Truth-telling and the Role of Limited Liability in Costly State Verification Loan Contracts

P Simmons & G Garino
University of York and University of Leicester

January 17, 2003

Abstract. Recent literature has considered the form of loan contract between two or more risk neutral parties where the revelation principle is inappropriate due to the lack of commitment to an auditing policy by the lender. The privately informed debtor has a stochastic return; once he knows the state realisation, auditing and cheating are determined as Nash equilibria. The literature assumes that this leads to randomised cheating and auditing. In this paper we verify that the contract may involve this randomisation; but that it may also involve truthtelling with random auditing and one or more investors in line with Persons (1996); or a single state independent repayment with no auditing. We define conditions on the state observation cost and the distribution of returns which determine which of these three forms of contract is optimal. We find that under unlimited liability when the loan size is fixed the two investor truthtelling contract dominates all the other forms; and that this is also true when the loan size is optimally chosen. On the other hand under limited liability if the cost of observation is large relative to the lowest state revenue, the random auditing contract or a constrained two investor truthtelling contract may be optimal. The limited liability condition in the constrained truthtelling contracts forces the level of finance to be higher than under unlimited liability.

Keywords: loan contracts, costly state verification, commitment, limited liability.

JEL Nos: C7, G0

Corresponding author:

P Simmons

Department of Economics

University of York

York Y01 5DD, UK

ps1@york.ac.uk

1. Introduction

Consider a risk neutral agent - an entrepreneur or a consumer - who wishes to borrow funds to finance a project or consumption from alternative competitive risk neutral lenders. Borrower income varies with states of nature and the loan(s) size; but once realised, the state remains private information to the borrower, who makes an income report to the lenders. In a scenario in which the prospective borrower has private information about their income at repayment and where it is common knowledge that income at that time may be insufficient to repay a fair (safe) return on the debt, then no loan could be made since the borrower would always cheat offering low repayments; and the lenders would be unable to get the fair return on their loan. However the costly state verification approach lets any lender undertake a costly audit at repayment stage in order to discover the true income of the borrower. The contract stipulates a loan size and feasible repayments in each state to each lender [1] which can be conditioned on the report of the borrower and/or the act of auditing (Townsend, 1979, 1989; Gale-Hellwig, 1985, 1989, Hellwig 2001; Mookherjee and Png, 1989). Sometimes the contract will respect limited liability on lenders so that all repayments in all states must be non-negative; but in other scenarios if lenders have unlimited liability repayments can be negative.

The simplest case of the above scenario would be a two period model; in the first period the loan contract is signed and in the second period repayments according to the contract are carried out. However, unless there is an external enforcement device to make the lenders perform the necessary auditing, generally the second best optimal contract will not result in truthtelling (Hart, 1995; Khalil, 1998; Choe, 1998) since the auditing required is not credible. In other words, there is a commitment problem for the lender in that if the contract is written to give a loan size and repayments that ensure truthful reports by the borrower at repayment, then since the lenders know that there is honest borrower behaviour, there is no incentive for them to undertake costly policing of the borrower.

[1] That is, the borrower must have sufficient resources to cover the relevant repayment.

If commitment were possible (ie monitoring were contractible and enforceable) or there were some means of making monitoring compulsory the optimal repayment contract would minimise the incentive for the borrower to cheat by making the returns in different states of second period borrower income as close together as possible. In particular, in a two state world this would mean taking all the resources of the borrower in the low state and leaving them all of the surplus in the high state after satisfying the lenders participation constraints.

To cope with the commitment problem, there are two ways in which the optimal contract can be formulated. First, it could still impose truthtelling but require that the repayments are structured so that the lender has an incentive to carry out costly policing sufficiently often to stop the borrower from cheating. In other words, a sequential rationality constraint could be imposed on the contract to give credible lender auditing at a level which will induce truthtelling by the borrower (Jost, 1996, Krasa-Villamil, 2000).This requires random policing by the lender, and a premium repayment to the lender when he polices that just covers the observation cost. Hence whenever the borrower tries to cheat, the lender is indifferent between policing or not and can randomise the monitoring decision. The probability of monitoring must be sufficiently high to deter the borrower from ever trying to cheat. The cost of this policy is that sometimes in the low state the borrower is left with some surplus (when he tries to cheat and is not monitored) although in the low state when monitored he can be left with a zero surplus.

The second approach is to allow for a contract that leads to an "equilibrium" amount of cheating by the borrower. That is, cheating and auditing could both be determined expost as Nash equilibria in simultaneous play mixed strategies, as in Khalil and Parigi (1998 - hereafter KP - who show that this has underinvestment relative to the first best but claim that it has overinvestment relative to the second best with the loan size being used as a way of motivating the lender to audit). More specifically, equilibrium is taken to mean that the probability of being monitored when trying to cheat and the probability of cheating are selected by the lender and borrower respectively to be mutual best responses, so that formally there is a Nash equilibrium in mixed strategies in cheating and monitoring between the lender and borrower. In this case it is possible to leave the borrower with zero rent whenever he tries to cheat since the lender gets his fair return out of the punishment repayments he can impose whenever the borrower cheats and is detected by monitoring.

In this type of setup Persons (1996) - hereafter P - considers a fixed loan size and allows for the possibility of more than a single lender. He makes various feasibility assumptions, the most important of which for our purposes are that the low state borrower income exceeds the observation cost of the lender and that all prospective lenders have limited liability so that any repayments that they receive must be non-negative. In addition to this he assumes that each party to the contract faces an initial fixed cost just of writing the contract and that monitoring is only sometimes successful in yielding any information about the true circumstances of the borrower (that is, there is a fixed probability q that the monitoring succeeds). Under these assumptions he shows that for different required investment levels and contract writing costs, any of three possible contract forms can be optimal: a truthtelling contract with two investors, a truthtelling contract with a single investor, and a single investor contract with no truthtelling but with a mixed strategy Nash equilibrium, as in KP. As we shall see it is crucial to his results that there are fixed contract writing costs *per person*. We call the truthtelling contracts "hybrid" contracts for reasons seen below (see section ?), and label them $H1$ and $H2$ for the two cases of one and two lenders respectively; while we label the mixed strategy contract KP.

Within this framework, we show that:

(1) the two approaches to managing the commitment problem (either imposing a sequential rationality constraint in the contract problem, or allowing cheating/auditing to be determined as mutual best responses in the second period) lead to the same optimal contract when that contract has truthtelling.

(2) If the loan size is fixed exogenously and there is unlimited liability, Persons as-

sumption on the contract writing costs is instrumental in setting the alternative forms for the optimal contract. If the fixed cost on each party of writing the contract is zero, a truthtelling contract with two investors, $H2$, *always* dominates the two other contract forms, $H1$ and KP. The intuition for this result is that the only reason that the single investor truthtelling contract is less efficient than the second best is that it has to leave some surplus to the borrower who cheats but is not monitored, in order to be able to make the premium repayment to cover the observation cost which just gives enough incentives to monitor. If it were possible to leave the borrower with zero surplus in the low state whether or not he is monitored then the truthtelling contract would be equivalent to the second best commitment contract. But with two investors and unlimited liability then since repayments can be negative, the passive investor can effectively pay the observation cost which then allows us to leave zero rent to the borrower in the low state whether he is monitored or not. It also follows that with a zero contract writing cost or a per contract rather than per person cost, the two investor truthtelling hybrid achieves the second best when the loan size is fixed and dominates the other contract forms. If the contract writing cost is not per capita but per contract so that there is a common fixed contract writing cost to a two or three party contract then again this result holds. More generally one might expect the contract writing cost to vary with the role of the lender-the monitoring lender has more complicated arrangements to make eg fixing the audit procedures and ensuring the result of the audit is public. It is perhaps more natural to expect that fixed contract cost is higher for the monitoring than passive investor. If so then we show that the two investor contract also dominates so long as the fixed cost on the passive investor is not too high.

(3) With limited liability and a fixed loan size the two investor truthtelling contract $H2$ is also dominant so long as low state revenues exceed the observation cost corrected for the success rate of monitoring. However if the observation cost is higher than this and there is limited liability then neither of the truthtelling contracts are feasible. Of the

three contracts in this case we are left just with the KP contract.

(4) We let the loan size vary; for example an entrepreneurial loan can be used to finance a technology in which the incomes in each state may plausibly vary with the loan size; a consumer loan may be used to finance real or portfolio investments in which case again the revenues arising from the loan in each state will vary with loan size. A prime example might be education loans. In this context, we reformulate the optimal contract problems and show that with unlimited liability $H2$ still dominates the other forms of contract. However with limited liability in order for a truthtelling hybrid contract to be feasible, the loan size may have to be constrained to ensure that low state revenues are sufficiently high in relation to the effective observation cost. The constrained version of $H2$ (where the optimal loan size is set equal to the value that will equate low state income to the observation cost corrected for the success rate of monitoring) may be better or worse than the KP contract, depending on different parameter values. We include a simulation that shows that the constrained $H2$ can dominate KP when the loans size is optimally chosen. Moreover we show that that all the contracts considered have underfinance relative to the first best, but by means of an example, show that the two investor truthtelling contract may have more finance than the Khalil-Parigi contract.

(5) We briefly consider the renegotiability of these contracts and show that if it is the lender who makes any offer after the borrower knows their type but before determination of reporting and auditing strategies, then the contracts will tend to be renegotiation proof.

The paper is organised as follows: section 2 contains our working assumptions; section 3 lays out the second period's game setup; section 4 characterises the optimal forms of contract when the level of investment is fixed and compares their levels of repayment; section 5 considers the contract when investment is optimally chosen and section 6 concludes.

2. Assumptions

An investment project requires initial finance in the first period and yields subsequent risky revenues in the second period. The finance required may either take the form of a fixed cost (e.g. a set up cost or entry fee), or the amount of finance may influence the quality of the project in the sense that returns are higher in every state with greater finance. The risk neutral borrower can choose to secure the finance from either one or two risk neutral lenders (we will see there is no gain to be had from having more than two lenders). There are only two states s in the second period: revenues can be high ($s = H$ and revenue is $y_H(B)$ where B is the total amount of finance) or can be low ($s = L$ and revenues are $y_L(B)$). The high state occurs with probability p. The realised state in the second period is private information to the borrower, but one possible investor can audit the state of the borrower by paying a fixed cost of ϕ. Following realisation of the state the borrower makes a public report of the state to lenders. The audit technology is imperfect: with probability q an audit reveals the true state but with probability $1 - q$ it reveals no additional information. The result of any audit and the fact that there has been an audit are public knowledge so if there are two lenders, it is inefficient to have more than one of them monitoring, the second lender has no monitoring role and is purely passive. The borrower has no additional outside resources and has limited liability so that repayments to investor(s) are made out of the realised revenues and in each state cannot exceed the revenues in that state.

Finance is organised through a contract: in period 1 the contract specifies B and the repayments to investor(s) conditional on the information available at the repayment stage. We take the contract to be written by the borrower-this has some advantages-Choe. The available information is the report of the firm and the result of any audit. If low state revenues are sufficient to pay a fair return to investor(s), there is no incentive problem, otherwise to ensure investors do receive a fair return, repayments must vary by state being higher when the firm has declared a high state. It follows that high state reports will never be audited, if the state is low the borrower will truthfully declare it. But if the true state is high the borrower may have an incentive to cheat and falsely report low. Hence we can define the repayments as: R_H, P_H to the monitoring investor and the pure investor respectively following a high state report of the borrower; R_L, P_L following a low state report that is not successfully monitored (either not monitored or monitored but monitoring fails); R_{LL}, P_{LL} following a low state report that is successfully monitored and is found to be truthful and R_{HL}, P_{HL} following a low state report which is successfully monitored and where the borrower is discovered to have cheated in their report.

If the project has only a fixed cost then incentive problems only arise if the fixed cost exceeds the low state revenue, otherwise it would be possible to set $R_s + P_s = (1+r)B$ for all s and avoid any incentive to cheat or need to monitor whilst giving investors a fair return.. We call a contract with repayments that are independent of state a Single Repayment Contract (SRC). We assume this is impossible: when revenues are independent of B, $y_L < (1+r)B$ where r is the safe interest rate. In this case it follows that the investors must include the monitor if the project is to go ahead- otherwise the borrower would just cheat for sure and the investor would not get back a fair return.

