Storage Behaviour of Cournot Duopolists over the Business Cycle

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Abstract

I disentangle the "commitment effect" from the "smoothing effect" of inventories in a model of two periods Cournot competition to show their dependence to the cost of production and the size of market demand. I show that one is more likely to observe the constitution of strategic inventories when the market enters in a downturn: firms can credibly flood the market and exert some leadership. In that case the leader's inventories cover the entire second period sales and the equilibrium of the Cournot game is asymmetric. In an upturn symmetric inventories are more likely to prevail in equilibrium.

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1 Introduction

On many mineral or non-mineral commodity markets producers are in imperfect competition. The highly standardized nature of those products regularly triggers aggressive commercial behaviour. Moreover those markets are often cyclical: upturns alternate to downturns, inducing modifications in firms strategies. One frequently reported behaviour is the search for a large "customer base": firms price aggressively in downturns to maintain or enlarge their presence on the market, with the obvious goal to reach new customers. For example the U.S. uranium spot market has faced an important downturn at the beginning of the 1990's[1]: during that period Kazakh producers have been the object of an anti-dumping duty conducted by the U.S. Department of Commerce[2]. A similar inquiry has been conducted on the market for PC memories (DRAMs). On this market, South-Korean producers have been suspected to sell the good at less-than-fair value when the market was in downturn, in order to enlarge their market share relatively to their American competitors[3]. The business cycle effect on firms strategies on the DRAM market is quoted explicitly in the report of the U.S. department of Commerce[4],

> The DRAM industry is highly cyclical in nature [...]. In the past, the DRAM industry has been characterized by dumping during periods of significant downturn. [...] Because DRAMs are a commodity product, DRAM producers/resellers must price aggressively during a downturn period in order to [...] maintain their customer base. This is especially true during the lowest point in the downturn.

On most if not all of these markets, storage behaviour of producers or consumers are playing a crucial role in the price formation process. On the uranium market for example, "on ground" storage (extracted from the mine but kept out of the market) has presumably contributed to the decrease of spot prices in the 1990's.

[1] The re-estimation of the growth rate of the number of nuclear powerplants and the decrease of the large strategic inventories (constituted by western buyers in the 1980's to face the prophesied scarcity of this natural resource) seemed to be the main drivers of this downturn.

[2] see U.S. department of Commerce (1999) for the summary of the investigations started in 1991.

[3] The inquiry started with a denunciation by American producers. See the panel report of the World Trade Organization, WT/DS99/R 29 January 1999.

[4] US/DOC Preliminary Results Third Review, 62 Fed. Reg. at 12796 (Ex. USA-20), par. 16.

The impact of storage strategies on the DRAMs market price has also been pointed out, in particular when demand started to decrease. Market analysts emphasized the strategy of South-Korean producers, consisting in stockpiling the product as the demand decreases to sell those quantities later on, inducing an even worse reduction of the market price. The goal of this paper is to provide a theoretical justification of this strategy, which goes against the standard economic view that firms are storing when expecting either a shortage of their own future supply, or a sudden increase of their future demand. I show that firms in Cournot competition find profitable to build up inventories to enlarge their market shares only when the market enters in a downturn, in particular when it reaches its lowest, for two main reasons. First, when demand at the target period where inventories are released is low, a firm owning large inventories is committed to sell those quantities on the market. Consequently to a low expected average revenue, its rival does not find profitable to produce any other quantity than a follower's one. If demand is low, the marginal cost of storage incurred to conduct this strategy is limited. Second if the market demand at the initiating period where inventories are constituted is low, and if the storage strategy cannot be separated from the first period strategy, the increase in the marginal cost of production resulting from storage is also limited. When demand decreases from period to period, and if it is low enough, firms are not deterred from using inventories to exert some leadership.

The present paper falls into the literature in which the Stackelberg leadership is endogenous. It contributes to the other studies of storage in Cournot competition (Arvan [1985], Ware [1985], Allaz [1991], Moellgaard [1994], Moellgaard, Poddar, and Sasaki [2000], Thille [2003]) by disentangling the *"smoothing effect"* from the *"commitment effect"* of inventories, and relating their use to the Business Cycle that affects the market demand. Moreover it evaluates their consequences on the existence of pure strategies Nash equilibria. The smoothing effect of inventories results from the convexity of the cost function: storing allows a firm to replicate its technology of production across time, and therefore enables one to work on the lowest part of its marginal cost of production. Integrating this reduction in its optimization, a firm is more aggressive than its opponent on subsequent markets: storage has the desirable equilibrium property to reduce the sales of any opponent by making a firm more efficient on the production side. The total smoothing effect of inventories includes this second marginal effect, together with the direct reduction of the marginal cost of production.

The commitment effect of inventories results from the fact that a firm is able to force its opponent to react as a follower to the quantity it has stockpiled, when its inventories are large and the follower's one are low. This effect is obviously connected to Dixit (1980), but is limited by the size of the market firms are sharing. Indeed building up inventories consists in sinking costs in the first period to enlarge one's future capacity to market the good, by a quantity -the inventories- from which a firm can sell at a zero marginal cost: the firm has the possibility to flood the market. However this threat is credible and profitable for the leader only when the market demand in target period is low enough. As said before, building up inventories and keeping them for later sale is not too costly both in terms of cost of production and of cost of storage. More strikingly, the follower does not deter the leader to stockpile, neither by increasing its sales in the period subsequent to the constitution of inventories, nor by increasing its inventories in the initiating period. The former is due to the fact that flooding the market when the market price is already low enough leaves any competitor who has to produce with an even lower expected average revenue. The latter follows from the same argument sustaining the Stackelberg result: sticking to a follower's quantity when inventories are constituted is always a profit maximizer when one's opponent acts as a leader. The unique difference with a Stackelberg game is that the commitment in the Cournot game with inventories is endogenous: firms have to pay for it, a natural limitation to the asymmetry in market shares.

In his path-breaking paper, Arvan [1985] shows, in a model with stationary demands, that the Cournot equilibrium of a game where firms are able to store before selling a good on the market may be asymmetric. He argues however that a firm may find profitable to increase its inventories to force the leader to give up the search for a bigger market share[5], leading to the non-existence of an asymmetric equilibrium. By disentangling all the various effects interacting, I am able to reach a more complete characterization of the Cournot game with inventories, in particular regarding the existence and the likelihood of asymmetric outcomes. I show that

[5] In his terminology "firms may be put with redundant inventories".

the equilibrium of the game is symmetric only when demand is high. When it is low, firms build up inventories to enlarge their market share. A first period demand reduces the incentives to conduct this strategy: storing increases also the marginal cost of production of all the units sold immediately. I find that building inventories is a profitable strategy only in when the market reaches the neighbourhood of a downturn. When the market is in upturn, firms do not store under linear costs of production. Under a convex cost of production the most likely issue is that firms start first to build up symmetric inventories to smooth their cost of production across time, then stop to store when the demand reaches its highest.

These findings contrast with Rotemberg and Saloner [1989], who show that inventories may be used to maintain collusion when market demand is high by enlarging an existing capacity of production. Moreover it may be connected closely to the qualitative evidence drawn from the DRAM market: without taking into account the fidelity or goodwill implicit aspect of a "customer base", that is without any other incentive than strategic to build up inventories, I show that firms find strategic storage profitable in downturns. Adding the "customer base" incentive to the model would simply reinforce the effect.

In a setting deeply grounded on exhaustible resource models, Ware [1985] studies the entry deterrence strategy that a firm could conduct by stockpiling and then releasing units on the market from this large deposit. While the dynamic of his model is longer than mine, the incumbent firm does not renew its inventories once sales have started. Entry is deterred until the entire inventories are sufficiently low to induce an accommodating behaviour by the leader. The asymmetry of the equilibrium he is describing comes from the timing of the game, and the importance of the size of the market is not studied.

This paper focuses on the trade-off between the "smoothing" effect and the "commitment" effect: it differs with previous studies by Allaz [1991], who shows that storage and forward trading are used simultaneously to enlarge the market share under decreasing return-to-scale, and to Moellgaard [1994], and Moellgaard, Poddar, and Sasaki [2000], who introduce conjectural variations in Arvan's model while keeping stationary demands and neglecting the search for a leader market share. The model studied here is a generalization of Saloner [1987], Pal [1991] and Pal [1996], who study two-period constant returns Cournot competition, assuming that all units produced in the first period are automatically sold on the market in second period. The trade-off between the smoothing and the commitment effect of inventories does not appear, as well as the importance of the market demand.

Finally Thille [2003] studies a infinite horizon model in which Cournot competitors produce the good with a convex technology of production while suffering risks on the marginal cost and on the linear demand interecept. He characterizes the volatility of the market price, which depends deeply on the source of the uncertainty in imperfect competition. The result presented here is a step towards the understanding of the effect of non stationary demands on the constitution of strategic inventories, effect that cannot be analyzed in a long dynamic settings since it is not possible to define value functions for non stationary systems.

Section 2 presents the model. Then section 3 analyzes the sales sub-game and shows how the effect of storage depend on costs and demand in quantity competition. In section 4, the existence of a pure strategy Nash equilibrium is discussed. Then I look at the effect of a non-stationary demand, in the context of a linear cost and of a convex cost of production. In section 5, I conclude.

2 The model

Let two producers $i = 1, 2$ of an homogeneous and non-perishable good compete during two periods. In the first period, they choose simultaneously the quantities they want to produce and sell. Every unit not sold is kept in inventories, and those inventories are perfectly observed by both firms at the end of the first period. Then in the second period they choose simultaneously the quantities they want to sell and the quantities they want to produce, given their inventories. Firms are therefore in two periods Cournot competition.

Let q_t^i be the production and s_t^i be the sales chosen by firm i in period t, for $t = 1, 2$. Firms cannot borrow the good in period 1: each has to produce at least the quantity it sells in each period. Moreover they do not hold inventories at the beginning of the game, $x_0^i = 0$ for any $i = 1, 2$. Let $x_t^i = q_t^i + x_{t-1}^i - s_t^i$ be the inventories hold by firm i at the end of period t. Then for $t = 1, 2$ and $i = 1, 2$, sales, productions and inventories satisfy the following inequalities

$$x_t^i \geq 0, \quad q_t^i \geq 0, \quad s_t^i \geq 0, \quad s_t^i \leq x_{t-1}^i + q_t^i \tag{1}$$

Firms perfectly observe the first period pair of inventories (x_1^1, x_1^2) before making second period decisions. This assumption is crucial for inventories to affect future competition. Indeed inventories commit firms second period decisions and modify the second period Cournot equilibrium. In particular since the costs of producing and carrying inventories are sunk when taking second period decisions, they lead a firm to be more aggressive on the second period market than a rival who has not stored. Several arguments support this assumption. First, from a theoretical point of view, the non-observability of a past choice does not mean that it will have no impact on current decisions. For example if a firm holds private information and its opponent receives a signal on its choice, the value of a commitment can be restored (Maggi [1999]). Moreover if agents use a "tatonnement" process to determine their equilibrium actions, they are able to reconstruct a past commitment, and will do it, provided that this commitment affect their payoffs. Second, from a practical point of view, the main actors on many markets are collecting precise information on existing inventories either through market studies (realized by independent analysts, brokers, regulation authorities on organized commodity markets,...), or by monitoring closely their rivals. A more complex theoretical setting would allow for imperfect observability of inventories and incomplete information on costs of production, but it would make the effect of the Business Cycle on storage behaviour in Cournot competition less clear and of course less tractable.

