RATIONAL CHOICE: THE CASE OF PATH DEPENDENT PROCEDURES

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ABSTRACT

Describing a procedure in which choice proceeds in a sequence, we propose two alternatives ways of resolving the decision problem whenever the outcome is sequence-sensitive. One way yields a rationalizable choice set, and the other way produces a weakly rationalizable choice set that is equivalent to von Neumann-Morgenstern's stable set. It is shown that for quasi-transitive rationalization, the maximal set must coincide with its stable set.

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1. INTRODUCTION

When an agent has to choose from a set of many alternatives, usually the final choice is the outcome of a process in which elements are compared in a sequence of alternatives or smaller sets. A choice procedure is a combination of a choice and a process. Since a process is involved, the set from which the final outcome emerges is not necessarily fixed. This led Plott (1973), Sertel and Van der Bellen (1980) and others to study rational choice in terms of a path independent choice procedure. They showed that a path independent choice function satisfying the Sen condition, which requires that an alternative being chosen from each subset must be chosen from their union, is necessarily rational. If the final choice is an outcome of a process, in which alternatives are compared in a sequence of alternatives or smaller sets, the requirement of path independence may turn out to be somewhat stringent in the sense that very few choices in real life are routewise invariant. The purpose of this paper is to characterize the outcome of a choice procedure that may turn out to be sequence sensitive. In other words, it would be interesting to know whether rationalization of the resulting choices can be tied to sequence dependence.

We propose two alternative ways of resolving the decision problem when the outcome of a choice procedure is path dependent. One way is to consider the set of alternatives that are chosen at the end of a choice process for every sequence. The other way is to consider all the alternatives that are chosen for at least one sequence. The set of alternatives that emerges under any other procedure for resolving the decision problem for a sequence sensitive choice process must be in between the sets of alternatives that emerge under our two proposed methods. So, our two proposed procedures provide lower and upper bounds on the final choice set in the sense that the final outcome corresponding to all other ways of resolving the decision problem must be in between these two sets. Clearly, for a path independent choice function, these two sets would coincide.

We first consider the set of alternatives that are chosen at the end of a choice process for every sequence and call it the *core* of a set presented for choice. We show that every element of the core dominates or is preferred to every element of its complementary set. That is, the core is indeed the set of best elements. Next, consider the union over the set of alternatives that we obtain at the end of a choice process for every sequence. If the procedure is path dependent, this set may include alternatives that are rejected in one of these sequential comparisons. Since a chooser, facing the difficulty in making a choice from a larger set, adopts a procedure of comparing smaller set of alternatives at a time, in that case it may be prudent for a chooser to repeat the procedure over the union of the chosen elements for every sequence, until there is no further contraction of the set. This irreducible set of alternatives is called the *upper core* of a set presented for choice. We show that if a choice function has some specific structure, then every element of the set, which is a complement of the upper core, is dominated by or is inferior to some element of the support of that set. That is, the upper core is indeed the von Neumann - Morgenstern's stable set.[1] It is shown that for quasi-transitive rationalization, the set of best elements a choice function must coincide with its stable set.[2]

2. PRELIMINARIES

Let X be a finite set of alternatives. For every $A \subseteq X$, $[A]$ denotes the set of all nonempty subsets of A. We consider the situation where there is a well-specified family S of subset of $[X]$; i.e., $S \subseteq [X]$.

DEFINITION 1. A *choice function* is a rule $C(.)$ that, for every $A \in S$, specifies a nonempty subset of A, i.e., $\emptyset \neq C(A) \subseteq A$.

