulties in relying on numerical derivatives in the maximisation procedure and so we use alternative maximisation methods in the estimation.

Throughout the analysis we assume that the price distribution is the same in all periods and that draws from that distribution are *iid.* To simplify the calculations we consider H discrete values from ***F*** and these discrete values are naturally the grid points used in the numerical solution.

Thus we avoid numerical integration in calculating the expectations over price[11]. Equation 6.3 shows the calculation of the conditional expectation and variance of second period expenditure under these assumptions, for *h=1,……H* draws of price in period *t* denoted by p_t, in *t+1*, denoted by π_{t+1}. The expectations are now calculated as simply weighted averages of the outcome for each price draw.

$$
\begin{aligned}
m\left(p_t c_t;\theta\right) &= E_{p_t}\left[E_{\pi_{t+1}}\left(p_{t+1}c_{t+1};p_t c_t,\theta\right)\right] \\
&= \frac{1}{H^2}\sum_{h=1}^{H}\sum_{h=1}^{H} f_{t+1}\left(\left(1+\zeta\right)\left(f_t^{-1}\left(p_t c_t;p_h,\theta\right)-p_t c_t\right),\pi_h;\theta\right)
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
v\left(p_t c_t;\theta\right) &= \left(E_{p_t}\left[E_{\pi_{t+1}}\left(\left(\left(p_{t+1}c_{t+1};p_t c_t,\theta\right)\right)-m\left(p_t c_t;\theta\right)\right)^2\right]\right) \\
&= \frac{1}{H^2}\sum_{h=1}^{H}\sum_{h=1}^{H}\left(\left(f_{t+1}\left(\left(1+\zeta\right)\left(f_t^{-1}\left(p_t c_t;p_h,\theta\right)-p_t c_t\right),\pi_h;\theta\right)-m\left(p_t c_t;\theta\right)\right)^2\right)
\end{aligned}
$$

These conditional moments are then substituted into (5.1) to give the PMLE.

We carry out a small Monte Carlo study to investigate the properties of the PMLE in this framework using the same parameters values as previously: β = 0.95, ρ = 2.5, δ = 0.4% and r = 0. The parameter vector in the estimation procedure could include $\beta,\rho,\delta,r,\overline{d}$ and possibly the parameters of the price distribution also, but we do not try to identify all those parameters from the data and instead restrict the estimation to the coefficient of risk aversion ρ. The replications are carried out for three values of ρ: (1.5, 2.5, 5), and for two different price distributions (normal and uniform). In the case of the uniform distribution, prices can take on any of six equally spaced discrete values between 0.75 and 1.25, for which the expectation is one. We use a normal distribution $N(1,0.17^2)$ so that it has the same support as the uniform distribution but twice the variance. This is discretised by six points, each point being the mid-points of the interval within which 1/6 of the density lies. (See Deaton and Laroque, 1996). For each of these scenarios we draw 100 datasets of either 500 or 1000 observations. When estimating just one parameter, we use a simple golden-section procedure to find the maximum and find this very fast and reliable. The results are presented below.

The empirical distribution of the estimator is very close to the asymptotic variance given by (5.2) and in all cases the estimated parameter is within two standard deviations of the true value.

[11] Therefore it is possible that ***F*** is a discrete distribution or is a continuous distribution approximated

These results suggest that there are no obvious problems within the estimation procedure and all the maxima are robust to changing the starting values.

Table 2
Monte Carlo results

Uniform Distribution of Prices

ρ	1.5		2.5		5	
Sample Size	500	1000	500	1000	500	1000
Mean Estimate	1.5025	1.5030	2.5088	2.5053	4.9637	5.1430
Sample SE	0.0199	0.0138	0.1022	0.0733	0.2170	0.1552
Asymptotic SE	0.0189	0.0134	0.0984	0.0692	0.2473	0.1359

Normal Distribution of Prices

ρ	1.5		2.5		5	
Sample Size	500	1000	500	1000	500	1000
Mean Estimate	1.4746	1.5032	2.4956	2.5108	5.0861	4.985
Sample SE	0.0141	0.0150	0.0876	0.0758	0.1477	0.1981
Asymptotic SE	0.0134	0.0144	0.0734	0.0756	0.1253	0.1756

