Cycles + Semi-endogenous Growth = Endogenous Growth*

Chol-Won Li
University of Glasgow

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Abstract

The scale effect prediction of a growth model with a single type of technological progress (usually modelled as variety expansion or quality improvement) is empirically inconsistent. This is due to the "knife-edge" assumption that new ideas created are linear in the stock of knowledge. If this assumption is dropped to make a one-sector R&D-based model consistent with data, growth becomes semi-endogenous in the sense that public policy and consumer preferences do not affect growth in the long run. This view is predominant among researchers. This paper challenges this Consensus View, using an otherwise very standard one-sector R&D-based model. Specifically, we demonstrate that the rate of technological progress may exhibit endogenous cycles in the long run, and it is no longer pinned down by exogenous structural parameters. In the long run, public policy and consumer preferences will affect productivity growth. This result is obtained when the knife-edge assumption is violated. Our paper departs from the dominant "knife-edge" perspective based upon the degree of externality in the production of knowledge, and turn to the nature of long-run equilibria, cyclical or not.

*Correspondence: Dept. of Economics, Univ. of Glasgow, Adam Smith Building, Glasgow G12 8RT, UK; (Tel.) ++44-(0)141-330-4654; (Fax) ++44-(0)141-330-4940; (Web) http://www.gla.ac.uk/economics/cwli/; (E-mail) cw.li@socsci.gla.ac.uk..

1 Introduction

Endogenous growth means that long-run growth is driven by the deliberate activity of agents who are motivated by private incentives. By implication, public policy and consumer preferences which affect private incentives have a permanent impact on long-run growth.

Although this result sounds appealing, its theoretical mechanisms are still controversial in R&D-based growth models in particular. Early R&D-based models assume, explicitly or implicitly, an increase in knowledge is linear (at least in the long run) in the stock of knowledge.[1] This assumption, often called a "knife-edge" condition, plays an essential role in endogenizing technical progress. However, because of this assumption, early R&D-based models produced an empirically inconsistent prediction of positive scale effects, i.e. a larger labor force implies a higher growth rate (see Jones (1995a)).[2]

On the other hand, Jones (1995b) shows that if the knife-edge condition is dropped, scale effects disappear. But a serious "side-effect" is that growth is no longer endogenous in the sense that public policy and consumer preferences do not have a permanent effect on growth.[3] Jones termed such long-run growth "semi-endogenous," since technological progress is still driven by economic incentives.

There are now several studies which demonstrate how to re-endogenize R&D-driven growth. A common approach is to introduce a second source of long-run growth. One strand of studies, pioneered by Young (1998), is to assume two types of R&D in the form of variety expansion and quality improvement.[4,5] In

[1] This assumption is made explicit in expanding-variety models of Romer (1990) and Grossman and Helpman (1991, Ch.3). It is implicit in the quality index of quality-ladder models of Aghion and Howitt (1992) and Grossman and Helpman (1991, Ch.4).

[2] The reason is that in a larger economy, a sales volume, hence profits are higher, creating greater incentives to do R&D.

[3] The models of Segerstrom (1998) and Kortum (1997) also exhibit the same property.

[4] See Jones (1999) for more references. He argues that a variant of the knife-edge assumption is behind endogenous growth in those two-sector R&D-based models. In fact, Li (2000) shows that growth is in general semi-endogenous in a two-R&D-sector model once inter-R&D knowledge spillovers are explicitly incorporated. In particular, Li (2002) demonstrates that the number of knife-edge restrictions regarding the production technology of knowledge, which are required for endogenous growth, increases with the number of R&D sectors.

[5] Cozzi (2001), which introduces an imitative activity of innovative goods, may be added

a different approach, Jones (2001) demonstrates that endogenous fertility decisions will make R&D-based models exhibit endogenous growth. Note that these studies explicitly depart from the one-sector framework, by introducing an additional engine of growth. The current state of the literature, therefore, suggests that the following view has emerged:

> **Consensus View:** *In one-sector R&D-based models, growth is endogenous only with scale effects, and it becomes semi-endogenous once scale effects are removed.*[6]

We call it "consensus," as the literature developed on the basis of this premise. At a positive level, this dominant view questions the robustness of one-R&D-sector models as a system generating endogenous growth. One-R&D-sector models would cease to be a useful tool for those who accept endogenous growth as a plausible representation of economic dynamics. At a normative level, policy implications derived from those models may require re-assessment.

The main objective of this paper is to challenge the Consensus View. We will demonstrate that long-run growth can be endogenous in a one-sector R&D-based model *without* scale effects. This result will be established in a growth model of expanding variety. It can also be easily established that the same result carries over to a quality ladder model.

To be more specific, we will demonstrate the possibility of endogenous cycles in the rate of technological progress in the long run, and such cycles are the source of endogenous growth without scale effects. As the model exhibits a cycle, the rate of technological progress oscillates from one period to the following period *in the long run*. Since the rate of technological progress fluctuates, it is no longer pinned down by population growth. More importantly, public policy and consumer preferences will affect cyclical rates of technological progress. The same result may carry over to the trend growth rate. These results clearly go

to this class of models.

[6]In this paper, a phrase "one-sector" is used to refer to the number of R&D sectors. For example, the quality ladder model of Aghion and Howitt (1992) is a one-sector R&D-based model, since quality improvement is a sole driving force of growth.

against the Consensus View.[7]

To understand the nature of endogenous growth through cycles, note that an important trait of R&D activity is that expectations of future profits and R&D intensity affect the decisions of the current R&D effort level. Therefore, endogenous cycles that arise in our model are interpreted as self-fulfilling *expectational* equilibria. We consider two different assumptions regarding expectations: perfect foresight and "sunspot" beliefs. These alternative assumptions of rational expectations give rise to two different types of endogenous cycles, deterministic and stochastic.

Consider first perfect foresight that is often assumed in R&D-based models. In a steady state equilibrium where the R&D share of workers is constant, growth is semi-endogenous. However, this result arises only if such equilibrium is "selected" in the long run. Under certain parameter values, a steady state coexists with cyclical equilibria, i.e. multiplicity of long-run equilibria. If a long-run equilibrium is expected to be a steady state, it will be achieved in the long run. But, if cyclical equilibria are expected in the long run, such expectations are self-fulfilled. Private agents' beliefs determine which equilibrium is selected, and in this sense, long-run equilibria are indeterminate.

With perfect foresight, agents "know" in advance what will happen in future. Although this assumption may be justified in a stationary or repetitive state, it is less convincing for the analysis of dynamics before such a state is reached. An alternative assumption of expectations is "sunspot" beliefs. Sunspot refers to some exogenous random signal which has no influence upon the fundamentals of the economy. Nevertheless, agents expect sunspot to affect the economy, and such belief becomes a self-fulfilling prophecy, even though they are aware that sunspot is irrelevant to the determination of the fundamentals. When sunspot-driven stochastic cycles exist, growth is not pinned down by population growth, and growth becomes endogenous.

[7] In our model, a growth cycle is induced by technology. This appears to be consistent with empirical evidence. Greenwood, *et al.* (2000) show that the introduction of new products or technologies can account for 30% of US GDP fluctuations in 1954-90. Jovanovic and Lach (1997) also report a similar result.

To develop our argument in a familiar framework, we introduce only three minimal modifications into the standard R&D-based model of Grossman and Helpman (1991, Ch3): (i) no knife-edge condition, (ii) (exogenous) population growth and (ii) discrete time. Assumptions (i) and (ii) are essential for an R&D-based model without scale effects. An assumption (iii) represents a minor departure, given that continuous time is assumed in the first-generation R&D-based models. Although this assumption is not essential for our key results,[8]it is useful to make the following observation. The use of continuous time limits the possibility of complicated non-linear dynamics. Cycles of arbitrary length can arise in one-dimensional difference equations, whereas they are not possible in one-dimensional differential equations. A cyclical equilibrium in differential equations requires a dimension of at least two.[9,10]The use of discrete time allows us to highlight the possibility of endogenous growth through cycles in a more simple way.

Does our results suggest that semi-endogenous growth is “dead” in one-sector R&D-based models? The answer is “not completely.” This is because cyclical endogenous growth is likely to arise only for a sub-set of the parameter space. Nevertheless, our analysis demonstrates an important qualification to the long-run analysis of the current literature.

The present paper is structured as follows. Section 2 briefly reviews the Consensus View and develops the basic ideas of our paper in an informal way. It also mentioned some related studies. The basic model of expanding variety

[8]The use of discrete time enables us to identify three channels through which growth becomes endogenous: (i) a deterministic cycle due to a Hopf bifurcation, (ii) a deterministic cycle due to a Flip bifurcation, and (iii) a stochastic cycle due to sunspot beliefs. In continuous-time models, the second channel does not exist, and establishing the last channel would be more involved than the present paper shows.

[9]Aperiodic cycles (i.e. “chaos”) can arise in one-dimensional difference equations. For differential equations, a dimension of at least three is required for aperiodic equilibria. See Nishimura and Sorger (1999) for a survey of non-linear dynamics and the qualitative differences between the discrete and continuous time systems.

[10]Jones (1995b) showed that a steady state in his continuous-time one-R&D-sector model without permanent scale effects is stable. However, this might be due to an assumption used in his local stability analysis that the R&D share of labour and the physical investment rate are held constant. This assumption will reduce dimensionality of his model, thereby reducing the possibility of periodic equilibrium. In non-scale growth models of Eicher and Turnovsky (1999, 2001), cyclical equilibria do not arise. Again, restrictions imposed by the use of continuous time may be operating.

is described in Section 3, followed by the analysis of stability of a steady state and semi-endogenous growth in Section 4. Section 5 establishes that our one-sector R&D-based model exhibits endogenous growth through a deterministic growth cycle. This is followed by Section 6 where long-run growth is shown to be endogenous due to a sunspot cycle. Section 7 summarizes key differences between the prevailing "knife-edge" perspective of the current literature and our approach to the issue of robustness of one-sector R&D-based models in generating endogenous growth. 8 concludes.

2 Preliminary

2.1 The Consensus View

What was "wrong" with the Consensus View? To answer this, suppose that there are L_t workers, and the labor force grows at a rate of $n > 0$. Of these, M_t workers are used in manufacturing, and R_t workers are used in R&D:

$$L_t = M_t + R_t. \tag{1}$$

The stock of knowledge is denoted by N_t, and it augments according to

$$N_{t+1} - N_t = \delta R_t N_t^{\phi}, \qquad \delta > 0, \qquad 1 > \phi. \tag{2}$$

ϕ is a measure of externalities, capturing two opposing effects. One is the "standing on shoulders effect" where accumulated knowledge improves R&D productivity. The other is the "diminishing technological opportunities" where the past technological innovations make the current R&D project more difficult. When the first effect dominates, we have $1 > \phi > 0$.

Note that the first-generation one-sector R&D-based models assume $\phi = 1$. For this knife-edge value of ϕ, technology improves at a rate of $N_{t+1}/N_t = \delta R_t + 1$, which exhibits scale effects, i.e. a larger number of R&D workers makes technology progress at a faster rate. In particular, a positive population growth would imply that R&D workers rises over time, resulting in a sustained increase in the rate of technological progress.[11] This is clearly inconsistent with an average TFP growth of 1.4% in the US with a positive population growth over the past fifty years. To get around this problem, ϕ need be less than one.

Using (1) and (2), the rate of technological progress is written as

$$\frac{N_{t+1}}{N_t} - 1 \equiv g_t = \delta\left(l_t - m_t\right) \tag{3}$$

where $l_t = L_t/N_t^{1-\phi}$ and $m_t = M_t/N_t^{1-\phi}$. We call l_t the "effective" worker-knowledge ratio, interpreting $M_t N_t^{\phi}$ as the effective number of workers. m_t is similarly interpreted. It is easy to establish that the effective worker-knowledge ratio is related to technological progress according to

$$\left(g_t + 1\right)^{1-\phi} = (n+1)\,\frac{l_t}{l_{t+1}}. \tag{4}$$

The Consensus View *assumes* that the rate of technological progress g_t is constant in the long run. This assumption leads us to the conclusion that growth is semi-endogenous growth, since $g + 1 = (n+1)^{\frac{1}{1-\phi}}$. Technological progress is pinned down by population growth, and public policy and consumer preferences do not affect growth. However, note that this analysis precludes the possibility of cyclical g_t (and l_t) as long run equilibria, since constancy is imposed on g_t. The current literature fails to give clear reasons for doing so. Moreover, disregarding the possibility of a growth cycle is not convincing, given that many studies in the nonlinear dynamics literature demonstrate that it can arise in the long run.[12] Data also show a persistent growth cycle over more than a century.[13] We argue that the current literature neglected the role of endogenous cycles in making growth endogenous, and this contributed to the Consensus View.

[11] Theoretically, R_t can be constant in the presence of population growth. However, this would mean an ever decreasing proportion of R&D workers, which is empirically inconsistent.

[12] See Section 2.3 for references.

[13] See Figure 10 for U.S. TFP growth in the second half of the last century. See also Figure 17.4 of Baumol and Wolff (1983, p.371) which shows persistent fluctuations of a labor productivity growth rate in the US for the period of 1880-1986.

2.2 Basic Ideas

This section presents the basic ideas of our argument in a less formal way, assuming perfect foresight. Using (3) and (4), one obtains

$$g_{t+1} + \delta m_{t+1} = (1+n)\frac{(g_t + \delta m_t)}{(g_t + 1)^{1-\phi}} \equiv z(g_t, m_t). \quad (5)$$

This is the equation of motion for g_t, given the evolution of the effective manufacturing worker-knowledge ratio m_t. Since it is derived from the full-employment condition (1), we refer to it as the labor market condition.