In words, a choice function is a rule that specifies a subset from a fixed set of alternatives presented for choice. Note that the subset so specified may contain more than one element. For example, demand functions can be regarded as special cases of choice functions. *We assume that for all $A \in [X]$ such that $|A| \leq 2$, a choice function is defined.*[3]

DEFINITION 2. Let Q be a binary relation defined over X, and also let Q* be a subrelation of Q such that for all $x_1,x_2 \in X$, $[x_1Q^*x_2 \leftrightarrow (x_1Qx_2 \;\&\; \sim x_2Qx_1)]$. Q is said to be: (i) *reflexive* iff, for all $x_1 \in X$, x_1Qx_1; (ii) *connected* iff, for all distinct $x_1,x_2 \in X$, x_1Qx_2 or x_2Qx_1; (iii) *acyclic* iff, for all $x_1,x_2,\ldots,x_n \in X$, $[(x_1Q^*x_2 \;\&\; x_2Q^*x_3 \;\&\ldots\&\; x_{n-1}Q^*x_n) \rightarrow x_1Qx_n]$; (iv) *quasi-transitive* iff, for all $x_1,x_2,x_3 \in X$, $[x_1Q^*x_2 \;\&\; x_2Q^*x_3) \rightarrow x_1Q^*x_3]$; (v) *suborder* iff it is reflexive, connected and acyclic.

DEFINITION 3. Let $A \in [X]$. For $x \in A$, x is said to be a *best element* of A with respect to a binary relation Q iff xQy for all $y \in A$. Then the *set of best elements* in A with respect to Q, $M(A,Q) = \{x|x \in A \text{ and } xQy \text{ for all } y \in A\}$.

DEFINITION 4. A choice function $C(.)$ is said to be *rational* if there is a suborder Q on X such that for all $A \in S$, $C(A) = M(A,Q)$. In addition, if Q is quasi-transitive, then $C(.)$ is said to be *quasi-transitive rational*.

In other words, a suborder Q on X is said to rationalize a choice function $C(.)$ if the set of chosen elements is always the set of Q-best elements of any set $A \in [X]$. The relation Q is then called the rationalization of $C(.)$. If a choice function $C(.)$ is rational with Q being quasi-transitive, we say Q is a quasi-transitive rationalization of $C(.)$.

DEFINITION 5. Let $C(.)$ be a choice function. For all $x,y \in X$, a binary relation R is defined: xRy iff $x \in C(\{x,y\})$.[4] Then xPy iff [xRy and ~ yRx], and xIy iff [x = y] or [xRy and yRx]. Let a binary relation G be defined on X by: xGy iff [xIy] or [xPy].

3. RATIONALIZATION AND WEAK RATIONALIZATION

The works of Afriat (1967), Plott (1973) and Sertel-Van der Bellen (1980) and others showed that the path independence of a choice procedure does not guarantee that the final choice at the end of the process is necessarily rational. Plott's two-stage choice procedure is a mechanism of "divide and conquer," where a set of alternatives is divided into subsets, a choice is made from each of these subsets and then a final choice is made over the chosen elements in the first round. Afriat and Sertel-Van der Bellen introduced a notion of sequential choice procedure. According to Sertel-Van der Bellen (SV), choice proceeds in a sequence of a finite set of alternatives by considering the first two alternatives over which a choice is made, then choosing from the union of the chosen element(s) with the third alternative, then choosing from the union of the chosen element(s) with the next alternative, and so on, until all the alternatives have been considered.[5] It is well-known that the path independence of neither the two stage procedure nor the SV procedure guarantees rational choice.[6]