If the project has revenues that vary with B it is possible (by choosing B suitably in the contract) to avoid incentive problems if there is a value of B such that $y_L(B) \geq (1+r)B$, we allow for this possibility so in this case a SRC may be optimal. Again if the optimal loan size involves $y_L(B) < (1+r)B$ then the investor(s) must include the monitor.

Since monitoring is not contractible, either the contract itself must give the monitoring investor the incentive to monitor so that the probability of audit is specified in the contract and the repayments in the contract are set to pay a premium to the monitoring investor to give them the incentive to audit, or the monitor must get the incentive to audit from the differences between R_{HL}, R_L, R_{LL} i.e. from the premium repayment that they get by catching and punishing a cheating borrower. In this case the monitoring probability is determined outside the contract and is a matter of choice for the monitor after the state has been realised. Here after the borrower knows their state, cheating and monitoring probabilities are selected as mutual best responses in a noncooperative game.

The time line is thus either :

$t = 1$: - Contract signed including setting m the probability of audit

$t = 2$: - State realised and observed by borrower

- The borrower chooses a report of the state which is public

- If the report is low the monitor randomly audits with the contracted m

- Repayments are made

or:

$t = 1$: - Contract signed including setting only repayments and loan size

$t = 2$: - State realised and observed by borrower

- The high state borrower and monitor simultaneously choose a probability of reporting low l and of monitoring a low report m respectively.

- Following the report and the result of any monitoring repayments are made

The investors may have limited or unlimited liability: that is with limited liability investors cannot be called upon to provide additional finance to the borrower at repayment stage and so $R_s, P_s \geq 0$ for all s. But with unlimited liability repayments in some contingencies may be negative. We thus have essentially four cases to consider: that in which investors have limited or unlimited liability for a project that either involves a fixed financial requirement or in which the financial requirement is a variable which is endogenously determined. In each case we have to consider whether it is better to use one or two investors, and to write the level of monitoring into the contract with a monitoring implementation constraint or to leave monitoring and cheating to be determined as a simultaneous Nash equilibrium at the start of the second period.

We make some technical assumptions as follows.

2.1. Technical Assumptions. On the project we have alternative assumptions depending on whether revenues are finance dependent:

(A.1) when revenues are independent of the level of finance ($\partial y_s/\partial B \equiv 0$), $y_L < (1+r)B < y_H$ and $py_H + (1-p)y_L - (1+r)B - \phi/q > 0$

(A.2) when revenues vary with the level of finance

$$y_H(B) > y_L(B);\ y_H^{'}(B) > y_L^{'}(B) \geq 0;\ y_H^{''}(B) < y_L^{''}(B) < 0; \forall B \tag{A.2.1}$$

$$y_L(B_1) < (1+r)B_1 \quad where \quad py_H^{'}(B_1) + (1-p)y_L^{'}(B_1) = 1+r \tag{A 2.2}$$

$$y_s(0) = 0, y_s^{'}(0) > 1+r, \lim_{B\to\infty} y_s^{'}(B) < 1+r \tag{A 2.3}$$

$$If \quad y_L(B_{SR}) \quad = \quad (1+r)B_{SR} \quad and \quad py_H^{'}(B_1) + (1-p)y_L^{'}(B_1) = 1+r \tag{A 2.4}$$

$$then \quad B_1 \quad > \quad B_{SR.}$$

$$py_H(B) + (1-p)y_L(B) - (1+r)B - \phi/q > 0 \quad for \quad B \geq B_{SR} \tag{A 2.5}$$

$$If \quad y_L(B_L) = \phi \quad then \quad B_L < B_1 \tag{1}$$

(A.1) says that when revenues are independent of the loan size, the fair return on the investment is attainable from the project even after observation costs have been paid, but that low state revenues alone are insufficient to achieve this.

(A.2.1) says that revenues in each state are positive and increasing but strictly concave in the loan size, revenues in the high state are higher for any given loan size than in the low state and that the high state is more productive in the sense that at any loan size the marginal productivity of additional investment is higher in the high than the low state. However the marginal productivity falls faster in the high state. An example that satisfies this is $y_s(B) = a_s f(B)$ where $a_H > a_L$ and $f(B)$ is increasing and strictly concave. (A.2.2) says that at the first best loan size B_1 which maximises $Ey_s(B) - (1+r)B$, low state revenues are insufficient to give a fair return on the loan so that a state-independent repayment system cannot achieve the first best; this is related to (A2.4) which says that the highest loan size which can repay the fair return with state independent repayments is below the first best loan size. Note that since (A2.4) holds at B_{SR} it implies that $p(y_H(B_{SR}) - y_L(B_{SR})) - \phi/q > 0$ and so for all $B > B_{SR}$ also

$$p(y_H(B) - y_L(B)) - \phi/q > 0 \tag{2}$$

(A2.3) limits the marginal productivities in each state in relation to the interest rate and also says that zero finance yields zero revenue. Under this assumption there will be sufficiently low values of B which will allow a SRC. Finally (A 2.5) says that for every loan size above the level which will just generate revenue equal to the fair return in the low state, the project is socially desirable in that the expected revenues exceed the fair cost net of the observation cost. (A 2.6) says that the level of finance which generates just sufficient low state revenue to cover the observation cost is below the first best level. If this condition failed then as we shall see, under limited liability, the truthtelling contracts would automatically have overinvestment compared with the firs best.

Fig 1 indicates some of these properties.

Putting all this together it is helpful to write the expected returns of the various parties. For the borrower the expected return is

$$\begin{aligned} EU &= p[y_H(B) - (1-l)(R_H + P_H) - lmq(R_{HL} + P_{HL}) - l(1-mq)l(R_L + P_L)] \quad (3) \\ &\quad +(1-p)[y_L - mq(R_{LL} + P_{LL}) - (1-mq)(R_L + P_L)] \end{aligned}$$

For the monitoring investor who supplies say a share α of the finance

$$\begin{aligned} E\Pi_1 &= p[(1-l)R_H + l\{mqR_{HL} + (1-mq)R_L - m\phi\}] \\ &\quad +(1-p)[mqR_{LL} + (1-mq)R_L - m\phi] - \alpha(1+r)B \end{aligned} \tag{4}$$

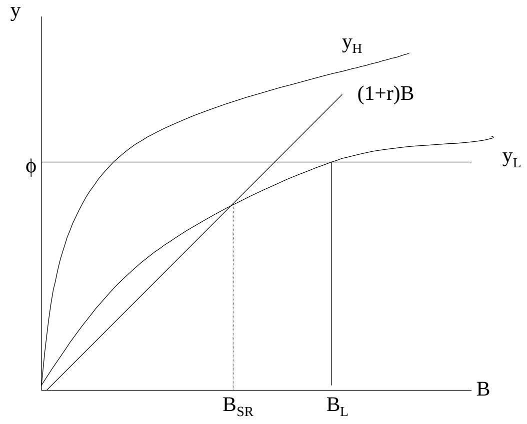

Figure 1: Technological assumptions

and for the pure investor who provides a share $1-\alpha$ of the finance

$$\begin{aligned} E\Pi_2 &= p[(1-l)P_H + l\{mqP_{HL} + (1-mq)P_L\}] \\ &\quad +(1-p)[mqP_{LL} + (1-mq)P_L - (1-\alpha)(1+r)B \end{aligned} \tag{5}$$

3. Setting the Monitoring Strategy in the Contract

3.1. Compulsory Monitoring. We know that without commitment the second best truthtelling contract will generally be unachievable. The monitoring investor knows that the contract induces truthful reports, so faced with a low state report they will be unwilling to incur the monitoring costs. The borrower knows this and so if this contract were signed, the borrower would just cheat for sure whenever the high state occurred. Nevertheless the second best contract provides a benchmark against which to judge the alternatives.

If monitoring is contractible and enforceable then a level of auditing that yields truthtelling can be written into the contract and this level will subsequently be enforced without having to give any interim incentive to the monitor. Using truthtelling ($l=0$) in the payoffs of each party the contract problem with a single investor and a fixed investment level is

$$\max_{m,R_{HL},R_H,R_L,R_{LL}} p(y_H - R_H) + (1-p)[y_L - (1-qm)R_L - mqR_{LL}] \tag{6}$$

s.t.

$$R_H \square mqR_{HL} + (1-mq)R_L \tag{7}$$

$$(1+r)B \square pR_H + (1-p)[(1-mq)R_L + mq(R_{LL} - \phi)] \tag{8}$$

$$y_H - R_H > 0; y_H - R_{HL} > 0; y_L - R_L > 0; y_L - R_{LL} > 0 \tag{9}$$

For a given loan size the solution to this problem has maximum punishment and zero rent to the borrower: $R_{HL} = y_H, y_L = R_{LL} = R_L$, a binding truthtelling constraint so that $mq = (R_H - y_L)/(y_H - y_L)$ and then R_H is set to satisfy the participation constraint of the lender. Using these in the borrowers payoff, the contract yields a maximal value to the borrower of[2]

$$p(y_H - R_H) = Ey_s(B) - (1+r)B - \frac{(1-p)(\phi/q)[(1+r)B - y_L]}{p(y_H - y_L) - (1-p)\phi/q} \tag{10}$$

With two investors it is very similar:

$$\max_{m,R_{HL},R_H,R_L,R_{LL}} p(y_H - R_H - P_H) + (1-p)[y_L - (1-mq)(R_L + P_L) - mq(R_{LL} + P_{LL})] \tag{11}$$

s.t.

$$R_H + P_H \square mq(R_{HL} + P_{HL}) + (1-mq)(R_L + P_L) \tag{12}$$

$$\alpha(1+r)B \square pR_H + (1-p)[(1-mq)R_L + mq(R_{LL} - \phi)] \tag{13}$$

$$(1-\alpha)(1+r)B \square pP_H + (1-p)[(1-mq)P_L + mqP_{LL}] \tag{14}$$

[2]Details of proofs are in the appendix

$$y_H - R_H > 0; y_H - R_{HL} > 0; y_L - R_L > 0; y_L - R_{LL} > 0 \tag{15}$$

In fact if we combine the participation constraints of the two investors we have the single investor contract again so that $R_{HL} + P_{HL} = y_H, y_L = R_{LL} + P_{LL} = R_L + P_L$, a binding truthtelling constraint so that $mq = (R_H + P_H - y_L)/(y_H - y_L)$. There is an interplay between α and the division of repayments between the two investors to satisfy each investors participation constraint. But the maximal payoff to the borrower is identical to that with one investor, so on incentive grounds there is no gain in having diversified investors when the monitoring strategy can be costlessly enforced.

3.2. Contracted Monitoring with a Sequential Rationality Constraint. Without commitment to monitoring the contract can still be designed to elicit truthful reports and ensure that the monitoring is carried out but it needs a monitoring implementation constraint. If the monitoring investor knows that the contract elicits truthful reports he will only be prepared to monitor a low state report if the expected repayment he gets $qR_{LL} + (1 - q)R_L$ exceeds the repayment he would get from not monitoring R_L by at least the observation cost ϕ. Hence the relevant sequential rationality constraint is $R_{LL} \geq R_L + \phi/q$; this must be added to the contract problem. However here we face a potential problem if low state revenues are low. Suppose that either the chosen B or the fixed finance requirement yields low state revenues below the effective observation cost: $y_L < \phi/q$. To ensure that $R_{LL} \geq R_L + \phi/q$ then requires $R_{LL} > R_L + y_L$. But the borrower always has limited liability so that if there is a single investor (which must be the monitor) $R_L, R_{LL} \square y_L$ and hence we must have $R_L < 0$ if there is just a single investor. So long as the single investor has unlimited liability this is possible, but with limited liability it fails. A way out of the problem might be to add a second investor (see Persons): the implementation constraint would be the same but the limited liability constraint of the borrower would become $R_L + P_L, R_{LL} + P_{LL} \square y_L$. If the second investor had unlimited liability we could set $P_{LL} = P_L - \phi/q, R_{LL} = R_L + \phi/q$ and $P_{LL} + R_{LL} = P_L + R_L = y_L$.