Since m_t enters the labor market condition (5), we require an equation of motion for m_t to fully determine the time trajectory of g_t. This is where the private incentives for R&D come into play. Innovators conduct R&D, motivated by future profits. Facing a trade-off between future profits and R&D costs in the current period, they determine the profit-maximizing number of R&D workers they employ. This R&D decision is influenced by the discount factor and public policy, such as R&D subsidy. In the next section, we will formally derive the R&D incentive condition which reflects all these considerations. However, it is sufficient at this stage to consider what would happen when the effective manufacturing worker-knowledge ratio m_t is constant or cyclical for our purposes.

Initially suppose that m_t is constant in the long run. (5) immediately tells us that technological progress is pinned down by population growth, i.e. $g_t + 1 = (n+1)^{\frac{1}{1-\phi}}$. Since the effective manufacturing worker-knowledge ratio m_t is cancelled out in (5), growth is independent of a private incentive for R&D, hence the discount factor and public policy. This is a familiar situation of semi-endogenous growth, as discussed above. However, this result is valid if and only if a steady state with a constant g_t is expected in the long run. Indeed, there is no reason why agents *should* expect a steady state to be a long-run equilibrium, if other types of long-run equilibria exist. Agents can expect g_t to follow a cyclical trajectory in the long run, and a constant g_t may not be achieved. In this case, the nature of long-run growth dramatically changes.

To illustrate this point, suppose that the effective manufacturing worker-knowledge ratio m_t exhibit persistent cycles. Cycles can be deterministic or stochastic. But what is important is that persistent fluctuations can be a long-run equilibrium rather than transitional dynamics. The first question that may arise is "Can g_t be constant when m_t oscillates?" The answer is negative.[14]

When m_t exhibits persistent cycles in the long run, so does g_t necessarily. Consequently, cyclical technological progress cannot be pinned down by population growth. This fact leads to the possibility of endogenous growth where government policy and consumer preferences affect long-run growth. This is the main point that the present paper emphasizes.

To diagrammatically represent this result, let us consider the simplest case of a deterministic period-2 cycle where m_t takes values m_0 and m_1, $m_0 \neq m_1$. (5) shows that g_t must satisfy the following two conditions

$$g_{t+1} + \delta m_1 = z(g_t, m_0), \qquad g_t + \delta m_0 = z(g_{t-1}, m_1). \tag{6}$$

These two functions are either increasing or decreasing in the lagged value of g, depending on the values of m_0 and m_1. Figure 1 shows the case where both curves are increasing.[15]Cyclical technological progress (g_0 and g_1) are determined by those curves in the figure. g_t alternates in each period, creating waves of technological innovations. A private incentive for R&D follows an oscillatory pattern in the long run, and this translates into cyclical rates of technological progress.

Note that g_0 and g_1 are determined by the values of m_0 and m_1, which in turn depend on model parameters including public policy. As the policy changes, m_0 and m_1 alter. In turn, the curves in Figure 1 change their positions, affecting equilibrium g_0 and g_1. That is, public policy (and consumer preferences) affect the long-run rate of technological progress. This is the mechanism which re-endogenizes growth in our one-sector R&D-based model. Note that those cyclical rates of technological progress are obtained as a *long-run* equilibrium. This finding goes against the Consensus View.

[14]It can be easily established, as follows. According to (4), a constant g_t means a constant l_t. In turn, (3) implies a constant m_t.

[15]Since the curves do not intersect on the 45 degree line, a constant g_t cannot arise.

2.3 Some Related Studies of R&D-driven Cycles

Baumol and Wolff (1983) is an early study which examines the possibility of an endogenous deterministic cycle in productivity growth driven by R&D. However, it is not based on microfoundations in an optimizing framework.

Aghion and Howitt (1992) is an important contribution to the growth literature. Although a growth cycle was not a main issue of the study, it indicated the possibility of endogenous cycle. Deissenberg and Nyssen (1998) and Francois and Shi (1999) demonstrated the existence of a growth cycle in a variant of the quality ladder model. The role of a financial market is emphasized in the former, and the latter study introduced a time lag in R&D activity. In a more recent attempt, Francois and Lloyd-Ellis (2002) show a growth cycle due to entrepreneurs' herd behavior fuelled by "animal spirits."

In an expanding variety model, Matsuyama (1999) showed the existence of a deterministic cycle in growth. His main interest was to show that an endogenous cycle is driven by changes in phases where growth is mainly driven by capital accumulation or technological progress. In this sense, it is related to the study of Jovanovic and Rob (1990), where what they call "extensive" and "intensive" search in R&D oscillate, though it is a partial equilibrium industry model.

Boldrin and Levine (2002) is closest to our paper in spirit, though their study concerns technology adoption rather than technology creation. What the authors term the slow growth regime arises where long-run growth is exogenous along a balanced growth path. Since technical adoption is endogenous in this regime, this equilibrium configuration is equivalent to semi-endogenous growth in our terminology. On the other hand, endogenous growth occurs along a cyclical growth regime where consumer preferences affect a long run growth path. However, this model departs from a well-established R&D-based models of growth in modelling approach.

Turning to a sunspot growth cycle, Evans, et al. (1999) demonstrated that it exists in an R&D-driven model of expanding variety. An economy switches between high and low growth phases, and multiple steady states are generated due to complementarity of accumulation of differentiated physical capital.

An endogenous stochastic cycle was also shown to exist in a continuous-time quality-ladders model of Drugeon and Wigniolle (1996). A driving mechanism is complementarity in R&D activity across firms.

When the present paper is compared with endogenous growth models mentioned above,[16]three important points can be made. First, all of those models predict positive scale effects. Hence they have to face a scale effect criticism directed at the first-generation R&D-based models. Second, none of the above studies consider endogenous cycles as a channel through which growth becomes endogenous. Third, the existence of an endogenous cycle is established by introducing an array of mechanisms into the first-generation R&D-based models. In contrast, our model keeps extensions of the Grossman and Helpman (1991, Ch3) model to minimum. Our analysis also shows that an endogenous cycle (hence endogenous growth) is possible even when externalities in R&D are almost non-existent.

3 The Model of Expanding Variety

Initially we assume perfect foresight, and postpone the discussion of sunspot beliefs to Section 6. There are three sectors; (i) final output produced using intermediate goods, (ii) intermediate goods produced using workers, and (iii) R&D conducted by workers.

A representative consumer maximizes his intertemporal utility function $\sum_{t=0}^{\infty} (\beta(n+1))^t \left(y_t^{1-\sigma}-1\right)/(1-\sigma)$ subject to a usual budget constraint where $1 > \beta > 0$ is consumers' discount factor and y_t is per capita consumption. The utility maximization implies

$$\left(\frac{y_t}{y_{t-1}}\right)^{\sigma} = \beta(r_t+1) \tag{7}$$

where r_t is the interest rate between periods t and $t-1$.

[16]That is, except for Baumol and Wolff (1983) and Jovanovic and Rob (1990).

Consumption goods are produced according to

$$Y_t = \left(\int_0^{N_t} x_{it}^{\alpha} \right)^{1/\alpha}, \qquad 1 > \alpha > 0 \tag{8}$$

where Y_t is final consumption goods and x_{it} is differentiated (perishable) intermediate goods. The final output sector is perfectly competitive, and the demand function of x_{it} is given by

$$x_{it} = \frac{p_{it}^{-\frac{1}{1-\alpha}}}{\int_0^{N_t} p_{it}^{-\frac{\alpha}{1-\alpha}} di} Y_t \tag{9}$$

where p_{it} is the price of a variety input.

The intermediate goods sector is monopolistically competitive. Facing the demand function (9), intermediate goods producers maximize profits $\pi_{it} = p_{it}x_{it} - w_t x_{it}$ where w_t is the wage and one worker is assumed to produce one unit of x_{it}. Given the price elasticity of demand being $-1/(1-\alpha)$, monopolists charge $p_t \equiv p_{it} = w_t/\alpha$, and earn

$$\pi_t \equiv \pi_{it} = \frac{1-\alpha}{\alpha} \frac{w_t M_t}{N_t}. \tag{10}$$

Turning to R&D, blueprints of new intermediate goods are produced according to (2). Successful innovators who engage in research at time t will start producing a new variety in the following period. The present value of future profit flows, v_t, is defined as

$$v_t = \beta \left(\pi_{t+1} + v_{t+1} \right). \tag{11}$$

Innovators determine the number of R&D workers by maximizing $v_t \delta R_t N_t^{\phi} - (1-s) w_t R_t$ where s is the rate of R&D subsidies for $s > 0$ and taxes for $s < 0$. This maximization problem gives the first-order condition

$$v_t = \frac{(1-s) w_t}{\delta} \text{ for } R_t > 0. \tag{12}$$

Using (7), (8), (9), (10), (11) and (12), one can derive the following difference equation

$$\frac{m_{t+1}^{\sigma}}{\frac{(1/\alpha - 1)\delta}{1-s} m_{t+1} + 1} = \beta m_t^{\sigma} \left(g_t + 1 \right)^{\theta} \tag{13}$$

where $\theta = (1-\sigma)(1/\alpha - 1 - \phi) - \sigma$. This is the R&D incentive condition.

4 Is Growth Semi-endogenous?

In the literature, semi-endogenous growth is established on the basis of local stability analysis. This section aims to review the reasoning employed in the analysis. In particular, we will point out a potential pitfall of this reasoning.

The evolution of our economy is defined by the system of difference equations (5) and (13). A unique steady state solution is given by

$$g^* + 1 = (1+n)^{\frac{1}{1-\phi}}, \ m^* = \frac{1-s}{\delta(1/\alpha - 1)} \left[\frac{1}{\beta(1+g^*)^\theta} - 1 \right]. \tag{14}$$

Linearizing (5) and (13) around the steady state yields

$$\begin{pmatrix} \widetilde{g}_{t+1} \\ \widetilde{m}_{t+1} \end{pmatrix} = J \begin{pmatrix} \widetilde{g}_t \\ \widetilde{m}_t \end{pmatrix} \tag{15}$$

where $\widetilde{m}_t \equiv m_t - m^*$, $\widetilde{g}_t \equiv g_t - g^*$, and J is the associated Jacobian matrix. Its determinant and trace, labelled D and T respectively are

$$D = \sigma\lambda_1\lambda_2 - \lambda_1\lambda_3, \qquad T = \sigma\lambda_1 + \lambda_2 - \lambda_1\lambda_3, \tag{16}$$

$$\lambda_1 = \frac{1}{\beta(1+g^*)^\theta - (1-\sigma)}, \tag{17}$$

$$\lambda_2 = 1 - (1-\phi)\frac{g^* + \delta m^*}{1+g^*}, \qquad \lambda_3 = \frac{\theta\delta m^*}{g^*+1}. \tag{18}$$

For local stability analysis, consider the space spanned by the determinant and trace of the Jacobian. We can define seven regions, separated by three lines, $D = T - 1$, $D = -T - 1$ and $D = 1$, as shown in Figure 2.[17]

To interpret the stability properties of the steady state, we emphasize two features of the model. First, the linearized system (15) takes the form of *forward*-looking perfect foresight difference equations (time running forward). Therefore, we should interpret m_{t+1} as an expected value which is consistent with pre-determined m_t, g_t and g_{t+1}. With perfect foresight, those future values are self-fulfilled in each period. Second, in our model both $m_0 = M_0/N_0$ and $g_0 = N_1/N_0 - 1$ (the initial conditions) are not pre-determined. In the initial

[17] Eigenvalues below (above) the dotted curve are real (complex). See Azariadis (1993, pp.62-7) for further details.

period, N_0 is inherited from the past, but M_0 and N_1 are determined in period 0 on the basis of expectations of private agents. Therefore, m_0 and g_0 can take "any" values in the initial period, depending on expectations of agents. Putting it differently, the initial conditions m_0 and g_0 are endogenous, since they are determined by a type of perfect foresight equilibria that agents expect to hold in the long run.

Bearing this in mind, consider the region Source 1 and 2. Since both eigenvalues are located outside the unit circle, an economy diverges from the steady state if it starts at its vicinity. In the R&D-based growth literature, this is often taken as an indication that g_t goes off to zero or infinity in the long run, which must be ruled out as rational expectation equilibria. This is justified as follows. Infinite g_t means that $R_t/N_t^{1-\phi}$ becomes arbitrarily large (see (2)), which in turn requires R_t grows faster than $N_t^{1-\phi}$, i.e. $n > (1-\phi)\, g_t$. This clearly contradicts an infinite g_t. Regarding trajectories towards $g_t = 0$, it contradicts the the perfect foresight assumption. That is, along those divergent paths, research firms do not conduct R&D in the long run despite the fact that they know for sure that R&D is profitable.[18]This suggests that an economy must start at the steady state. In this sense, the initial condition is endogenously determined, like many other models with perfect foresight. A "long-run" equilibrium is reached from the initial period, and growth is semi-endogenous.[19]

However, this no-zero-no-infinity-growth reasoning to determine the initial condition implicitly assumes that there are only three candidate equilibrium configurations in the long run: $g_t = g^*$, $g_t = 0$, and $g_t = \infty$. This analysis disregards the possibility of (deterministic) endogenous cycles in g_t as long-run equilibria. If there are oscillatory equilibria, semi-endogenous growth may not be attained, depending on what agents expect. In this sense, the no-zero-no-infinity-growth reasoning would be flawed *if* cyclical equilibria coexist with the steady state. Therefore, it is important to examine the possibility of an endogenous cycle in order to assess the validity of the no-zero-no-infinity-growth reasoning, hence

[18]This is formally shown in Appendix.

