The Impact of Multiperiod Planning Horizons on Portfolios and Asset Prices in a Dynamic CAPM*

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Abstract

This paper investigates a financial market in which heterogeneous investors with linear mean-variance preferences and multiperiod planning horizons of arbitrary finite length interact. Given subjective beliefs, market clearing prices are calculated explicitly. The classical capital market line result of CAPM theory is extended to the case with multiperiod planning horizons by proving that portfolios of investors with homogeneous beliefs and identical planning horizons contain equal proportions of risky assets. The existence of a perfect forecasting rule which generates rational expectations is established, the properties of the induced processes of prices and portfolios are analyzed. Numerical evidence is provided that different planning horizons provide a natural source of clustered volatility, empirically observed in financial data.

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1 Introduction

A typical feature of financial markets is that traders will generally have different planning horizons when investing their wealth in assets. It is intuitively clear that the length of the planning horizon will affect investors' risk taking behavior and thus their portfolio decisions. Investors with a long planning horizon are likely to invest more wealth into risky assets than those with short horizons. In particular, institutional investors will pursue long-term strategies rather than trying to follow a momentary trend. An open issue is how multiperiod planning horizons affect the individual portfolio decisions, if investors are allowed to revise their portfolio plans through the course of time. Moreover, it is unclear what the impact of different planning horizons on the dynamics of asset prices, asset returns, and portfolio holdings is. In view of institutional investors, it would be interesting to have a tractable model which allows to study the effects of planning horizons of arbitrary but finite length.

Starting with the work of Markowitz (1952) and Tobin (1958), economists have investigated portfolio decisions which, given a certain expected return, minimize the risk of future wealth fluctuations. Based on this portfolio theory, Sharpe (1964), Lintner (1965) and Mossin (1966) developed the famous *Capital Asset Pricing Model (CAPM).* The CAPM has been extended by Stapleton & Subrahmanyam (1978) to the case in which investors face a multiperiod rather than a single-period planning horizon. These models, however, remain inherently static and the results depend crucially on the assumption that beliefs of all investors be homogeneous and rational. All investors face the *same* multiperiod planning horizon. These assumptions preclude an analysis of how *distinct* planning horizons with possibly heterogeneous beliefs affect individual portfolio decisions and how the trading behavior of investors with different planning horizons affects asset prices. Furthermore, most studies of multiperiod portfolio decisions consider an essentially static one-shot optimization, e.g., see Chen, Jen & Zionts (1971), Hakansson (1970, 1983), Ingersoll (1987) or Pliska (1997). Hence no analysis of asset prices and portfolios can be conducted when portfolios may be re-optimized over time. Considering situations in which portfolio decisions are permanently revised are particularly important for scenarios in which investors update subjective beliefs as to incorporate new information.

The present paper addresses the above issues and is based on work of Böhm, Deutscher & Wenzelburger (2000), Böhm & Chiarella (2000), Wenzelburger (2004), and Hillebrand (2003). The key feature of these models is that asset prices are endogenously determined by the demand behavior of traders. This allows for a fully explicit dynamic analysis of a financial market where investors may be arbitrarily heterogeneous with respect to their individual beliefs as well as their usual microeconomic characteristics like preferences, endowments, etc. The trading behavior of all agents is described by asset demand functions which derived from individual optimization problems. From these demand functions an explicit temporary equilibrium map is derived determining market clearing prices in each period. By employing the concept of a forecasting rule, the expectations formation of all investors is made explicit. Combined with these forecasting rules, this yields an explicit time-one map of a so-called *random dynamical system* (Arnold 1998) in which expectations feed back into the actual evolution of asset prices, portfolios, and expectations.

In this spirit our analysis proceeds in three steps: First, we consider the individual portfolio choice problem of a single investor with a planning horizon of arbitrary finite length given subjective beliefs. Assuming linear mean-variance preferences, we compute an explicit asset demand function depending on beliefs. This provides a first insight how planning horizons of different length affect the demand behavior of investors. In a second step, a temporary equilibrium map determining market-clearing prices is derived from the aggregate excess demand function. The classical capital market line result of CAPM theory is extended to the case with multiperiod planning horizons by showing that portfolios of investors with homogeneous beliefs and identical planning horizons contain equal proportions of risky assets while different planning horizons will generally lead to structurally different portfolios. In a third step, the individual demand functions and the price law are embedded into a sequential model, taking proper account of how the individual demand behavior changes with new information and observations. By allowing for re-optimization of portfolio decisions, we thus obtain a dynamic description of how prices and portfolios evolve over time. Within this framework, a fully explicit dynamic analysis of the influence of multiperiod planning horizons on portfolios and the dynamics of asset prices and portfolios can be carried out. We show that different planning horizons provide a natural source of clustered volatility, empirically observed in financial data.