We restrict our attention to second period strategies that depend only on the pair of inventories chosen in the first period, and not on previous sales or poduction[6]: strictly speaking we are considering Markovian strategies in inventories. For any $i = 1, 2$, firm i strategy ω^i is

$$\omega^i = \{x_1^i, s_1^i, q_1^i, x_2^i(x_1^i, x_1^j), s_2^i(x_1^i, x_1^j), q_2^i(x_1^i, x_1^j)\} \tag{2}$$

Let $C(q)$ be the cost of producing the quantity q for firm i, identical across firms and across periods. We assume that there are neither sunk nor non-sunk fixed cost of production, i.e.

$$C(0) = 0 \text{ and } \lim_{q \to 0^+} C(q) = 0 \tag{3}$$

Remark that relaxing the second assumption (i.e. allowing for non-sunk fixed cost of

[6]Past sales could matter in an infinite horizon game.

production) would give a strong incentive to stockpile, since by producing everything in the first period a firm avoids to pay the fixed cost again in the second period[7]. However the most striking result of this paper does not rely on this effect. Therefore I do not consider decreasing average cost of production in this analysis. To end up the characterization of the production technology, the marginal cost of production is positive and increasing with the quantity produced,

$$\frac{dC}{dq} \equiv C'(q) \geq 0, \ \lim_{q \to 0^+} C'(q) = C'(0) > 0, \ \text{and} \ \frac{d^2C}{dq^2} \equiv C''(q) \geq 0 \text{ for } q \geq 0 \tag{4}$$

where $C'(0)$ is the additional cost of producing the first units of output.

Let $H(x)$ be the cost of holding total inventories x from one period to the other, paid by firm i at the end of period in which inventories are constituted. I assume that the storage capacity exists at the time where firms decide to stockpile. Consequently there are neither sunk nor non-sunk fixed cost of storage,

$$H(0) = 0 \text{ and } \lim_{x \to 0^+} H(x) = 0 \tag{5}$$

Moreover this cost is convex in the quantity stored

$$\frac{dH}{dx} \equiv H'(x) \geq 0, \ \text{and} \ \frac{d^2H}{dx^2} \equiv H''(x) \geq 0 \text{ for } x \geq 0 \tag{6}$$

In order not to create extra incentives to release inventories on the market in period 2, I assume that firms are able to destroy at no cost unsold inventories at the end of this period. This assumption guaranties that the "commitment" effect of inventories is not a last period effect[8].

Assuming that storage and production costs are identical between firms is useful. Indeed part of this study consists in stressing the differences in firm inventories and market shares in equilibrium: assuming *symmetric payoffs* allows to characterize unambiguously *asymmetric equilibria*. The results established here are easily generalizable to asymmetric payoffs. Finally the model may be interpreted as a short-run storage game, where all the long-run investments (e.g. the production tool, the storage technology...) are given.

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[7] Arvan [1985] provides some remarks on the effect of a non-sunk fixed cost on the storage strategy.

[8] See Saloner [1986] for a study of the commitment effect of the cost to dispose of unsold units. For example if the output were loosing its value at the end of the game, because of a "fashion effect", firms would be more willing to sell it when its value is still high.

To analyze the effect on storage behaviour of changes in demand from period to period, the inverse demand in each period is assumed to be a function of the aggregated quantity sold as well as of the period considered. Let $P(Q,\mathbf{t})$ be the inverse demand function, where Q denotes the aggregate quantity sold on the market, and $t = 1, 2$ denotes the period. Both the first and second derivative of $P(Q,\mathbf{t})$ are negative and independent of the period t considered,

$$\frac{\partial P}{\partial Q}(Q,\mathbf{t}) \equiv P'(Q) < 0 \text{ and } \frac{\partial^2 P}{\partial Q^2}(Q,\mathbf{t}) \equiv P''(Q) \leq 0, \text{ for } Q \geq 0 \text{ and } t = 1, 2 \quad (7)$$

In addition, I assume that it exists a finite maximal price consumers are ready to pay to obtain the good. This price, denoted $\overline{P}_t = P(0,\mathbf{t})$, is such that a monopolist would find profitable to trade in each period,

$$C'(0) \leq \overline{P}_t < +\infty \quad (8)$$

$\overline{Q}_t$ denotes the finite maximal quantity for which consumers are satiated, i.e. such that the price of the product is equal to 0 whenever this quantity is sold. Then

$$P(Q_t,\mathbf{t}) = 0 \text{ if } Q_t > \overline{Q}_t \text{ for } t = 1, 2 \quad (9)$$

The difference between the two periods comes therefore from parallel shifts in demand from period to period, modifying solely the pair $(\overline{P}_t, \overline{Q}_t)$.

Let δ be the discount rate at which firms are working, $\delta < 1$. Firm i's total profit discounted in the first period is

$$\Pi^i(\omega^i,\omega^j) = P(s_1^1 + s_1^2, \mathbf{1})\ s_1^i - C(q_1^i) - H(x_1^i) + \delta\ [(P(s_2^1 + s_2^2, \mathbf{2})\ s_2^i - C(q_2^i)] \quad (10)$$

The two periods Cournot storage game in which two firms (1 and 2) interact using respectively strategies (ω^1, ω^2) and earning respectively profits $(\Pi^1(\omega^1,\omega^2), \Pi^2(\omega^1,\omega^2))$ is denoted $\mathcal{G}$.

3 Storage effect on second period sales

In second period, firms maximize their current profits given first period inventories. Since the game ends up at this period, there is no point for them to build up extra inventories. Therefore, for any s_2^i the firm is ready to sell, the production

q_2^i is determined in such a way that together with inventories x_1^i, it matches sales s_2^i exactly. If x_1^i is lower than the planned level of sales s_2^i, then production is positive and equal to the difference between sales and inventories, $q_2^i = s_2^i - x_1^i$, and if inventories are larger than planned sales then the production is equal to 0. It remains to describe the effect of inventories on firms market behaviour s_2^i.

As already seen by Arvan and Ware, when storing before selling, a firm sinks the cost of production of these units. Consequently it endows itself with a capacity x_1^i from which it may sell at a marginal cost equal to 0 later on. Once past inventories have been exhausted by current sales, the firm has to produce again to be able to sell more. First, selling in addition to its inventories obliges the firm to start its production tool and therefore to pay a strictly positive incremental cost of production. Second, when the marginal cost of production is strictly increasing, selling an amount larger than its inventories is feasible at a total cost strictly lower than the cost it would suffer to produce this quantity in a single period. Under a convex cost of production, firms have an incentive to store in order to replicate the technology of production across time and reduce the total cost of producing a given quantity. This cost reduction may be used strategically by Cournot competitors: therefore the higher the inventories cumulated in previous periods are, the more aggressive is a firm on the current period market. To fix the ideas, it is helpful to stress the difference between the game $\mathcal{G}$ and the static Cournot game by introducing the cost of production faced by a firm when selling s_2^i given inventories x_1^i, $\Gamma^i(s_2^i, x_1^i)$. It is equal to

$$\Gamma^i(s_2^i, x_1^i) = \left| \begin{array}{lll} 0 & \text{if} & s_2^i \leq x_1^i \\ C(s_2^i - x_1^i) & \text{if} & s_2^i > x_1^i \end{array} \right. \tag{11}$$

Whereas firms costs of production are identical, firms costs to sell s_2^i when they own x_1^i in inventories differ from each other when inventories differ: asymmetric equilibria may therefore appear in this symmetric setting. Figure 1 represents this cost function and compare it with the no inventories case. Inventories shift the marginal cost of producing a given level of sales to the South-East of the graph: a Cournot competitor with inventories is more aggressive than a Cournot competitor without inventories. Moreover whenever $C(q)$ is discontinuous or non differentiable at $\tilde{q}$, $\Gamma(s_2^i, x_1^i)$ is discontinuous or non differentiable at $s_2^i = \tilde{q} + x_1^i$.

[INSERT FIGURE 1 FROM APPENDIX (A.) HERE]

The best responses of firms when choosing sales are continuous but, as a consequence of the strictly positive marginal cost of production, present two kinks. For small sales of her opponent, firm i's sales in the second period are a function of its opponent's sales $f_2^i(s_2^j, x_1^i)$ solution in s_2^i of

$$P'(s_2^i + s_2^j)\ s_2^i + P(s_2^i + s_2^j, \mathbf{2}) - C'(s_2^i - x_1^i) = 0 \quad \Leftrightarrow \quad s_2^i \equiv f_2^i(s_2^j, x_1^i) \tag{12}$$

When its opponent sells a larger quantity on the market, selling more than its inventories may be too costly for firm i: it may generate a marginal revenue always lower than the marginal cost of production, no matter the quantity produced. However if selling exactly its inventories generates a strictly positive marginal revenue given firm j's sales, then firm i will sell exactly this quantity,

$$\left.\begin{array}{l} P'(x_1^i + s_2^j)\ x_1^i + P(x_1^i + s_2^j, \mathbf{2}) - C'(0) < 0 \\ P'(x_1^i + s_2^j)\ x_1^i + P(x_1^i + s_2^j, \mathbf{2}) > 0 \end{array}\right\} \quad \Leftrightarrow \quad s_2^i = x_1^i \tag{13}$$

Finally when its opponent sells a very large quantity on the market, the marginal revenue of selling exactly its inventories may be strictly negative. In that case firm i sells strictly less than its inventories to maximize its profits. This choice is a function of the sales of its opponent only, $g_2(s_2^j)$, and is not firm specific contrary to f_2^i which depends on inventories: g_2 is the solution in s_2^i of

$$P'(s_2^i + s_2^j)\ s_2^i + P(s_2^i + s_2^j, \mathbf{2}) = 0 \quad \Leftrightarrow \quad s_2^i \equiv g_2(s_2^j) \tag{14}$$

Note that this description of a firm best choice is valid when its inventories x_1^i are small enough[9]. Without loss of generality we restrict the description of the sub-game equilibria of the game to this case, since it encompasses the two other cases.

Let us introduce some useful notations: $(s^1_{\varphi,\varphi'}, s^2_{\varphi,\varphi'})$ denotes the equilibrium when it results from the intersection of functions φ for firm 1 and φ' for firm 2, where φ and φ' can be equal to functions $f_2^i(s_2^j, x_1^i)$ or $g_2(s_2^j)$[10]. Similarly $(s^1_{\varphi,x}, x_1^2)$

[9] If inventories x_1^i are larger than s_2^M solution of $P'(s_2^M)\ s_2^M + P(s_2^M, \mathbf{2}) - C'(0) = 0$, then obviously the first inequality in (13) is always satisfied for any values of firm j's sales, and firm i will not produce again in that period: its best choice is given either by (13) or (14). If inventories are larger than $g_2(0)$, the firm always sell less than its inventories by reacting always only according to (14).

[10] For example $(s^1_{g,f}, s^2_{g,f})$ results from the intersection of $g_2(s_2^2)$ and $f_2^2(s_2^1, x_1^2)$, and $(s^1_{f,g}, s^2_{f,g})$ results from the intersection of $f_2^1(s_2^2, x_1^1)$ and $g_2(s_2^1)$.

denotes the intersection between function φ for firm 1 and x_1^2 for firm 2[11]. Finally (x_1^1, x_1^2) is the equilibrium when both firms sell exactly their inventories. As lemma 1 below establishes, the extent to which a Cournot competitor may use its inventories to act as a leader on the target market depends on the size of the market demand. Indeed whenever the market demand is low enough, the market price resulting from sales by a firm benefiting from a null marginal cost may be lower than the marginal cost of production of any other firm who has to produce to serve the market. In that case the latter will not find profitable to produce, meaning that the former owning large inventories may flood the market and obtain a leader market share. For some levels of inventories an asymmetric equilibrium may appear in the sub-game, such as $(x_1^1, s_{x,f}^2)$.