[19]In their continuous-time model, Grossman and Helpman (1991,Ch3) describe this type of equilibrium configuration as "an instantaneous jump" to a steady state at time 0.

semi-endogenous growth. This observation is particularly relevant to the region Source 1, as we will see.

Next consider the regions Saddle 1a, 1b, 2a and 2b. If an economy is not initially located on a saddle path (or a steady state), g_t diverges from the steady state once it starts around it. Again this could be taken as an indication that g_t goes off to zero or infinity in the long run. The afore-said no-zero-no-infinity-growth reasoning could be used to eliminate such trajectories, so that the initial condition is such that the economy starts on a saddle path (or a steady state).[20] This analysis depends on the assumption that cyclical equilibria do not exist. In this sense, the nature of equilibria when a steady state is a saddle does not differ much from that of equilibria when a steady state is a source. Note that this observation is relevant to the region Saddle 1a, as we will show.

Finally, consider the region Sink. Irrespective of the initial conditions, the economy always converges to the steady state if it starts at its vicinity. There are an infinite number of equilibrium trajectories leading to the steady state, i.e. indeterminacy. Does this mean that the growth is necessarily semi-endogenous? On the contrary, the nonlinear dynamics literature demonstrated that indeterminacy is closely related to the existence of cycles. Indeed, we will establish that deterministic cycles of different periodicity coexist under certain conditions.[21] To appreciate the importance of this result, recall that the initial conditions g_0 and m_0 are endogenous, but they cannot be uniquely determined. Given multiplicity of (g_0, m_0), it is not clear whether the initial conditions leading to the steady state is "chosen". This model simply does not have a mechanism which allows us to pin down a unique initial condition in the presence of multiple equilibria. Therefore, in the presence of endogenous cycles, there is no guarantee that a steady state, i.e. semi-endogenous growth is attained.

An upshot of the above discussion is that establishing semi-endogenous

[20] Note that the endogenously determined initial condition is not unique. According to the no-zero-no-infinity reasoning, the economy can start at any point on the saddle path or a steady state.

[21] Moreover, a stochastic sunspot cycle of any finite periodicity also exists when a steady state is indeterminate.

growth as a long-run equilibrium crucially depends on the absence of cyclical equilibria. An important issue, therefore, is whether or not cyclical equilibria exist under the perfect foresight assumption, to which we turn in the next section.

5 Growing through Deterministic Cycles

5.1 The Framework of Analysis

In Section 4, the linearized system (15) was used to examine stability properties of a steady state. Indeed, the same framework can be applied to detect endogenous cycles around a steady state. This section aims to briefly discuss this method.[22]

A bifurcation occurs when the qualitative nature of a steady state(s) changes when a given parameter alters only slightly. To describe this concept, suppose that parameters are such that the economy is located in the region Sink in Figure 2. Further suppose that an externality parameter ϕ changes with other parameters being constant. If the economy is well inside the triangular region, it stays in the region even after a small change in ϕ. However, if the economy is close enough to or on the triangular boarder line, a slight change in ϕ (a bifurcation parameter) may move the economy into an adjacent region, and stability properties abruptly alter. These changes may also reveal the "emergence" of endogenous cycles around a steady state. There are two types of bifurcations that are relevant to this paper.

First, suppose that an economy which is initially in the region Sink in Figure 2 crosses the $D = -T - 1$ line due to a slight change in a bifurcation parameter. In this case, a Flip bifurcation occurs (one of the eigenvalues which were only slightly greater than -1 crosses the value -1). It indicates either the emergence of a stable period-2 cycle in the region Saddle 1 or the existence of a period-2 cycle with a saddle property in the region Sink. The former is called supercrit-

[22]See Azariadis (1993) and Grandmont et. al. (1998) for more details.

ical, and the latter is subcritical. Note that in either case, a Flip bifurcation reveals the existence of a period-2 cycle.

Second, if an economy, starting from the region Sink, crosses the $D = 1$ line from "below" in Figure 2 after a small parameter change, a Hopf bifurcation occurs (one of the eigenvalues which were within the unit circle in the complex plane goes outside the circle). It signifies either the appearance of a stable invariant closed curve in the region Source 1 or the existence of an unstable invariant closed curve in the region Sink. As above, they are called supercritical and subcritical, respectively. The equilibrium dynamics along this orbit does not need to be periodic, since the economy never visits the same point twice, although it remains on the closed curve forever.

This local bifurcation analysis demonstrates that a cycle can exist in the regions Sink, Source 1 or Saddle 1a, but not any other regions.[23]

5.2 The Case of $\sigma \geq 1$

This case includes the logarithmic utility function, i.e. $\sigma = 1$, which is often assumed in the literature. First note that bifurcations require that a steady state is a sink or indeterminate. Also note that $D > T - 1$ is necessary for indeterminacy. This necessary condition is equivalent to

$$1 > \sigma\lambda_1, \tag{19}$$

using the determinant and trace in (16) and noting $\lambda_2 > 0$. If this inequality is not satisfied, the economy is located in one of the regions Saddle 2a, Saddle 2b or Source 2 in Figure 2 where no cyclical equilibria exist.

To show that this is indeed the case, note that the consumer's intertemporal utility function must be bounded in a steady state. This requirement is met for

$$1 > \beta\left(1 + g^*\right)^{\theta+1+(1-\phi)\sigma}. \tag{20}$$

[23]If an economy crosses the $D = T - 1$ line in Figure 2 due to a slight change in a parameter, a saddle-node bifurcation is said to occur. However, this happens only when there are multiple steady states. A saddle-node bifurcation cannot occur in our model, given that a steady state is unique.

Since this condition implies $1 > \beta (1+g^*)^\theta$,[24]we have $\sigma\lambda_1 > 1$, given (17). Therefore, there are no cyclical equilibria, and growth is always semi-endogenous for $\sigma \geq 1$. In general, endogenous cycles require $\sigma < 1$, on which the rest of the paper focus.

5.3 Identifying Indeterminacy

ϕ is taken as a bifurcation parameter, holding other parameters constant. However, it is difficult to conduct analysis for arbitrary values of parameters. Therefore, we initially assume

$$\frac{1}{\alpha} = \{0.5,\ 0.7,\ 0.9\}, \quad \beta = \frac{1}{1.05}, \quad n = 0.02, \quad \delta = 0.01 \quad s = 0. \tag{21}$$

The discount factor chosen means the long-run interest rate of 5%, which is slightly larger than a long-run rate of a free risk asset. But it is smaller than the rate of return on the stock market, which is based on the average real return of 7% on the stock market for the last century in the U.S. (see Mehra and Prescott (1985)). Given that the return on the stock market is also relevant to our model, the choice of β does not seem too high. In any case, we mention the impact of changes in β on the possibility and nature of equilibrium cycles.[25]

Regarding α, we can give it at least three interpretations. First, its inverse measures a monopoly markup over marginal cost.[26]Second, α is related to the

[24]This also ensures that $m^* > 0$.

[25]δ is a scale factor, and it turns out that it does not affect the possibility of endogenous growth.

[26]Empirical studies distinguish between gross output markup and value added markup. They differ due to the use of intermediate goods in final output production. Our $1/\alpha$ corresponds to gross output markup (See Rotemberg and Woodford (1994)). Empirical estimates of gross output markup vary from 1.041 (the lowest estimate of Basu and Fernald (1997)) to 1.7 (the highest estimate of Domowitz *et al.* (1988)). However, more recent studies seem to point to a lower range of this interval after estimation is improved. For example, Basu and Fernald (2000), correcting utilization of production factors, obtain value added markup of 5%, which implies an even lower gross output markup. Gali, *et al.* (2001) refer to the interval $(1.1, 1.4)$ as a range of plausible estimates of price markup, i.e. $\alpha \in (0.714, 0.909)$.

elasticity of output with respect to the stock of knowledge.[27] Third, α determines the elasticity of substitution between any two variety inputs.[28] Since empirical estimates based on these interpretations point to the range of $\alpha \in (0.5, 1)$, we focus on this interval by considering three values given in (21).[29]

We use parameter values in (21) to identify the parameter combinations (ϕ, σ) which give rise to a sink. An advantage of this approach is that the values of σ relevant to endogenous cycles can be identified without prior restrictions. Once those parameter combinations are found, we conduct bifurcation analysis.

Given (21), we can write the determinant and trace in (16) as a function of ϕ and σ, i.e. $D(\phi, \sigma)$ and $T(\phi, \sigma)$. Then, consider the following three equations:

$$D(\phi, \sigma) - T(\phi, \sigma) = -1, \quad (C1)$$

$$D(\phi, \sigma) = 1, \quad (C2)$$

$$D(\phi, \sigma) + T(\phi, \sigma) = -1. \quad (C3)$$

These equations correspond to the three lines drawn in Figure 2. They are depicted in the (ϕ, σ) space in Figures 3. The parameter space is divided into seven regions by three curves, just like Figure 2. The shaded region is infeasible, as the intertemporal utility will not be bounded in a steady state. The nature of a steady state is indicated in the same way as in Figure 2.[30]

[27] The production function (8) can be re-written as $Y_t = A_t^{\frac{1-\alpha}{\alpha}} M_t$, showing that the elasticity of output with respect to the stock of knowledge is $(1-\alpha)/\alpha$. Griliches (1992, p.44) suggests 0.3 as being representative of estimates of the elasticity, implying $\alpha = 0.769$. This value is also used by Kortum (1997) to calculate the social rate of return from R&D in his model.

[28] Using an R&D-based model, Caballero and Jaffe (1993) gave an estimate which implies $\alpha = 0.463$. In a different strand of macroeconomics Rotemberg and Woodford (1995) reported the elasticity of substitution being 7.88, which translates into $\alpha = 0.8726$.

[29] Indeed, endogenous cycle is very unlikely for $\alpha < 0.4$, given the above parameter values.

[30] This is confirmed by simulation which shows the following properties. We have $D < T - 1$ in the area above the curve $C1$, and the inequality is reversed below the curve. Regarding the curve $C2$, it pivots anti-clockwise around the intersection point with the curve $C1$, as

There are four things worth mentioning. First, indeterminacy is possible, but it requires a low value of σ. Second, indeterminacy does not require a high degree of externality. Indeed, if it is too high, a steady state becomes a saddle. Third, the size of the region Sink is not monotonically related to α. High or low values of α tend to make a sink less likely. In fact, Sink disappears for $\alpha < 0.2829$ approximately. Fourth, simulation indicates that a lower discount factor enlarges the size of the region Sink. This is shown only in the case of $\alpha = 0.7$ in Figure 3-(d). The same property is also found in other cases.

5.4 Endogenous Cycles

We are in a position to examine the possibility of endogenous deterministic cycles. Note that in Figures 3, Sink is adjacent to Source 1 and Saddle 1a. This implies that Flip and Hopf bifurcations are possible.[31] Indeed, any small parameter change which moves the economy from Sink to either Source 1 or Saddle 1a will lead to the "appearance" of a cycle.

Choose any point on the $C3$ curve between Sink and Saddle 1a in one of Figures 3. A slight increase in ϕ will tip the economy into Saddle 1a, and a Flip bifurcation occurs. It means the "emergence" of a period-2 cycle either in the region Saddle 1a or Sink, depending on its stability. Similarly pick any point on the $C2$ curve between Sink and Source 1. As ϕ falls, a steady state turns into a source, and a Hopf bifurcation occurs. An invariant closed curve appears in the region Source 1 or Sink, depending on the stability of a cycle.[32] This confirms the existence of endogenous cycles in the rate of technological progress.

Cyclical equilibria are interpreted as an endogenous growth cycle, driven by self-fulfilling expectations. Intuitively a period-2 cycle can be explained by the inverse relationship between the current and future R&D intensity. As more

D decreases from 1. It will eventually converge to the curve $C1$. On the other hand, as D increases from 1, the curve $C2$ turns clockwise and approaches the curve $C1$. Next consider the curve $C3$. It turns clockwise as $D + T$ increases from -1, and converges to the curve $C1$. The opposite happens when $D + T$ falls from -1.

[31] Note that Sink is not directly connected to Saddle 2a, Saddle 2b and Source 2. This means that the economy cannot cross the $D = T - 1$ line in Figure 2. See foonote 23.

[32] Obviously, bifurcations can occur by changing σ, but holding ϕ.

varieties of goods are introduced, profits of a firm will drop. Therefore, if future R&D intensity is expected to be high, then future profits are expected to fall, discouraging the current R&D efforts. The opposite story holds when R&D intensity is expected to be low. This pattern of expectational oscillations continues in the long run equilibrium. On the other hand, it is less straightforward to give an intuitive account of a cycle that emerged through a Hopf bifurcation.

Regarding the stability of a cycle, it is difficult to confirm analytically whether a bifurcation is super- or subcritical.[33]Instead, we resort to simulation to examine the stability property of a cycle through examples.

5.4.1 Flip Bifurcations

Example 1 *For* $\alpha = 0.5$, *pick a bifurcation point of* $\sigma = 0.025$ *and* $\phi^{Flip} \simeq 0.2908$. *A period-2 cycle with a saddle property occurs at* $\phi \simeq \phi^{Flip} - 0.01$ *with* $g_0 \simeq 0.0321$, $g_1 \simeq 0.0237$, $m_0 \simeq 2.65$, *and* $m_1 \simeq 3.48$.

Example 2 *For* $\alpha = 0.7$, *pick a bifurcation point of* $\sigma = 0.02$ *and* $\phi^{Flip} \simeq 0.49669$. *A period-2 cycle with a saddle property occurs at* $\phi \simeq \phi^{Flip} - 0.001$ *with* $g_0 \simeq 0.0430$, $g_1 \simeq 0.0371$, $m_0 \simeq 12.21$, *and* $m_1 \simeq 12.78$.