The remainder of this paper is organized as follows. Section 2 is concerned with the multiperiod portfolio choice problem of investors with linear mean-variance preferences. An explicit temporary equilibrium map describing market-clearing prices is computed. Section 3 treats the case with homogeneous expectations. Section 4 are concerned with the existence of forecasting rules which generate rational expectations, the dynamics of prices and portfolios under rational expectations is studied in Section 5. Conclusions are found in Section 6, the mathematical proof of the main theorem is placed in the appendix.

2 The Model

2.1 Overlapping generations of investors

Consider a financial market in which a population of overlapping generations (OLG) of investors trades in discrete time trading periods. The set of investors in each period is composed of $J+1$ different generations. In each trading period $t \in \mathbb{N}$, a new young generation enters the market and trades for $J+1$ consecutive periods before its members exit the market to consume terminal wealth in period $t+J$. Each generation will be identified by the index $j = 0, 1, \ldots, J$ describing lifetime, i.e., the number of periods they remain in the market until their members exit. In particular, $j = J$ refers to the young and $j = 0$ to the old generation. Each generation j consists of I types of investors characterized by risk preferences and subjective beliefs regarding the future evolution of the market. More precisely, a single investor in an arbitrary period is identified by the pair (i, j) describing his type $i \in \{1, \ldots, I\}$ and his generation $j \in \{0, 1, \ldots, J\}$. Excluding the old generation $j = 0$, the set of investors trading in the market in each period is given by $\mathbb{I} := \{1, \ldots, I\} \times \{1, \ldots, J\}$. The population structure in an arbitrary trading period $t \in \mathbb{N}$ is depicted in Figure 1.

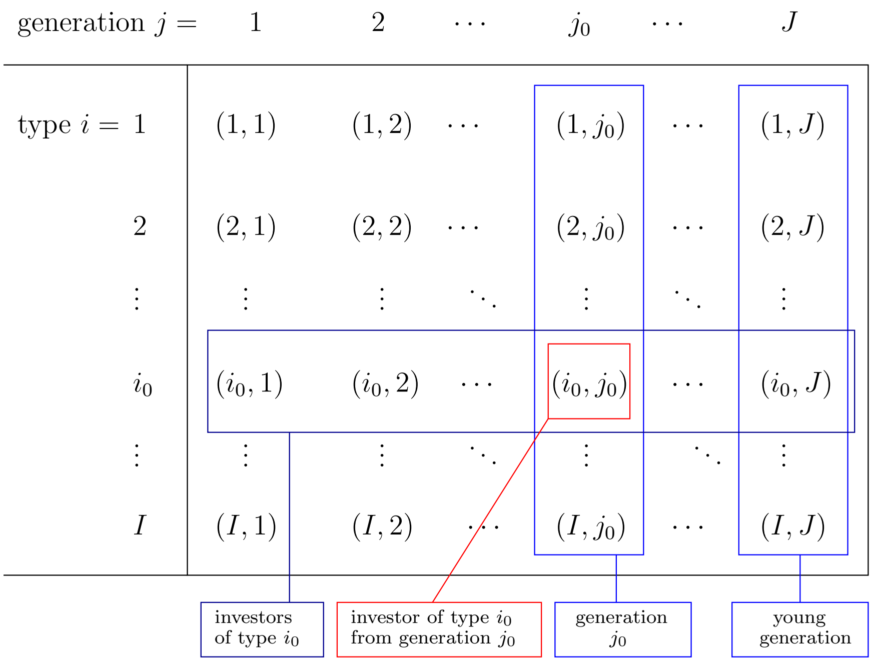

Figure 1: The investors in an arbitrary trading period t.