On the other hand when the market demand is high enough, the owner of inventories cannot deter its opponent with no inventories to produce in second period: its marginal revenue may be higher than its marginal cost of production for some sales level. This firm will indeed produce and sell the good, leaving the owner of inventories with unsold quantities. The higher is the second period demand the larger will be the set of inventories (x_1^1, x_1^2) for which a firm will find profitable to produce and sell a positive amount when its opponent owns large inventories. Therefore the sale levels $s_{g,f}^1$ and $s_{f,g}^2$ represent the upper bounds on the leadership each firm can exert in second period through the inventories it has constituted in first period. Remark that the greater is the second period demand relatively to the cost $C'(0)$ the smaller is the region in which firms exert a leadership or sell exactly their inventories: the bound $s_{g,f}^i$ tends to $s_{f,f}^i$ for both firms. The commitment power of inventories vanishes when the market demand increases.

Lemma 1 *When the second period demand is low enough, $P(g_2(0), 2) - C'(0) \leq 0$, a firm that builds up large inventories in the first period to obtain the leadership cannot be obliged to sell less than this quantity by its opponent who has not stored. The second period sales sub-game admits 7 types of equilibria in which each firm*

[11] For example $(s_{g,x}^1, x_1^2)$ results from the intersection of $g_2(s_2^2)$ and x_1^2.

sells more than, less than or exactly her inventories,

$$
\begin{array}{rlllll}
(s_2^1(x_1^1,x_1^2), s_2^2(x_1^1,x_1^2)) = & (s_{g,g}^1, s_{g,g}^2) & if & x_1^1 \geq s_{g,g}^1, & x_1^2 \geq s_{g,g}^2 & \texttt{R1} \\
& (s_{g,x}^1, x_1^2) & if & x_1^1 > s_{g,x}^1, & x_1^2 < s_{g,g}^2 & \texttt{R2} \\
& (x_1^1, s_{x,g}^2) & if & x_1^1 < s_{g,g}^1, & x_1^2 > s_{g,x}^2 & \texttt{R3} \\
& (x_1^1, x_1^2) & if & x_1^1 < s_{g,x}^1, & x_1^2 < s_{x,g}^2 & \texttt{R4} \\
& & & x_1^1 > s_{f,x}^1, & x_1^2 > s_{x,f}^2 & \\
& (s_{f,x}^1, x_1^2) & if & x_1^1 \leq s_{f,x}^1, & x_1^2 > s_{f,f}^2 & \texttt{R5} \\
& (x_1^1, s_{x,f}^2) & if & x_1^1 \geq s_{f,f}^1, & x_1^2 \leq s_{x,f}^2 & \texttt{R6} \\
& (s_{f,f}^1, s_{f,f}^2) & if & x_1^1 < s_{f,f}^1, & x_1^2 < s_{f,f}^2 & \texttt{R7}
\end{array}
$$

When the second period demand is high enough, $P(g_2(0), \mathbf{2}) - C'(0) > 0$, a firm that builds up large inventories to obtain the leadership in the first period may be obliged to sell less than its inventories by an opponent producing and selling starting from small inventories. Regions ***R1*** *and* ***R4*** *are not affected, but two new equilibria appear,*

$$
\begin{array}{lllll}
(s_{g,f}^1, s_{g,f}^2) & if & x_1^1 > s_{g,f}^1, & x_1^2 < s_{g,f}^2 & \texttt{R8} \\
(s_{f,g}^1, s_{f,g}^2) & si & x_1^1 < s_{f,g}^1, & x_1^2 > s_{f,g}^2 & \texttt{R9}
\end{array}
$$

Frontiers of regions ***R2****,* ***R3****,* ***R5****, and* ***R6*** *are obviously modified.*

Proof. Available upon request (Referees may find the proof in appendix B.)‖

It may be helpful for the reader to refer to the following figures.

[INSERT FIGURES 2 AND 3 FROM APPENDIX (A.) HERE]

The last section is devoted to the analysis of the first period and discuss the effect of the business cycle on the constitution of inventories in Cournot competition.

4 Storage choice in first period

4.1 Existence of an equilibrium in pure strategies

Let $\Pi_2^i(x_1^1, x_1^2)$ be the value of firm i 's second period profit ($i = 1, 2$) as a function of the pair of inventories,

$$\Pi_2^i(x_1^1,x_1^2) = P(s_2^1(x_1^1,x_1^2) + s_2^2(x_1^1,x_1^2), \mathbf{2})\ s_2^i(x_1^1,x_1^2) - \Gamma(s_2^i(x_1^1,x_1^2), x_1^i) \tag{15}$$

This profit is continuous, but even if it is concave on each of the regions defined in lemma 1 under some standard assumptions, it is non concave on the full domain and non differentiable at certain pairs of inventories with respect to x_1^i. As pointed out also by Arvan, these non-concavities arise because a player may find profitable to cease to smooth its cost of production to behave as a leader. Facing a leader the other player prefers to reduce its sales on the second period market: payoffs may have a local maximum in regions `R5` and `R6` as well as in `R7`. Firm i's total profit to optimize with respect to x_1^i and s_1^i is

$$\Pi^i(s_1^1, s_1^2, x_1^1, x_1^2) = P(s_1^1 + s_1^2, \mathbf{1})\ s_1^i - C(s_1^i + x_1^i) - H(x_1^i) + \delta\ \Pi_2^i(x_1^1, x_1^2) \qquad (16)$$

Without any inventories at the beginning of the game, a close look at this profit shows immediately that firms trade-off is to decide how to allocate first period production between revenue in period 1 thanks to sales s_1^i, and profit in period 2 thanks to the marginal effect of inventories on profits net of the storage cost. As discussed later, if a firm is storing and selling in period 1, the marginal revenue coming from period 1 sales has to be equal to the marginal profitability of inventories in period 2 net of storage costs.

Some pairs of inventories lead to strictly dominated second period sub-games for any first period sales (s_1^1, s_1^2). When deciding how much to store, and since producing is costly, a firm never chooses to carry more to period 2 than what it will sell. Moreover, since it is costly to produce and store, both firms never choose to store and sell exactly their inventories: whenever one firm tries to store a large amount to act as a Stackelberg leader, the other stores a small amount. As in games with exogenous commitment and strategic substitutes, the follower is better-off selling a small quantity when its opponent sells a large quantity.

Proposition 1 *Any pair of inventories (x_1^1, x_1^2) such that at least one firm sells less than its inventories in second period, or both firms sell exactly their inventories in second period, leads to a strictly dominated second period sub-game for any first period sales (s_1^1, s_1^2). Therefore pairs of inventories belonging to regions R1, R2, R3, R4, R8, and R9 cannot be part of an equilibrium strategy of the game $\mathcal{G}$.*

Proof. See appendix C.‖

The consequence of this lemma is that firms storage choices - when they exist in pure strategy - is either for both to sell more than their inventories, or for one to sell exactly its inventories, the other acting as a follower in second period. To put it differently if there is an equilibrium candidate in which one firm stores a large quantity and its opponent a small quantity, the latter never deviates unilaterally from that equilibrium by storing more. Therefore if one firm finds profitable to store a large quantity, the game may end up in an asymmetric equilibrium. Moreover firms will not use those pairs of inventories when designing mixed strategies.

The marginal profitability of inventories allow us to draw another important but intuitive feature of the Cournot game with inventories.

Lemma 2 *When firms sell $(s^1_{f,f}, s^2_{f,f})$ in second period, an increase in first period inventories increases their second period profit. The marginal profitability of inventories of firm i in second period is given by*

$$\frac{\partial \Pi^i_2}{\partial x^i_1} = \frac{\partial s^j_{f,f}}{\partial x^i_1}\ P'(s^1_{f,f} + s^2_{f,f})\ s^i_{f,f} + C'(s^i_{f,f} - x^i_1)$$

$$= \frac{1}{\Delta} C''(s^i_{f,f} - x^i_1)[P''(s^1_{f,f} + s^2_{f,f})\ s^j_{f,f} + P'(s^1_{f,f} + s^2_{f,f})]\ [P'(s^1_{f,f} + s^2_{f,f}) s^i_{f,f} - C'(s^i_{f,f} - x^i_1)] > 0$$

where $\Delta > 0$ is defined in appendix D.

Proof. The proof is immediate (referees may look at appendix D.)‖

If the marginal cost of production is constant, $C''(q) = 0$, inventories are not used to smooth the cost of production. The marginal profitability of an increase in inventories in that case is equal to 0: small inventories do not modify firms future behaviour. However large inventories can still modify firms behaviour no matter the convexity of the cost of production, as appear in lemma 3 below. When one firm (the leader) stores a large amount and the other (the follower) produces in second period, the follower's marginal benefit to carry inventories comes only from the reduction of the marginal cost of production, while the leader enjoys a limited leadership. Again if the marginal cost of production is constant the follower earns no profit from storing.

Lemma 3 *When firm 1 produces in second period while firm 2 sells exactly its inventories, $(s^1_{f,x}, x^2_1)$, the marginal effect of inventories are*

$$\frac{\partial \Pi^1_2}{\partial x^1_1} = \frac{\partial x^2_1}{\partial x^1_1} \ P'(s^1_{f,x} + x^2_1) \ s^1_{f,x} + C'(s^1_{f,x} - x^1_1) = C'(s^1_{f,x} - x^1_1) > 0$$

and

$$\frac{\partial \Pi^2_2}{\partial x^2_1} = (1 + \frac{\partial s^1_{f,x}}{\partial x^2_1}) \ P'(s^1_{f,x} + x^2_1) \ x^2_1 + P(s^1_{f,x} + x^2_1, \boldsymbol{2})$$

The effects are reversed when firms are selling $(x^1_1, s^1_{x,f})$ in period 2.

Proof. The proof is immediate (referees may look at appendix D.)‖

In the first period, firms simultaneously maximize their profits (16) with respect to first period sales s^i_1 and inventories x^i_1. Let J^i be the Jacobian vector of the program, composed of the first order derivatives,

$$J^i = \begin{pmatrix} \frac{\partial \Pi^i}{\partial s^i_1} \\ \frac{\partial \Pi^i}{\partial x^i_1} \end{pmatrix} = \begin{pmatrix} P'(s^1_1 + s^2_1) \ s^i_1 + P(s^1_1 + s^2_1, \mathbf{1}) - C'(s^i_1 + x^i_1) \\ -C'(s^i_1 + x^i_1) - H'(x^i_1) + \delta \ \frac{\partial \Pi^i_2}{\partial x^i_1}(x^1_1, x^2_1) \end{pmatrix} \tag{17}$$

where the effect of inventories on second period profit depends on the regions to which inventories belong. The Hessian matrix H^i of the program is

$$H^i = \begin{pmatrix} P''(s^1_1 + s^2_1) \ s^i_1 + 2 \ P'(s^1_1 + s^2_1) - C''(s^i_1 + x^i_1) & -C''(s^i_1 + x^i_1) \\ -C''(s^i_1 + x^i_1) & -C''(s^i_1 + x^i_1) - H''(x^i_1) + \delta \frac{\partial^2 \Pi^i_2}{\partial (x^i_1)^2} \end{pmatrix} \tag{18}$$

Since the first element on the diagonal is obviously negative, we need to check that the determinant of the Hessian matrix H^i is positive to be insured that H^i is negative semi-definite and therefore that the objective function is concave in each region. This is equivalent to verify that

$$\delta \ \frac{\partial^2 \Pi^i_2}{\partial (x^i_1)^2}(x^i_1, x^j_1) < H''(x^i_1) + \frac{P'' s^i_1 + 2P'(s^1_1 + s^2_1)}{P'' s^i_1 + 2P'(s^1_1 + s^2_1) - C''(s^i_1 + x^i_1)} \ C''(s^i_1 + x^i_1) \tag{19}$$

If the second period profit is concave with respect to inventories x^i_1 in every region not excluded by proposition 1, and if moreover the cross partial derivative of this profit with respect to x^i_1 and x^j_1 is also negative, then for every expression of $\Pi^i_2(x^1_1, x^2_1)$ it exists local strictly decreasing best response functions. The main difficulty is however to characterize the global properties of these best response functions, and the consequences these properties may have on the existence of an equilibrium in pure strategies.