The simulated dataset gives us an idea of what we should expect to see in the actual dataset if it is to be described by our model. Some problems were identified earlier in this section when we found very small and very large observations and observations below the minimum predicted by the model. However comparison of the simulated and actual datasets shows that the change in expenditure between weeks is very large in the actual data relative to what the model predicts in the simulated dataset. This is because for large ranges of wealth, the conditional expectation given in 5.3 can be closely approximated by a straight line with a slope less than one. Figure 3a shows a histogram of the proportional change in expenditures between weeks two and three from a simulated dataset. The mean is close to 0.05% with variance of 0.126. This is in sharp contrast to the same histogram for the actual dataset from the FES where although the mean

using a method of quadrature.

change is 16% the variance is 0.838, giving rise to the large swings evident in the data. This arises either because the variance in prices in the actual dataset if far larger than we allow for or alternatively there is substantial measurement error in the data. In addition, a simple linear regression of expenditure in *t+1* on *t* gives a coefficient close to 0.75 in the simulated dataset, while in the actual data the coefficient is closer to 0.35. This is symptomatic of the attenuation bias resulting from classical measurement error and so we extend the estimation procedure to take account of that.

As a crude attempt to examine the robustness of the model we select a subsample of data which omits observations which have expenditure as a proportion of income greater than 0.8 in any week, less than 0.2 in week and have changes in expenditure between the two weeks greater than 100%. This leaves 377 data points and results of applyin the estimation method above are presented in Table 3 for different assumptions for the limit on borrowing as a proportion of income and for the two price distributions considered in the Monte Carlo results.

Table 3

Results of Estimation

	Uniform Distribution			*Normal Distribution*		
$\bar{d}$	**0.33**	**0.66**	**0.9**	**0.33**	**0.66**	**0.9**
ρ	3.9694	4.1774	3.8869	4.3827	4.666	4.3281
Asymptotic SE	0.2887	0.3191	0.3662	0.4395	0.3997	0.2262

Figure 3a Simulated data: proportional change in expenditure between second and third week

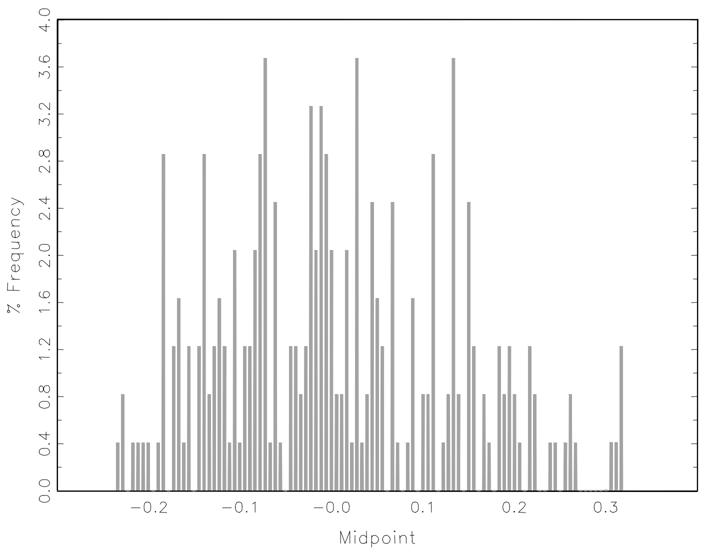
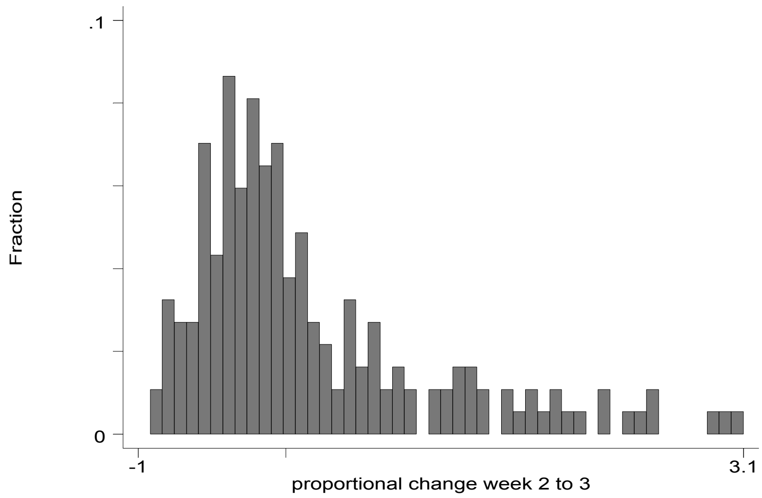


Figure 3b FES data: proportional change in expenditure between second and third week



While we do emphasize that these results are not definitive, they give an indication of how robust the model is. Estimates of the coefficient of risk aversion close to 4 are well within those found previously in the literature and this is very reassuring. We believe that once measurement error in the data is taken account of, the estimates will be very robust.