We now introduce an alternative choice procedure, which is a variant of a binary sequential procedure introduced by Bandyopadhyay (1988). In that procedure choice proceeds in a sequence of elements of a set of alternatives presented for choice. After choosing from the first two elements of the sequence (or path), consider the third element of the sequence and compare it pairwise with every chosen element of the first pair, and collect these chosen elements. Then for every chosen element of the earlier round, compare with the next element of the sequence, and repeat the procedure until all the alternatives have been considered.[7] We make the following observations in the description of the binary sequential procedure. Consider a sequence $\langle x,y,z,w \rangle$. Suppose $C(\{x,y\}) = \{x,y\}$. Also suppose $C(\{x,z\}) = \{x\}$ and $C(\{y,z\}) = \{z\}$. Clearly, z is rejected in the pairwise comparison with x. However, following the binary sequential choice procedure, once again z is to be compared with w. Clearly, this sequential procedure allows a rejected alternative to be carried over to the next round for further consideration. Since Chernoff (1954) showed that a rational choice set never contains an element that is rejected in a pairwise comparison, this procedure clearly allows some inefficiency in making a decision. Furthermore, the binary sequential procedure requires every alternative chosen in an earlier round is to be compared pairwise with the next element of the sequence even after it is revealed that the new element of the sequence is dominated by some chosen alternatives of the earlier round, which in turn facilitates the inclusion of a dominated alternative in the surviving set at a particular stage. Clearly, the procedure is not only inefficient, it is costly as well. So we propose a refinement in which choice proceeds in a sequence (or path) of elements of a set of alternatives presented for choice. After choosing from the first two elements of the sequence, consider the third element of the sequence and compare it pairwise with every chosen element of the first pair. However, if the third element is rejected by any alternative chosen from the first pair, then ignore the third element and go on to the fourth element. The intuition is that if both are chosen from the first pair then they are considered to be equally preferred, and since the third element of the path is dominated by one of the chosen elements of the first pair, it is simply an unnecessary cost of comparing the third element with the remaining equally preferred surviving alternative of the first round. Otherwise collect all the alternatives chosen in the pairwise comparisons with the third element of the sequence. Then for every chosen element of the earlier round, compare sequentially with the next element of the path, and repeat the procedure until all the elements of the path have been considered. Note that, at any stage, the process of pairwise comparisons continues until the new element of the path is not defeated by any surviving alternative of the earlier round. Clearly, in the description of this procedure some degree of optimality is built-in. For example, comparing pairwise every element chosen in the (i-1)th round with the ith element of the sequence, if the ith element is rejected in any pairwise comparison, then the procedure suggests to ignore the ith element of the sequence and tells us to consider the (i+1)th element of the sequence. Furthermore, following the refined binary sequential choice procedure, a chooser can avoid the cost of comparing the new element of a sequence with all surviving alternatives of the earlier round once it is revealed that the new element is dominated by someone chosen earlier.

For a formal description of a choice procedure, we require some additional notation. For an $A \in [X]$, let $T_k = \langle x_1(k), x_2(k), \ldots, x_{|A|}(k) \rangle$ be an ordered set of elements of A. T_k will be called a *path* for A where, k denotes a particular path. Let $\Sigma(A)$ be the set of all paths for A.

DEFINITION 6. Let $C(.)$ be a choice function. The *refined binary sequential choice procedure* is defined to be a function g, which for all $A \in [X]$ and all $T_k \in \Sigma(A)$, specifies a subset $g(T_k)$ of A such that $g(T_k) = T_k T^{|A|}$ where, $T_k T^2 = C(\{x_1(k), x_2(k)\})$ and for $i \in \{3,4,\ldots,|A|\}$, $T_k T^i = \bigcup_{a \in T_k T^{i-1}} C(\{x_i(k), a\})$ whenever $x_i(k) \in C(\{x_i(k), a\})$ for all $a \in T_k T^{i-1}$; otherwise, $T_k T^i = T_k T^{i-1}$.[8]

Note that for a single-valued choice function, the refined binary sequential choice procedure is equivalent to the SV choice procedure.

Suppose for a set $A \subseteq [X]$ and for the paths $\sigma_k, \sigma_{k'} \in \Sigma(A)$, $\sigma_k T^{|A|} \neq \sigma_{k'} T^{|A|}$. In that case, one may propose two alternative ways of resolving the decision problem. One way is to consider the set of alternatives that are obtained at the end of a choice procedure for every path that belongs to the set $\Sigma(A)$, i.e., the set $\bigcap_{\sigma_k \in \Sigma(A)} \sigma_k T^{|A|}$.

DEFINITION 7. Let $C(.)$ be a choice function, and g be the refined binary sequential choice procedure. For all $A \subseteq [X]$, the *core* of a set A, $\underline{C}(A)$, is defined iff $\underline{C}(A) = \bigcap_{\sigma_k \in \Sigma(A)} \sigma_k T^{|A|}$.[9]

DEFINITION 8. For all $A \in S$, a choice function $C(.)$ satisfies the *core property* (CP) iff $C(A) = \underline{C}(A)$.