Theorem 1 below shows, under fairly general conditions on the demand function but under the assumption of a linear cost of production, that the two-periods Cournot game with inventories $\mathcal{G}$ has a pure strategy Nash-Perfect equilibrium. Why does the linearity of the cost of production matter for the existence of a pure strategy equilibrium ? Because even if the non-concavities in the second period payoffs $\Pi_2^i(x_1^i, x_1^j)$ are innocuous regarding the properties of the best response when x_1^i is the unique strategy of the firm in the first period[12], they create upward jumps in the choice of s_1^i when the firms have two strategies. Indeed if the choice of s_1^i cannot be separated from the choice of x_1^i, as it is the case with any cost of production other than linear[13], a downward jump in x_1^i decreases the cost of production and allow to increase brutally sales s_1^i, as appear in the first element of (17). As in any game where actions are strategic substitutes, this type of jump jeopardizes the existence of a pure strategy Nash equilibrium: the locus (s_1^j, x_1^j) at which the jump happens is evolving with the parameters of the model, making impossible to insure the existence of a pure strategy Nash equilibrium for all their possible values.

Theorem 1 *If the second order derivative of the inverse demand $P''(Q)$ is constant, and if the cost of production $C(q)$ is linear, $C'(q) = C'(0)\ q = c\ q$, then the game $\mathcal{G}$ has a pure strategy Nash-Perfect equilibrium.*

Proof. See appendix E.$\|$

The equilibrium in pure strategies exists presumably for not too convex costs of production: the upward jumps when choosing sales are limited. Note that multiple equilibria may arise: for some parameter values the asymmetric and symmetric issues may coexist simultaneously. In the next section I derive two corollaries from this theorem, then I deepen the analysis of the first period first order conditions to try derive some properties of the equilibrium when it exists in pure strategy.

[12]Mitraille [2003] shows in a version of $\mathcal{G}$ without any demand in the first period that it always exist a pure strategy Nash equilibrium. The issue of the game is either symmetric, or asymmetric, depending on second period demand, the cost of storage and the degree of convexity of the cost of production. In particular the two players may have a local influence on second period payoffs through their inventories.

[13]As appear in the expression of the Jacobian J^i, when the cost of production is linear, $C'(s_1^i + x_1^i) = C'(0) = c$, the two first order conditions can be treated separately since they depend only on one endogenous variable when considering the choice of the opponent as given.

4.2 Effect of the Business Cycle

The first corollary examines the case of a high second period demand.

Corollary 1 *If the cost of production is linear, and if the second period demand is high enough for some given discount factor and storage cost, then the game $\mathcal{G}$ has a unique symmetric Nash-Perfect equilibrium, in which both firms are not storing and selling the one-shot Cournot quantity in each period.*

From theorem 1 we know that it exists an equilibrium in pure strategy. Consider the asymmetric candidate x_1^L for firm 1. It solves

$$-c - H'(x_1^L) + \delta\,\frac{\partial \Pi_2^i}{\partial x_1^i}(x_1^L, 0) = \left| \begin{array}{l} 0 \text{ if } s_{f,f}^1 \leq x_1^L \leq s_{g,f}^1 \\ > 0 \text{ if } x_1^L = s_{g,f}^1 \\ < 0 \text{ if } x_1^L \leq s_{f,f}^1 \end{array} \right. \tag{20}$$

where $s_{f,f}^1 (= s_{f,f}^2)$ designates the Cournot symmetric outcome, which can be an equilibrium of the sub-game in the absence of the cost smoothing effect. The profit obtained by firm 1 when storing x_1^L has to be compared with the Cournot profit in second period obtained when not storing at all. Depending on the degree of convexity of the storage cost and on the discount factor, even if x_1^L is interior to the asymmetric region, a too high demand makes the exercise of some leadership too costly. On the other hand if the second period demand is low enough the cost of exerting the leadership is low enough to make the asymmetric choice x_1^L close to the Stackelberg quantity which maximizes the profit when the commitment is exogenous. The comparison between the Stackelberg-like and the Cournot profits will turn in favour of the asymmetric issue.

Corollary 2 *If the cost of production is linear, and if the second period demand is low enough, then the game $\mathcal{G}$ has a two asymmetric Nash-Perfect equilibria, in which one firm stores in first period to obtain the leadership on the second period market. Its competitor does not store.*

Together with corollaries 1 and 2, theorem 1 extends Arvan's theorem on the existence of a *symmetric* Nash equilibrium in this game (see Arvan (1985) theorem 1, p. 572): if, as stated in the assumptions sustaining Arvan's result, the second period profit is globally concave in inventories, then there are no discontinuities in the choice of inventories and therefore no upward jumps in the sales choice in first period. The game has naturally an equilibrium in pure strategy which is symmetric. However it is possible to find symmetric equilibria in pure strategy even in the game with non concave second period profit $\Pi_2^i(x_1^i, x_1^j)$. In the particular context of a linear cost the two corollaries to theorem 1 stated above illustrate that the driver of the symmetry of the equilibrium is not the concavity of $\Pi_2^i(x_1^i, x_1^j)$, but the importance of second period demand. To put it differently the concavity of $\Pi_2^i(x_1^i, x_1^j)$, which is of course sufficient to have the symmetry of the equilibrium, is not necessary. This analysis may be generalized to the case of a convex cost of production, keeping in mind that the pure strategy equilibrium may not exist. In that case one has to look at mixed strategy Nash equilibria[14].

Consider again the system (17). In a pure strategy Nash equilibrium (if it exists), s_1^i and x_1^i have to verify

$$P'(s_1^1 + s_1^2)\ s_1^i + P(s_1^1 + s_1^2, \mathbf{1}) - C'(s_1^i + x_1^i) \left| \begin{array}{l} = 0 \text{ if } s_1^i > 0 \\ < 0 \text{ if } s_1^i = 0 \end{array} \right. \tag{21}$$

and

$$-C'(s_1^i + x_1^i) - H'(x_1^i) + \delta\ \frac{\partial \Pi_2^i}{\partial x_1^i}(x_1^1, x_1^2) \left| \begin{array}{l} = 0 \text{ if } x_1^i > 0 \\ < 0 \text{ if } x_1^i = 0 \end{array} \right. \tag{22}$$

Some key features on the effect of differences in demands between periods may be established when the costs of production are convex. First, the higher second period benefit a player can hope is a profit resulting from a limited leadership. However by storing this player increases its first period cost of production and looses some market shares to the benefit of its opponent. Looking for exerting some second period leadership is equivalent to accept to suffer some first period "followership". If first period demand is high enough compared to second period demand, the marginal revenue from selling may be always higher than the marginal profit from storing in first period, and the two equations above cannot be equated. In that case no inventories will be constituted. It is consequently unlikely to observe the constitution of strategic inventories when the market is at the peak of an upturn. If on

[14]It presumably exists mixed strategy equilibria in which firms put some weight on a pair (high sales,low inventories) and some weight on a pair (low sales, high inventories). One has to check whether a symmetric pair may be part of the equilibrium strategy, but the analysis will have to be done in particular cases.

the contrary first period demand is low enough, two favourable situations for the constitution of inventories may appear. Either the market is reaching the bottom of a downturn, with a first period demand higher than second period demand, and one firm only will presumably exert some leadership (the other one does not store since smoothing costs does not matter when demand decreases), or the market demand start to increase from a very low first period level, and in that case a symmetric equilibrium may appear if the difference between demands is high enough. In this latter case both firms will enjoy a local influence on second period profits through their inventories.

5 Discussion and extensions

I have shown in a two periods model that one is more likely to observe in downturns the use of strategic inventories to allow one firm to exert some limited leadership. This result has been established by restricting the analysis to a separable problem across periods in order to insure the existence of a Nash equilibrium in pure strategies. When the two periods are not separable, the non-existence of a pure strategy equilibrium comes from the fact that firms have to constitute their inventories in one single period. Since only the cumulated inventories count when coming to the period at which they are released, a way to proceed could possibly be to allow firms to stockpile the good on several periods instead of one. This much more realistic setting would allow firms to separate the two problems - determining optimal sales and building up inventories in order to get the leadership on the market- by stockpiling small amounts in each period. In that case the cost of production would be close enough to the cost of sales in each period, but of course the cumulated inventories would be big enough to allow firms to get a larger market share, provided that the cost of storage is not too high. From the theoretical evidence drawn before, my conjecture is however that the search for a leader market share cannot happen in each period: under a convex cost of production firms will find profitable to stockpile by adding little quantities to their cumulated inventories in each period, in order not to loose too much in each period, and then release those quantities on the most favourable target market. I have shown in a two period game that the most profitable period to release inventories on a market is during a downturn in terms of the potential gain in market shares. In a longer dynamic, the same may be true: firms may build up small inventories at a low cost, those inventories having the higher marginal effect in periods of low demand. This conjecture seems consistent with the qualitative empirical evidence presented in introduction.

The use of strategic inventories in downturns pointed out in this paper may be connected to dumping practices: indeed storage consists in a capacity from which it is rational for firms to sell even at a price equal to 0. There is no cost of production suffered to release these units. Moreover depending on the market demand firms are committed to flood the market once inventories are constituted. The results established here are a possible explanation of why in several cases, Public Authorities involved in the control of International Trade have observed large inventories before downturns, and received in return complaints from firms obliged to reduce their sales in those period (see WTO (1999)). By increasing competition, storage lowers the price in downturns. Therefore if the firm who stores is better-off, it worsens the situation for the firm who does not store. Some producers may clearly be against the use of inventories, while others may be in favour. In the short run, the overall effect on the Social Welfare is positive, for the same reason that the Social Welfare in Stackelberg competition is higher than in Cournot competition. In the long run, if the consequence of asymmetric market shares is to provoke the exit of one producer from the market, and if for example the "customer base" effect locks buyers to their current suppliers, the effect on the Social Welfare may be negative, which is a strong rationale for Public Authorities to look carefully at strategic storage behaviour.

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Appendices

A. Figures

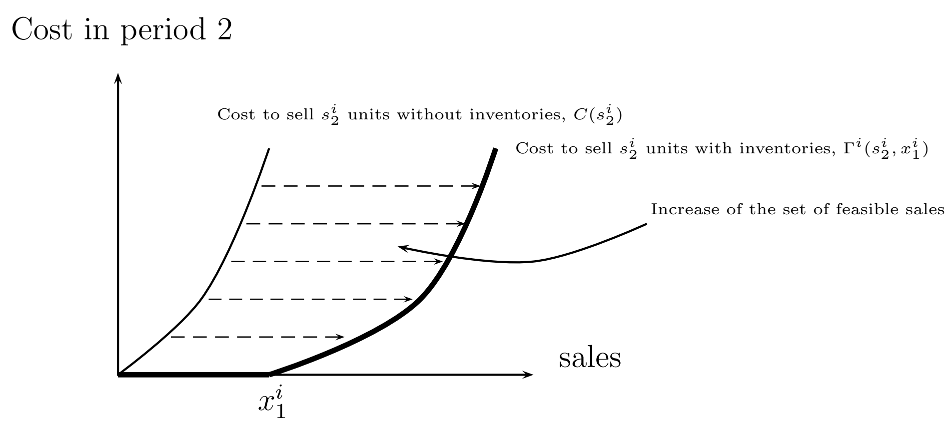

Figure 1: Storage and cost manipulation in the sales subgame

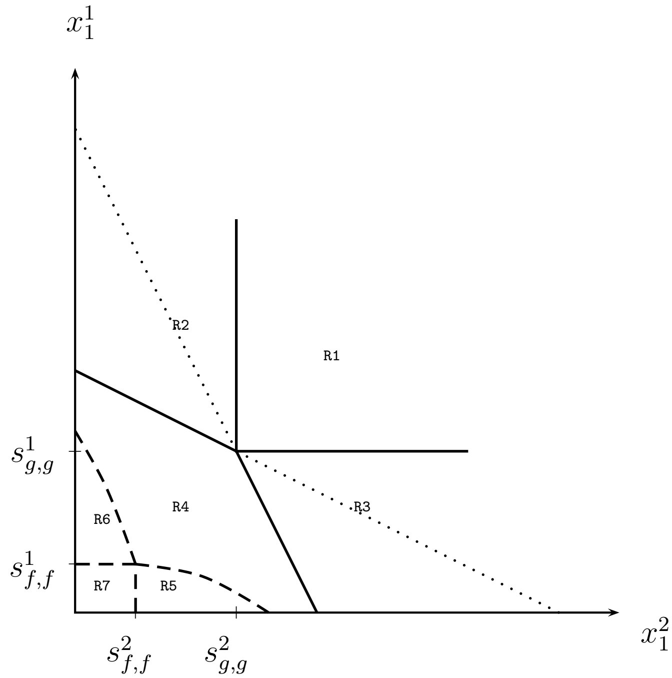

Figure 2: Subgame equilibrium when $P(g_2(0), \mathbf{2}) - C'(0) \leq 0$.