Then $P_{LL} = y_L - \phi/q - R_L < 0$ and the role of the passive investor is then to "finance" the monitoring costs.. This is only possible if the second investor has unlimited liability; if they have limited liability then with either one or two investors the feasible set to the contract problem is empty if $y_L < \phi/q$. With the project only requiring finance to cover a fixed cost this means that there cannot be a truthful reporting contract with monitoring determined at the contract writing stage. On the other hand if revenues vary with the loan size, it means that the contract must constrain the loan size so that $y_L(B) \geq \phi/q$.

4. Determining the Monitoring Strategy Outside the Contract

The alternative to setting the monitoring strategy at the contractual stage is to allow it to be chosen by the monitoring investor as a best response to the report of the firm. Here we are in the scenario in which l and m are determined as mutual best responses. Recalling the time line, the cheating-auditing game is played after the contracted repayments have been agreed. Thus the optimal contract will have repayments that maximise the borrowers expected gain within the investor(s) participation constraints and the constraint that the probabilities of cheating and auditing will assume Nash equilibrium values that vary with the repayments. It follows that if the game has alternative forms of Nash equilibria for different repayments then the optimal contract will select repayments that lead to a Pareto efficient Nash equilibrium-otherwise the investor(s) can still be held to their reservation levels but the gain to the borrower increased. For the analysis of the second period Nash equilibrium whether the loan size is endogenous or not is immaterial so we first analyse second period behaviour picking out efficient Nash equilibria.

4.1. The Second Period Game. Given the loan size and repayments, in period 2 income is realised and the debtor chooses an income report. The debtor will always make truthful reports of low income to the active lender (since $R_H + P_H > R_L + P_L, R_{LL} + P_{LL}$) but with high income the debtor may make a false low income report to the monitoring lender with probability l. In period 2 this lender chooses the probability m with which to monitor any low income report by the debtor. This lender's best response m to a low income report maximises his expected profit conditional on the low state report:

$$\begin{aligned} E\pi_1|\widehat{L} \quad = \quad & plm[q(R_{HL}-\phi)+(1-q)(R_L-\phi)]+(1-p+pl)(1-m)R_L \\ & +(1-p)m[q(R_{LL}-\phi)+(1-q)(R_L-\phi)]-\alpha(1+r)B \end{aligned} \tag{16}$$

Depending on the sign of $\partial E\pi 1/\partial m$ the best response is either a pure strategy ($m=0$ if $\partial E\pi_1|\widehat{L}/\partial m<0, m=1$ if $\partial E\pi_1|\widehat{L}/\partial m>0$) or a mixed strategy $0<m<1$ (if $\partial E\pi_1|\widehat{L}/\partial m=0$).

On the other hand the borrower's best reporting strategy l maximises his expected utility:

$$\begin{aligned} EU = p(1-l)(y_H-R_H-P_H)+plm[q(y_H-R_{HL}-P_{HL})+(1-q)(y_H-R_L-P_L)] \\ +pl(1-m)(y_H-R_L-P_L)+(1-p)(1-m)(y_L-R_L-P_L) \\ +(1-p)m[q(y_L-R_{LL}-P_{LL})+(1-q)(y_L-R_L-P_L)] \end{aligned} \tag{17}$$

(here with two investors; with a single investor we have the same expression but $P_s=0$ for all s) and again depending on the sign of $\partial EU/\partial l$ the best response is either a pure strategy ($l=0, l=1$) or a mixed strategy $0<l<1$.

For different repayments and loan sizes the Nash equilibrium may be a pure strategy equilibrium, a mixed strategy equilibrium or a hybrid equilibrium in which just one of the game players randomises.

Since the sum of the expected payoffs $E\pi_1+E\pi_2+EU$ is equal to the difference between $E_s y_s(B)-(1+r)B$ and the expected observation cost $\Phi=(1-p+pl)m\phi$ in the equilibrium, an optimal contract will set the repayments and loan size to maximise $E_s y_s(B)-(1+r)B-\Phi$. So we can select repayments that lead to efficient Nash equilibria by comparing the expected observation costs in the Nash equilibrium.

When B is fixed the optimal contract must either lead to a mixed strategy equilibrium or to the hybrid equilibrium $l=0, 0<m<1$. The details are in the appendix; the table below shows the expected observation costs in different forms of Nash equilibrium. In each row the precise values of $m.l$ differ but we can establish relations between them which give the results of the final column. Some of the solutions are infeasible as they involve the borrower always repaying $R_L + P_L$ which is insufficient to meet the investor(s) participation constraints. Others are feasible but are inefficient since they involve higher Φ than a feasible alternative.

	Φ	
$l = 1, m = 0$	0	infeasible
$l = 1, m = 1$	ϕ	inefficient
$l = 0, m = 0$	0	not best responses
$l = 0, m = 1$	$(1 - p)\phi$	inefficient
$l = 0, 0 < m < 1$	$(1 - p)m\phi$	possibly efficient
$l = 1, 0 < m < 1$	$m\phi$	inefficient
$m = 0, 0 < l < 1$	0	infeasible
$m = 1, 0 < l < 1$	$(1 - p + pl)\phi$	inefficient
$0 < m < 1, 0 < l < 1$	$(1 - p + pl)m\phi$	possibly efficient

Table 1: Properties of Alternative Nash Equilibria

Looking at the Nash equilibrium values of m, in the truthtelling hybrid cases which may be efficient

$$m = \frac{R_H + P_H - R_L - P_L}{q(R_{HL} + P_{HL} - R_L - P_L)} \tag{18}$$

with two investors or

$$m = \frac{R_H - R_L}{q(R_{HL} - R_L)} \tag{19}$$

with one investor.

When revenues are finance dependent, an optimal contract will include choice of B. Moreover contracts with different levels of finance and repayments will induce different types of Nash equilibria. In particular by setting the loan size sufficiently low the subsequent pure strategy equilibrium $l = 1, m = 0$ (the single repayment contract) can arise. Nevertheless we can still deduce:

Proposition 1. *The optimal contract must lead to a Nash equilibrium which is one of* *pooling* $l = 1, m = 0)$, *interior mixed strategy* $(0 < l < 1, 0 < m < 1)$ *or the particular hybrid case* $(0 < m < 1, l = 0)$ *with either one or more investors (see appendix A.1).*

The emphasis is on minimising monitoring costs whilst preserving incentives for the borrower to tell the truth. If observation costs are very high, monitoring fails as a control mechanism and it is best just to let the borrower cheat. For intermediate observation costs the chance of cheating can be held down to below unity. For lower observation costs it is possible to randomly monitor just enough to prevent the borrower cheating at all.

4.2. The Optimal Repayments Given B. We start with assumptions (A.1) where revenues are independent of the level of finance. The optimal contract problem conditional on a particular form of outcome of the monitoring/cheating game and on B is:

$$\max_{R_s, P_s} \quad EU \tag{20}$$

$$s.t. \quad E\pi \geq 0$$

$$l, m \quad are\ Nash\ equilibria$$

where EU and $E\pi = E\pi 1 + E\pi 2$ are given by (3),(4),(5) and $s = H, L, HL, LL$. Here (following Persons), we have combined the participation constraints of the two investors-if there are two- so that the relative investment shares α disappear. For our purposes this is enough since with risk neutrality of lenders the distribution of debt is unimportant although if we were to consider a two investor misrepresentation contract the shares of investment would matter (see Menichini-Simmons). The first constraint will always bind: the optimal contract for a given form of the subsequent game equilibrium must always give a zero expected return to the lenders. Otherwise it would be possible to reduce the repayments to one of the investors in all states and preserve the lenders participation constraint.

The KP Contract: A Single Investor. If the second period Nash equilibrium is in mixed strategies then the borrower is indifferent between truthfully reporting the high state and cheating whilst the monitoring lender is indifferent between accepting or auditing a low state report. Using these indifference conditions in the payoffs of the borrower and monitoring lender, the contract problem is to choose R_s to maximise

$$p[y_H - R_H] + (1-p)y_L - mq(R_{LL} + P_{LL}) - (1-mq)(R_L + P_L)]$$

st

$$p[(1-l)R_H + lR_L] + (1-p)R_L \geq (1+r)B$$

$$y_s \geq R_S \tag{21}$$

$$l = \frac{(1-p)(\phi + R_L - R_{LL})}{p(R_{HL} - R_L - \phi/q)}, m = \frac{R_H - R_L}{q(R_{HL} - R_L)} \tag{22}$$

The optimal repayments and strategies are (see appendix A.1 or Khalil-Parigi, 1998):

$$R_L^{KP} = y_L \tag{23}$$

$$R_{LL}^{KP} = y_L \tag{24}$$

$$R_{HL}^{KP} = y_H \tag{25}$$

$$R_H^{KP} = \frac{(1+r)B(y_H - y_L - \phi/q) - y_L(y_H - y_L)(1-p)}{p(y_H - y_L) - \phi/q} \tag{26}$$

$$m^{KP} = \frac{[(1+r)B - y_L][y_H - y_L - \phi/q]}{q[p(y_H - y_L) - \phi/q][y_H - y_L]} \tag{27}$$

$$l^{KP} = \frac{(1-p)\phi/q}{p(y_H - y_L - \phi/q)} \tag{28}$$

The borrower receives zero surplus in the low state whether or not he is audited. There is maximum punishment for detected cheating. The incentive to audit does not come from a premium $R_{LL} > R_L$ but from the rewards from catching a borrower who cheats-these are set to make the monitoring lender just indifferent between monitoring or not. A high state borrower who truthfully reports receives some surplus. Cheating increases with ϕ but falls with q and the spread of repayments $y_H - y_L$; monitoring rises with ϕ/q and with the loan size but for given y_L falls with the spread of repayments. It is surprising that monitoring increases with ϕ/q; we will find this repeatedly and interpret it as arising from the requirements of a mixed strategy equilibrium in which one parties' indifference condition determines the Nash equilibrium strategy of the other party ie higher ϕ/q requires a higher level of monitoring to make the borrower indifferent between cheating and truthfully reporting.

In order for the mixed strategy to be a possible form of Nash equilibrium we must have

$$y_L < (1+r)B \quad and \quad p(y_H - y_L) - \phi/q > 0 \tag{29}$$

to ensure that $0 < l, m < 1$. Both these are assured by (A.1).

The $H1$ Contract: Truthtelling with a Single Investor. On the other hand if the subsequent Nash equilibrium is the hybrid with $l = 0$ and one investor and either if the investor has unlimited liability or if $y_L > \phi/q$, the optimal contract problem will then have the form: choose R_s to maximise

$$p[y_H - R_H] + (1-p)[y_L - mqR_{LL} - (1-mq)R_L]$$

subject to:

$$pR_H + (1-p)[mqR_{LL} + (1-mq)R_L - m\phi] \geq (1+r)B$$

$$R_H \square\, mqR_{HL} + (1-mq))R_L \tag{30}$$

$$R_{LL} \geq R_L + \phi/q \tag{31}$$

$$R_s \square y_s \tag{32}$$

where $mq = (R_H - R_L)/(R_{HL} - R_L)$.

The optimal repayments and strategies are (see appendix A.1):

$$R_{HL}^{H1} = y_H \tag{33}$$

$$R_L^{H1} = y_L - \phi/q \tag{34}$$

$$R_{LL}^{H1} = y_L \tag{35}$$

$$R_H^{H1} = \frac{[(1+r)B - q(y_L - \phi)]}{p(y_H - y_L + \phi/q)} y_H \tag{36}$$

$$+\frac{[E_s y_s - (1+r)B - f_1 - (1-p)\phi/q]}{p(y_H - y_L + \phi/q)} q(y_L - \phi) \tag{37}$$

$$m^{H1} = \frac{(1+r)B - (y_L - \phi/q)}{p(y_H - y_L + \phi/q)} \tag{38}$$

Again there is maximum punishment for detected false low state reports (this is never actually paid in this contract) and the low income borrower who reports truthfully and is audited receives zero rent. However the low type who is not audited retains some surplus, this is necessary to give the monitor the incentive to audit sufficiently often to stop cheating (so that $qR_{LL}+(1-q)R_L-\phi = R_L$). The monitor is just indifferent between auditing a low state report or not. The idea of the contract is that there is just sufficient monitoring to ensure truthtelling and the monitor is willing to perform this because of the premium that he gets ($R_{LL} > R_L$). Notice that $R_L^{H1} < 0$ if $y_L < \phi/q$ - this is feasible if the investor has unlimited liability so in these circumstances the lender may optimally make a transfer to the borrower. Again monitoring rises with the loan size.