Example 3 *For* $\alpha = 0.9$, *pick a bifurcation point of* $\sigma = 0.01$ *and* $\phi^{Flip} \simeq 0.1525$. *A period-2 cycle with a saddle property occurs at* $\phi \simeq \phi^{Flip} - 0.0001$ *with* $g_0 \simeq 0.0251$, $g_1 \simeq 0.0222$, $m_0 \simeq 46.01$, *and* $m_1 \simeq 46.24$.

These examples confirm that the Flip bifurcations are subcritical.[34]They confirm that a growth cycle exists when a steady state is indeterminate. That is, there exists a period-2 cycle around a stable steady state, and there are "saddle paths" leading to the cyclical equilibrium. Put differently, there are long-run multiple equilibria: one is a steady state, and the other is a growth cycle. It is, therefore, misleading to conclude that growth is semi-endogenous by merely showing that a steady state is stable. Indeed, depending on expectations,

[33] See Guckenheimer and Homes (1983, Sec.3.5) for stability analysis.

[34] To obtain these results, time should run forward for (5) but backward for (13) in simulation, given a saddle property of the cycles.

the economy may end up in cyclical equilibria where growth is endogenous even when a steady state is stable.

One may object to this conclusion by observing that being on a "saddle path" towards cyclical equilibria is possible only on a narrow set of initial conditions. However, given the absence of an equilibrium-selection mechanism which determines the initial condition, we cannot exclude the possibility of being on a "saddle path". Moreover, we will show in Section 5.7 that there can exist multiple cycles when a steady state is a sink. This is particularly so when the government intervenes in R&D activities.

In fact, this point is related to the emergence of more complicated cycles when ϕ falls further from ϕ^{Flip} in the above examples. Simulation indicates that a higher ϕ makes the amplitude of a cycle larger, so that g_t tends to hit zero before a cycle of higher order arises without government intervention. On the other hand, when the government subsidizes R&D, a cycle of higher order can arise. This point will be discussed in detail in Section 5.7.

5.4.2 Hopf Bifurcations

Example 4 *For $\alpha = 0.7$, pick a bifurcation point of $\sigma = 0.01$ and $\phi^{Flip} \simeq 0.00657$. A stable invariant closed curve arises $\phi = \phi^{Flip} - 0.00001$, as shown in Figure 4.*

This example shows that the Hopf bifurcation is supercritical. Indeed, the same result was found in other examples considered (but not reported) for different parameter values. In Figure 4, the economy starting off the steady state but inside the variant closed curve converges to the latter. Therefore, it is wrong to *assume* that the rate of technological progress will reach 0 or infinity in finite time if a steady state is shown to be a source with complex eigenvalues. Moreover, when the economy starts outside the invariant closed curve, it is actually moving "closer" to a steady state, as shown in Figure 4. Although a steady state can be reached if it is expected in the initial period, the likelihood of cyclical equilibria is far greater in this case. Endogenous growth is more likely than semi-endogenous growth.

Another important observation is that the degree of externality does not need to be very large for endogenous cycles. In the above example, the value of ϕ for a long-run cycle is nearly zero. A similar result is obtained for other parameter combinations, as can be seen in Figures 3.

5.5 Endogenous Growth

Having established the existence of growth cycles, our next task is to examine the impacts of parameter changes on long-run growth. Their effects are realized via m_t. Unfortunately, it is in general difficult to examine their qualitative impacts unless specific parameter values are assumed. Therefore, we demonstrate that growth is endogenous through examples. For a period-2 cycle, results are summarized in Table 1. We call g_0 and g_1 a good and bad state, respectively.

As expected, the rate of technological progress changes in response to parameter changes. The first feature that one can easily notice is that the direction of changes in g_t is typically asymmetrical between a good and a bad state. As an example, consider a lower discount factor. It discourages technological progress in a bad state, but encourages it in a good state. An intuition goes as follows. A lower discount factor directly discourages R&D in both states. However, there is an indirect effect. When R&D is discouraged in a bad state, it means less creative destruction in that state. Since a good state precedes a bad state, R&D becomes relatively more profitable in a good state. Simulation shows that the indirect effect dominates the direct effect in a good state, but the opposite happens in a bad state.

However, technological progress does not respond in a "consistent" way in a given state. For example, the R&D subsidy stimulates growth in a good state in Example 2, whereas the opposite happens in other examples. This is due to a highly nonlinear nature of the model. R&D subsidy promote R&D, but it depends on parameter values as well as the state of the economy.

Regarding population growth, its impact can be positive or negative. Therefore, there is no clear positive link between technological progress and population growth. This appears consistent with empirical studies which often found the relationship statistically insignificant. This property starkly contrasts with the existing semi-endogenous growth models.

Another feature worth mentioning is that parameter changes can alter the nature of an equilibrium. That is, the economy can stop exhibiting a cycle following a parameter change. This happens in Example 3 of Table 1 where g_t becomes constant at 2.36% due to R&D subsidy.[35]In this case, a steady state is a long-run equilibrium. This observation suggests that a policy changes can make growth semi-endogenous.[36]

Turning to a cycle along an invariant closed circle (via a Hopf bifurcation), consider the effect of an increase in the discount factor in Example 4. The amplitude of the cycle gets smaller, as confirmed in 5, although the frequency of a cycle seems affected little. Similar results are obtained for R&D subsidy and population growth. A small increase in R&D subsidy or population growth reduces the amplitude of a cycle. Note that like a period-2 cycle, there is no clear link between population growth and technological progress.

On the other hand, if β, s or n increases sufficiently, the amplitude of a cycle becomes arbitrarily small, and a steady state becomes a long-run equilibrium. Conversely, a sufficient fall in β, s or n makes the amplitude of a cycle so large that g_t hits zero. In that case, a long-run equilibrium is a steady state where growth is semi-endogenous.[37]

[35]In simulation, this occurs when g_t hits zero, which is not consistent with rational expectations. See Appendix A

[36]On the other hand, the public policy can enlarge the possibility of endogenous growth as long as g_t does not hit zero. The point is explored in Section 5.7.

[37]Note that a supercritical Hopf bifurcation in Example 4 occurs just above the $D = 1$ line between Sink and Source 1 in Figure 2. A fall in β, s or n pulls the economy closer to the line, and the parameter change can put the economy back in the triangle Sink. Conversely, a rise in β, s or n pushes the economy further away from the $D = 1$ line, and the amplitude of an invariant cycle gets larger.

5.6 Average Growth Rate

The main finding of the preceding analysis is that the cyclical period-to-period rates of technological progress are obtained as long-run equilibria, and they are affected by public policy and consumers' preferences. In this sense, long-run growth is endogenous. The following question, then, naturally arises. Is the average growth rate, around which the rate of technological progress oscillates, endogenous or not?

First, consider the arithmetic average of fluctuating rates of technological progress. In a period-2 cycle, the trend or average growth rate is given by

$$g_{\text{arith}} = \frac{g_0 + g_1}{2}. \tag{22}$$

We already established that g_0 and g_1 are affected by model parameters. Hence, it should be clear that g_{arith} also depends on them in general. Therefore, growth is endogenous even at the level of arithmetic means. The same result clearly carries over to a growth cycle of a higher order or along an invariant closed curve.

Next consider the geometric mean of cyclical rates of technological progress. In a period-2 cycle, it is

$$g_{\text{geo}} + 1 = \sqrt{(g_0 + 1)(g_1 + 1)}. \tag{23}$$

Now substituting (4) into this expression yields $g_{\text{geo}} + 1 = (n+1)^{\frac{1}{1-\phi}}$. The average growth rate is pinned down by population growth. This implies that R&D subsidy is ineffective on the trend growth measured by a geometric average.[38]Note that this result is unaffected by the periodicity of endogenous growth cycles.[39]

[38]Recall that technological progress alters asymmetrically between a good and bad state in response to the industrial policy (see Table 1). It promotes R&D in one state, but discourages in another state. A semi-endogenous geometric average growth rate means that the industrial policy affects period-to-period rates of growth in such a way that a net effect is zero. That is, a policy-induced increase in productivity growth in one period is exactly offset by its fall in the following period if those effects are measured using a geometric mean. This observation applies to changes in any parameter, except for population growth n and the measure of externality ϕ.

[39]This statement is true in the limit in the case of an invariant closed curve of g_t that occurs via a Hopf bifurcation.

How do we reconcile these seemingly conflicting observations? Note that g_t is typically a small number. Therefore, an arithmetic average can be taken as a good approximation of an geometric mean, i.e. $\ln(g_{\text{geo}} + 1) \simeq g_{\text{arith}}$ or[40]

$$\frac{\ln(g_0+1)+\ln(g_1+1)}{2} \simeq \frac{g_0+g_1}{2}. \tag{24}$$

This means that the impact of R&D subsidy on the average growth would be quantitatively small. However, note that the error of this approximation increases more than linearly in g. Therefore, the impact of, e.g. R&D subsidy seems greater as the rates of technological progress are larger.

5.7 Multiplicity of Cycles

5.7.1 "Long-run" Analysis

The aim of this section is to establish the existence of multiple cycles under certain conditions. This interesting result will be established for the special case of $\theta = 0$.[41] A consequence of this restriction is that the R&D incentive condition is reduced to

$$m_t = \frac{m_{t+1}}{\left[\beta\left(\frac{(1/\alpha - 1)\,\delta}{1-s}m_{t+1} + 1\right)\right]^{1/\sigma}} \equiv f(m_{t+1}). \tag{25}$$

Since it is independent of g_t, the evolution of m_t is determined by (25) only. This condition is depicted in Figure 6. Although the determination of g_t requires analysis of both difference equations (5) and (25), we initially focus on the behavior of m_t. By doing so, we aim to isolate long-run cyclical equilibria, postponing the discussion of its stability property, i.e. dynamics around the long-run equilibria. Our strategy is to examine the long-run behavior of m_t first, and then impose it on the equation of motion of g_t (5) to derive the long-run dynamics of g_t, sideskipping the stability issue of the equilibria that arises. In a sense, this approach is similar to a familiar long-run steady state analysis

[40] $\ln(1+g_0)$ can be approximated by g_0 when g_0 is small.

[41] This restriction is equivalent to $\sigma = \frac{\frac{1-\alpha}{\alpha}-\phi}{1+\frac{1-\alpha}{\alpha}-\phi}$.

of a dynamic model without examining its stability property. The *full* stability property of long-run equilibria will be discussed later.

The condition (25) is initially interpreted as a *backward*-looking difference equation, i.e. the current value m_t depends on its future value m_{t+1}. This interpretation is appropriate, since the current R&D effort is determined on the basis of what innovators expect profits will be in the future (see (11) and (12)). An obvious problem with this approach is that time is running backwards. However, note that an equilibrium cycle that emerges when time goes to infinite past from now also exists when time goes to infinite future. We will discuss later how the results based on a backward-looking equation (25) may be interpreted when time goes forward.

5.7.2 Existence of Cycles

One can easily check that the slope of $f(.)$ at a steady state m^* is $f'(m^*) = -\frac{\alpha-(1-\alpha\phi)\beta}{1-\alpha(1+\phi)}$ where the denominator is positive for $\theta = 0$. When it is modulus less than one, the steady state is stable, hence no cycle arises once an economy starts at its vicinity.[42]When this slope becomes less than -1, the steady state turns unstable, and the economy diverges from it if it starts from its neighborhood. However, note that m_t cannot go to zero, since the slope of $f(m_t)$ at $m_t = 0$ is greater than one.[43]m_t cannot go to infinity either, since m_t cannot escape from the interval $(0, f(\widetilde{m}))$ once it enters the region, as seen in Figure 6. Therefore, an economy must converge to a periodic (or aperiodic) cycle.

In general, cyclical equilibria require $f'(m^*) < -1$, which means

$$\phi \geq \frac{1}{\alpha} - \frac{2}{1+\beta}. \tag{26}$$

This condition defines the lower bound of the degree of externality for cyclical m_t. On the other hand, there is the upper bound defined by $\theta = 0$, which requires $\sigma \geq 0$ or

$$\frac{1}{\alpha} - 1 \geq \phi. \tag{27}$$

[42]Stability of m_t is analyzed in a backward-looking sense in what follow until we consider forward-looking dynamics later.

[43]One can eaily verify $f'(0) = \beta^{-\frac{1-\alpha\phi}{1-\alpha(1+\phi)}} > 1$.

Note that the set of values of ϕ that satisfy (26) and (27) is non-empty. This confirms that there exist long-run cyclical dynamics of m_t.

To demonstrate this result graphically, Figure 7 is created for $\alpha = 0.7$ with other parameters given in (21). The horizontal axis measures the degree of externality ϕ starting from 0.4, holding other parameter values fixed. On the vertical axis, the long-run values of m_t, generated after many iterations are measured for each value of ϕ. For $\phi \precsim 0.4042$, a steady state is stable, and there is no cycle. But once ϕ passes this critical value, the steady state bifurcates into a period-2 cycle, followed by a period-4 cycle, a period-8 cycle, and so on. Given parameter values used, the range of ϕ where long-run cycles arise is rather narrow. But within this range, a rich variety of equilibrium cycles exist.[44] Also note that simulation used to create Figure 6 can identify stable cycles only in a backward-looking sense.