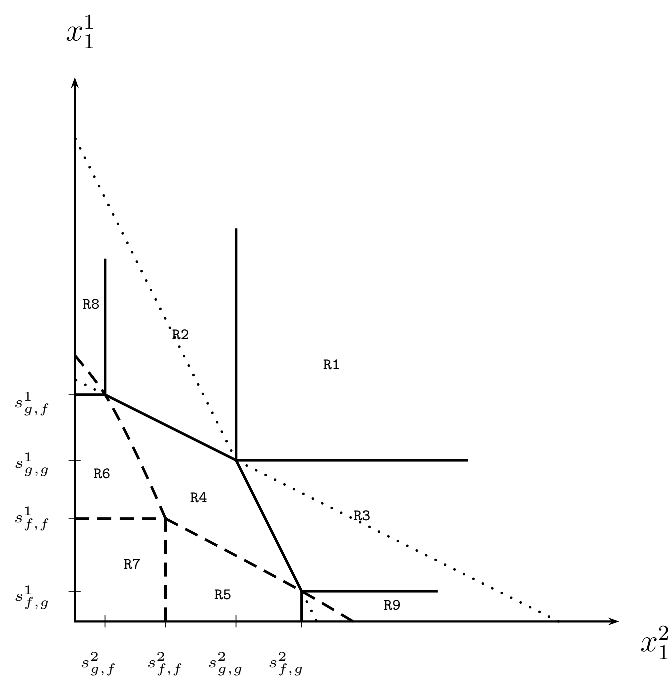

Figure 3: Subgame equilibrium when $P(g_2(0), \mathbf{2}) - C'(0) > 0$.

B. FOR REFEREES ONLY - Proofs of lemma 1

In the second period each firm chooses its sales by maximizing the second period revenue net of the total cost of production at this period,

$$\Pi_2^i(s_2^i, s_2^j) = P(s_2^1 + s_2^2, \mathbf{2})s_2^i - \Gamma^i(s_2^i, x_1^i) \tag{B.1}$$

where $\Gamma^i(s_2^i, x_1^i)$ is given by (11), for non-negative sales and non-negative inventories. This maximization problem depend on the inventories of firm i in the following way.

First of all remark that the gross revenue of each firm admits a unique maximum given the sales of the opponent, $g_2(s_2^j)$, such that

$$g_2(s_2^j) = \arg\max_{s_2^i} P(s_2^1+s_2^2, \mathbf{2})s_2^i \text{ , solution of } P'(g_2(s_2^j)+s_2^j)g_2(s_2^j)+P(g_2(s_2^j)+s_2^j, \mathbf{2}) = 0 \tag{B.2}$$

This first order condition depicted above is sufficient to determine the maximum $g_2(s_2^j)$, since the gross revenue is strictly concave,

$$P''(g_2(s_2^j) + s_2^j)g_2(s_2^j) + 2P'(g_2(s_2^j) + s_2^j) < 0 \tag{B.3}$$

and as long as $s_2^j \leq \overline{Q}_2$, the solution is interior, $g_2(s_2^j) > 0$. In the remaining we omit the last part of this piece of the reaction function, namely $g_2(s_2^j) = 0$ for $s_2^j \geq \overline{Q}_2$ since it deals with a strictly dominated strategy for firm j, i.e. selling more than the maximal quantity consumers are ready to buy, at a null price. Let us construct the best response $B_2^i(s_2^j)$ of a firm.

The first case for the inventories x_1^i is when they exceed the quantity that firm i will sell to maximize its profit when alone on the market. Reacting to s_2^j according to $g_2(s_2^j)$ is obviously the best choice. Indeed the function $g_2(s_2^j)$ is strictly decreasing with s_2^j: by applying the implicit function theorem to the first order condition above, it comes

$$\frac{\partial g_2}{\partial s_2^j}(s_2^j) = -\frac{P''(s_2^i + s_2^j)s_2^i + P'(s_2^i + s_2^j)}{P''(s_2^i + s_2^j)s_2^i + 2P'(s_2^i + s_2^j)} < 0 \tag{B.4}$$

This slope is strictly lower than one in absolute value. The higher firm j's sales are, the lower firm i's sales will be: whenever the inventories x_1^i are higher than $g_2(0)$, firm i will react according to $g_2(s_2^j)$. Since $g_2(s_2^j)$ does not depend on x_1^i, note that the condition $x_1^i \geq g_2(0)$ is strictly equivalent to $P'(x_1^i)x_1^i + P(x_1^i, \mathbf{2}) \leq 0$: as soon as the inventories are such that the marginal revenue of exhausting x_1^i when alone on the market is negative, firm i will always prefer to maximize its gross revenue and sell less than x_1^i. Therefore

$$B_2^i(s_2^j) = g_2(s_2^j) \text{ if } P'(x_1^i)x_1^i + P(x_1^i, \mathbf{2}) \leq 0 \tag{B.5}$$

Remark now that when producing again in the second period, firm i will sell on the market according to the function $f_2^i(s_2^j, x_1^i)$, which is firm-specific since it depends on x_1^i. This function verifies

$$f_2^i(s_2^j, x_1^i) = \arg\max_{s_2^i} P(s_2^1 + s_2^2, \mathbf{2})s_2^i - C(s_2^i - x_1^i) \tag{B.6}$$

solution of

$$P'(f_2^i(s_2^j, x_1^i) + s_2^j)f_2^i(s_2^j, x_1^i) + P(f_2^i(s_2^j, x_1^i) + s_2^j, \mathbf{2}) - C(f_2^i(s_2^j, x_1^i) - x_1^i) = 0 \tag{B.7}$$

and is again strictly decreasing with s_2^j, from the same argument than before

$$\frac{\partial f^i}{\partial s_2^j}(s_2^j) = -\frac{P''(s_2^i + s_2^j)s_2^i + P'(s_2^i + s_2^j)}{P''(s_2^i + s_2^j)s_2^i + 2P'(s_2^i + s_2^j) - C''(s_2^i - x_1^i)} < 0 \tag{B.8}$$

Note that this solution is unique when it exists, and that the slope of this function is strictly lower than one in absolute value.

Consider now the case where $P'(x_1^i)x_1^i + P(x_1^i, \mathbf{2}) > 0$. Selling exactly x_1^i is therefore a possible solution for some values of s_2^j. However if $f_2^i(0, x_1^i) \leq x_1^i$, then selling more than its inventories will never be a solution for firm i for any s_2^j. Note that this condition is equivalent to $P'(x_1^i)x_1^i + P(x_1^i, \mathbf{2}) - C'(0) \leq 0$: as soon as the net marginal profit of selling one unit in addition to the inventories is positive, firm i will produce in the second period when alone on the market. In the case where $P'(x_1^i)x_1^i + P(x_1^i, \mathbf{2}) > 0$ and $P'(x_1^i)x_1^i + P(x_1^i, \mathbf{2}) - C'(0) \leq 0$, firm i will sell according to $g_2(s_2^j)$ as long as s_2^j is such that $g_2(s_2^j) < x_1^i$, i.e. $P'(x_1^i + s_2^j)x_1^i + P(x_1^i + s_2^j, \mathbf{2}) \leq 0$. When $g_2(s_2^j) \geq x_1^i$, i.e. when $P'(x_1^i + s_2^j)x_1^i + P(x_1^i + s_2^j, \mathbf{2}) \geq 0$, firm i would be willing to sell more than its inventories since the marginal profit of selling exactly x_1^i is strictly positive. However since $P'(x_1^i)x_1^i + P(x_1^i, \mathbf{2}) - C'(0) \leq 0$, producing again is not profitable at the margin: therefore when s_2^j is such that $P'(x_1^i + s_2^j)x_1^i + P(x_1^i + s_2^j, \mathbf{2}) \geq 0$, firm i will sell exactly x_1^i. To conclude, if $P'(x_1^i)x_1^i + P(x_1^i, \mathbf{2}) > 0$ and $P'(x_1^i)x_1^i + P(x_1^i, \mathbf{2}) - C'(0) \leq 0$

$$B_2^i(s_2^j) = \left| \begin{array}{lll} x_1^i & \text{if } g_2(s_2^j) \geq x_1^i & \text{i.e. } P'(x_1^i + s_2^j)x_1^i + P(x_1^i + s_2^j, \mathbf{2}) \geq 0 \\ g_2(s_2^j) & \text{if } g_2(s_2^j) < x_1^i & \text{i.e. } P'(x_1^i + s_2^j)x_1^i + P(x_1^i + s_2^j, \mathbf{2}) < 0 \end{array} \right. \tag{B.9}$$

Let us now consider the last case, $P'(x_1^i)x_1^i + P(x_1^i, \mathbf{2}) - C'(0) > 0$. We have seen that firm i could find profitable to produce again to sell more than its inventories. The decision will again be simply based on the sales of its opponent s_2^j. If $g_2(s_2^j) < x_1^i$, then firm i will sell according to $g_2(s_2^j)$. If s_2^j is such that $g_2(s_2^j) \geq x_1^i$ and $f_2^i(s_2^j, x_1^i) < x_1^i$, then firm i will sell exactly x_1^i, and finally if s_2^j verifies $f_2^i(s_2^j, x_1^i) \geq x_1^i$, then i sells $f_2^i(s_2^j, x_1^i)$. Indeed in this last case, the profit is strictly increasing as long as $s_2^i \leq x_1^i$, since by construction $g_2(s_2^j) > f_2^i(s_2^j, x_1^i)$. It is also still increasing until s_2^i reaches $f_2^i(s_2^j, x_1^i)$ by definition of this function, and decreasing after. To summarize, if $P'(x_1^i)x_1^i + P(x_1^i, \mathbf{2}) - C'(0) > 0$

$$B_2^i(s_2^j) = \left| \begin{array}{lll} f_2^i(s_2^j, x_1^i) & \text{if } f_2^i(s_2^j, x_1^i) \geq x_1^i & \text{i.e. } P'(x_1^i + s_2^j)x_1^i + P(x_1^i + s_2^j, \mathbf{2}) - C'(0) > 0 \\ x_1^i & \text{if } f_2^i(s_2^j, x_1^i) < x_1^i & \text{i.e. } P'(x_1^i + s_2^j)x_1^i + P(x_1^i + s_2^j, \mathbf{2}) - C'(0) \leq 0 \\ & \text{and } g_2(s_2^j) \geq x_1^i & \text{and } P'(x_1^i + s_2^j)x_1^i + P(x_1^i + s_2^j, \mathbf{2}) \geq 0 \\ g_2(s_2^j) & \text{if } g_2(s_2^j) < x_1^i & \text{i.e. } P'(x_1^i + s_2^j)x_1^i + P(x_1^i + s_2^j, \mathbf{2}) < 0 \end{array} \right. \tag{B.10}$$

Again this best response is decreasing but continuous.