But if $y_L < \phi/q$ and the investor has limited liability this solution is infeasible. Then we have to add the constraints $R_s \geq 0$ to the contract. The constraint set is then empty:

combining (31) and low state feasibility $y_L \geq R_{LL} \geq R_L + \phi/q > R_L + y_L$ which would imply $R_L < 0$. Hence under limited liability and $y_L < \phi/q$ there is no single lender contract which will generate truthtelling as part of a Nash equilibrium.

The $H2$ Contract: Truthtelling with Two Investors. Finally if the subsequent Nash equilibrium is the hybrid with $l = 0$ and two investors and again either there is unlimited liability on investors or $y_L > \phi/q$, the optimal contract problem is to choose R_s, P_s to maximise

$$p[y_H - R_H - P_H] + (1-p)[y_L - mq(R_{LL} + P_{LL}) - (1-mq)(R_L + P_L)]$$

subject to:

$$p(R_H + P_H) + (1-p)[mq(R_{LL} + P_{LL}) + (1-mq)(R_L + P_L) - m\phi] \geq (1+r)B$$

$$R_H + P_H \Box mq(R_{HL} + P_{HL}) + (1-mq)(R_L + P_L) \tag{39}$$

$$R_{LL} \geq R_L + \phi/q \tag{40}$$

$$P_S + R_s \Box y_s, s = H, L, LL, HL \tag{41}$$

where $mq = (R_H + P_H - R_L - P_L)/(R_{HL} + P_{HL} - R_L - P_L)$.

The optimal repayments and strategies are (see appendix A.1):

$$R_{LL}^{H2} + P_{LL}^{H2} = R_L^{H2} + P_L^{H2} = y_L \tag{42}$$

$$R_{LL}^{H2} = R_L^{H2} + \phi/q$$

$$P_{LL}^{H2} = P_L^{H2} - \phi/q$$

$$R_{HL}^{H2} + P_{HL}^{H2} = y_H \tag{43}$$

$$R_H^{H2} + P_H^{H2} = \frac{(1+r)B - y_L}{p(y_H - y_L) - (1-p)\phi/q} y_H + \frac{E_s y_s - (1+r)B}{p(y_H - y_L) - (1-p)\phi/q} y_L \tag{44}$$

$$m^{H2} = \frac{(1+r)B - y_L}{q[p(y_H - y_L) - (1-p)\phi/q]} \tag{45}$$

We still have maximum punishment (which is never paid) for false low state reports and the low type of borrower gets zero surplus whether or not he is audited. The monitor receives a premium if he audits a low state report (paid by the passive investor) which just covers the effective monitoring cost. The borrower receives a zero rent in all states that actually occur except if he truthfully reports a high state. Monitoring rises with loan size and with ϕ, falls with the spread of repayments and with q. Note that if $y_L < \phi/q$ we have $R_L + P_{LL} = y_L - \phi/q$ so at least one of R_L, P_{LL} must be negative.

Again it then follows that if the investors have limited liability and $y_L < \phi/q$ there is no feasible contract with two investors which yields truthtelling. So long as investors have limited liability this is also true with any number of investors[3].

Maximum Utility Functions. Defining $EU_{SR}(B) = E_s y_s(B) - (1+r)B$, replacing the optimal repayments for each contract in the debtor's expected utility and rearranging we obtain:

$$EU_{KP}(B) = EU_{SR}(B) - \frac{(1-p)(\phi/q)[(1+r)B - y_L]}{p(y_H - y_L) - \phi/q} \tag{46}$$

$$EU_{H1}(B) = EU_{SR}(B) - \frac{(1-p)(\phi/q)[(1+r)B - y_L + \phi/q]}{p(y_H - y_L + \phi/q)} \tag{47}$$

$$EU_{H2}(B) = EU_{SR}(B) - \frac{(1-p)(\phi/q)[(1+r)B - y_L]}{p(y_H - y_L) - (1-p)\phi/q} \tag{48}$$

[3] At first glance one might think that even if low state income is below the observation cost, we can still get a zero surplus multiple lender and truthtelling outcome by having say n passive lenders with each of them paying a share $1/n$ of the observation cost. However this will not work since if each passive lender gets P_{LL}, P_L respectively, we need
$R_{LL} + nP_{LL} = R_L + nP_L$, $R_{LL} \geq R_L + \phi$, $R_s, P_s \geq 0$. If we again set $R_L = 0$ and $R_{LL} = \phi$ we are back to the condition that low state income must be above the observation cost to ensure $P_{LL} \geq 0$.

Welfare Comparisons under Fixed B. We can derive our first main result using expressions (47),(65),(48). Firstly contrast the $H2$ contract with the $H1$ contract. These are both well defined under (A.1) either when $y_L > \phi/q$ or when investors have unlimited liability. We have:

$$\begin{aligned} EU_{H1} - EU_{H2} &= \frac{(1-p)(\phi/q)[(1+r)B - y_L]}{p(y_H - y_L) - (1-p)\phi/q} \\ &\quad - \frac{(1-p)(\phi/q)[(1+r)B - y_L + \phi/q]}{p(y_H - y_L + \phi/q)} \quad (49) \\ &= -\frac{(\phi/q)^2(1-p)(Ey - (1+r)B - (1-p)\phi/q)}{(p(y_H - y_L) - (1-p)(\phi/q))\, p(y_H - y_L + \phi/q)} \quad (50) \end{aligned}$$

Under (A.1) this is always negative, so the two investor contract with truthtelling always dominates the single investor contract with truthtelling (irrespective of whether there is limited liability or not). However if $y_L < \phi/q$ and investors have limited liability neither of these contracts is well defined.

On the other hand:

$$EU_{H2} - EU_{KP} = \frac{(1-p)(\phi/q)[(1+r)B - y_L]}{p(y_H - y_L) - (1-p)\phi/q} - \frac{(1-p)(\phi/q)[(1+r)B - y_L]}{p(y_H - y_L) - \phi/q} \quad (51)$$

$$= (\phi/q)(1-p)\left[(1+r)B - y_L\right] \frac{p(\phi/q)}{[p(y_H - y_L) - \phi/q]\,[p(y_H - y_L) - (1-p)\phi/q]} \quad (52)$$

Here again under (A.1) both denominator and numerator are positive so we have $EU_{H2} > EU_{KP}$ whenever the H2 contract is well defined (ie when either $y_L > \phi/q$ or when the investors have unlimited liability.

If the hybrid forms of contract are not well defined (if $y_L < \phi/q$ and there is limited liability on investors) we can only have the KP form.

Thus we have the result:

Proposition 2. *For any fixed value of the loan size B that satisfies $p(y_H - y_L) > \phi/q$ and $y_L > \phi/q$ the two investor truthtelling contract, $H2$, dominates both the single person truthtelling contract, $H1$, and the contract which leads to a Nash equilibrium level of cheating, KP. For any fixed value of the loan size B that satisfies $p(y_H - y_L) > \phi/q$ and $y_L < \phi/q$ the KP contract is the only feasible form under limited liability.*

From this result it follows that the relative merits of the H1 and KP contracts are of little interest-both are dominated by H2 when the hybrid truthtelling contracts are feasible. This is of interest since Persons finds that there are cases in which either the H1 or the KP contract can dominate the H2 contract. We come back to this in the section on extensions below; essentially it arises because he uses a fixed cost per agent of writing a contract so that cet par a two investor contract is more costly than a single investor contract.

5. Sequentially Rational Contracts or Games?

So far when revenues are independent of the amount of finance we have examined the optimal repayments and outcomes if the audit policy together with the borrowers cheating policy is determined after the contract setting repayments has been signed and after the borrower has discovered its type in a Nash equilibrium. In this section we show that in the hybrid truthtelling cases these noncooperative solutions are identical to those that would emerge if the audit policy were set together with the repayments within the contract. In this case the contract would have to include a sequential rationality constraint ensuring that the lender/monitor has the incentive to undertake the audit and also a truthtelling constraint ensuring that there is sufficient auditing to prevent any cheating.

5.1. The Single Lender Truthtelling Case. To ensure truthtelling by the high type of borrower requires $R_H \leq mqR_{HL} + (1 - m + m(1 - q))R_L$ and to ensure that the auditing will be undertaken we require $R_{LL} \geq R_L - \phi/q$. The optimal contract problem will then have the form: choose R_s, m to maximise

$$p[y_H - R_H] + (1 - p)[y_L - mqR_{LL} - (1 - mq)R_L]$$

subject to:

$$pR_H + (1 - p)[mqR_{LL} + (1 - mq)R_L - m\phi] \geq (1 + r)B$$

$$R_H \leq mqR_{HL} + (1 - mq))R_L \tag{53}$$

$$R_{LL} \geq R_L + \phi/q \tag{54}$$

$$R_s \leq y_s \tag{55}$$

There must be maximum punishment, otherwise R_{HL} could be increased which would make both the lenders participation constraint and the truthtelling constraint more slack, allowing a reduction in R_H which would make the borrower better off. Then given this the truthtelling constraint must bind at the optimum, otherwise m could be reduced which would reduce the expected observation costs and also make the lenders participation constraint slacker. Thus the solution must have

$$l = 0, m = (R_H - R_L)/(q(y_H - R_L))$$

But these are identical to the Nash equilibrium values so that the optimal repayments in the sequentially rational contract must coincide with the optimal repayments when audit/cheating strategies are determined as a post-contract simultaneous Nash equilibrium.

5.2. The Two Lender Truthtelling Contract. The same considerations for ensuring truthtelling and the incentive to audit apply here as in the single lender case so the optimal contract problem has the form: choose R_s, P_s, m to maximise

$$p[y_H - R_H - P_H] + (1 - p)[y_L - mq(R_{LL} + P_{LL}) - (1 - mq)(R_L + P_L)]$$

subject to:

$$p(R_H + P_H) + (1 - p)[mq(R_{LL} + P_{LL}) + (1 - mq)(R_L + P_L) - m\phi] \geq (1 + r)B$$

$$R_H + P_H \square mq(R_{HL} + P_{HL}) + (1 - mq)(R_L + P_L) \tag{56}$$

$$R_{LL} \geq R_L + \phi/q \tag{57}$$

$$P_S + R_s \square y_s, s = H, L, LL, HL \tag{58}$$

The same arguments apply here. There must be maximum punishment, otherwise $R_{HL} + P_{HL}$ could be increased which would make at least one of the lenders participation constraint and the truthtelling constraint more slack, allowing a reduction in $R_H + P_H$ which would make the borrower better off. Then given this the truthtelling constraint must bind at the optimum, otherwise m could be reduced which would reduce the expected observation costs and also make the lenders participation constraint slacker. Thus the solution must have

$$l = 0, m = (R_H + P_H - R_L - P_L)/(y_H - R_L - P_L)$$

But these are identical to the Nash equilibrium values so that the optimal repayments in the sequentially rational contract must coincide with the optimal repayments when audit/cheating strategies are determined as a post-contract simultaneous Nash equilibrium.

These results are interesting as they tell us that a "decentralised" solution in which auditing is not written into the contract but is determined noncooperatively in a game gives the same outcome as the "centralised" solution when auditing is determined within a sequential rationality constraint.

6. The Optimal Choice of B

All the above is for the fixed finance case where revenues are independent of the scale of finance. When revenues vary with the level of finance as under assumptions (A.2), there are new questions and considerations. these are:

-will the various contract forms lead to overfinancing or to underfinancing of projects?