5.7.3 Multiplicity of Cycles

A more striking result can be obtained if we consider the so-called Sarkovskii ordering of all positive integers

$$\begin{aligned} 3 &\Rightarrow 5 \Rightarrow 7 \Rightarrow \cdots \Rightarrow 2\times 3 \Rightarrow 2\times 5 \Rightarrow 2\times 7 \Rightarrow \cdots \\ &\Rightarrow 2^2\times 3 \Rightarrow 2^2\times 5 \Rightarrow 2^2\times 7 \Rightarrow \cdots \Rightarrow 2^3\times 3 \Rightarrow 2^3\times 5 \Rightarrow 2^3\times 7 \Rightarrow \cdots \\ &\Rightarrow 2^5 \Rightarrow 2^4 \Rightarrow 2^3 \Rightarrow 2^2 \Rightarrow 2 \Rightarrow 1. \end{aligned}$$

Sarkovskii (1964) proved that if a map has an orbit of period k, which precedes k' in the Sarkovskii ordering, then the map also has a periodic orbit of k'. This result reveals the coexistence of cycles of different periodicity. Remarkably, if a map exhibits a period-3 cycle, then there coexist infinitely many cycles with every possible period. Figure 7 shows that a period-3 cycle exists.

Example 5 *For parameter values used in Figure 7, a period-3 cycle exists at* $\phi = 0.4133$ *with* $m_0 \simeq 3.123$, $m_1 \simeq 0.126$ *and* $m_2 \simeq 33.067$.

[44] Figure 7 does not show a full range of values that ϕ can take. However, it should be obvious that a steady state m^* is stable for $\phi < 0.4$, but complicated cycles exist for $0.415 < \phi \lesssim 0.4286$ where the upper bound value is defined by (27).

Now suppose that m_t exhibits a period-k cycle. There are k different equilibrium values of m_t in the long run. To translate the dynamics of m_t into a cycle of g_t, we need to solve a system of k different equations obtained from (5), i.e. $g_{i+1} + \delta m_{i+1} = z(g_i, m_i)$, $i = 0, 1, ...k - 1$. Solving these equations yields long-run cyclical equilibria in the (m_t, g_t) space.

However, simulation shows that g_t often hits zero when m_t exhibits a cycle of order 4 or higher in the Sarkovskii ordering.[45] In that case, a steady state is the long-run equilibrium, and growth is semi-endogenous. On the other hand, if R&D subsidy is sufficiently high, more complex dynamics in g_t becomes possible.

Example 6 *For parameter values used in Figure 7,* $\phi = 0.40939$ *and* $s = 0.62$*, the economy exhibits a period-4 growth cycle with* $g_0 \simeq 0.0682$*,* $g_1 \simeq 0.0015$*,* $g_2 \simeq 0.0689$ *and* $g_3 \simeq 0.00004$*.*

Example 7 *For parameter values used in Figure 7,* $\phi = 0.4133$ *and* $s = 0.84$*, the economy exhibits a period-3 growth cycle with* $g_0 \simeq 0.0495$*,* $g_1 \simeq 0.0538$ *and* $g_2 \simeq 0.0005$*.*

In Example 7, an infinite number of orbital paths coexist in the long run, along which growth is endogenous. Those examples also indicate that R&D subsidy can be a source of multiplicity of cyclical equilibria.

These results admittedly require an implausibly high rate of R&D subsidy.[46] However, it seems too premature to dismiss this intriguing result on the basis of calibrated values, given that our model lacks other realistic features of an aggregate economy (e.g. capital and human capital accumulation).[47]

[45] Simulation also shows that a cycle of order 4 or even higher order can occur without g_t being zero, if n gets small enough.

[46] The lowest rate of R&D subsidy required for complex cycles becomes higher or lower depending on parameter combinations. For example, consider $\alpha = 0.654$. A period-4 cycle in g_t occurs with $s = 0.57$, and a period-3 cycle arises with $s = 0.76$.

[47] Indeed, if we introduce other types of policy, e.g. production subsidy related to differentiated goods, the required rate of R&D subsidy would be lower. Simulation also shows that a cycle of higher order can arise without a much lower subsidy rate when a population growth rate n is low.

5.7.4 Stability

We examine the stability of a cycle in *forward*-looking dynamics in three steps. First, we consider the stability property of m_t alone (i.e. along the $m-$axis) in *backward*-looking dynamics. For this, consider the Schwarzian derivative:

$$Sf\left(m_{t+1}\right) = \frac{f'''\left(m_{t+1}\right)}{f''\left(m_{t+1}\right)} - \frac{3}{2}\left(\frac{f''\left(m_{t+1}\right)}{f'\left(m_{t+1}\right)}\right)^2. \tag{28}$$

An important result in the nonlinear dynamics literature is the following. Given that $f\left(m_{t+1}\right)$ is single-peaked, there is *at most* one weakly stable cycle if the Schwarzian derivative (28) is negative for the range of $(0, f(\widetilde{m}))$ in Figure 6, except $\widetilde{m}$ where $f'(\widetilde{m}) = 0$.[48] That is, when there exist multiple cycles in m_t, as shown above, at most one cycle is stable and other cycles are unstable.

Second, note that cycles (as well as a steady state) which are stable in *backward*-looking dynamics are unstable in *forward*-looking dynamics.[49] This means that when the Schwarzian derivative is negative for a backward-looking difference equation (25), there is at most one weakly unstable cycle and other cycles that exist are all stable in a *forward*-looking sense. Therefore, if there is a period-3 cycle in *backward*-looking dynamics with a negative Schwarzian derivative, then there exists an infinite number of cycles of every other possible periodicity, all of which are stable in forward-looking dynamics.

Example 8 *$Sf\left(m_{t+1}\right) < 0$ for the range of $(0, f(\widetilde{m}))$, except $\widetilde{m}$, whenever long-run cycles arise in Figure 7.*

In the third step, note that the above discussion concerns the stability property of m_t or stability of the model along the m_t-axis. Therefore, it is not clear whether equilibrium growth cycles are a sink, a source or a saddle, just like a local bifurcation analysis above. This approach may be justified, since the issue of whether or not growth is endogenous can be answered by examining *long-run* dynamics of the model, rather than dynamics around the long-run equilibria.

[48] See Grandmont (1992) for details.

[49] Therefore, cycles in Figure 7 are in fact unstable in a forward-looking sense. On the other hand, backward- and forward-looking difference equations may share the same stability property, once a learning process is explicitly introduced. We do not explore this interesting subject here. See Evans and Honkapohja(1999) for details of this topic.

However, the full stability property of a specific case can be easily verified by simulation. A period-3 cycle in Example 7 is a saddle.[50] This should not be surprising, given that all examples of Flip bifurcations above are subcritical. In Example 7, there exists an infinite number of saddle-type cycles of different periodicity in addition to a stable steady state.

6 Growing through Stochastic Cycles

In the above analysis, endogenous growth arises due to deterministic cycles. The aim of this section is to demonstrate that growth can also be endogenous due to stochastic cycles, which occur endogenously. A key difference between deterministic and stochastic cycles lies in the type of expectations.

With perfect foresight, agents correctly anticipate what will happen in future. An alternative assumption is sunspot beliefs in which agents base their expectations on exogenous random signals, which they know are not related to the fundamentals of the economy. Such sunspot beliefs are self-fulfilled, depending on parameter values. The question is: Is semi-endogenous growth robust to sunspot beliefs?

6.1 The Stochastic Model

To simplify analysis, we assume the risk-neutral consumers, i.e. $\sigma = 0$. In this case, modification is required only in the value of innovation (11), which is the sum of the expected future profit flows:

$$v_t = \beta \mathbf{E} \left(\pi_{t+1} + v_{t+1} \right) \tag{29}$$

where $\mathbf{E}$ is an expectation operator. Note that expectations are taken over future profits, but uncertainty arises due to entrepreneurs' beliefs about sunspot, which has no information about endowments, technology and preferences. Sunspots are assumed to follow a stationary Markov process with a sufficiently small

[50] In simulation, time runs backward for (25) and forward for (5) with an initial condition involving $m_0 = \widetilde{m}$ in Figure 5.

support on a given interval. Entrepreneurs observe the random variable in each period before they make their R&D decisions.

Using (29) and other conditions, we can derive the R&D incentive condition:

$$\frac{1-s}{\delta\left(1/\alpha-1\right)}\left[\frac{\left(g_t+1\right)^{\phi-\frac{1-\alpha}{\alpha}}}{\beta}-1\right]=\mathbf{E}\left(m_{t+1}\right). \quad (30)$$

The long run distribution of (m_t, g_t) is determined by two conditions (5) and (30). Linearizing those equations around a steady state gives

$$\begin{pmatrix} \widetilde{g}_{t+1} \\ \widetilde{m}_{t+1} \end{pmatrix} = J \begin{pmatrix} \widetilde{g}_t \\ \widetilde{m}_t \end{pmatrix} + \begin{pmatrix} -\delta \\ 1 \end{pmatrix} \widetilde{\varepsilon}_{t+1} \quad (31)$$

where $\widetilde{\varepsilon}_{t+1} \equiv m_{t+1} - \mathbf{E}(m_{t+1})$ and the Jacobian matrix J is equivalent to (16) for $\sigma = 0$.

Next, we invoke the following famous result in the nonlinear dynamics literature. Stochastic sunspot equilibria arise around a steady state if it is a sink.[51] That is, endogenous stochastic cycles exist if both eigenvalues of J are modulus less than one. The idea goes as follows. If a steady state is a sink, the economy converges to it over time, if it is located close enough to the steady state. On the other hand, sunspot beliefs dislocate the economy off the convergent path. The economy can move away from the steady state, but not "too far" from it, given the assumption of a sufficiently small finite support of the random process (i.e. $\widetilde{\varepsilon}_{t+1}$ is not too large). This assumption is essential for a linearized system to be a good approximation of the nonlinear model. In the long run, those opposing forces balance out, and there is an invariant distribution of (m_t, g_t) in the long run.

According to this result, we only need to examine the stability property of a steady state, using Figure 2. First note that $D > T - 1$ that is necessary for indeterminacy is equivalent to $1 > \lambda_2$, which is always satisfied. Given this, our strategy is to characterize parameter combinations (ϕ, α), using parameter values in (21). An advantage of this approach is that we do not impose prior restrictions on ϕ and α, while other parameters take reasonably plausible values.

[51] See Woodford (1986) and Guesnerie and Woodford (1992).

In Figure 8, two curves are drawn, representing $D(\phi, \alpha) = 1$ and $D(\phi, \alpha) + T(\phi, \alpha) = -1$. The intertemporal utility is not bounded in the shaded region. The figure demonstrates that the region of indeterminacy is not trivial. Endogenous stochastic cycles can occur in a wide range of parameter combinations. An interesting observation is that a higher degree of externality tends to limit an interval of α compatible with indeterminacy. Indeed, if ϕ is too high, a steady state becomes a saddle, and a stochastic cycle will not arise.

6.2 An Example

To illustrate the existence of a stationary sunspot equilibrium, consider the following example. Agents believe that there are two states, i.e. $m_{t+1} + e_{t+1}$ where $e_{t+1} = \{e_0, e_1\}$ and $e_0 \neq e_1$. Using P to denote the probability that e_0 will occur, $Pe_0 + (1 - P)e_1 = 0$ is assumed. Then, the R&D incentive condition (30) becomes

$$\frac{1-s}{\delta(1/\alpha - 1)}\left[\frac{(g_t+1)^{\phi - \frac{1-\alpha}{\alpha}}}{\beta} - 1\right] = m_{t+1} + e_{t+1}. \tag{32}$$

If e_0 and e_1 are sufficiently small, the conditions (5) and (32) define the degenerate distribution of (g_t, m_t).

Example 9 *Using parameter values of* $\alpha = 0.7$, $\phi = 0.6$, $e_0 = 1$, $e_1 = -0.25$ *and* $P = 0.2$, *Figure 9 shows the stochastic dynamics of* g_t *in the long run.*

In the figure, sudden jumps represent extrinsic uncertainty which has nothing to do with the fundamentals. Given no clear periodicity, it is not possible to examine the effect of parameter changes on each growth rate. Instead, we consider the standard deviation of g_t as an alternative indicator demonstrating endogenous growth. The results are summarized in Table 2. Note that the standard deviation of g_t is zero for semi-endogenous growth.

Regarding the average growth rate, simulation indicates that it little differs from a semi-endogenous growth rate up to the third decimal point, whether it is calculated as an arithmetic or geometric average.

7 Departing from the Knife-edge Perspective

The present paper examines the robustness of a one-sector R&D-based model as an economic system generating endogenous growth. It is worth stressing what differentiates our approach from the dominant perspective of the current literature.

Section 2.1 showed that, given the R&D technology (2), stationary growth is endogenous for $\phi = 1$ with scale effects, but semi-endogenous for $1 > \phi$ without scale effects. Note that $\phi = 1$ is obtained only on a measure-zero subset of $1 \geq \phi$. This means that if a value of ϕ is randomly chosen, the probability of endogenous growth is zero, while the probability of semi-endogenous growth is one. In this sense, endogenous growth is highly special according to this externality criteria. The robustness issue hinges only on ϕ.

The approach adopted in this paper departs from this knife-edge perspective. We examine the robustness issue on the basis of the nature of long-run equilibria, stationary or cyclical. Growth can be endogenous or semi-endogenous even if a knife-edge condition is violated. Consider Figures 3, which show combinations of (ϕ, σ). Endogenous deterministic cycles exist in the area near the border lines between the regions Sink and Source 1/Saddle 1a. Therefore, if a combination (ϕ, σ) is randomly picked, the probability of endogenous growth is strictly positive. Similarly, Figure 8 shows combinations of (ϕ, α). Given that endogenous stochastic cycles exist in the region Sink, the probability of endogenous cycles is again strictly positive. In short, endogenous growth ceases to be a "knife-edge" case and becomes more likely once the possibility of endogenous cycles is taken into account.