5.2 Measurement Error

If we assume that classical measurement error exists in the data, then the observed expenditure is the sum of true expenditure plus noise i.e. $\tilde{p}_t c_t = p_t c_t^* + \varepsilon_t$ such that $E\left(p_t c_t^*, \varepsilon_\tau\right) = 0$. Therefore the conditional expectation of interest is now

$$m\left(\tilde{p}_t c_t;\theta\right) = E_{p_t}\left[E_{\pi_{t+1}}\left(\tilde{p}_{t+1} c_{t+1}; \tilde{p}_t c_t, \theta\right)\right]$$

and the variance

$$v\left(\tilde{p}_t c_t;\theta\right) = E_{p_t}\left[E_{\pi_{t+1}}\left(\left(\left(\tilde{p}_{t+1} c_{t+1};\tilde{p}_t c_t,\theta\right)\right) - m\left(\tilde{p}_t c_t;\theta\right)\right)^2\right].$$

Thus the parameters are estimated by minimising the distance between the observed (noisy) expenditure in *t+1* and the expectation of expenditure in *t+1* period based on a contaminated observation in *t*. However, the methodology outlined above relates true expenditure observations in different weeks. Nevertheless if we assume a joint normal distribution between the contaminated observation and the error itself, then the expectation can still be calculated. Assume that the joint distribution is

$$\left(\begin{pmatrix}\varepsilon_t \\ \tilde{p}_t c_t\end{pmatrix} \sim N\left(\begin{pmatrix}0 \\ \mu_{p_t c_t}\end{pmatrix}\begin{pmatrix}\sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2 + \sigma^2_{p_t c_t}\end{pmatrix}\right)\right)$$

and thus the conditional distribution of the error given a noisy observation is

$$\varepsilon \mid \tilde{p}_t c_t \sim N\left(\frac{\sigma^2}{\sigma^2 + \sigma^2_{p_t c_t}}\left(\tilde{p}_t c_t - \mu_{p_t c_t}\right), \frac{\sigma^2 \sigma^2_{p_t c_t}}{\sigma^2 + \sigma^2_{p_t c_t}}\right).$$

For any observation and a corresponding value of $\varepsilon_t \mid p_t c_t$, $\tilde{p}_t c_t - \left(\varepsilon \mid \tilde{p}_t c_t\right)$ gives the "true" observation and so the procedure outlined in 5.1 can be used to calculate the conditional expectation as

$$m\left(\tilde{p}_t c_t;\varepsilon \mid \tilde{p}_t c_t,\theta\right) = m\left(\tilde{p}_t c_t - \left(\varepsilon \mid \tilde{p}_t c_t\right);\varepsilon \mid \tilde{p}_t c_t,\theta\right)$$

This can however be integrated over the conditional distribution of ε_t and therefore the conditional expectation allowing for the measurement error becomes

$$m\left(\tilde{p}_t c_t;\theta\right) = E_{\varepsilon_t \mid \tilde{p}_t c_t} m\left(\tilde{p}_t c_t - \left(\varepsilon \mid \tilde{p}_t c_t\right);\theta\right)$$

and similarly the variance can be written as

$$v\left(p_t c_t;\theta\right) = E_{\varepsilon \mid \tilde{p}_t c_t}\left(E_{\pi_t,\pi_{\tau+1}}\left(p_{t+1} c_{t+1}{}^2;\varepsilon \mid \tilde{p}_t c_t,\theta\right)\right) + \sigma^2 - m\left(\tilde{p}_t c_t;\theta\right)^2.$$

Thus the first and second moments conditional on observed expenditure can be calculated provided an expectation is also taken over the conditional distribution of ε_t. We use Gauss-Hermite quadrature to approximate the conditional distribution of ε_t. The parameter vector now consists of the risk aversion parameter ρ and the variance of the measurement error σ^2.