-under what conditions will it be best to constrain the finance level to ensure that a SRC is possible which will avoid the incentive problems and costly state observation?

-allowing for choice of the optimal level of finance within any given contract form, is it still true that the two investor hybrid truthtelling contract dominates the other forms?

-we have seen that with finance-independent revenues, if investors have limited liability and $y_L < \phi/q$ neither of the hybrid contracts are feasible. But with finance-dependent revenues, the level of finance could be constrained to ensure that $y_L(B) \geq \phi/q$ which generates the idea of a constrained one or two investor truthtelling contract. Then we want to see how these constrained hybrid contracts compare in welfare terms with a KP contract.

First we describe the detail of the SRC contract. Next since each contract requires some restrictions on the range of B to ensure that monitoring and cheating probabilities are in the unit interval and that participation constraints can hold, we define these ranges. Here we show that each of the three main contract forms is feasible iff $B > B_{SR}$ while the SRC contract is feasible iff B □ B_{SR}. Then we consider comparisons of finance levels and welfare between the contract forms under unlimited liability, and finally analyse the limited liability case.

6.1. The SR Contract. When revenues vary with the level of finance and (A.2) holds it is possible that B is set so low that the subsequent Nash equilibrium is in pure strategies $l = 1, m = 0$. We call this a SR contract since there is always a low report which results in a single repayment of R_L. In this case the contract problem has the form:

$$\max E_s y_s - R_L \tag{59}$$

$$R_L \geq (1 + r)B \tag{60}$$

$$R_L \square y_L \tag{61}$$

The first constraint must bind since otherwise R_L could be reduced raising debtor expected utility. Since $y_s(.)$ is increasing and concave, the solution for B in the SR contract will be at B_{SR}. We have to contrast this payoff with those arising from either a KP solution or a $H1, H2$ solution in which $l = 0$.

6.2. Feasibility of the Contracts. The SR contract is feasible only for:

$$B \square B_{SR}$$

Note that at $B_{SR}, EU_{KP} = EU_{SR} > EU_H$ so long as EU_{KP} is well defined. And for $B > B_{SR}$, $EU_{SR} > EU_{KP}$ again so long as EU_{KP} is well defined. We also know that by assumption $B_1 > B_{SR}$.

The KP contract (65) is only valid for $0 < l^{KP} < 1$ (which requires $B > B_{KP}$) and $0 < m^{KP} < 1$ (which requires $B > B_{SR}$), where B_{KP} is defined by:

$$p[y_H(B_{KP}) - y_L(B_{KP})] = \phi/q \tag{62}$$

From the assumption about the efficiency of the investment (A 2.5) we know that $B_{KP} < B_{SR}$ so that the KP contract is certainly feasible at or above B_{SR}. In particular the condition $m^{KP} < 1$ can be expressed as:

$$(y_H - y_L)[(1+r)B - py_H - (1-p)y_L + \phi/q] < [(1+r)B - y_L]\phi/q \tag{63}$$

This is always true at $B \geq B_{SR}$ since from (A 2.5) the LHS is negative whist the RHS is positive.

The $H1$ contract (47) is only well defined when $0 < m^{H1} < 1$; this requires $(1 + r)B - y_L + \phi/q > 0$ and $(1+r)B < y_H$. Generally there may exist two values of B, B_{HS1} and B_{HS2}, at which $(1 + r)B - y_L + \phi/q = 0$; i.e. such that for $B_{HS2} < B < B_{HS1}$, $(1 + r)B + \phi/q < y_L$ although for sufficiently high ϕ there will be no solutions to this equation. Define ϕ_h, B_h by $(1 + r)B_h + \phi_h/q = y_L(B_h); (1 + r) = y_L^{'}(B_h)$ For $\phi < \phi_h$, they both satisfy $B_{HS_i} < B_{SR}$ $(i = 1, 2)$. So the $H1$ contract is feasible at any B such that:

$$B > B_{SR}$$

From the form of (47) it follows that below B_{SR} the SR contract dominates the $H1$ contract in regions where the latter is defined. However, since we are concerned with situations where the first best level of investment cannot be implemented by a SR contract, our second best optima will be at or above B_{SR}.

Finally, the $H2$ contract is feasible iff $0 < m^{H2} < 1$; this requires $B > B_{H2}$ where B_{H2} satisfies:

$$p[y_H(B_{H2}) - y_L(B_{H2})] = (1-p)\phi/q \tag{64}$$

as well as $B > B_{SR}$. Note that these two conditions together are sufficient to ensure that, in particular, $m^{H2} < 1$, i.e:

$$E_s y_s - (1+r)B < (1-p)\phi/q$$

Also $B_{H2} < B_{KP} < B_{SR}$ so that the $H2$ contract is feasible at any $B \geq B_{SR}$.

We can summarise this discussion by noting that the two forms of hybrid truthtelling contract and the KP contract are all well defined iff $B \geq B_{SR}$.

6.3. The Unlimited Liability Case. Here with unlimited liability for each type of contract form, $H2, H1$ or KP, the optimal loan size will maximise the respective value function (48),(47) or (46) within the constraint $B \geq B_{SR}$. So long as $EU_s'(B_{SR}) > 0$ and the value function is concave[4], the constraint will not bind but in either unproductive technologies or where the observaton cost is high, the optimum may be at B_{SR}.

It is then easy to show that:

[4]Sufficient conditions for concavity which also guarantee $EU_s'(B_{SR}) > 0$ for each value function are available on request. The simulations below do not respect these conditions but nevertheless the simulated value functions are each concave.

Proposition 3. *With unlimited liability the superiority of the two investors truthtelling contract over the other two forms of contract carries over into the scenario in which the loan size is endogenous.*

The reason is that each of the three contract forms is feasible at any finance level at or above B_{SR}. Hence it follows that the $H2$ contract is feasible at the optimal finance levels of the $H1$ and KP contracts. From Proposition 2, fixing the level of finance at respectively the optimal level of the $H1$ or KP contracts, these contracts are worse than $H2$. Hence a fortiori these contracts are worse than the $H2$ contract at its optimal level of finance.

However it is still possible that the optimal single repayment contract which is at B_{SR} dominates the optimal two investor truthtelling contract. We might expect this to be the case if either the observation cost is sufficiently high or the technology relatively unproductive, so that the first best investment level is only marginally above B_{SR}. Since global properties of the technology are involved, in each case an explicit comparison has to be made between these two contracts.

The two investor truthtelling contract (48) has expected utility of identical form to the compulsory monitoring (second best) contract (6)-(9); so they both have the same optimal investment level. We also know that B_{SR} is below the optimal investment level of each contract.

There is a question about the optimal level of finance compared with the (unattainable) first best. In fact we can show that all the contracts have underfinance relative to the first best. Thus we have:

Proposition 4. *With unlimited liability all the contracts SRC, KP and the two hybrid truthtelling contracts have underfinance compared with the first best if each value function is strictly concave. The optimal finance level in the two investor hybrid coincides with that of the compulsory monitoring second best; the SRC contract has less finance than* *the second best.*

The comparisons between the optimal finance level in the KP and the two investor truthtelling contracts are unclear. For example if we use functions $y_s = a_s\sqrt{B}$ then if $a_L = 1, r = 0.04$ and $p = 0.05$ (which yield $B_{SR} = 0.924$) then for varying a_H and ϕ we have the following results

ϕ/q	a_H	B^*_{Kp}	B^*_{H2}	EU^*_{KP}	EU^*_{H2}
1	4.0	0.924	0.924	1.442	1.442
1	5.0	1.68	1.608	2.000	2.033
1	7.0	3.37	3.356	3.653	3.671
2	7.0	2.983	3.023	3.374	3.477
3	7.0	2.866	2.65	2.862	3.255

6.4. The Limited Liability Case. Under limited liability all repayments must be nonnegative. This means that in either the one or two investor truthtelling contracts we need $y_L(B) \geq R_{LL} \geq R_L + \phi/q$. Since $R_L \geq 0$ this means a truthtelling contract with limited liability is feasible only for values of B such that $y_L(B) \geq \phi/q$. Define B_L by the equation $y_L(B) = \phi/q$.

The problems with limited liability arise only if $B^*_{H2} < B_L$. If $B^*_{H2} > B_L$ we can use the unlimited liability argument to show that the $H2$ contract dominates the KP contract, it also dominates the $H1$ contract whether or not $B^*_{H1} > B_L$.

If $B^*_{H2}, B^*_{H1} < B_L$ then both limited liability truthtelling contracts have their optimal investment levels at B_L and so $EU_{H1}(B_L) < EU_{H2}(B_L)$. On the other hand the comparison between the $H2$ and the KP contract is ambiguous .

If $B^*_{H1} > B_L > B^*_{H2}$ then the optimal limited and unlimited liability $H1$ contracts coincide whereas the optimal limited liability $H2$ contract is at B_L. Again here the comparison between the constrained finance $H2$ contract and either of the unconstrained finance $H1, KP$ contracts is ambiguous.

Proposition 5. *With limited liability if $B_L > B^*_{H1}, B^*_{H2}$ the two investor truthtelling contract dominates the single investor truthtelling contract and may still be better than the KP contract. Moreover the two truthtelling contracts have the same investment level*

B_L. If $B^*_{H1} > B_L > B^*_{H2}$ the two investor truthtelling contract may dominate the KP and the single investor hybrid truthtelling contract. If $B_L < B^*_{H2}$ the limited and unlimited liability contracts have the same investment level in each contract form and so the two investor hybrid truthtelling contract dominates the alternatives.

To see the relation between the contracts under limited liability we have simulated some numerical examples taking $y_H = 7\sqrt{B}, y_L = \sqrt{B}, p = 0.5, r = 0.04$. This gives $B_{SR} = 0.924$ and a first best loan size of 3.698. We use two alternative values of ϕ/q :

ϕ/q	B_L	B^*_{H2}	B^*_{KP}	B^*_{H1}	EU^*_{H2}	$EU_{H2}(B_L)$	EU^*_{H1}	EU^*_{KP}
1.6	2.56	3.164	3.132	3.406	3.558*	3.525	3.424	3.501
1.7	2.89	3.129	3.095	3.399	3.538*	3.533	3.388	3.471
1.8	3.24	3.094	3.057	3.393	3.518	3.516*	3.351	3.441
1.9	3.61	3.059	3.020	3.388	3.498	3.475*	3.313	3.408

In the first two rows all unlimited liability finance levels are above B_L and so the limited liability constraint does not bind and the two investor hybrid is optimal. In the third row $B^*_{H2} < B_L < B^*_{H1}$ so the limited liability constraint binds only for $H2$ contract. Nevertheless the constrained $H2$ with finance equal to B_L still dominates the other two unlimited liability, unconstrained optima. Finally in the last row $B^*_{H2}, B^*_{H1} < B_L$ so the limited liability constraint binds for both the truthtelling contracts, hence the $H2$ contract is better than the $H1$ contract and also again in the example the constrained $H2$ contract dominates the KP contract. The example satisfies the technical assumptions (A.2), other features of interest are that always B^*_{H1} involves the highest finance level and although not shown here, the$H2$ contract has lower monitporing than the other two contracts[5].

7. EXTENSIONS

Persons includes a uniform per capita cost on investors of actually writing the contract: he assumes that each lender faces an initial fixed cost just of writing the contract so that, for example, a contract with three parties is 50% more costly to write than a contract with two. Given this, he shows that for different required investment levels and contract writing costs, any of the $H1, H2$ or KP contract forms can be optimal. Generally as

[5]These seem to be general features in the simulations which we conjecture to be anlytically true.

the required investment level rises, the optimal contract switches from the KP form to a two investor truthtelling contract and then finally for high required investment to a single investor truthtelling contract. On the other hand we have shown that if the fixed cost on each party of writing the contract is zero, a truthtelling contract with two investors, $H2$, *always* dominates the two other contract forms, $H1$ and KP, under unlimited liability even if the finance level is endogenous and also with limited liability if the fixed finance requirement generates high enough low state revenues.