There is an additional advantage of our cyclical approach. Our model can potentially offer an explanation of cyclical behaviors of some R&D-related time-series data. First, Figure 10 shows a TFP growth rate in the U.S. An extremely oscillatory pattern can be due to short-run shocks or may represent uncorrected factors such as monopoly markups and capital utilization. However, it is widely accepted that TFP exhibits a persistent cyclical trajectory over a long period of time. This cyclical feature of TFP growth cannot be explained by R&D-based models in which a non-cyclical steady state is the only type of long-run equilibria.

Second, in Figure 2 of Jones (1995b, p.764), he showed that the share of R&D workers in total labor force increased in the U.S. for four decades since 1950. A look at the figure immediately suggests the existence of a cycle. The share steadily increased from 1950 to late 1960s. Then, it showed a steep fall which continued for about 10 years, followed by a steady rise to the end of the data. Since the trend is increasing, Jones (2002) argues that the U.S. economy has not achieved a long-run equilibrium. Our cyclical model offers an alternative interpretation. As far as the share of R&D workers is concerned, it can be accounted for as a long-run cyclical equilibrium behavior.[52]

Third, Figure 2 of Howitt (1999, p.726) shows the proportion of R&D expenditure in GDP in the U.S. from 1953 to mid-1990s. It showed a steep rise in the first decade, followed by a fall which continued for more than a decade. Then, it picked up for about five year, and showed a drop again. Since the data appears trendless, Howitt argued that a non-cyclical steady state of his model is consistent with the data.[53]Our model suggests that a cycle that seems to exist in the data can be explained as a long-run equilibrium behavior.[54]

8 Conclusion

A hallmark of R&D-based endogenous growth models is that public policy affects long-run growth. An empirical study of Jones (1995a) challenged this theoretical prediction derived from early one-sector R&D-based models. Those models predict scale effects, whereas Jones demonstrated a clear lack of correlation between a falling TFP trend growth and an increasing number of scientists

[52] Jones (2002) regarded an increase in educational attainment as an additional indication that the economy is not in a non-cyclical long-run equilibrium.

[53] He also observed that "the ratio reached a peak in 1964 that was never again reached in 32 years."

[54] In our model, the share of R&D workers in total population is $g_t/(g_t + \delta m_t)$, and the proportion of R&D expenditure in GDP is $\alpha g_t/\delta m_t$. They exhibit cycles when the economy is in oscillatory equilibria.

and engineers over decades in developed economies. In an attempt to reconcile the empirical evidence with one-sector R&D-based models, Jones (1995b) proposed a semi-endogenous growth model with limited externalities in the R&D sector. This result led to the Consensus View that is currently predominant in the literature. This View stands on the ground that growth is endogenous in one-sector R&D-based models only when a parameter capturing externalities in R&D takes a knife-edge value ($\phi = 1$), i.e. new ideas created are linear in the stock of knowledge.

This paper challenged this Consensus View. We departed from this dominant knife-edge perspective, and turned to the issue of whether or not long run equilibria are cyclical (when the knife-edge condition is violated). Our argument was developed, using a discrete-time version of an otherwise very standard one-sector R&D-based model. Specifically, we demonstrated that the rate of technological progress may exhibit an endogenous cycle, deterministic or stochastic, in the long run, hence it is no longer pinned down by parameters which are often considered to be exogenous. In the long run, public policy and consumer preferences will affect productivity growth. Growth is re-endogenized in our cyclical framework, and endogenous growth ceases to be a "knife-edge" case in one-sector R&D-based models. Our "cyclical" approach makes endogenous growth more probable in one-sector R&D-based models than previously thought.

Appendix

This appendix shows that a divergent path towards $g_t = 0$ contradicts perfect foresight of private agents. Suppose that $g_t = 0$ in the long run. This means that $\overline{N} \equiv N_t$ and $L_t = M_t$, which in turn imply $Y_t = \overline{N}^{\frac{1-\alpha}{\alpha}} L_t$ from (8). These also imply $w_t = \overline{w} = \alpha \overline{N}^{\frac{1-\alpha}{\alpha}}$ from (9) and $p_{it} = w_t/\alpha$. Moreover, since $R_t = 0$, we must have $v_{t+1} < \frac{(1-s)w_t}{\delta}$ instead of (12). Making use of (10), this inequality means $\sum_{\tau=1}^{\infty} \beta^{\tau} \frac{1-\alpha}{\alpha} \frac{\overline{w} L_{t+\tau}}{\overline{N}} < \frac{(1-s)\overline{w}}{\delta}$, which can be re-written as

$$\frac{1-\alpha}{\alpha} \frac{L_t}{\overline{N}} \sum_{\tau=1}^{\infty} [\beta(1+n)]^{\tau} < \frac{1-s}{\delta}. \tag{33}$$

For $\beta(1+n) \geq 1$, this inequality does not hold. For $\beta(1+n) < 1$, the inequality is violated in finite time, since L_t grows at a rate of n. This contradicts perfect foresight, allowing us to rule out $g_t = 0$ as a long-run equilibrium.

References

[1] Aghion, Philippe and Howitt, Peter (1992). "A Model of Growth through Creative Destruction." *Econometrica*, vol. 60, pp. 323-351.

[2] Azariadis, Costas. (1993). *Intertemporal Macroeconomics*. Blackwell.

[3] Baumol, William J. and Wolff, Edward N. (1983). "Feedback from Productivity Growth to R&D." *Scandinavian Journal of Economics*, vol.85, pp.147-57.

[4] Basu, S. and Fernald, J.G., (1997). "Returns to Scale in US Production: Estimates and Implications.". *Journal of Political Economy*, vol.105, pp.249-283.

[5] Basu, S. and Fernald, J.G., (2000). "Why Is Productivity Procyclical? Why Do We Care?". *NBER* Working Papers, No.7940.

[6] Boldrin, Michele, and David K. Levine (2002). "Factor Saving Innovation." *Journal of Economic Theory*, 105, 18-41.

[7] Caballero, Ricardo J. and Jaffe, Adam B., (1993). "How High Are the Giants' Shoulders: An Empirical Assessment of Knowledge Spillovers and Creative Destruction in a Model of Economic Growth," *NBER Macroeconomics Annual*, 1993, 8, 15-74.

[8] Cozzi, Guido. (2001) "Inventing or Spying? Implications for Growth." *Journal of Economic Growth*, 6, 55-77.

[9] Deissenberg, Christophe and Nyssen, Jules (1998). "A Simple Model of Schumpeterian Growth with Complex Dynamics." *Journal of Economic Dynamics and Control*, vol.22, pp.247-66.

[10] Domowitz, I., G. Hubbard, and B. Petersen. (1988). "Market Structure and Cyclical Fluctuations in U.S. Manufacturing." *Review of Economics and Statistics*, vol.70, pp.55-66.

[11] Drugeon, Jean-Pierre and Wigniolle, Bertrand (1996). "Continuous-Time Sunspot Equilibria and Dynamics in a Model of Growth." *Journal of Economic Theory*, 69, 24-52.

[12] Eicher, Theo S. and Turnovsky, Stephen J. (2001). "Transitional Dynamics in a Tow-sector Non-scale Growth Model." *Journal of Economic Dynamics and Control*, 25, 85-113.

[13] Eicher, Theo S. and Turnovsky, Stephen J. (1999). "Non-scale Models of Economic Growth." *Economic Journal*, 109, 394-415.

[14] Evans, George W. and Honkapohja, Seppo (1999). "Learning Dynamics." In John B. Taylor and Michael Woodford (eds.), *Handbook of Macroeconomics*, vol.1A, pp.449-542.

[15] Evans, George W.; Honkapohja, Seppo, and Romer, Paul (1999). "Growth Cycles." *American Economic Review*, vol.88, pp.495-515.

[16] Francois, Patrick and Lloyd-Ellis, Huw (2002). "Animal Spirits through Creative Destruction." Queen's University, unpublished.

[17] Francois, Patrick and Shi, Shouyong (1999). "Innovation, Growth, and Welfare-Improving Cycles." *Journal of Economic Theory*, vol.85, pp.226-257.

[18] Gali, Jordi, Mark Gertler, and David Lopez-Salido (2001). "European Inflation Dynamics." vol.45, pp.1237-70.

[19] Grandmont, Jean-Michel. (1992). "Periodic and Aperiodic Behavior in Discrete One-Dimensional Dynamical Systems." In: Benhabib, Jess., (ed.) *Cycles and Chaos in Economic Equilibrium*, Princeton University Press, Princeton, NJ, pp. 44-63.

[20] Grandmont, Jean-Michel; Pintus, Patrick, and de Vilder, Robin. (1998). "Capital-Labor Substitution and Competitive Nonlinear Endogenous Business Cycles." *Journal of Economic Theory*, vol.80, pp.14-59.

[21] Greenwood, Jeremy, Hercowitz, Zvi and Krusell, Per (2000) "The Role of Investment-specic Technological Change in the Business Cycle." *European Economic Review*, vol. 44, pp. 91-115.

[22] Griliches, Z. (1992). "The Search for R&D Spillovers". Scandinavian Journal of Economics, vol. 94, pp. 29-47.

[23] Grossman, Gene and Helpman, Elhanan. (1991). *Innovation and Growth in the Global Economy.* Cambridge MA: MIT Press.

[24] Guckenheimer, John and Holmes, Philip. (1983). "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields." New York: Springer-Verlag.

[25] Guesnerie, R. and Woodford, M. (1992). "Endogenous Fluctuations." In Laffont, J.J. (ed.), *Advances in Economic Theory*, Vol. 2, Cambridge University press, pp. 289-412.

[26] Howitt, Peter (1999). "Steady Endogenous Growth with Population and R&D Inputs Growing." *Journal of Political Economy*, vol.107, no.4, pp.715-730.

[27] Jones, Charles I. (2002). "Sources of U.S. Economic Growth in a World of Ideas." *American Economic Review*, vol. 92, pp. 220-239.

[28] Jones, Charles I. (2001). "Population and Ideas: A Theory of Endogenous Growth." Unpublished, Stanford University.

[29] Jones, Charles I. (1999). "Growth: With or Without Scale Effects?" *American Economic Review Papers and Proceedings*, vol.89, pp. 139-144.

[30] Jones, Charles I. (1995a). 'Time series tests of endogenous growth models.' *Quarterly Journal of Economics*, vol. 110, pp. 494-525.

[31] Jones, Charles I. (1995b). "R&D-based Models of Economic Growth." *Journal of Political Economy*, vol. 103, pp. 759-83.

[32] Jovanovic, B. and Lach, S. (1997). "Product Innovation and the Business Cycle." International Economic Review, vol.38, pp.3-22.

[33] Jovanovic, B. and Rob, R. (1990). "Long Waves and Short Waves: Growth through Intensive and Extensive Search", *Econometrica*, Vol.58, pp.1391-1409.

[34] Kortum, Samuel S. (1997). "Research, Patenting, and Technological Change." *Econometrica*, vol. 65, pp. 1389-1419.

[35] Li, Chol-Won. (2002). "Growth and Scale Effects: the Role of Knowledge Spillovers." *Economics Letters*, 74, 177-86.

[36] Li, Chol-Won. (2000). "Endogenous vs. Semi-endogenous Growth in a Two-R&D-sector Model." *Economic Journal*, 110, C109-C122.

[37] Matsuyama, Kiminori. (1999). "Growing Through Cycles." *Econometrica*, vol.67. pp.335-47.

[38] Mehra, R. and E.C. Prescott. (1985). "The Equity Premium: A Puzzle." *Journal of Monetary Economics*, vol.15, pp.145-61.

[39] Nishimura, Kazuo and Sorger, Gerhard. (1999). "Non-linear Dynamics in the Infinite Time Horizon Model." *Journal of Economic Survey*, vol.13, pp. 617-52.

[40] Romer, Paul. (1990). "Endogenous Technological Change." *Journal of Political Economy*, 98, S71-S102.

[41] Rotemberg, J. and Woodford, M., (1995). "Dynamic General Equilibrium Models with Imperfectly Competitive Product Markets." In: Cooley, T.F., (ed.) *Frontiers of Business Cycle Research*, Princeton University Press, Princeton, NJ, pp. 243-293.

[42] Segerstrom, Paul. (1998). "Endogenous Growth Without Scale Effects." *American Economic Review*, vol. 88, pp.1290-1310.

[43] Woodford, Michael. (1986). "Stationary Sunspot Equilibria in a Finance Constrained Economy." *Journal of Economic Theory*, 40, 128-27.

[44] Young, Alwyn. (1998). "Growth Without Scale Effects." *Journal of Political Economy*, vol. 106, pp. 41-63.

		Before Changes	R&D Subsidy $s = 0.01$	Discount Factor $\beta = 1/1.0501$	Population Growth $n = 0.0201$
Example 1	g_0	0.0321	0.0302 $(-)$	0.0332 $(+)$	0.0309 $(-)$
$(\alpha = 0.5)$	g_1	0.0237	0.0257 $(+)$	0.0227 $(-)$	0.0252 $(+)$
Example 2	g_0	0.0430	0.0563 $(+)$	0.0533 $(+)$	0.0447 $(+)$
$(\alpha = 0.7)$	g_1	0.0371	0.0241 $(-)$	0.0270 $(-)$	0.0358 $(-)$
Example 3	g_0	0.0251	0.0236 $(-)$	0.0263 $(+)$	0.0254 $(+)$
$(\alpha = 0.9)$	g_1	0.02220	0.0236 $(+)$	0.0210 $(-)$	0.02216 $(-)$

Table 1: The effects of parameter changes in period-2 cycles. "+" and "-" signs indicate an increase and decrease following parameter changes.