The difference between the conditional expectation with and without measurement error is shown in Figure 4 where the variance of the measurement error is assumed to be 0.1. Clearly this is significant. Again a Monte Carlo study was carried out to investigate the performance of the estimator and the (preliminary and incomplete) results are given in Table 4.

Table 4

Monte Carlo results

Uniform Distribution of Prices

ρ	1.5		2.5	
σ	0.01	0.05	0.01	0.1
Sample Size	1000	1000	1000	1000
ρ				
Mean Estimate	1.52205	1.4999	2.6561	2.7354
Sample SE	0.0357		0.1184	
Asymptotic SE	0.0559		0.202	
σ				
Mean Estimate	0.0098	0.0501	0.0092	0.0973
Sample SE	0.0038	0.0018	0.0090	0.0042
Asymptotic SE	0.0049	0.0033	0.0148	0.0040

These results are very encouraging: except for the first estimate of $\rho = 2.5$, all of the estimates are within a standard error of the true values. The discrepancy between the sample and asymptotic standard error warrants further investigation. However, we offer two possible explanations for this. Firstly, we have only 100 replications in each case and while this was sufficient in the simpler case which ignored measurement error, this may not be the case here and so a larger number of replications may solve the problem. In addition, given the non-differentiabilities in the likelihood, the calculation of the asymptotic variance-covariance is relies on numerical derivatives and is sensitive to the step size used in the calculations and so it may be that at each maximum a different step size would lead to a different value for the asymptotic standard error. Clearly these results are incomplete and we intend to extend the Monte Carlo to other values for the parameters etc.

When applying these estimation techniques to the actual data to hand, we chose to firstly to reparameterise the model. This ensures that the values of ρ do not go towards one because the calculation of the numerical solution then breaks down. Also the estimated standard deviation of the measurement error must be positive and less than the value for the standard deviation of the observed expenditure $\left(\sigma^2_{\tilde{p}_t c_t} = \sigma^2 + \sigma^2_{p_t c_t}\right)$ and so for this reason we re-parameterised the model so that the estimated parameter vector θ is defined by

$$\rho = \underline{\rho} + \left(\left(\bar{\rho} - \underline{\rho}\right)/1 + \exp\left(-\theta_1\right)\right)$$

$$\sigma = \underline{\sigma} + \left(\left(\bar{\sigma} - \underline{\sigma}\right)/1 + \exp\left(-\theta_2\right)\right)$$