THEOREM 1. *Every choice function $C(.)$ is rational if and only if it satisfies the core property.*

When the outcome of a choice procedure is path dependent, the other possible way to resolve the decision problem is to consider all alternatives that are obtained at the end of a choice procedure for at least one sequence, i.e., the set $\bigcup_{\sigma_k \in \Sigma(A)} \sigma_k T^{|A|}$. Let $B^0 = \bigcup_{\sigma_k \in \Sigma(A)} \sigma_k T^{|A|}$. Since we set out to make a choice from the set A, that led us to adopt a choice procedure, and as a consequence we end up with the set B^0. We repeat the procedure to choose from the set B^0 as we chose from the set A, and obtain the set B^1, where $B^1 = \bigcup_{\sigma_k \in \Sigma(B^0)} \sigma_k T^{|B0|}$. If $B^1 \neq B^0$, we repeat the procedure until we obtain the set B^i such that $B^i = B^{i+j}$ for $j \geq 0$ where $B^i = \bigcup_{\sigma_k \in \Sigma(B^{i-1})} \sigma_k T^{|Bi-1|}$. Clearly, B^i is a set such that $\bigcup_{\sigma_k \in \Sigma(B^{i-1})} \sigma_k T^{|Bi-1|} = \bigcup_{\sigma_k \in \Sigma(B^{i+j-1})} \sigma_k T^{|Bi+j-1|}$ for $j \geq 0$. The set B^i is said to be the *upper core* of a set A, and will be denoted by $\mathcal{U}(A)$. Formally,

DEFINITION 9. Let $C(.)$ be a choice function, and g be the refined binary sequential choice procedure. For an integer $i > 0$, let $B^i = \bigcup_{\sigma_k \in \Sigma(B^{i-1})} \sigma_k T^{|Bi-1|}$ where, $B^0 = \bigcup_{\sigma_k \in \Sigma(A)} \sigma_k T^{|A|}$. For all $A \subseteq [X]$, the *upper core* $\mathcal{U}(A)$ is defined as follows: $\mathcal{U}(A) = B^i$, if $B^i = B^{i+j}$ for $j \geq 0$.

DEFINITION 10. For all $A \in S$, a choice function $C(.)$ satisfies the *upper core property* iff $C(A) = \mathcal{U}(A)$.

To characterize the upper core of a choice function $C(.)$, we introduce the notion of a stable set, originally introduced by von Neumann - Morgenstern (1947).

DEFINITION 11. Let $A \in [X]$ and let Q be a given reflexive and connected binary relation defined over A. Given Q, a *stable set* of A is a subset V of A such that (i) for all $x,y \in V$, xQy; and (ii) for all $z \in A \backslash V$, there exists $y \in V$ such that yQ$^*z$. The set $V(A$,Q) denotes the stable set.

DEFINITION 12. For all $A \in S$, a choice function is said to be *weakly rationalizable* iff there exists a reflexive and connected binary relation Q defined on X such that $C(A) = V(A$,Q).

von Neumann and Morgenstern (1947) showed that for any suborder Q on X, there exists a unique stable set $V(A$,Q) for every A in $[X]$. However, the example below shows that if for A in $[X]$, $M(A$,Q) = ∅, then $V(A$,Q) is not unique.

EXAMPLE 1. Let $C(.)$ be a choice function. For $A = \{x,y,z,w\}$, let $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, $C(\{z,w\}) = \{z\}$, $C(\{x,z\}) = \{x,z\}$ and $C(\{y,w\}) = \{y,w\}$. Clearly, $M(\{x,y,z,w\}$,G) = ∅, however, both $\{x,z\}$ and $\{y,w\}$ are the stable sets. This example also shows that the requirement of $M(A$,Q) $\neq$ ∅ is not necessary for the existence of a stable set $V(A$,Q).

The next result establishes relationship between the stable set and the upper core of a choice function.