Example 9	Before Changes	R&D Subsidy $s = 0.1$	Discount Factor $\beta = 1/1.06$	Population Growth $n = 0.022$
Standard Deviation	3.2221×10^{-5}	3.0223×10^{-5} $(-)$	3.0486×10^{-5} $(-)$	3.2082×10^{-5} $(-)$

Table 2: The effects of parameter changes on the standard deviation in a stochastic growth cycle of Example 9. Each simulation involves 10100 iterations, and each standard deviation is calculated using the last 10000 observations. "+" and "-" signs indicate an increase and decrease following parameter changes.

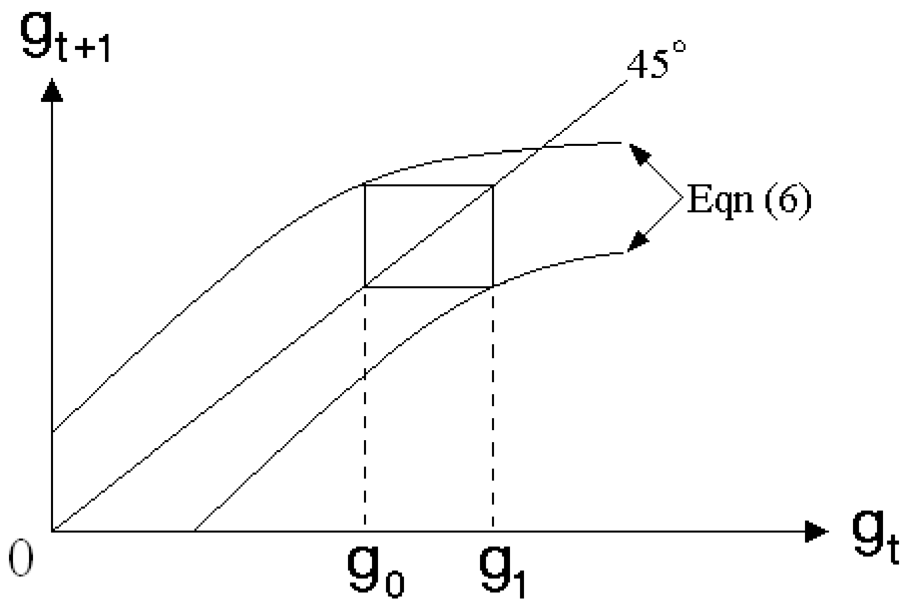

Figure 1: A period–2 deterministic cycle.

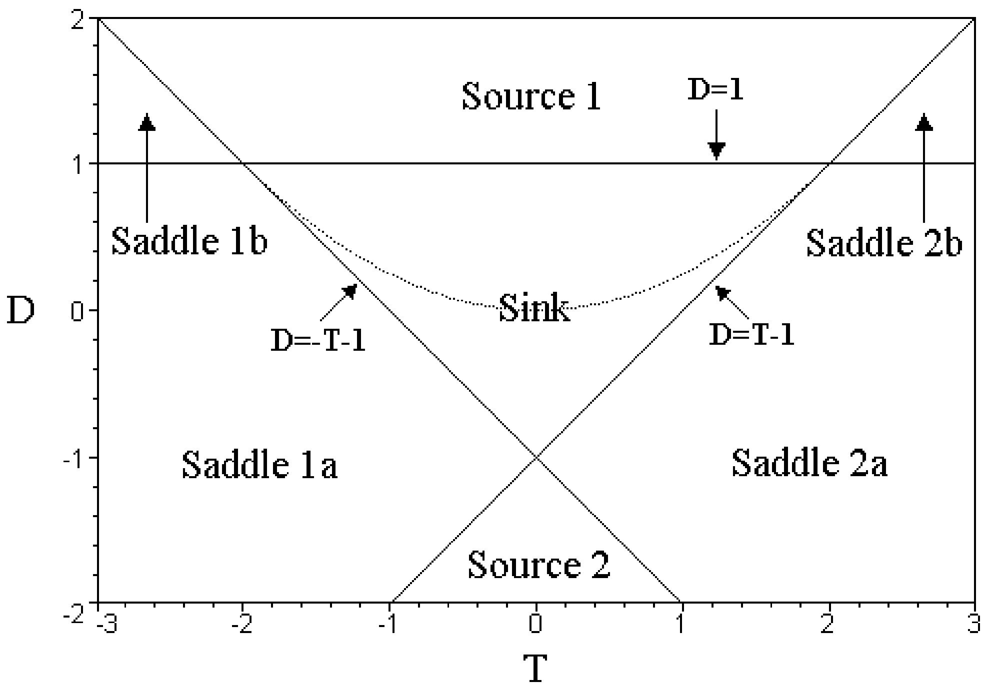

Figure 2: Local stability analysis.

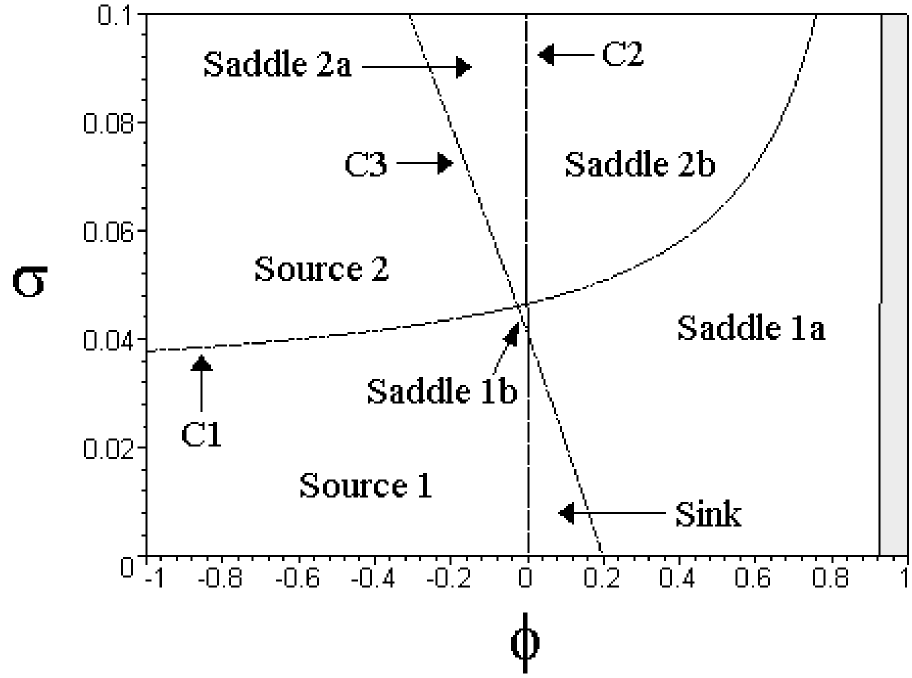

Figure 3-(a): Local stability analysis for $\alpha = 0.9$
($\beta = 1/1.05,\ \delta = 0.01,\ n = 0.02$ and $s = 0$).

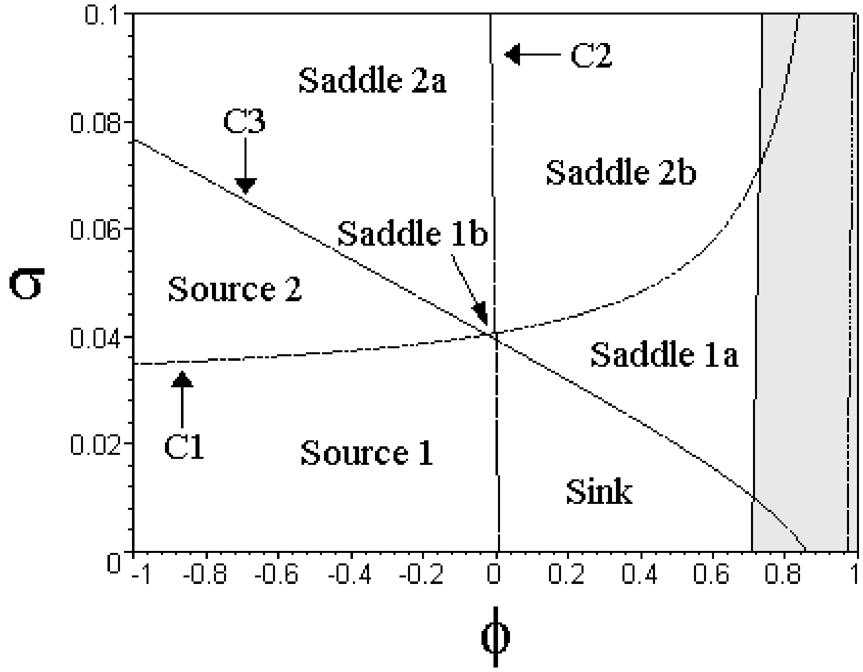

Figure 3-(b): Local stability analysis for $\alpha = 0.7$
($\beta = 1/1.05,\ \delta = 0.01,\ n = 0.02$ and $s = 0$).

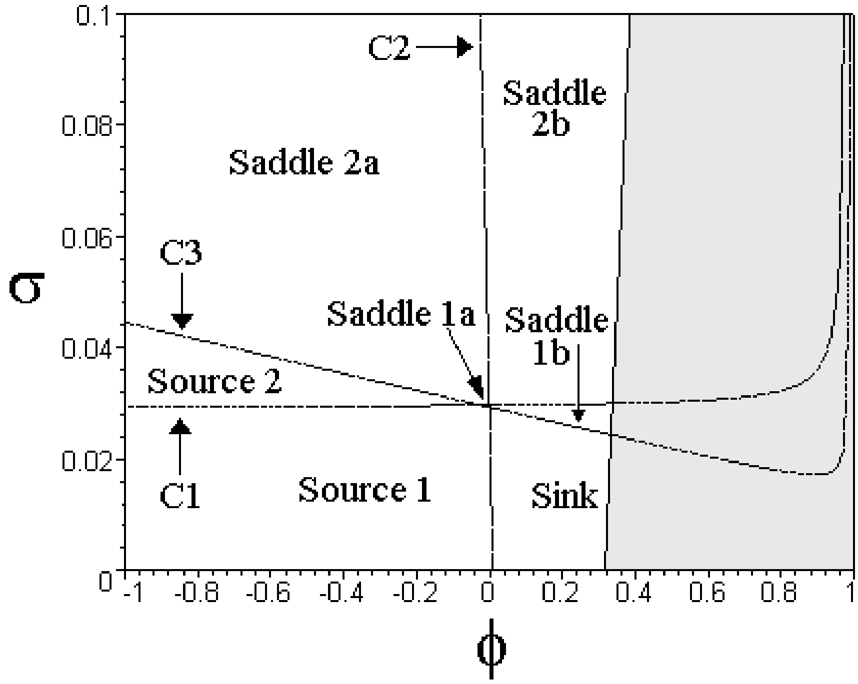

Figure 3-(c): Local stability analysis for $\alpha = 0.5$
($\beta = 1/1.05,\ \delta = 0.01,\ n = 0.02$ and $s = 0$).

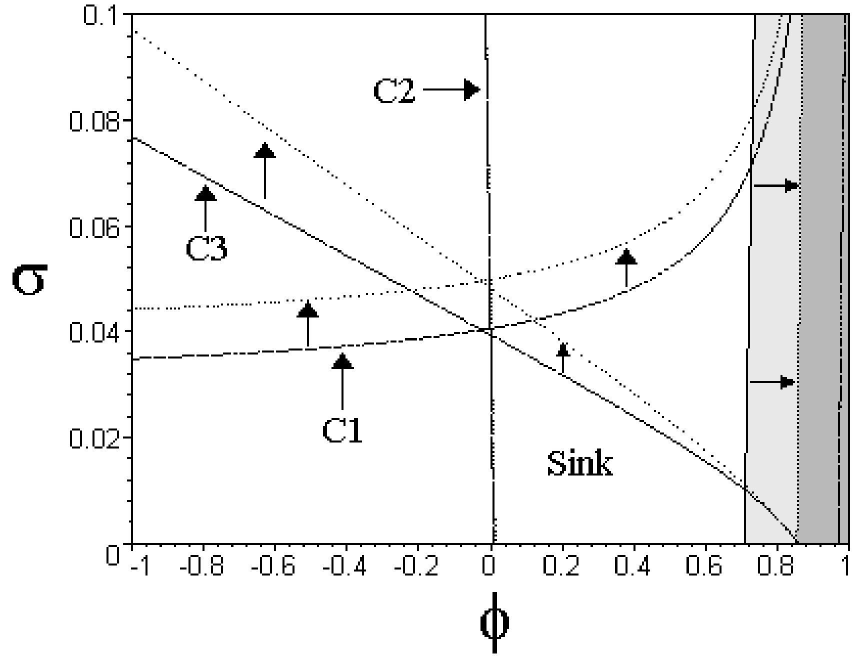

Figure 3-(d): Local stability analysis when β changes from 1/1.05 to 1/1.06 for $\alpha = 0.7$ ($\delta = 0.01,\ n = 0.02$ and $s = 0$).