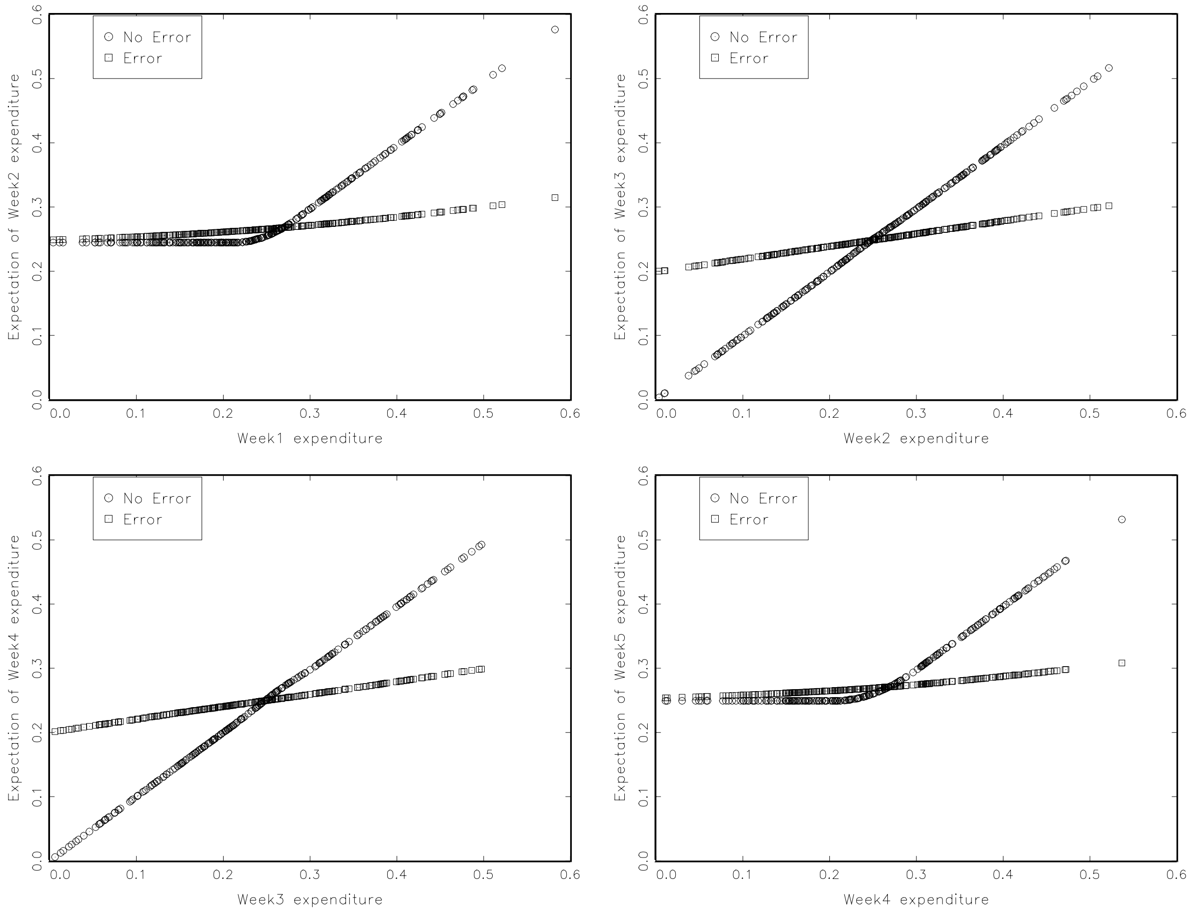
such that $\left[\underline{\rho}, \bar{\rho}\right] = [1.5, 11]$ and $= [\bar{\sigma}, \underline{\sigma}] = \left[1e-05, \sigma^2_{\tilde{p}_t c_t}\right]$. Preliminary results for these estimations are shown in Table 5. We assume that the borrowing limit is set to 90% of disposable income and that prices are normally distributed N(1, 0.1225). We exclude all observations where the proportion of expenditure is greater than total disposable income in any one week (29 observations). These observations triple the standard deviation of the data while increasing the mean of expenditure by 0.06. In the first column of Table 6 we also excluded observations for which expenditure in week 1 is less than 10% of disposable income (59). The last row of the table shows the standard deviation of the expenditure data which we condition on and so from the estimates for σ above, we see that measurement error accounts for close to 50% of it. These results are very preliminary and further estimation is ongoing.

<u>Table 5</u>

<u>Results of Estimation</u>

Price Distribution	N(1,0.122)	N(1,0.122)	N(1,0.04)	N(1,0.01)
N	594	653	653	653
mean	2.92411	2.90767	2.8158	2.75589
loglikelihood				
ρ	6.89213	10.305	6.5503	7.098
Asymptotic SE				
σ	0.07143	0.07768	0.1083	0.1177
Asymptotic SE				
$\sigma^2_{\tilde{p}_t c_t}$	0.1532	0.1515	0.1515	0.1515

Figure 4 Conditional expectation with measurement error.



6. Conclusion

In this paper, we have used the basic prediction of consumption theory: that consumers will make the discounted utility value of expenditure equal in every period, to characterise behaviour in a general framework. We can exactly predict consumer behaviour when there are market imperfections with respect to the quantity available and price of credit and when there is uncertainty with respect to expenditure levels. The model examined proposed is one of optimal consumer behaviour of an individual who makes consumption decisions over the short-run when there are credit market constraints and when income is received either at the same time as when consumption decisions are made, or when there is periodic receipt of income. We arrive at the stationary solution following Deaton and Laroque, (1992) and Bailey and Chambers, (1996), and that solution allows us to characterise the behaviour of any consumer for given interest rates, discount rate, real wealth level and set of expectations. The solution suggests that individuals are prudent when they face a large amount of uncertainty but as the uncertainty is revealed over time, they increase consumption. The pattern of consumption observed in the data is consistent with the predictions of the model for even a relatively simple utility function. However, we believe that measurement error is a significant problem in the data and allow the estimation to take account of this. The effect on the first conditional moment of the model is significant and preliminary estimates suggest that measurement error accounts for about 50% of the standard deviation in the data. Our estimated values of ρ are quite large but further estimation is ongoing.