THEOREM 2. *For every choice function C(.) and every A in S, there exists a suborder* Q *such that* &(*A*)= $V(A$,Q) iff $\underline{C}(A) \neq$ ∅.

We now present the weak rationalization result.

THEOREM 3. *For every A in S, let* $\underline{C}(A) \neq$ ∅. *Then a choice function C(.) is weakly rational if and only if it satisfies the upper core property.*

Now a remark is in order. The examples below show that for any set A in S, the requirement of nonempty core of a choice function cannot be dispensed with.

EXAMPLE 2. Let $C(.)$ be a choice function. For $X = \{x,y,z\}$, suppose C($\{x,y\}$) = $\{x\}$, C($\{y,z\}$) = $\{y\}$, C($\{x,z\}$) = $\{z\}$ and C($\{x,y,z\}$) = $\{x,y,z\}$. It is easy to check that, for all $A \in S$, $C(A)$ = &(A). However, there does not exist any suborder Q such that $C(.)$ is weakly rationalizable.

EXAMPLE 3. Let $C(.)$ be a choice function. For $X = \{x,y,z,w\}$, let $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y,z\}$, $C(\{z,w\}) = \{w\}$, $C(\{y,w\}) = \{y\}$, $C(\{x,w\}) = \{x,w\}$, $C(\{x,z\}) = \{z\}$, $C(\{x,y,z,w\}) = \{x,w\}$ and for $A \in S$ such that $|A| = 3$, $C(A) = \underline{C}(A)$. It is easy to check that $C(A) = V(A,G)$ for all $A \in S$ and $C(\{x,y,z,w\}) = \emptyset$. Clearly, &$(\{x,y,z,w\}) = \{x,y,z,w\} \neq C(\{x,y,z,w\})$.

Finally, we characterize the quasi-transitive rationalization of a choice function.

THEOREM 4. *A choice function $C(.)$ is quasi-transitive rational if and only if the core coincides with the stable set.*

Clearly, for a single valued choice function, the conditions that are necessary and sufficient respectively for quasi-transitive and acyclic rationalization are all equivalent to the condition that is necessary and sufficient for weak rationalization, i.e., all of these rationalizable set is identical to the von Neumann - Morgenstern's stable set. Furthermore, all of our results can be generalized utilizing the notion of refined sequential choice procedures.

4. SUMMARY

In the event that the set of alternatives survived at the end of the choice process is not the same for every sequence, theorem 1 shows that, for a fixed set of alternatives, if a choice procedure specifies only the core of a set of alternatives, then those alternatives are the set of best elements. In a similar situation, theorem 3 shows that, for a fixed set of alternatives, if a choice function specifies only the upper core of a set of alternatives, then it is indeed a unique stable set, provided the choice function is rationalizable. Theorem 2 establishes the relation between the stable set and the set of best elements. Finally, theorem 4 shows that the stable set of a refined binary sequential choice procedure will be identical to the set of best elements if and only if the choice function is quasi-transitive rational. All of these results serve to characterize the outcome of a choice process whether or not it is sequence sensitive.

5. PROOF OF THE THEOREMS

PROOF OF THEOREM 1. Let Q rationalize a choice function, $C(.)$, so for all $A \in S$, $C(A) = M(A,Q)$. We first show that every rational choice function satisfies CP. Let $x \in C(A)$. Then xQy for all $y \in A$. Clearly, in that case, $x \in T_k T^{|A|}$ for all $T_k \in \mathsf{S}(A)$. Now, let $x \in \bigcap_{T_k \in \mathsf{S}(A)} T_k T^{|A|}$ and suppose $x \notin C(A)$. Then, by rationality, there exists $y \in A$ such that yQ^*x. Consider a path $T_k \in \mathsf{S}(A)$ such that the first two elements of the path T_k are y and

x. Clearly, $x \notin \tau_k T^{|A|}$. Therefore, $x \notin \bigcap_{\tau_k \in \Sigma(A)} \tau_k T^{|A|}$, a contradiction.