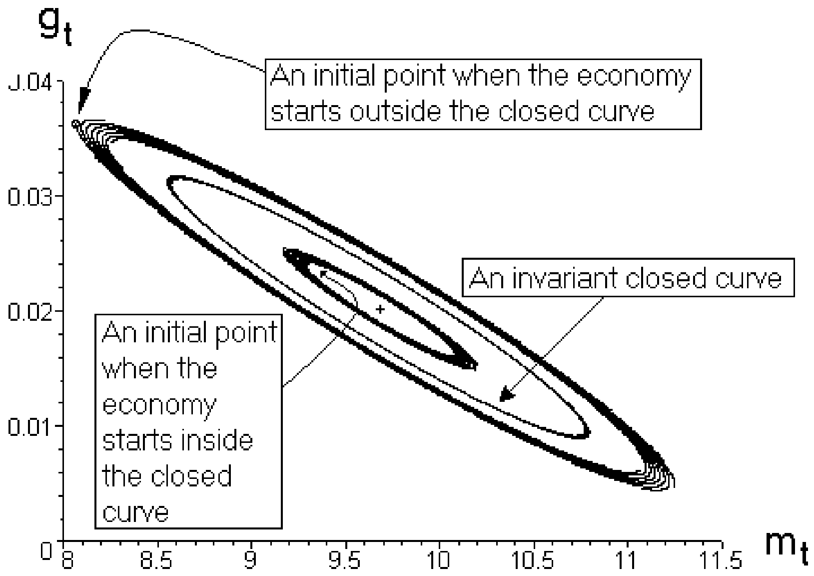

Figure 4: A stable invariant closed curve due to a Hopf bifurcation is created from a series of 10000 points (i.e. 10000 iterations). Convergent paths from outside and inside the curve consist of 10000 and 20000 points, respectively.

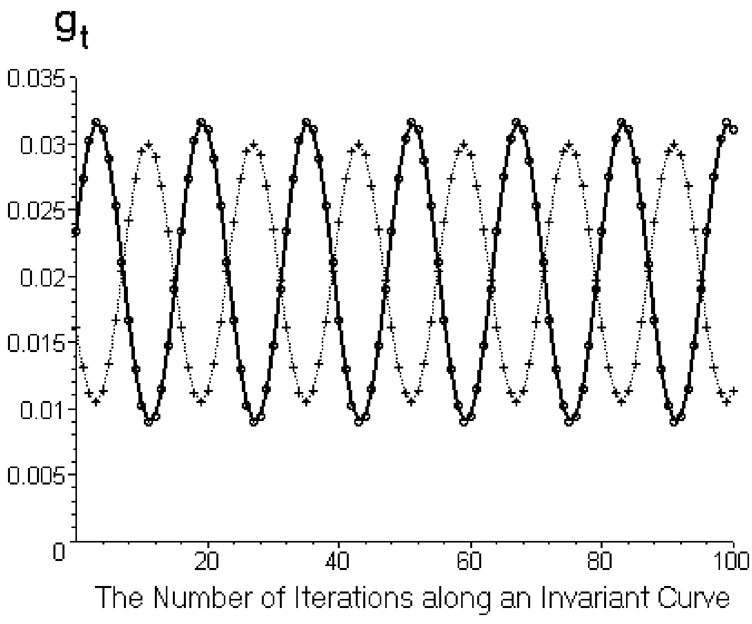

Figure 5: The effect of a higher discount factor on technological progress. $\beta = 1/1.05$ for a solid curve and $\beta = 1/1.04999$ for a dotted curve.

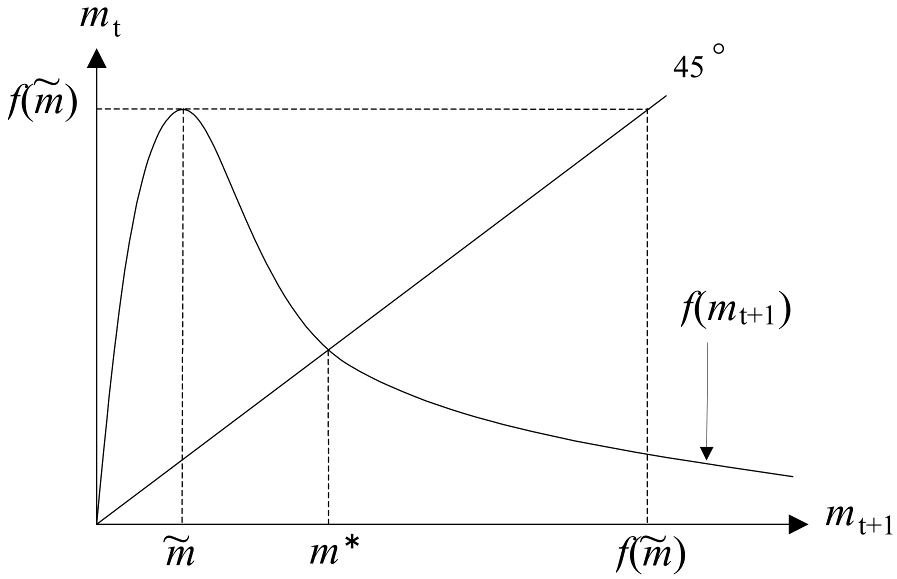

Figure 6: A backward-looking dynamics of m_t.

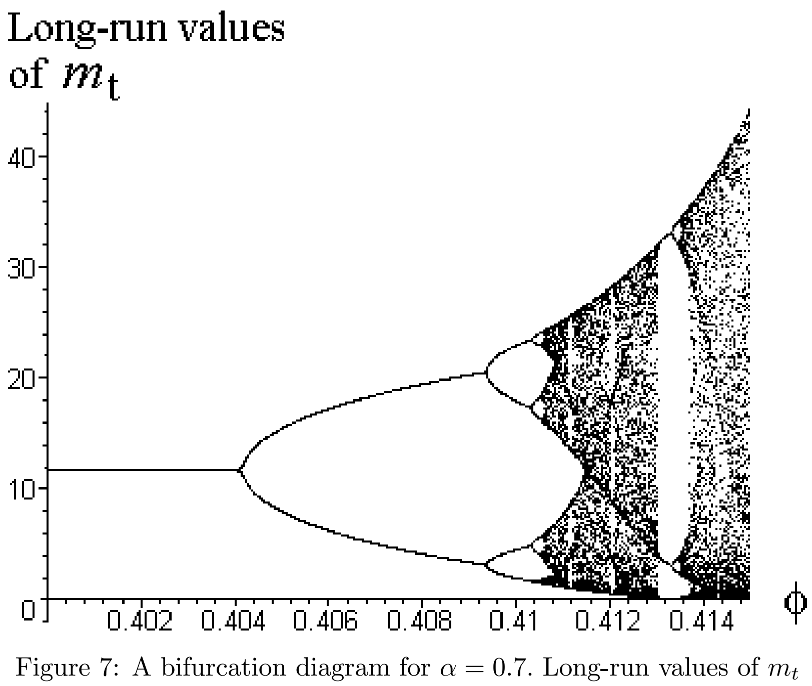

Figure 7: A bifurcation diagram for $\alpha = 0.7$. Long-run values of m_t are obtained after 300-500 iterations for each value of ϕ.

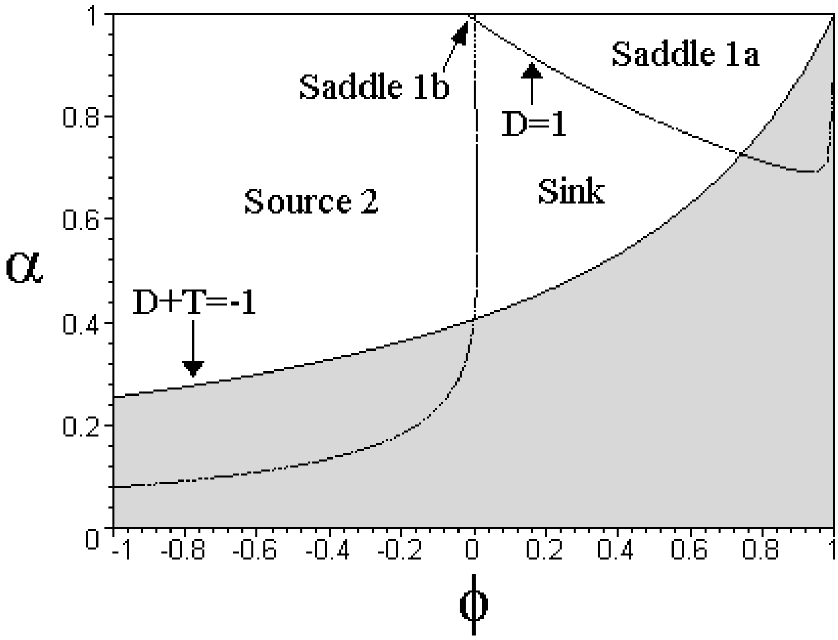

Figure 8: A steady state is indeterminate in the region Sink where there is a continuum of sunspot equilibria, and growth is endogenous.

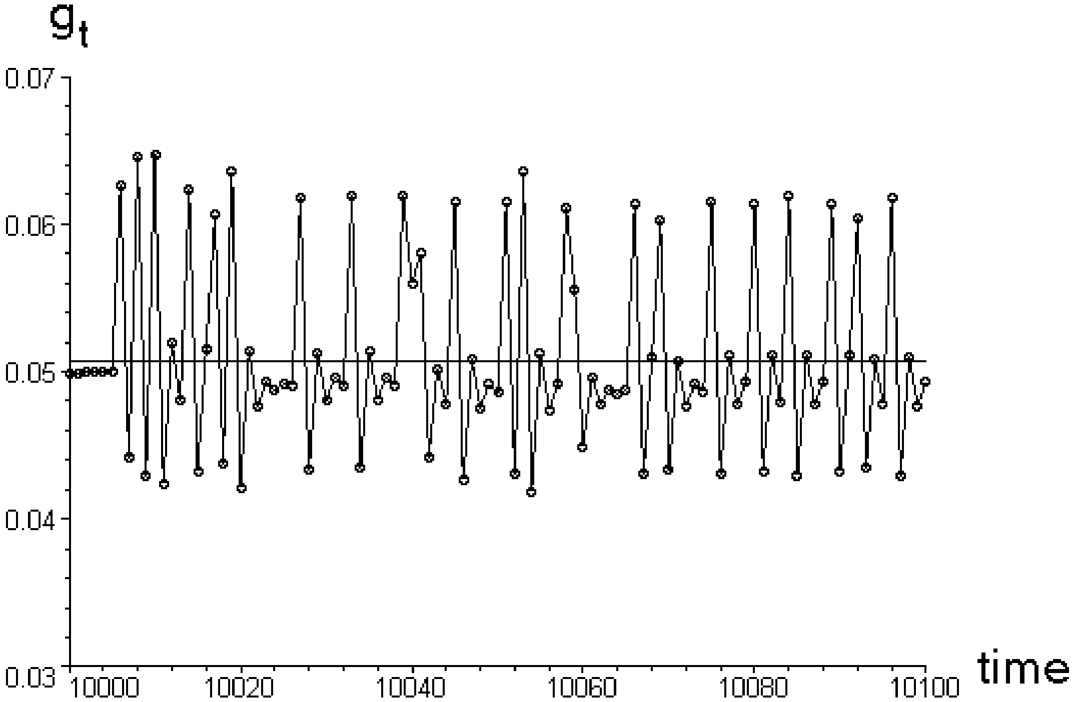

Figure 9: A time series of g_t after 10000 iterations in sunspot equilibria.

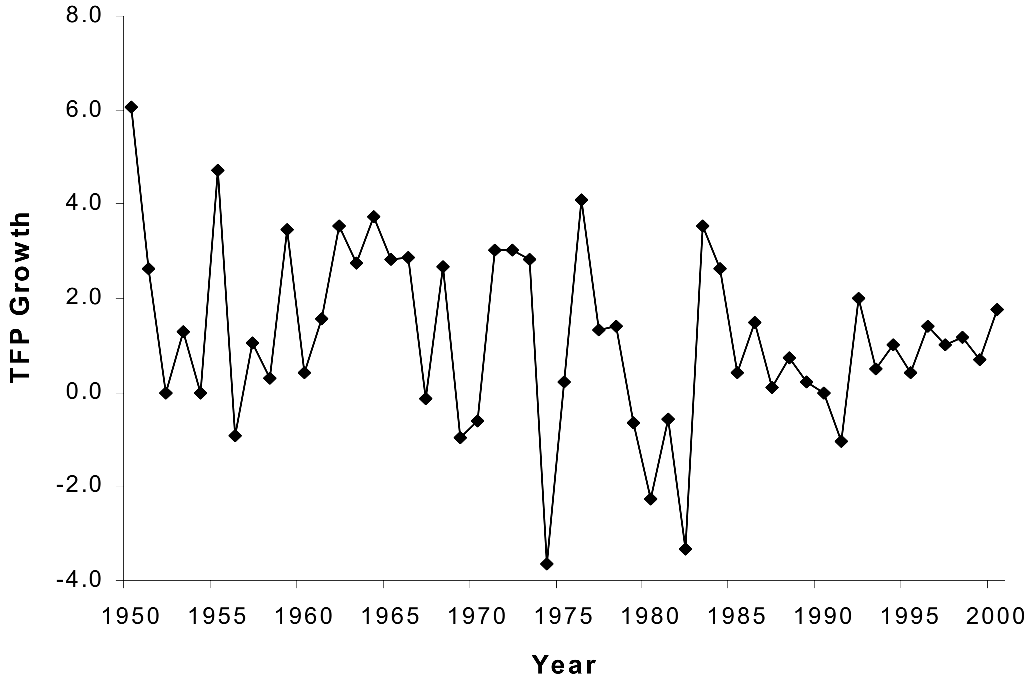

Figure 10: Private non-farm business TFP growth rates in the U.S.
(Source: The Bureau of Labor Statistics)