7. References:

Ausubel, L.M., (1991), "The Failure of Competition in the Credit Card Market", *American Economic Review*, March, Vol. 81 No. 1, pp. 50-81.

Benveniste L.M. and J.A. Scheinkman, (1979), "On the Differentiability of the Value Function in Dynamic Models of Economics", *Econometrica*, Vol. 47, No. 3, pp. 727-32.

Blackwell, D. (1965), "Discounted dynamic programming", *Annals of Mathematical Statistics*, Vol. 36, pp. 331-336.

Blundell R. and T. Stoker, (1999), "Consumption and the Timing of Income Risk", *European Economic Review*, Vol. 43, No. 3, pp. 475-507.

Browning M. and A. Luscardi, (1996), "Household Saving: Micro Theories and Micro Facts", *Journal of Economic Literature*, Vol. 24, pp. 1797 – 1855.

Calem P. S., and L. J. Mester, (1995), "Consumer Behaviour and the Stickiness of Credit-Card Interest Rates", *American Economic Review*, Vol. 85, No. 5, pp. 1327-1336.

Chambers M.J. and R. Bailey, (1996), "A Theory of Commodity Price Fluctuations", *Journal of Political Economy,* Vol. 104, No. 5, pp. 924-956.

Carroll C. and M. Kimball, (2001), "Liquidity constraints and precautionary saving", NBER Working Paper no. 8496.

Deaton, A., (1991), "Saving and Liquidity constraints", *Econometrica,* Vol. 59, No. 5, pp. 1221-1248.

Deaton, A. and G. Laroque, (1992), "On the Behaviour of Commodity Prices", *Review of Economic Studies,* Vol. 59, pp. 1-23.

_________, (1995), "Estimating a Non-Linear Rational Expectations Commodity Price Model with Unobservable State Variables", *Journal of Applied Econometrics,* Vol. 10, pp. S9 – S40.

_________, (1996), "Competitive Storage and Commodity Price Dynamics", *Journal of Political Economy,* Vol. 104, pp. 897-923.

Gourieroux, C., A. Monfort, and A Trognon, (1984), "Pseudo Maximum Likelihood Methods: Theory", *Econometrica*, Vol. 52, No. 3, pp. 681–700.

Hartley, P. R., (1996), "Value function approximation in the presence of uncertainty and inequality constraints: An application to the demand for credit cards", *Journal of Economic Dynamics and Control*, Vol. 20, pp. 63-92.

Hall R. E. and F. S. Mishkin, (1982), "The Sensitivity of Consumption to Transitory Income: Estimate from Panel Data on Households", *Econometrica*, 50, pp. 461-481.

Japelli, T., J.S Pischke and N.S. Souleles (1998), "Testing for liquidity constraints in Euler equations with complementary data sources", *The Review of Economics and Statistics,* Vol. 80, pp 251 – 262.

Kimball, M. S., (1990), "Precautionary Saving in the Small and in the Large", *Econometrica,* Vol. 58, No. 1, pp. 53-73.

Laroque, G. and B. Salanie, (1994), “Estimating the canonical disequilibrium model: asymptotic theory and finite sample properties”, *Journal of Econometrics,* vol. 62, pp. 165-210.

Michaelides, A. and S. Ng, (2000). “Estimating the rational expectations model of speculative storage: A Monte Carlo comparison of three simulation estimators”, *Journal of Econometrics,* vol.96, pp. 231-266.

Parker, J. A., (1999), “The Reaction of Household Consumption to Predictable Changes in Social Security Taxes”, *American Economic Review*, Vol. 89, No. 4, pp. 959-973.

Skinner, J. (1998), “Risky Income, Life Cycle Consumption and Precautionary Savings” *Journal of Monetary Economics*, Vol. 22, pp. 237-255.

Souleles, N. S. (1999), “The Response of Household Consumption to Income Tax Refunds”, *American Economic Review*, Vol. 89, No. 4, pp. 947-958.

Zeldes, S. (1989a), “Optimal Consumption with Stochastic Income Deviations from Certainty Equivalence”, *Quarterly Journal of Economics*, May, pp. 275-297.

Zeldes, S. (1989b), “Consumption and Liquidity constraints: An Empirical Investigation*”,* *Journal of Political Economy*, Vol. 97, No.2, pp. 305-346.