As shown before, the slope of the reaction function $B_2^i(s_2^j)$ is always strictly lower than 1 under our assumptions on the cost and the inverse demand functions. Consequently the Cournot equilibrium will be unique for every pair of inventories and will always exist. The remaining part of this proof discuss the type of equilibrium that arises depending on the inventories. We show that the second period sales subgame will have at least 7 and at most 9 intersections, depending on the parameters of the model and in particular on the size of the market of that period.

The first and obvious case is when both inventories are large enough to rule out any production in the second period. In that case the equilibrium is

$$(s_2^1(x_2^1, x_2^2), s_2^2(x_2^1, x_2^2)) = (s_{g,g}^1, s_{g,g}^2) \text{ if } P'(x_1^1)x_1^1 + P(x_1^1, \mathbf{2}) \leq 0 \text{ and } P'(x_1^2)x_1^2 + P(x_1^2, \mathbf{2}) \leq 0$$

where $(s_{g,g}^1, s_{g,g}^2)$ is the intersection of $g_2(s_2^1)$ and $g_2(s_2^2)$, and solves the system

$$\begin{cases} P'(s_{g,g}^1 + s_{g,g}^2)s_{g,g}^1 + P(s_{g,g}^1 + s_{g,g}^2, \mathbf{2}) = 0 \\ P'(s_{g,g}^1 + s_{g,g}^2)s_{g,g}^2 + P(s_{g,g}^1 + s_{g,g}^2, \mathbf{2}) = 0 \end{cases}$$

Note that the quantities sold in equilibrium are not a functional form of the inventories, but are only a function of the market demand. The dependance to the inventories comes from the region of the plan (x_1^1, x_1^2) in which this equilibrium is valid.

Consider now the case $P'(x_1^1)x_1^1 + P(x_1^1, \mathbf{2}) \leq 0$, $P'(x_1^2)x_1^2 + P(x_1^2, \mathbf{2}) > 0$ and $P'(x_1^2)x_1^2 + P(x_1^2, \mathbf{2}) - C'(0) \leq 0$. The equilibrium will be either $(s_{g,g}^1, s_{g,g}^2)$ or $(s_{g,x}^1, x_1^2)$, depending on the inventories of both firms. Indeed if x_1^2 is such that the marginal revenue of selling the entire inventories while the opponent reacts according to $g_2(s_2^j)$ is negative, then firm 2 will reduce its sales: if $P'(g_2(x_1^2) + x_1^2)x_1^2 + P(g_2(x_1^2) + x_1^2, \mathbf{2}) \leq 0$ the equilibrium is $(s_{g,g}^1, s_{g,g}^2)$. On the other hand if $P'(g_2(x_1^2) + x_1^2)x_1^2 + P(g_2(x_1^2) + x_1^2, \mathbf{2}) > 0$, firm 2 could still realize a positive profit by selling more than its inventories, but since it has to produce again, it suffers an extra cost. We are considering the case where $P'(x_1^2)x_1^2 + P(x_1^2, \mathbf{2}) - C'(0) \leq 0$, therefore the marginal benefit of producing again is negative when firm 1 is selling $g_2(x_1^2)$. Firm 2 will stick to sell x_1^2 and firm 1 will react using $g_2(x_1^2)$. To summarize

$$(s_2^1(x_2^1, x_2^2), s_2^2(x_2^1, x_2^2)) = \left| \begin{array}{ll} (s_{g,g}^1, s_{g,g}^2) & \text{if } P'(x_1^1)x_1^1 + P(x_1^1, \mathbf{2}) \leq 0 \\ & \text{and } P'(g(x_1^2) + x_1^2)x_1^2 + P(g(x_1^2) + x_1^2, \mathbf{2}) \leq 0 \\ (s_{g,x}^1, x_1^2) & \text{if } P'(x_1^1)x_1^1 + P(x_1^1, \mathbf{2}) \leq 0 \\ & \text{and } P'(g(x_1^2) + x_1^2)x_1^2 + P(g(x_1^2) + x_1^2, \mathbf{2}) > 0 \end{array} \right.$$

where $(s_{g,x}^1, x_1^2)$ solves $P'(s_{g,x}^1 + x_1^2)s_{g,x}^1 + P(s_{g,x}^1 + x_1^2, \mathbf{2}) = 0$. Note that the condition $P'(g(x_1^2) + x_1^2)x_1^2 + P(g(x_1^2) + x_1^2, \mathbf{2}) \leq 0$ is equivalent to $x_1^2 \geq g_2(g_2^1(x_1^2))$. We may establish the symmetric case

$$(s_2^1(x_2^1, x_2^2), s_2^2(x_2^1, x_2^2)) = \left| \begin{array}{ll} (s_{g,g}^1, s_{g,g}^2) & \text{if } P'(x_1^1 + g_2(x_1^1))x_1^1 + P(x_1^1 + g_2(x_1^1), \mathbf{2}) \leq 0 \\ & \text{and } P'(x_1^2)x_1^2 + P(x_1^2, \mathbf{2}) \leq 0 \\ (x_1^1, s_{x,g}^2) & \text{if } P'(x_1^1 + g_2(x_1^1))x_1^1 + P(x_1^1 + g_2(x_1^1), \mathbf{2}) > 0 \\ & \text{and } P'(x_1^2)x_1^2 + P(x_1^2, \mathbf{2}) \leq 0 \end{array} \right.$$

where $(x_1^1, s_{x,g}^2)$ solves $P'(x_1^1 + s_{x,g}^2)s_{x,g}^2 + P(x_1^1 + s_{x,g}^2, \mathbf{2}) = 0$.

In the case $P'(x_1^1)x_1^1 + P(x_1^1, \mathbf{2}) \leq 0$, $P'(x_1^2)x_1^2 + P(x_1^2, \mathbf{2}) - C'(0) > 0$, 3 equilibria may appear. Again if $P'(g_2(x_1^2) + x_1^2)x_1^2 + P(g_2(x_1^2) + x_1^2, \mathbf{2}) \leq 0$, firm 2 will reduce its sales and the equilibrium will be $(s_{g,g}^1, s_{g,g}^2)$. If $P'(g_2(x_1^2) + x_1^2)x_1^2 + P(g_2(x_1^2) + x_1^2, \mathbf{2}) > 0$, the equilibrium will be either $(s_{g,x}^1, x_1^2)$ or $(s_{g,f}^1, s_{g,f}^2)$, depending on the sign of

$P'(g_2(x_1^2)+x_1^2)x_1^2 + P(g_2(x_1^2)+x_1^2,\mathbf{2}) - C'(0)$. First remark that if

$$P'(g_2(x_1^2)+x_1^2)x_1^2 + P(g_2(x_1^2)+x_1^2,\mathbf{2}) - C'(0)\Bigg|_{x_1^2=0} < 0$$

then the expression will always be negative for any x_1^2 since it is decreasing with x_1^2: indeed the derivative with respect to x_1^2 is

$$(1+\frac{dg_2}{dx_1^2})P''(\cdot)x_1^2 + P'(\cdot) + (1+\frac{dg_2}{dx_1^2})P'(\cdot)$$

which is negative under our assumptions. The condition above reduces to $P(g_2(0),\mathbf{2})-C'(0)<0$, where $g_2(0)$ solves $P'(g_2(0))g_2(0)+P(g_2(0),\mathbf{2})=0$. Since they are identical, this condition is valid for both firms simultaneously. Therefore we have to distinguish two cases: either $P(g_2(0),\mathbf{2})-C'(0)<0$, in which case $g_2(s_2^i)$ and $f_2^j(s_2^i,x_1^i)$ cannot have an intersection, and $P(g_2(0),\mathbf{2})-C'(0)\geq 0$, in which case $g_2(s_2^i)$ and $f_2^j(s_2^i,x_1^i)$ may have an intersection for some pair of inventories. Before detailing the equilibria, let us first interpret the condition. $P(g_2(0),\mathbf{2})-C'(0)<0$ states that when the opponent is selling from his inventories, there is no point in producing to put him with redundant inventories, since even if he behaves as a monopolist, the price falls below the marginal cost $C'(0)$. It does not imply however that firms are not able to sell anything: indeed $P_2(f_2^i(0,0))$ is still higher than $C'(0)$.

When $P(g_2(0),\mathbf{2})-C'(0)<0$, only two equilibria may appear: $(s_{g,g}^1,s_{g,g}^2)$ or $(s_{g,x}^1,x_1^2)$, with the same conditions than before. When $P(g_2(0),\mathbf{2})-C'(0)\geq 0$, $(s_{g,f}^1,s_{g,f}^2)$ may appear if $x_1^2 > f_2^2(g_2(x_1^2),x_1^2)$, which is equivalent to $P'(g_2(x_1^2)+x_1^2)x_1^2 + P(g_2(x_1^2)+x_1^2,\mathbf{2}) - C'(0) > 0$. To summarize, if the parameters are such that $P(g_2(0),\mathbf{2})-C'(0)\geq 0$, then

$$(s_2^1(x_2^1,x_2^2),s_2^2(x_2^1,x_2^2)) = \left|\begin{array}{ll} (s_{g,g}^1,s_{g,g}^2) & \text{if } P'(x_1^1)x_1^1+P(x_1^1,\mathbf{2})\leq 0 \\ & \text{and } P'(g(x_1^2)+x_1^2)x_1^2+P(g(x_1^2)+x_1^2,\mathbf{2})\leq 0 \\ (s_{g,x}^1,x_1^2) & \text{if } P'(x_1^1)x_1^1+P(x_1^1,\mathbf{2})\leq 0, \\ & P'(g(x_1^2)+x_1^2)x_1^2+P(g(x_1^2)+x_1^2,\mathbf{2})>0 \\ & \text{and } P'(g(x_1^2)+x_1^2)x_1^2+P(g(x_1^2)+x_1^2,\mathbf{2})-C'(0)\leq 0 \\ (s_{g,f}^1,s_{g,f}^2) & \text{if } P'(x_1^1)x_1^1+P(x_1^1,\mathbf{2})\leq 0 \\ & \text{and } P'(g(x_1^2)+x_1^2)x_1^2+P(g(x_1^2)+x_1^2,\mathbf{2})-C'(0)>0 \end{array}\right.$$

The symmetric case $P'(x_1^1)x_1^1+P(x_1^1,\mathbf{2})-C'(0)>0$, $P'(x_1^2)x_1^2+P(x_1^2,\mathbf{2})\leq 0$ may be established immediately.