Suppose that we generalise Persons contract writing costs by assuming that the borrower faces a fixed cost of f_0 to write the contract whilst the lender/monitor faces a cost of f_1 and the passive investor faces a contract writing cost of f_2. f_0 is immaterial since essentially in all the analysis it just amounts to subtracting f_0 from the revenues of each state-we can reinterpret all of our assumptions and results in terms of revenues net of f_0. Similarly if $f_2 = 0$ then the role of f_1 is solely to raise the fair return of investors from $(1+r)B$ to $(1+r)B+f_1$ and since every contract has at least the monitoring investor, it makes no difference to the comparison of the merits of the different contracts[6]. However the presence of f_2 does make a potential difference: It puts the two investor truthtelling contract at a relative disadvantage, and if it is sufficiently high we lose the result that the $H2$ contract is always preferable. Skipping over the details of the repayments, the value functions of the borrower in each contract form with contract writing costs become

$$EU_{KP}(B) = EU_{SR}(B) - \frac{(1-p)(\phi/q)[(1+r)B + \sum_{i=0}^{1} f_i - y_L]}{p(y_H - y_L) - \phi/q} \tag{65}$$

$$EU_{H1}(B) = EU_{SR}(B) - \frac{(1-p)(\phi/q)[(1+r)B + \sum_{i=0}^{1} f_i - y_L + \phi/q]}{p(y_H - y_L + \phi/q)} \tag{66}$$

$$EU_{H2}(B) = EU_{SR}(B) - \frac{(1-p)(\phi/q)[(1+r)B + \sum_{i=0}^{2} f_i - y_L]}{p(y_H - y_L) - (1-p)\phi/q} \tag{67}$$

[6]One further difference is in the definition of B_{SR} : now it is defined by $y_L(B_{SR}) - f_0 = (1+r)B_{SR} + f_1 + f_2$

Even if $f_2 = 0$ it may be that there are no roots to this equation in which case the SRC contract is not feasible at any loan size; or there may be two roots, in which case we use the higher root in our analysis.

and the comparisons between contracts for a fixed investment size become

$$EU_{H1} - EU_{H2} = \frac{p(y_H - y_L)f_2}{p(y_H - y_L) - (1-p)(\phi/q)} - \frac{(\phi/q)^2(1-p)(Ey - (1+r)B - f_1 - (1-p)\phi/q)}{(p(y_H - y_L) - (1-p)(\phi/q))\, p(y_H - y_L + \phi/q)} \tag{68}$$

and

$$\begin{aligned} EU_{H2} - EU_{KP} = & \frac{(\phi/q)(1-p)\left[(1+r)B + \sum_{i=0}^{1} f_i - y_L\right] p(\phi/q)}{[p(y_H - y_L) - \phi/q]\,[p(y_H - y_L) - (1-p)\phi/q]} \\ & - \frac{(1-p)(\phi/q)f_2}{p(y_H - y_L) - (1-p)\phi/q} \end{aligned} \tag{69}$$

So if f_2 is high enough, the $H2$ contract may no longer dominate. However there are reasons for taking f_2 to be relatively small. It represents the additional contract writing cost of adding a passive lender to a two party contract with an auditing lender. First if contract writing costs are per contract and not per party then we can set $f_1 = f_2 = 0$; if the additional costs of an extra party amount to just photocopying the contract terms that seems reasonable. Second one can argue that the arrangements that the auditing lender has to make in preparing the contract (e.g. providing a mechanism for an audit and public announcement of the result of an audit) are likely to be quite a lot higher than the costs of the passive lender. Under these circumstances the $H2$ contract will continue to dominate.

We have shown that there are benefits to sharing finance essentially on liquidity and incentive grounds when the borrower has limited liability-the second lender can be used to cover the observation costs of the monitor, keeping down the incentive of the borrower to cheat. Another reason for sharing the provision of finance obviously arises if lenders are risk averse.

Generally the literature argues that stopping the spread of information to uninformed parties (here lenders) will assist in ensuring renegotiation-proofness (Hart & Tirole (1988), Dewatripoint (1989), Fudenberg & Tirole (1990), Krasa & Villamil (2000).

All the contracts considered here optimally involve some costly monitoring, implicitly there is an incentive for the parties to the contract to eliminate this cost by renegotiation at the interim stage. Given that the audit and reporting strategies are determined in a simultaneous noncooperative game, any renegotiation must occur after the borrower knows their type but before the game is played. We will assume that it is the monitoring lender who makes an offer[7] In general from the original contract the high type expects

$$U_H = y_H - (1-l)R_H - l(mR_{HL} + (1-m)R_L) \tag{70}$$

the low type expects

$$U_L = y_L - (mR_{LL} + (1-m)R_L) \tag{71}$$

and the lender expects

$$E\Pi = p(1-l)R_H + (1-p+pl)(m(R_{LL} - \phi) + (1-m)R_L) \tag{72}$$

First consider the $H1$ contract. Suppose the lender makes an offer $m' = 0$ in exchange for a payment from the borrower of e, if it is accepted, then e is paid, there is no report and no audit. his is similar to an out of court settlement. The low type accepts if $(mR_{LL} + (1-m)R_L) > e$ while the high type accepts if $(1-l)R_H - l(mR_{HL} + (1-m)R_L) > e$ where l, m have their $H1$ NE values with repayments of the original contract. In $H1$ contract

$$U_H = y_H - R_H \tag{73}$$

$$U_L = y_L - (mR_{LL} + (1-m)R_L) \tag{74}$$

$$E\Pi = pR_H + (1-p)(m(R_{LL} - \phi) + (1-m)R_L) \tag{75}$$

So with the hybrid we would need $R_H > e$ for the high type to accept, $mR_{LL} + (1-m)R_L > e$ for the low type to accept and $e > pR_H + (1-p)(m(R_{LL} - \phi) + (1-m)R_L)$ for the

[7]This is the simpler scenario since we do not have to worry about the information that is transmitted by the offer to the uninformed lenders if the offer is made by the borrower. However our main result on renegotiation-proofness of the contracts extends to this case. For simplicity here we also assume $q = 1$.

lender to offer. Using the hybrid condition $R_{LL} - \phi = R_L$, these inequalities are mutually inconsistent:

$$y_L - (1-m)\phi = mR_{LL} + (1-m)R_L > e \tag{76}$$

$$> pR_H + (1-p)(m(R_{LL} - \phi) + (1-m)R_L) \tag{77}$$

$$= pR_H + (1-p)(y_L - \phi) \tag{78}$$

or $m\phi > p(R_H - y_L + \phi)$. Using the expression for m this gives $(1-p)\phi > p(y_H - y_L)$ which is false. Similarly we cannot find an offer e that only the low type will accept and that the lender will make, knowing that only the low type will accept (this would require $e < R_H, mR_{LL} + (1-m)R_L > e$ and $e > (m(R_{LL} - \phi) + (1-m)R_L)$. In part because $R_H > y_L \geq R_{LL}, R_L$). We can have offers that only the high type will accept: $R_H > e > mR_{LL} + (1-m)R_L$ but $e > pR_H + (1-p)(m(R_{LL} - \phi) + (1-m)R_L)$ but then the lender knows that with a rejected offer he will be facing a low type in the game and hence will not wish to monitor. The expected gain to the lender from making an offer that only the high type accepts is then $pe + (1-p)R_L$ (ie either the offer is accepted or since the lender learns that the type is low if the offer is rejected, the borrower reports low and there is no monitoring). It will be worth the lender doing this is if

$$pR_H + (1-p)R_L > \tag{79}$$

$$pe + (1-p)R_L > pR_H + (1-p)(m(R_{LL} - \phi) + (1-m)R_L) \tag{80}$$

which reduces to

$$R_L > R_{LL} - \phi \tag{81}$$

which contradicts having hybrid repayments.

Now consider the KP contract: here the H type will accept if

$$(1-l)R_H + l(mR_{HL} + (1-m)R_L) > e \tag{82}$$

and the L type will accept if

$$mR_{LL} + (1-m)R_L > e \tag{83}$$

whilst offers will be made if

$$e > p(1-l)R_H + mpl(R_{HL} - \phi) + (1-p)m(R_{LL} - \phi) + (1-p+pl)(1-m)R_L) \tag{84}$$

Combining these with the indifference condition of the monitor in mixed strategy $plR_{HL} + (1-p)R_{LL} - \phi = (1-p+pl)R_L)$ again no acceptable offer is possible: it would need

$$\begin{aligned} y_L &= mR_{LL} + (1-m)R_L > && (85)\\ e &> p(1-l)R_H + mpl(R_{HL} - \phi) + (1-p)m(R_{LL} - \phi) + (1-p+pl)(1-m)R_L && (86)\\ &= p(1-l)R_H + (1-p+pl)R_L > R_L = y_L && (87) \end{aligned}$$

With the truthtelling $H2$ contract the offer by the lender takes the form of $m' = 0$ in exchange for a repayment to the monitor of e and a repayment to the pure investor of P'. The pure investor expects to get

$$pP_H + (1-p)[mP_{LL} + (1-m)P_L] \tag{88}$$

Hence any offer must satisfy

$$P' > pP_H + (1-p)[mP_{LL} + (1-m)P_L] \tag{89}$$

as well as the conditions $R_H + P_H > e + P'$ for the high type to accept, $m(R_{LL} + P_{LL}) + (1-m)(R_L + P_L) > e + P'$ for the low type to accept and $e > pR_H + (1-p)(m(R_{LL} - \phi) + (1-m)R_L)$ for the monitoring lender to accept.

We can then mimic the $H1$ argument: there is no offer that is acceptable by both types of borrower and the pure lender: it would need

$$\begin{aligned} y_L &= m(R_{LL} + P_{LL}) + (1-m)(R_L + P_L) > e + P' && (90)\\ &> p(R_H + P_H) + (1-p)(m(R_{LL} + P_{LL} - \phi) + (1-m)(R_L + P_L)) && (91)\\ &= p(R_H + P_H) + (1-p)(y_L - m\phi) && (92) \end{aligned}$$

or $0 > p(R_H + P_H - y_L) - (1-p)m\phi$. Using the expression for m this gives $0 > m[p(y_H - y_L) - (1-p)\phi]$ which is a contradiction.

Hence all three contracts are renegotiation proof if the offer is made by the lender after the borrower knows their type but before the game[8].

In contrast to this Persons finds that $H1$ and $H2$ are renegotiable but his timing is different. If monitoring is written into the contract together with a sequential rationality constraint and a truthtelling constraint, then renegotiation can occur after the borrower knows his type and has made his report but before the monitoring has been implemented. In this context both Persons and Krasa and Villamil argue that a truthtelling contract with random monitoring is renegotiable: since the contract has truthtelling and random monitoring the indifference condition of the lender states that $R_{LL} = R_L + \phi$ and so following a low report, the lender will get R_L for sure. If the borrower offers a reduction in R_{LL} the lender will revise m to zero but will still get R_L for sure. This saving in monitoring costs reduces the borrowers repayments so the offer will be made, and accepted by the indifferent lender. However this argument has the weakness that the high type of borrower can argue that they know the low type will make this report and acceptable offer in which case they too should cheat, report low and make the same offer. The rational lender would realise this and hence should argue that there is a chance p that they will get R_{HL} after the low report and offer. Hence they should refuse the offer.

8. Conclusions

In a costly state verification framework for loan contracts without commitment we consider the result of Persons (1996) that there may be alternative forms of optimal contract that involve various degrees of misrepresentation. Whilst taking a formal postcontract game theoretic approach we also specialise his environment to cases where there is no independent fixed cost to writing a contract. We then compare the following forms of contract: firstly, a contract involving randomised cheating and randomised monitoring.

[8]This result also holds if it is the borrower who makes the offer.