Next, we show that a $C(.)$ satisfying CP is necessarily rational. For $A \in S$, let $C(A) = \bigcap_{\tau_k \in \Sigma(A)} \tau_k T^{|A|}$. We consider the binary relation G for possible rationalization. Connectedness and reflexivity of G are trivial; we show that the relation P is acyclic. Let $A = \{x_1, x_2, \ldots, x_n\}$. Let $x_1 P x_2$ & $x_2 P x_3$ & $x_3 P x_4$ &...$x_{n-1} P x_n$. We now show that if $x_n P x_1$, then, given CP, $\bigcap_{\tau_k \in \Sigma(A)} \tau_k T^{|A|} = \emptyset$. Suppose not. Let $x_i \in \bigcap_{\tau_k \in \Sigma(A)} \tau_k T^{|A|}$. Then there are two possibilities: (a) $i \neq 1$ or (b) $i = 1$. Consider (a). For $i \neq 1$, consider a path $\tau_k \in \Sigma(A)$ such that x_{i-1} and x_i are the first two elements of the path. Clearly, given $x_{i-1} P x_i$, $x_i \notin \tau_k T^{|A|}$. Therefore, $x_i \notin \bigcap_{\tau_k \in \Sigma(A)} \tau_k T^{|A|}$, a contradiction. Consider (b). For $i = 1$, consider a path $\tau_k \in \Sigma(A)$ such that x_n and x_1 are the first two elements of the path. Given $x_n P x_1$, clearly, $x_1 \notin \tau_k T^{|A|}$, which once again would lead to a contradiction.

Now, we show that G is a rationalization of $C(.)$, i.e., for any given set $A \in S$, the set $M(A,G) = C(A)$. Let $x \in M(A,G)$. This implies that xGy for all $y \in A$. In that case, $x \in \bigcap_{\tau_k \in \Sigma(A)} \tau_k T^{|A|}$. By CP, $x \in C(A)$. Therefore, $x \in M(A,G)$ implies $x \in C(A)$. Now, suppose $x \in C(A)$ and $x \notin M(A,G)$. Then there exists $y \in A$ such that yPx. Given yPx, consider a path $\tau_k \in \Sigma(A)$ such that the first two elements of the path are y and x. Clearly, $x \notin \tau_k T^{|A|}$ which implies $x \notin \bigcap_{\tau_k \in \Sigma(A)} \tau_k T^{|A|}$. Therefore, by CP, $x \notin C(A)$, a contradiction. Thus, $x \in C(A)$ implies $x \in M(A,R)$. Hence, $C(A) = M(A,G)$.■

To establish the relationship between the stable set and the upper core we need a preliminary result.

LEMMA 1. Let $C(.)$ be a choice function. Let $\underline{C}(A) \neq \emptyset$ for all $A \in [X]$. Then $A \backslash \underline{C}(A) \neq \emptyset$ implies that there exists an $x \in A \backslash \underline{C}(A)$ such that $x \notin \bigcup_{\tau_k \in \Sigma(A)} \tau_k T^{|A|}$.

PROOF. Since $\underline{C}(A) \neq \emptyset$, there exists a relation G over X such that $M(A,G) = \underline{C}(A)$. Clearly, if $A \backslash \underline{C}(A) \neq \emptyset$, there exists $x \in A \backslash \underline{C}(A)$ and $y \in \underline{C}(A)$ such that yPx. Now, for any $\tau_k \in \Sigma(A)$, $y \in \tau_k T^{|A|}$. Clearly, for any $\tau_k \in \Sigma(A)$, there are two possibilities: either y comes after x or x comes after y. Since yPx, in either case, $x \notin \tau_k T^{|A|}$. Hence, $x \notin \bigcup_{\tau_k \in \Sigma(A)} \tau_k T^{|A|}$.■