In the case $P'(x_1^1)x_1^1+P(x_1^1,\mathbf{2}) > 0$, $P'(x_1^1)x_1^1+P(x_1^1)-C'(0) \leq 0$, and $P'(x_1^2)x_1^2+P(x_1^2,\mathbf{2}) > 0$, $P'(x_1^2)x_1^2+P(x_1^2,\mathbf{2})-C'(0) \leq 0$, both reaction functions have two parts. As soon as $x_1^i < s_{g,g}^i$ for any agent, $(s_{g,g}^1, s_{g,g}^2)$ cannot be an equilibrium anymore. Therefore $(s_{g,g}^1, s_{g,g}^2)$ is an equilibrium if $x_1^1 > s_{g,g}^1$ and $x_1^1 > s_{g,g}^1$. Consider now that $x_1^1 < s_{g,g}^1$. Then the equilibrium will be either $(x_1^1, g_2(x_1^1))$ or (x_1^1, x_1^2). Obviously $(x_1^1, g_2(x_1^1))$ is an equilibrium if the marginal revenue of selling the entire inventories of firm 2 is negative, i.e. $P'(x_1^1+x_1^2)x_1^2+P(x_1^1+x_1^2,\mathbf{2}) \leq 0$, equivalent to $g_2(x_1^1) < x_1^2$. On the contrary, if $P'(x_1^1+x_1^2)x_1^2+P(x_1^1+x_1^2,\mathbf{2}) > 0$ and knowing that producing anything to be sold in addition of the inventories leads to a negative marginal profit, firm 2 will sell exactly its inventories. To summarize,

$$(s_2^1(x_2^1,x_2^2), s_2^2(x_2^1,x_2^2)) = \left| \begin{array}{ll} (s_{g,g}^1, s_{g,g}^2) & \text{if } x_1^1 > s_{g,g}^1,\ x_1^2 > s_{g,g}^2 \\ (s_{g,x}^1, x_1^2) & \text{if } P'(x_1^1+x_1^2)x_1^1 + P(x_1^1+x_1^2,\mathbf{2}) \leq 0, \\ & \text{and } x_1^2 \leq s_{g,g}^2 \\ (x_1^1, s_{x,g}^2) & \text{if } x_1^1 \leq s_{g,g}^1 \\ & \text{and } P'(x_1^2+x_1^2)x_1^2 + P(x_1^2+x_1^2,\mathbf{2}) \leq 0 \\ (x_1^1, x_1^2) & \text{if } P'(x_1^2+x_1^2)x_1^1 + P(x_1^2+x_1^2,\mathbf{2}) \leq 0 \\ & \text{and } P'(x_1^2+x_1^2)x_1^2 + P(x_1^2+x_1^2,\mathbf{2}) \leq 0 \end{array} \right.$$

In the case where $P'(x_1^1)x_1^1 + P(x_1^1,\mathbf{2}) > 0$, $P'(x_1^1)x_1^1 + P(x_1^1) - C'(0) \leq 0$, and $P'(x_1^2)x_1^2 + P(x_1^2,\mathbf{2}) - C'(0) > 0$, we have again to consider two cases separately. First if $P(g_2(0),\mathbf{2}) - C'(0) < 0$, $g_2(s_2^2)$ and $f_2^2(s_2^1, x_1^2)$ will never cross. In addition to the equilibria described before, $(x_1^1, s_{x,f}^2)$ may also appear. Let us investigate the conditions needed. If $x_1^1 > s_{g,g}^1$ and $x_1^2 > s_{g,g}^2$ then again the equilibrium is $(s_{g,g}^1, s_{g,g}^2)$.

If $x_1^1 < s_{g,g}^1$ then 3 cases arise. If $P'(x_1^2+x_1^2)x_1^2 + P(x_1^2+x_1^2,\mathbf{2}) \leq 0$ then firm 2 will not sell the whole inventories and the equilibrium is $(x_1^1, s_{x,g}^2)$. If $P'(x_1^2+x_1^2)x_1^2 + P(x_1^2+x_1^2,\mathbf{2}) > 0$ and $P'(x_1^2+x_1^2)x_1^2 + P(x_1^2+x_1^2,\mathbf{2}) - C'(0) < 0$, firm 2 will stick to sell exactly its inventories: the equilibrium is (x_1^1, x_1^2). Finally if $P'(x_1^2+x_1^2)x_1^2 + P(x_1^2+x_1^2,\mathbf{2}) - C'(0) \geq 0$, firm 2 finds profitable to produce again and the equilibrium is $(x_1^1, s_{x,f}^2)$.

If $x_1^1 \geq s_{g,g}^1$ and $x_1^2 < s_{g,g}^2$ then the equilibrium is $(s_{g,x}^1, x_1^2)$ if $P'(x_1^2+x_1^2)x_1^1 + P(x_1^2+x_1^2,\mathbf{2}) \leq 0$, (x_1^1, x_1^2) if $P'(x_1^2+x_1^2)x_1^1 + P(x_1^2+x_1^2,\mathbf{2}) > 0$ and $P'(x_1^2+x_1^2)x_1^1 + P(x_1^2+x_1^2,\mathbf{2}) - C'(0) \leq 0$, and $(x_1^1, s_{x,f}^2)$ if $P'(x_1^2+x_1^2)x_1^1 + P(x_1^2+x_1^2,\mathbf{2}) - C'(0) > 0$.

Finally if $x_1^1 < s_{g,g}^1$ and $x_1^2 < s_{g,g}^2$, the equilibrium is (x_1^1, x_1^2) if $P'(x_1^2+x_1^2)x_1^2 + P(x_1^2+x_1^2,\mathbf{2}) - C'(0) \leq 0$, and $(x_1^1, s_{x,f}^2)$ if $P'(x_1^2+x_1^2)x_1^2 + P(x_1^2+x_1^2,\mathbf{2}) - C'(0) > 0$.

Second if $P(g_2(0), \mathbf{2}) - C'(0) \geq 0$, $g_2(s_2^2)$ and $f_2^2(s_2^1, x_1^2)$ have an intersection. Again, the equilibrium is the same than in the first case when $\{x_1^1 \geq s_{g,g}^1, x_1^2 \geq s_{g,g}^2\}$, when $\{x_1^1 < s_{g,g}^1, x_1^2 \geq s_{g,g}^2\}$, and when $\{x_1^1 < s_{g,g}^1, x_1^2 < s_{g,g}^2\}$. The difference appears when $\{x_1^1 \geq s_{g,g}^1, x_1^2 < s_{g,g}^2\}$. Indeed if $x_1^1 \geq s_{g,g}^1$ and $x_1^2 < s_{g,g}^2$ then the equilibrium is $(s_{g,x}^1, x_1^2)$ if $P'(x_1^2+x_1^2)x_1^1+P(x_1^2+x_1^2, \mathbf{2}) \leq 0$, (x_1^1, x_1^2) if $P'(x_1^2+x_1^2)x_1^1+P(x_1^2+x_1^2, \mathbf{2}) > 0$, $(x_1^1, s_{x,f}^2)$ if $P'(x_1^2+x_1^2)x_1^2 + P(x_1^2+x_1^2, \mathbf{2}) - c'(0) > 0$, and $(s_{g,f}^1, s_{g,f}^2)$ if.

Let us consider the last case, $P'(x_1^1)x_1^1 + P(x_1^1, \mathbf{2}) - C'(0) > 0$ and $P'(x_1^2)x_1^2 + P(x_1^2, \mathbf{2}) - C'(0) > 0$, where again we have to distinguish $P(g_2(0), \mathbf{2}) - C'(0) < 0$ from $P(g_2(0), \mathbf{2}) - C'(0) \geq 0$.

When $P(g_2(0), \mathbf{2}) - C'(0) < 0$, $g_2(s_2^i)$ and $f_2^j(s_2^i, x_1^i)$ do not cross. Therefore there will be 7 possible equilibria depending on the pair of inventories. In addition to the equilibria described before, $(s_{f,f}^1, s_{f,f}^2)$, $(s_{f,x}^1, x_1^2)$, $(x_1^1, s_{x,f}^2)$ are possible equilibria.

If simultaneously $P'(x_1^1 + f_2^2(x_1^1, x_1^2))x_1^1 + P(x_1^1 + f_2^2(x_1^1, x_1^2), \mathbf{2}) - C'(0) > 0$ and $P'(f_2^1(x_1^2, x_1^1) + x_1^2)x_1^2 + P(f_2^1(x_1^2, x_1^1) + x_1^2, \mathbf{2}) - C'(0) > 0$ then both firms will find profitable to increase their sales. They will sell up to $(s_{f,f}^1, s_{f,f}^2)$. Note that the condition is equivalent to $x_1^1 < s_{f,f}^1$ and $x_1^2 < s_{f,f}^2$, since the pair $(s_{f,f}^1, s_{f,f}^2)$ is the unique intersection of $f_2^1(s_2^2, x_1^1)$ and $f_2^2(s_2^1, x_1^2)$.

If $P'(x_1^1+f_2^2(x_1^1, x_1^2))x_1^1+P(x_1^1+f_2^2(x_1^1, x_1^2), \mathbf{2})-C'(0) \leq 0$ and $P'(x_1^1+f_2^2(x_1^1, x_1^2))x_1^1+P(x_1^1 + f_2^2(x_1^1, x_1^2), \mathbf{2}) > 0$, then firm 1 will find profitable to sell exactly its inventories. Depending on firm 2, the equilibrium will be either $(x_1^1, s_{x,f}^2)$ if $P'(x_1^1+x_1^2)x_1^2 + P(x_1^1 + x_1^2, \mathbf{2}) - C'(0) > 0$ or (x_1^1, x_1^2) if $P'(x_1^1 + x_1^2)x_1^2 + P(x_1^1 + x_1^2, \mathbf{2}) - C'(0) \leq 0$ and $P'(x_1^1 + x_1^2)x_1^2 + P(x_1^1 + x_1^2, \mathbf{2}) > 0$. This condition is equivalent to $x_1^2 < s_{x,f}^2$.

When $P(g_2(0), \mathbf{2}) - C'(0) > 0$, $g_2(s_2^i)$ and $f_2^j(s_2^i, x_1^i)$ do cross. Therefore there will be 9 possible equilibria depending on the pair of inventories, the 7 described before, $(s_{f,g}^1, s_{f,g}^2)$ and $(s_{g,f}^1, s_{g,f}^2)$. They arise when one firm does not find profitable to sell exactly its inventories any more when its opponent increases its sales. Indeed if $P'(x_1^1+f_2^2(x_1^1, x_1^2))x_1^1+P(x_1^1+f_2^2(x_1^1, x_1^2), \mathbf{2}) < 0$ and $P'(g_2(x_1^2)+x_1^2)x_1^2+P(g_2(x_1^2)+x_1^2, \mathbf{2}) - C'(0) > 0$, firm 2 will place firm 1 with redundant inventories by increasing its sales. The equilibrium in that case is $(s_{g,f}^1, s_{g,f}^2)$. Note that the conditions are equivalent to $x_1^1 > s_{g,f}^1$ and $x_1^2 < s_{g,f}^2$.

C. FOR REFEREES ONLY - Proof of proposition 1

Since producing is costly, we may rule out from the set of best responses (and consequently from the set of plausible equilibria) the pair of inventories such that at least one firm sells less than its inventories (regions R1, R2, R3, R8 and R9): no firm deviates unilaterally to these regions. Firms never choose simultaneously to store and sell exactly their inventories (region R4). Indeed the second period profit in that case is equal to the total revenue of selling (x_1^1, x_1^2), i.e. $\delta\ P(x_1^1 + x_1^2, \mathbf{2})\ x_1^i - H(x_1^i) - C(s_1^i + x_1^i)$. In first period, the optimization of the total profit induces a choice x_1^i solution of $\delta(P'(x_1^1+x_1^2)x_1^i + P(x_1^1+x_1^2, \mathbf{2})) - H'(x_1^i) - C'(s_1^i + x_1^i) = 0$ for any given s_1^i, which is strictly lower than the frontier of this region, $P'(x_1^1 + x_1^2)\ x_1^i + P(x_1^1 + x_1^2, \mathbf{2}) - C'(0) = 0$. Therefore the profit in this region is strictly higher on the lower frontier $\{x_1^1 = s_{f,x}^1, x_1^2 = s_{x,f}^2\}$ than everywhere else inside the region. Consequently we may rule out regions R4 from the set of plausible equilibria.