This is an interior mixed strategy type contract that coincides with Persons' "misrepresentation" contract and has subsequently also been examined by Khalil and Parigi (1998); hence we call it the KP contract form. Secondly, a contract that involves a pooling type of solution in which investment and the loan size are sufficiently reduced to make a single non-contingent repayment a feasible means of repaying the loan: we call this the single repayment contract or SR form. And, thirdly, the contract may involve truthtelling and randomised monitoring as natural Nash equilibrium outcomes; these two contracts coincide with Persons' "truthtelling" contracts and because they merge truthtelling with random monitoring we call them hybrid contracts, $H1$ and $H2$ respectively. Persons shows that if the loan size is fixed each of the $KP, H1$ and $H2$ forms can be optimal, under given feasibility conditions. We show that if the loan size is fixed the $H2$ form always dominates the other two if there is unlimited liability and if there is limited liability but the fixed size of the loan is such that the low state income is still greater than the observation cost; otherwise the KP form will be optimal. If the loan size is optimally chosen and there is unlimited liability again the $H2$ form dominates the other two. On the other hand, again if there is limited liability under certain conditions either a KP form or a constrained $H2$ form may be optimal.

We see different loan circumstances, perhaps most often consumer loans have a single lender whilst entrepreneurial loans come from multiple lenders. The theory in this paper says that when there is unlimited liability or when the observation costs are small relative to the low state revenues then we can expect to observe two lenders and that the reason for this is essentially that the passive investor finances the monitoring costs of the active investor, which improves incentives for truthtelling by the borrower. The existence of the two investor hybrid contract is important since despite the noncooperative no commitment scenario, it implements the commitment solution. This may no longer be possible under limited liability, which means repayments in the lowest state must be positive so that for high observation costs a truthtelling contract feasible set will simply not exist. This is in accordance with Persons' argument that management misrepresentation may induce Pareto improvements when monitoring is not contractible, although it limits this argument to a very specific set of circumstances that must occur in the face of an optimally chosen loan size.

On the other hand, one might argue that consumer loans typically have high observation costs compared to entrepreneurial loans where there are legal standards for reporting. It is also arguable that worse things can happen to consumers than to entrepreneurs so that in bad states there is less profit to fight over and to finance audit. In fact the recovery rate on debt in personal bankruptcy after audit costs have been paid is very low in the UK.

In this paper we have shown that a simplification of the Persons framework allows us to get much stronger results about the relative superiority of different financing arrangements in asymmetric information contexts. Limited liability is important as is the size of low state revenues in relation to audit costs. Generally with unlimited liability or in situations where the observation cost is low relative to low state revenues in our framework it is better to have two investors rather than one.

A. APPENDIX

A.1. The Second Period Game Forms. For each possible game form we find the restrictions on repayments supporting that Nash equilibrium and show that when the lender is on their reservation level ($E\pi = 0$) the debtor payoff is the difference between $E_s y_s - (1+r)B$, $s = H, L$, and the expected observation cost.

Pure Strategy Second Period Equilibria.

(i) $l = m = 0$. This requires $\partial E\pi/\partial m|(l = 0) = q(R_{LL} - R_L - \phi/q) < 0$ and $\partial EU/\partial l|(m = 0) = p(R_H - R_L) < 0$ to be a Nash equilibrium. Of course The expected observation cost with this game outcome is zero. However it is of no interest since it gives incentives to cheat in the realised low income state.

(ii) $l = 0, m = 1$. This requires $\partial E\pi/\partial m|(l = 0) = (1-p)q(R_{LL} - R_L - \phi/q) > 0$ and $\partial EU/\partial l|(m = 1) = p(R_H - qR_{HL} - (1-q)R_L) < 0$ to be a Nash equilibrium. This is clearly very inefficient; the lender knows that there is no lying but can make money by monitoring so observation costs are always paid. Effectively the state with no monitoring and a low income report is discarded. The expected observation cost is $(1-p)\phi$.

(iii) $l = m = 1$. This requires $\partial E\pi/\partial m|(l=1) = q(pR_{HL} + (1-p)R_{LL} - R_L - \phi/q) > 0$ and $\partial EU/\partial l|(m=1) = p(R_H - qR_{HL} - (1-q)R_L) > 0$ to be a Nash equilibrium. Again this is very inefficient: since $R_H < qR_{HL} + (1-q)R_L$ the borrower has an incentive to always cheat; but on average the lender will gain by monitoring every low income report. Again there are effectively two states in the second period: the high income state with a false low income report which is monitored and punishment is paid and the low income state which is truthfully reported but always monitored. The expected observation cost is ϕ.

(iv) $l = 1, m = 0$. This requires $\partial E\pi/\partial m|(l=1) = q(pR_{HL} + (1-p)R_{LL} - R_L - \phi/q) < 0$ and $\partial EU/\partial l|(m=0) = p(R_H - R_L) > 0$ to be a Nash equilibrium. The borrower has an incentive to cheat but ϕ is too high for the lender to monitor. Whenever $l = 1$ we have a pooling equilibrium with identical repayment offers R_L from both types of debtor. The expected observation cost is zero.

Note that if ϕ is high enough the (present) pooling equilibrium form with $l = 1, m = 0$ dominates the first (and all the other) pure strategy equilibria.

Hybrid Equilibria. The second period game may have equilibria in which only one party plays a pure strategy and the other party randomises. This requires the randomising party to be indifferent between different values of her choice probability whilst the deterministic party must strictly prefer the corner.

(i) $m = 0; 0 < l < 1$. This requires $\partial E\pi/\partial m|(l = \frac{(1-p)(R_L - R_{LL} + \phi/q)}{p(R_{HL} - R_L - \phi/q)}) < 0$ and $\partial EU/\partial l|(m=0) = p(R_H - R_L) = 0$. Taken together these yield $R_H = R_L$ and any value of l satisfying $0 < l < \frac{(1-p)(R_L - R_{LL} + \phi/q)}{p(R_{HL} - R_L - \phi/q)} < 1$ will suffice. The expected observation cost is zero.

(ii) $m = 1; 0 < l < 1$. This requires $\partial E\pi/\partial m|(l = \frac{(1-p)(R_L - R_{LL} + \phi/q)}{p(R_{HL} - R_L - \phi/q)}) > 0$ and $\partial EU/\partial l|(m=1) = p(R_H - qR_{HL} - (1-q)R_L) = 0$. Taken together these yield $R_H = qR_{HL} + (1-q)R_L$ and any value of l satisfying $1 > l > \frac{(1-p)(R_L - R_{LL} + \phi/q)}{p(R_{HL} - R_L - \phi/q)} > 0$ will suffice. The expected observation cost is $(1 - p + pl)\phi$.

(iii) $l = 0; 0 < m < 1$. This requires $\partial E\pi/\partial m|(l=0) = (1-p)q(R_{LL} - R_L - \phi/q) = 0$ and $EU/\partial l|(m = \frac{R_H - R_L}{q(R_{HL} - R_L)}) < 0$. Taken together these yield $R_{LL} = R_L + \phi/q$ and any value of m satisfying $0 < m < \frac{R_H - R_L}{q(R_{HL} - R_L)} < 1$ will suffice. The expected observation cost is $(1-p)m\phi$.

(iv) $l = 1; 0 < m < 1$. This requires $\partial E\pi/\partial m|(l=1) = q(pR_{HL} + (1-p)R_{LL} - R_L - \phi/q) = 0$ and $EU/\partial l|(m = \frac{R_H - R_L}{q(R_{HL} - R_L)}) > 0$. Taken together these yield $pR_{HL} + (1-p)R_{LL} = R_L + \phi/q$ and any value of m satisfying $1 > m > \frac{R_H - R_L}{q(R_{HL} - R_L)} > 0$ will suffice. The expected observation cost is $(1-p)m\phi$.

Interior Mixed Strategy Equilibrium. A Nash equilibrium in mixed strategies requires:

$$\partial EU/\partial l = 0 = -p(y_H - R_H) + p[m(y_H - qR_{HL} - (1-q)R_L) + (1-m)(y_H - R_L)]$$

$$\partial E\pi/\partial m = 0 = plq(R_{HL} - R_L - \phi/q) + (1-p)q(R_{LL} - R_L - \phi/q)$$

Solving these for m and l respectively then gives:

$$m = \frac{R_H - R_L}{q(R_{HL} - R_L)}$$

$$l = \frac{(1-p)(R_L - R_{LL} + \phi/q)}{p(R_{HL} - R_L - \phi/q)}$$

For an interior intersection of the reaction curves, the above imply:

$q(R_{HL} - R_L) > R_H - R_L > 0$ to give $0 < m < 1$ and either:

(A): $R_L - R_{LL} + \phi/q > 0$ $\quad R_{HL} - R_L - \phi/q > 0$ $\quad pR_{HL} + (1-p)R_{LL} - R_L - \phi/q > 0$

or:

(B): $R_L - R_{LL} + \phi/q < 0$ $\quad R_{HL} - R_L - \phi/q < 0$ $\quad pR_{HL} + (1-p)R_{LL} - R_L - \phi/q < 0$

to give $0 < l < 1$

However, to rule out the case $l = 0, m = 1$ we must be in *(A)*. The expected observation costs are $(1 - p + pl)m\phi$.

Dominance Relations between Game Forms. Since $y_L < (1+r)B$, the contracts leading to the pure strategy ($l = 1, m = 0$) or to the hybrid form ($m = 0, 0 < l < 1$) are infeasible since the single repayment R_L does not cover $(1+r)B$.

Since the optimal contract will always hold the lender to his reservation level of zero, we can Pareto rank the contracts by the expected borrower's payoff. This amounts to ranking by the expected observation cost:

- the hybrid case $l = 0, 0 < m < 1$ dominates the pure strategy case $l = 0, m = 1$ because it has lower observation cost; in turn the latter dominates the hybrid case $m = 1, 0 < l < 1$;
- the hybrid case $l = 0, 0 < m < 1$ also dominates the hybrid case $l = 1, 0 < m < 1$ for the same reasons;
- the pure strategy case $l = m = 1$ is dominated by $l = 0, m = 1$.

  So there are two possible candidates for an optimal contract-game form: the hybrid case $l = 0, 0 < m < 1$ and the interior mixed strategy case.

Now suppose that B is variable: under our technological assumptions whilst the first best cannot be implemented, nevertheless, either the hybrid form $m = 0, 0 < l < 1$ or the pure strategy case $m = 0, l = 1$ can be made feasible by reducing B to the level satisfying $y_L(B) = (1 + r)B$ and are then indifferent to each other.

Take any other pair of H or pure strategy cases; at any fixed level of B we know the above rankings hold. So for example $E\{U(B)|l = 0, 0 < m < 1\} > E\{U(B)|l = 0, m = 1\}$ for any B. Letting $B_{01} = \arg\max E\{U(B)|l = 0, m = 1\}$, it follows that:

$$\begin{aligned}\max_B E\{U(B)|l &= 0, 0 < m < 1\} \geq E\{U(B_{01})|l = 0, 0 < m < 1\} \\ &> E\{U(B_{01})|l = 0, m = 1\}\end{aligned}$$

A.2. The Optimal Repayments Conditional on the Interior Mixed Strategy Solution. The borrower's payoff is:

$$\begin{aligned}EU &= p(1 - l)(y_H - R_H) + pl[m(q(y_H - R_{HL}) + (1 - q)(y_H - R_L)) + (1 - m)(y_H - R_L)] \\ &+ (1 - p)[m(q(y_L - R_{LL}) + (1 - q)(y_L - R_L)) + (1 - m)(y_L - R_L)] \\ &= p(1 - l)(y_H - R_H) + pl[m(y_H - qR_{HL} - (1 - q)R_L) + (1 - m)(y_H - R_L)] \\ &+ (1 - p)[m(y_L - qR_{LL} - (1 - q)R_L) + (1 - m)(y_L - R_L)]\end{aligned}$$

and there is only one lender whose payoff is:

$$\begin{aligned}E\pi &= p(1 - l)R_H + pl[m(q(R_{HL} - \phi) + (1 - q)(R_L - \phi)) + (1 - m)R_L] \\ &\quad +(1 - p)[m(q(R_{LL} - \phi) + (1 - q)(R_L - \phi)) + (1 - m)R_L] - (1 + r)B \\ &= p(1 - l)R_H + pl[m(qR_{HL} + (1 - q)R_L - \phi) + (1 - m)R_L] \\ &\quad +(1 - p)[m(qR_{LL} + (1 - q)R_L - \phi) + (1 - m)R_L] - (1 + r)B \\ &= pR_H + (1 - p)R_L - pl(R_H - R_L) + plmq(R_{HL} - R_L - \phi/q) \\ &\quad +(1 - p)mq(R_{LL} - R_L - \phi/q) - (1 + r)B\end{aligned}$$