PROOF OF THEOREM 2. Let $C(.)$ be a choice function. Recall from theorem 1, $\underline{C}(A) \neq \emptyset$ for all $A \in [X]$ implies $C(A)$ is rational. We first show that if $\mathcal{C}(A) = V(A,Q)$ for all $A \in [X]$, then $[x_1 Q^* x_2$ & $x_2 Q^* x_3$ &...& $x_{n-1} Q^* x_n]$ implies $x_1 Q x_n$. To the contrary suppose $x_n Q^* x_1$. In that case, it is left to the reader to check that $\bigcup_{\tau_k \in \Sigma(A)} \tau_k T^{|A|} = A$, which implies $\mathcal{C}(A) = A = V(A,Q)$, a contradiction. Now, we show that $\underline{C}(A) = M(A,Q)$. First consider $x \in \underline{C}(A)$. Then clearly, $x \in \bigcap_{\tau_k \in \Sigma(A)} \tau_k T^{|A|}$, which implies xQy for all $y \in A$. Next consider xQy for all $y \in A$. Clearly,

$x \in$:B $T_k T^{|A|}$.
$o_k \in S(A)$

Next, we show that if C(.) ≠ ∅ for all $A \in [X]$, then &(A) = V(A,G) for all $A \in [X]$. Let C(.) ≠ ∅ for all $A \in [X]$. By theorem 1, C(A) = M(A,G). We will show that &(A) = V(A,G) for all $A \in [X]$. Clearly, the hypothesis is true for |A| = 1, 2. Suppose it is true for $|A| \leq m$, and let |A| = m. If C(A) = A, we are done, since V(A,G) = A = &(A). By the rationality of C(.), xGy for all $x,y \in A$. Now consider the case C(A) ≠ A. By definition, we know, &(A) = &(B), where $B = \cup_{o_k \in S(A)} T_k T^{|A|}$. By lemma 1, we know that B ≠ A. Then, by induction, &(B) = V(B,G). Now, we have to show that &(B) = V(A,G) = &(A), that is, (a) xIy for all $x,y \in$ &(B) and (b) $x \in A\backslash$&(B) imply that there exists $y \in$ &(B) such that yPx. Consider (a). Since &(B) is the stable set for B, by definition, (a) holds. Next consider (b). Let $x \in A\backslash$&(B). Then there are two possibilities: (i) $x \in B\backslash A$ and (ii) $x \in A\backslash B$. If (i) holds, then we are done, since &(B) is the stable set for B. So let (ii) hold. For all $z \in$ &(B) if xGz, then consider the set $U = \{u \in A\backslash B | u\text{G}y$ for all $y \in$ &$(B)\}$. By the rationality of C(.), C(U) = M(U,G). Then, there exists $u^* \in U$ such that u^*Gu for all $u \in U$. Now, consider $T_k \in S(A)$ such that $T_k(i) \in$ &(B) for i = 1, 2,...,m and $T_k(m+1) = u^*$. Clearly, $u^* \in T_k T^{m+1}$ for this sequence. Therefore, $u^* \in \cup_{o_k \in S(A)} T_k T^{|A|}$, contradicting the fact that $u^* \in$ A\B.❍

PROOF OF THEOREM 3. Suppose a choice function, C(.), is weakly rationalizable for some binary relation Q. To show that C(A) = &(A) for all $A \in S$, we first note that the binary relation Q must be consistent with the binary relation G, since C(A) is a stable set for all $A \in S$ such that |A| = 2. Then C(A) = V(A,Q) = V(A,G). Moreover, since C(A) ≠ ∅ for all $A \in S$, by theorem 2, &(A) = V(A,G). Note, by von Neumann - Morgenstern (1947), V(A,G) is unique. Hence, &(A) = C(A). For the sufficiency part, let C(A) = &(A) for all $A \in S$, and the rest of the proof follows from theorem 2.❍