D. FOR REFEREES ONLY - Proofs of lemmas 2 and 3

When firm 1 sells more than her inventories and firm 2 sells exactly her inventories (regions R5 in lemma 1 and R7 in lemma ??), the marginal effect on second period sales of firm 1 are

$$\frac{\partial s_{f,x}^1}{\partial x_1^1} = -\frac{C''(s_{f,x}^1 - x_1^1)}{P''(s_{f,x}^1 + x_1^2)\ s_{f,x}^1 + 2P'(s_{f,x}^1 + x_1^2) - C''(s_{f,x}^1 - x_1^1)} > 0$$

and

$$\frac{\partial s_{f,x}^1}{\partial x_1^2} = -\frac{P''(s_{f,x}^1 + x_1^2)\ s_{f,x}^1 + P'(s_{f,x}^1 + x_1^2)}{P''(s_{f,x}^1 + x_1^2)\ s_{f,x}^1 + 2P'(s_{f,x}^1 + x_1^2) - C''(s_{f,x}^1 - x_1^1)} < 0$$

The symmetric effects are valid when firm 2 sells more than its inventories and firm 1 sells exactly its inventories (regions R6 in lemma 1 and R8 in lemma ??.) The effect of an increase of period 1 inventories of firm 1 on its own equilibrium second period sales may be interpreted as follows. When keeping part of its production in inventories, firm 1 splits the cost of producing $s_{f,x}^1$ between two periods. Since its cost of production is convex, producing in two periods instead of one reduces the marginal cost tomorrow and allows a Cournot competitor to be more aggressive on the market[15]. This "cost smoothing" turns however to be strategic, contrary to what it would be in perfect competition. The second equation shows that an increase in

[15] To say it differently, it allows to replicate the production tool.

the sales of the opponent decreases firm i sales: crucially, this effect does not rely on the convexity of the cost function but on the "commitment effect" allowed by inventories. If we allow for the second order derivative of the cost function to be equal to 0, increasing its own inventories has no effect on firm 1 second period sales, but an increase in firm 2's inventories reduces firm 1 sales.

When both firms sell strictly more than their inventories (regions R7 in lemma 1 and R9 in lemma ??), the effect of an increase in firm i and firm j inventories on firm i sales are given by

$$\frac{\partial s^i_{f,f}}{\partial x^i_1} = -\frac{C''(s^i_{f,f}-x^i_1)\ (P''(s^1_{f,f}+s^2_{f,f})s^j_{f,f}+2P'(s^1_{f,f}+s^2_{f,f})-C''(s^j_{f,f}-x^j_1))}{\Delta} > 0$$

and

$$\frac{\partial s^i_{f,f}}{\partial x^j_1} = \frac{C''(s^j_{f,f}-x^j_1)(P''(s^1_{f,f}+s^2_{f,f})s^i_{f,f}+P'(s^1_{f,f}+s^2_{f,f}))}{\Delta} < 0$$

where

$$\Delta = P'P''(s^1_{f,f}+s^2_{f,f})-P''s^1_{f,f}C''_2+3(P')^2-2P'C''_2-P''s^2_{f,f}C''_1-2P'C''_1+C''_1C''_2 > 0,$$

with $P'=P'(s^1_{f,f}+s^2_{f,f})$, $P''=P''(s^1_{f,f}+s^2_{f,f})$, and $C''_i=C''(s^i_{f,f}-x^i_1)$

When their inventories allow them to produce again in second period, both firms use it to benefit from a "cost smoothing" effect. The effect of inventories on their sales is again directly proportional to the convexity of the cost function, and disappear when $C''(q)$ is always nil. Increasing its inventories allows one firm to increase its sales, and on the other hand an increase in its opponent's inventories reduces its sales. The integration of those effects into the second period profit to measure the marginal effect of inventories is straightforward.

E. FOR REFEREES ONLY - Proof of theorem 1

Under the assumption that the cost of production is linear, the two first order conditions defined in J^i can be treated separately for the two players. The first period equilibrium is obviously symmetric regarding sales, but can be asymmetric regarding production. (s^{1*}_1, s^{2*}_1) satisfy

$$\begin{cases} P'(s^{1*}_1+s^{2*}_1)\ s^{1*}_1+P(s^{1*}_1+s^{2*}_1,\mathbf{1})-c=0 \\ P'(s^{1*}_1+s^{2*}_1)\ s^{2*}_1+P(s^{1*}_1+s^{2*}_1,\mathbf{1})-c=0 \end{cases}$$

and this system leads obviously to a unique pure strategy Nash equilibrium under our assumptions on the inverse demand. In fact we do not need $P''(Q)$ constant to characterize this equilibrium. If we are able to determine that it exists an equilibrium (x_1^{1*}, x_1^{2*}) then first period production will be equal to $(q_1^{1*}, q_1^{2*}) = (x_1^{1*}+s_1^{1*}, x_1^{2*}+s_1^{2*})$, a unique pair of second period sales resulting from this choice of inventories as stated in lemmas 1 and **??**. Consider the first order condition leading to the choice of x_1^i,

$$-c - H'(x_1^i) + \delta \, \frac{\partial \Pi_2^i}{\partial x_1^i}(x_1^1, x_1^2) = 0 \tag{E.1}$$

We have to insure first that it defines a unique choice x_1^i for each x_1^j and moreover we need to verify that the best response function in each region is downward sloping to be able to use the fact that downward jumps will lead to a pure strategy Nash equilibrium[16]. Therefore we have to verify that the objective function is concave with respect to x_1^i in each region, i.e. verify (19). Consider for example firm 1 (this reasoning can be applied symmetrically to firm 2). First, the expression of the profits on regions R5 and R7 obviously satisfy (19): on R5, $\frac{\partial \Pi_2^i}{\partial x_1^i} = c$ constant and therefore the second order effect is equal to 0, and on R7, $\frac{\partial \Pi_2^i}{\partial x_1^i} = 0$ since this region appears only because of the convexity of the cost of production. It remains to check that we have the property on region R6. We have to compute the second order effect of inventories in that case. Since

$$\frac{\partial \Pi_2^1}{\partial x_1^1} = (1 + \frac{\partial s_{x,f}^2}{\partial x_1^1}) \; P'(x_1^1 + s_{x,f}^2) \; x_1^1 + P(x_1^1 + s_{x,f}^2, \mathbf{2})$$

we have after simplifications

$$\frac{\partial^2 \Pi_2^1}{\partial (x_1^1)^2} = \frac{\partial^2 s_{x,f}^2}{\partial (x_1^1)^2} \; P'(x_1^1 + s_{x,f}^2) \; x_1^1 + (1 + \frac{\partial s_{x,f}^2}{\partial x_1^1}) \; [(1 + \frac{\partial s_{x,f}^2}{\partial x_1^1}) \; P'' \; x_1^1 + 2P'(x_1^1 + s_{x,f}^2)]$$

where

$$\frac{\partial s_{x,f}^2}{\partial x_1^1} = -\frac{P'' \; s_{x,f}^2 + P'(x_1^1 + s_{x,f}^2)}{P'' \; s_{x,f}^2 + 2P'(x_1^1 + s_{x,f}^2)}$$

and consequently after another round of simplifications

$$\frac{\partial^2 s_{x,f}^2}{\partial (x_1^1)^2} = -[P'' \; s_{x,f}^2 + 2P'(x_1^1 + s_{x,f}^2)]^{-2}$$
$$\times P'' \; [\frac{\partial s_{x,f}^2}{\partial x_1^1} \; P'(x_1^1 + s_{x,f}^2) - P'' \; s_{x,f}^2 (1 + \frac{\partial s_{x,f}^2}{\partial x_1^1})]$$

[16]See for example Tirole [1988] for a discussion of the existence of a pure strategy Nash equilibrium in Cournot games.

Since $(1 + \frac{\partial s^2_{x,f}}{\partial x^1_1}) > 0$, it follows immediately that when demand is linear ($P'' = 0$), the second order effect of inventories on profits is negative. We have obviously the property needed. Let us check that it is also true for $P'' < 0$. We have

$$1 + \frac{\partial s^2_{x,f}}{\partial x^1_1} = \frac{P'(x^1_1 + s^2_{x,f})}{P''\ s^2_{x,f} + 2P'(x^1_1 + s^2_{x,f})}$$

therefore

$$\begin{aligned}
\frac{\partial^2 \Pi^1_2}{\partial (x^1_1)^2} &= [P''\ s^2_{x,f} + 2P'(x^1_1 + s^2_{x,f})]^{-2} \\
\times\{-P''\ [\frac{\partial s^2_{x,f}}{\partial x^1_1}\ P'(x^1_1 + s^2_{x,f}) - P''\ s^2_{x,f}(1 + \frac{\partial s^2_{x,f}}{\partial x^1_1})]P'(x^1_1 + s^2_{x,f})x^1_1 \\
+(P'(x^1_1 + s^2_{x,f}))^2 P'' x^1_1 + 2(P'(x^1_1 + s^2_{x,f}))^2(P''\ s^2_{x,f} + 2P'(x^1_1 + s^2_{x,f}))\} \\
&= [P''\ s^2_{x,f} + 2P'(x^1_1 + s^2_{x,f})]^{-3} \\
\times\{-[-(P''\ s^2_{x,f} + P')P' - P''P's^2_{x,f}]P''P'x^1_1 \\
+(P''\ s^2_{x,f} + 2P')(P')^2 P'' x^1_1 + 2(P')^2(P''s^2_{x,f} + 2P')^2\}
\end{aligned}$$

The sign of this last expression turns to be always negative. Indeed the first part between square brackets is a negative expression power 3, therefore negative, and the expression between brackets is the sum of 3 positive expressions, therefore positive. Consequently the program on region R6 is indeed concave. The implicit function theorem applied to the first order condition (E.2) gives the slope of the best response function $x^1_1(x^2_1)$ in that region,

$$\frac{dx^1_1}{dx^2_1} = -\frac{\delta \frac{\partial^2 \Pi^1_2}{\partial x^1_1 \partial x^2_1}}{-H''(x^1_1) + \delta \frac{\partial^2 \Pi^1_2}{\partial (x^1_1)^2}}$$

When the cost is linear, the cross effect of inventories simplifies to give

$$\frac{\partial^2 \Pi^1_2}{\partial x^1_1 \partial x^2_1} = \frac{\partial^2 s^2_{x,f}}{\partial x^1_1 \partial x^2_1}\ P' x^1_1$$

which turns to be equal to zero. The maximum of firms profit is therefore unique in the region where one firm act as a leader, and depends of the cost of storage, the discount factor and the demand function. Therefore firms reaction functions are non increasing. Let us show now that they are downward jumping.

Since it is costly to store (remember that H is the cost of storage) and to produce before selling due to the discount factor (the discounted cost of production is

$(1+\delta)C(x)$ in second period), and since a firm cannot modify the behaviour of an opponent acting as a leader when its cost is linear, the best choice when facing a leader is to store nothing. Therefore whenever a firm finds profitable to produce and store a leader quantity to be released in second period there will be an asymmetric pure strategy Nash equilibrium (in fact two, since the roles of the two firms can be permuted). If the leader does not find profitable to exert its leadership, then both firms will not store at all. The pure strategy Nash equilibrium is symmetric and firms do not store. The payoffs functions are therefore non concave in the graph (x_1^1, x_1^2), with a peak at $x_1^i = 0$ in region `R7` and a peak given by equation(E.2) for $x_1^j = 0$ in regions `R5` or `R6`. The comparisons of the profits obtained when storing nothing and storing x_1^L solution of

$$-c - H'(x_1^L) + \delta \, \frac{\partial \Pi_2^i}{\partial x_1^i}(x_1^L, 0) = 0 \tag{E.2}$$

determines the nature of the equilibrium and is the straightforward goal of propositions 1 and 2.