The Nash equilibrium values of the probabilities of monitoring and lying are given by the reaction functions:

$$\partial EU/\partial l = 0 = -p(y_H - R_H) + p[m(y_H - qR_{HL} - (1 - q)R_L) + (1 - m)(y_H - R_L)]$$

$$\partial E\pi/\partial m = 0 = plq(R_{HL} - R_L - \phi/q) - (1 - p)q(R_{LL} - R_L - \phi/q)$$

Solving these for m and l respectively then gives:

$$m = \frac{R_H - R_L}{q(R_{HL} - R_L)}$$

$$l = \frac{(1-p)(R_L - R_{LL} + \phi/q)}{p(R_{HL} - R_L - \phi/q)}$$

There is zero borrower surplus in the low state:

$$R_{LL} = R_L = y_L$$

and there is maximum punishment when caught cheating in the high state:

$$R_{HL} = y_H$$

These values of the repayments allow us to rewrite the probabilities of monitoring and lying as:

$$m = \frac{R_H - y_L}{q(y_{HL} - y_L)}$$

$$l = \frac{(1-p)\phi/q}{p(y_H - y_L - \phi/q)}$$

We can then substitute the above values of the probabilities and of the repayments into the lenders participation constraint $E\pi = 0$ to find:

$$E\pi = 0 = pR_H + (1-p)R_L - (1-p)(R_H - y_L)(\frac{\phi/q}{y_H - y_L - \phi/q}) - (1+r)B$$

Solving for the repayment in the high state R_H then gives:

$$R_H = \frac{(1+r)B(y_H - y_L - \phi/q) - (1-p)y_L(y_H - y_L)}{p(y_H - y_L) - \phi/q}$$

and:

$$m = \frac{[(1+r)B - y_L](y_H - y_L - \phi/q)}{[p(y_H - y_L) - \phi/q](y_H - y_L)}$$

Finally substituting all values of the repayments and of the probabilities of monitoring and lying gives the borrower's expected utility:

$$EU = E_s y_s - (1+r)B - pqlm(y_H - y_L) = EU_{SR} - (1-p)(\phi/q)[\frac{(1+r)B - y_L}{p(y_H - y_L) - \phi/q}]$$

A.3. The Optimal Repayments Conditional on the One Investor Hybrid Solution. When $l = 0$ the borrower and lender's payoffs become:

$$\begin{aligned} EU &= p(y_H - R_H) + (1-p)[y_L - m(qR_{LL} + (1-q)R_L) - (1-m)R_L] \\ &= p(y_H - R_H) + (1-p)[y_L - mqR_{LL} + (1-mq)R_L] \end{aligned}$$

and:

$$\begin{aligned} E\pi &= E\pi_1 + E\pi_2 \\ &= pR_H + (1-p)[m(q(R_{LL} - \phi) + (1-q)(R_L - \phi)) + (1-m)R_L] - (1+r)B \\ &= pR_H + (1-p)[mqR_{LL} + (1-mq)R_L - m\phi] - (1+r)B \end{aligned}$$

respectively. The truthtelling constraint is:

$$R_H = m[qR_{HL} + (1-q)R_L] + (1-m)R_L = mqR_{HL} + (1-mq)R_L$$

There is zero borrower surplus when monitored in the low state:

$$R_{LL} = R_L + \phi/q = y_L$$

Hence:

$$R_L = y_L - \phi/q$$

There is maximum punishment when caught cheating in the high state:

$$R_{HL} = y_H$$

These two conditions imply that the truthtelling constraint becomes:

$$R_H = mqy_H + (1-mq)(y_L - \phi/q)$$

Then substituting these values for the repayments into the lenders participation constraint $E\pi = 0$ we solve for the probability of monitoring:

$$m = \frac{(1+r)B - y_L + \phi/q}{qp(y_H - y_L + \phi/q)}$$

which in turn gives the repayments in the high state:

$$
\begin{aligned}
R_H &= y_L - \phi/q + [\frac{(1+r)B - y_L + \phi/q}{p(y_H - y_L + \phi/q)}](y_H - y_L + \phi/q) \\
&= [\frac{(1+r)B - y_L + \phi/q}{p(y_H - y_L + \phi/q)}]y_H + [\frac{E_s y_s - (1+r)B - (1-p)\phi/q}{p(y_H - y_L + \phi/q}](y_L - \phi/q)
\end{aligned}
$$

note that $R_{HL} > R_H > R_{LL} > R_L$

We then substitute again all the repayment values and probability of monitoring into the borrowers expected utility EU to get:

$$EU = E_s y_s - (1+r)B - (1-p)\phi m = EU_{SR} - (1-p)(\phi/q)[\frac{(1+r)B - y_L + \phi/q}{p(y_H - y_L + \phi/q)}]$$

A.4. The Optimal Repayments Conditional on Two Investor Hybrid Solution. When $l = 0$ the borrower and lenders payoffs become:

$$
\begin{aligned}
EU &= p[y_H - (R_H + P_H)] + (1-p)[y_L - m(q(R_{LL} + P_{LL}) + (1-q)(R_L + P_L)) - (1-m)(R_L + P_L)] \\
&= p[y_H - (R_H + P_H)] + (1-p)[y_L - mq(R_{LL} + P_{LL}) + (1-mq)(R_L + P_L)]
\end{aligned}
$$

and:

$$
\begin{aligned}
E\pi &= E\pi_1 + E\pi_2 \\
&= p(R_H + P_H) + (1-p)[m(q(R_{LL} + P_{LL} - \phi) + (1-q)(R_L + P_L - \phi)) + (1-m)(R_L + P_L)] - (1+ \\
&= p(R_H + P_H) + (1-p)[mq(R_{LL} + P_{LL}) + (1-mq)(R_L + P_L) - m\phi] - (1+r)B
\end{aligned}
$$

respectively. The truthtelling constraint is:

$$
\begin{aligned}
R_H + P_H &= m[q(R_{HL} + P_{HL}) + (1-q)(R_L + P_L)] + (1-m)(R_L + P_L) \\
&= mq(R_{HL} + P_{HL}) + (1-mq)(R_L + P_L)
\end{aligned}
$$

There is zero borrower surplus when monitored in the low state:

$$R_{LL} + P_{LL} = R_L + \phi/q + P_L - \phi/q = y_L$$

Hence:

$$R_L + P_L = y_L$$

There is maximum punishment when caught cheating in the high state:

$$R_{HL} + P_{HL} = y_H$$

These two conditions imply that the truthtelling constraint becomes:

$$R_H + P_H = mqy_H + (1-mq)y_L$$

Then substituting these values for the repayments into the lenders participation constraint $E\pi = 0$ we solve for the probability of monitoring:

$$m = \frac{(1+r)B - y_L}{q(p(y_H - y_L) - (1-p)\phi/q)}$$

which in turn gives the repayments in the high state:

$$\begin{aligned} R_H + P_H &= y_L + [\frac{(1+r)B - y_L}{p(y_H - y_L) - (1-p)\phi/q}](y_H - y_L) \\ &= [\frac{(1+r)B - y_L}{p(y_H - y_L) - (1-p)\phi/q}]y_H + [\frac{E_s y_s - (1+r)B}{p(y_H - y_L) - (1-p)\phi/q}]y_L \end{aligned}$$

Note that $R_{HL} + P_{HL} > R_H + P_H > R_{LL} + P_{LL} = R_L + P_L$.

We then substitute again all the repayment values and probability of monitoring into the borrowers expected utility EU to get:

$$EU = E_s y_s - (1+r)B - (1-p)\phi m = EU_{SR} - (1-p)(\phi/q)[\frac{(1+r)B - y_L}{p(y_H - y_L) - (1-p)\phi/q}]$$

A.5. Unlimited Liability Optimal Contract. Proof. In each contract form the optimal investment is at or above B_{SR}. It follows that the $H2$ contract is feasible at the optimal investment levels of the KP and $H1$ contracts, B^*_{KP} and B^*_{H1} respectively. Hence for example $EU_{KP}(B^*_{KP}) < EU_{H2}(B^*_{KP}) < EU_{H2}(B^*_{H2})$. Similarly $EU_{H1}(B^*_{H1}) < EU_{H2}(B^*_{H1}) < EU_{H2}(B^*_{H2})$. ■

A.6. Optimal Investment Levels of $H1, H2, KP$ Are Below the First Best. Proof. For the H contract since $EU_H(B)$ is concave, this is true if $EU'_H(B_1) < 0$. However:

$$EU'_H(B_1) = \frac{(1-p)\phi(1+r-y'_L]}{p^2(y_H - y_L + \phi)^2}[(1+r)B - py_H - (1-p)y_L + \phi(1-p)]$$

which under the feasibility condition is negative.

Similarly for the KP contract:

$$\begin{aligned} EU'_{KP}(B_1) &= \frac{-[(1+r) - y'_L][p(y_H - y_L) - \phi] + p(y'_H - y'_L)[(1+r)B_1 - y_L]}{[p(y_H - y_L) - \phi]^2} \\ &= \frac{-[(1+r) - y'_L]\{Ey - \phi] - (1+r)B_1\}}{[p(y_H - y_L) - \phi]^2} < 0 \end{aligned}$$

because of the first best condition that $p(y'_H - y'_L) = (1+r) - y'_L > 0$ and the feasibility condition which gives the sign. ■

References

[1] C. Choe, "Contract Design and Costly State verification Games", *Journal of economic Behaviour and Organization(1988)*, 34, pp. 327-340.

[2] P. Dasgupta, P. Hammond and E. Maskin, "The Implementation of Social Choice Rules: Some General Results on Incentive Compatibility", *Review of Economic Studies* (1979), 46, pp. 185-216.

[3] M. Dewatripont, "Commitment Through Renegotiation Proof Contracts with Third Parties", *Review of Economic Studies (1988)*, 55, pp. 377-390.

[4] D. Gale and M. Hellwig, "Incentive Compatible Debt Contracts: The One Period Problem", *Review of Economic Studies* (1985), 52, pp. 647-663.

[5] D. Gale and M. Hellwig, "Reputation and Renegotiation: The Case of Sovereign Debt", *International Economic Review* (1989), pp. 3-31

[6] O. Hart, Firms, Contracts and Financial Structure, Oxford University Press (1995).

[7] M. Hellwig, "Risk aversion and incentive compatibility with ex post information asymmetry", *Economic Theory* (2001), Vol.18, No.2, pp.415-438

[8] P. Jost, "On the Role of Commitment in a Principal-Agent Relationship with an Informed Principal", *Journal of Economic Theory* (1996), 68, pp. 510-530.

[9] F. Khalil, "Auditing Without Commitment", *Rand Journal of Economics* (1997), 28, pp. 629-640.

[10] F. Khalil and B.M.Parigi, "Loan Size as a Commitment Device", *International Economic Review* (1998), 39, pp. 135-150.

[11] S. Krasa and A.P.Villamil, "Optimal Contracts When Enforcement is a Decision Variable", *Econometrica* (2000), 68, pp. 119-134.

[12] D.Mookherjee and I.Png, "Auditing,Insurance and Redistribution", *Quarterly Journal of Economics* (1989), pp.397-415.

[13] J.C. Persons, "Liars Never Prosper? How Management Misrepresentation Reduces Monitoring Costs", Journal of Financial Intermediation" (1996), 6, 269-306.

[14] P Simmons & G Garino, "The Form of Loan Contract with Risk Neutrality, No Commitment and Fixed Investment", mimeo, 2002.

[15] R. Townsend, "Optimal Contracts and Competitive Markets with Costly State Verification", *Journal of Economic Theory* (1979), 21, pp. 265-293.