PROOF OF THEOREM 4. It is obvious that quasi-transitive rationalization implies C(A) = &(A). So we need to establish the sufficiency part, i.e., we have to establish that if for any $A \in [X]$, C(A) = &(A), C(.) is quasi-transitive rational. Let G be the candidate for rationalization. Since &(A) = C(A), therefore, C(A) ≠ ∅. Thus C(.) is rational. To show quasi-transitivity, let xPy & yPz and suppose zIx. Given xPy and yPz, clearly, x ≠ z. Now, for a set of three distinct elements, A = {x,y,z}, consider a path $T_k \in S(A)$ such that $T_k(1) = x$, $T_k(2) = y$ and $T_k(3) = z$. In that case, $T_k T^3 = \{x,z\}$. Clearly, &(A) = {x,z}. However, for a path $T_{k'} \in S(A)$ such that $T_{k'}(1) = y$, $T_{k'}(2) = z$, and $T_{k'}(3) = x$, $T_{k'} T^3 = \{x\}$; i.e., C(A) = {x}, a contradiction.❍

FOOTNOTES

1. von Neumann and Morgenstern (1947) introduced a solution concept in terms of their dominance relation. The elements of a choice set form a von Neumann - Morgenstern solution means that every element outside the choice set is dominated by some chosen element and if two elements are chosen together, then neither dominates the other.

2. Wilson (1970) was one of the first who established, in the context of social choice, the Weak Axiom of Revealed Preference (Arrow; 1959) implies the solution concept for cooperative games proposed by von Neumann and Morgenstern. In an unpublished paper, Plott (1974) investigated the relationship between the notion of rational choice and the von Neumann - Morgenstern's solution concept.

3. Most of our results below will sail through without this assumption.

4. The relation R is called the *base relation* (Herzberger(1973)).

5. Given a choice function, $C(.)$, the *two-stage choice procedure* is defined to be a function h which, for every A 0 $[X]$, every integer n 0 $\{1,...,|A|\}$ and every $<A_1,A_2,...,A_n>$ 0 S(A), specifies a subset $h(<A_1,A_2,...,A_n>)$ of A such that for integer n 0 $\{2,...,|A|\}$, $h(<A_1,A_2,...,A_n>) = C(cB_i)$ where, for i 0 $\{1,2,...,n\}$, $B_i = C(A_i)$ and for $n = 1$, $h(<A>)= C(A)$. Similarly, given a choice function $C(.)$, the *SV choice procedure* is defined to be a function g^{sv} which, for every A 0 $[X]$ and every $<x_1(k),x_2(k),...,x_{|A|}(k)>$ 0 S(A), specifies a subset $g^{sv}(<x_1(k),x_2(k),...,x_{|A|}(k)>)$ of A such that $g^{sv}(<x_1(k),x_2(k),...,x_{|A|}(k)>)$ = T$_k$J$^{|A|}$ where T$_k$J$^1 = C(\{x_1\})$ and for $i = \{2,3,...,|A|\}$, T$_k$J$^i = C(\{x_i$cT$_k$J$^{i-1}\})$.

6. Under the restriction that every A^t is a proper subset of A in $[X]$, the condition of path independence with respect to the Plott procedure is equivalent to the condition of path independence with respect to the Sertel - Van der Bellen procedure (Bandyopadhyay; 1990).

7. Bandyopadhyay and Sengupta (1999) generalized the binary sequential choice procedure where choice proceeds in a sequence of subsets.

8. A generalized notion of a refined binary sequential choice procedure can be defined as follows: The *refined sequential choice procedure* is a function g which for every A 0 $[X]$, every integer n 0 $\{1,2,...,|A|\}$ and every $<A_1,A_2,...,A_n>$ 0 S(A) specifies a subset $g(<A_1,A_2,...,A_n>)$ of A such that $g(<A_1,A_2,...,A_n>) = T^n$ where, $T^1 = C(A_1)$ and for i 0 $\{2,..,n\}$, T^i = ^$_{a 0 T^{i-1}}$ $C(\{a\}$c$A_i)$ whenever $[C(\{a\}$c$A_i)$ _ $A_i]$ … ∅ for all a 0 T^{i-1}, otherwise, $T^i = T^{i-1}$.

9. Sertel - Van der Bellen, utilizing their choice procedure, called the set :B$_{o_k 0 S(A)}$ T$_k$J$^{|A|}$ the folding of a choice function.

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