There is a single consumption good in the economy which serves as numeraire for all prices and payments. At the beginning of each period, any young investor $(i, J) \in \mathbb{I}$ of type i receives an initial endowment of $e^{(i)} > 0$ units of the consumption good. These endowments may depend on the type i but are constant over time. Investors (i, j) with $j < J$ do not receive endowments. Assuming that the consumption good cannot be stored by consumers directly, each investor faces the problem of transferring wealth from the first to the last period of life in which they consume the proceeds of their investments. There exist $K+1$ retradeable assets in the economy, indexed by $k = 0, 1, \dots, K$. The first asset $k = 0$ is a riskless bond which pays a constant return $R > 0$ per unit invested in the previous period. The assets $k = 1, \dots, K$ correspond to risky shares of firms which are traded at prices $p_t = (p_t^{(1)}, \dots, p_t^{(K)})^\top \in \mathbb{R}^K$ of period t. For simplicity, we abstract from dividend payments.

2.2 Decision Problem

Consider first the portfolio choice problem faced by an investor $(i, j) \in \mathbb{I}$ in an arbitrary period t with planning horizon $t + j$. At the beginning of period t any investor forms beliefs regarding future prices $p_{t+1}, \dots, p_{t+j}$ which are relevant for her portfolio choices. These beliefs are given by a subjective joint probability distribution for the random variables $p_{t+1}, \dots, p_{t+j}$.[1] Given her beliefs, the investor's portfolio decision will depend on current prices as well on her wealth position in period t. We assume that the portfolio problem in period t is solved *prior* to trading, i.e., before the actual price p_t is observed. Current prices will therefore enter the decision problem as a parameter $p \in \mathbb{R}^K$. To determine the investors initial wealth position at time t we need to distinguish between young and non-young investors. Each young investor's wealth is equal to his initial endowment $e^{(i)}$. The wealth of any non-young investor $(i, j) \in \mathbb{I}$, $j < J$ at time t corresponds to the value of his portfolio $(x_{t-1}^{(i,j+1)}, y_{t-1}^{(i,j+1)})$ from the previous period at prices at prices of period t. We therefore set

$$w_t^{(ij)} = \begin{cases} e^{(i)} & j = J \\ R y_{t-1}^{(i,j+1)} + p_t^\top x_{t-1}^{(i,j+1)} & j = 1, \dots, J-1 \end{cases} \tag{1}$$

for initial wealth in period t. Note that the wealth of a non-young investor (i, j), $j < J$, depends on prices.

In order to obtain explicit demand schedules, we make specific assumptions regarding investors' preferences and beliefs. Investors beliefs in any period t are assumed to be given by multivariate a normal distribution for future prices $p_{t+1}, \dots, p_{t+j} \in \mathbb{R}^K$. Let $\mathcal{M}_{Kj}$, denote the set of all symmetric, positive definite $(Kj) \times (Kj)$ matrices. Recall that a *(multivariate) normal distribution* with parameters $(\mu, \Sigma) \in \mathbb{R}^{Kj} \times \mathcal{M}_{Kj}$ is given by the density function

$$f_{Kj}(q; \mu, \Sigma) := (2\pi)^{-\frac{N}{2}} [\det \Sigma]^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}(q-\mu)^\top \Sigma^{-1} (q-\mu) \right\}, \quad q \in \mathbb{R}^{Kj}, \tag{2}$$

cf. Tong (1990). As a further technical restriction, let $\mathcal{M}_{Kj}^\star \subset \mathcal{M}_{Kj}$ denote the class of all symmetric, positive definite $Kj \times Kj$ matrices which satisfy a certain invertibility condition which will be made explicit in Assumption 3, Appendix A. We are now ready to specify the assumptions on investors who maximize utility of terminal wealth.

[1] For notational simplicity we use the same notation for random variables and their realizations.

Assumption 1
Preferences and beliefs of investor $(i, j) \in \mathbb{I}$ are characterized by the following:

1. *Preferences of an investors of type i are described by an exponential utility function*

$$u(w; a^{(i)}) := -\exp\Big\{-a^{(i)} w\Big\}, \quad w \in \mathbb{R}, \tag{3}$$

where $a^{(i)} > 0$ denotes risk-aversion.

2. *The subjective beliefs of investor $(i, j) \in \mathbb{I}$ at time t regarding prices $p_{t+1}, \ldots, p_{t+j}$ are given by a normal distribution on $\mathbb{R}^{Kj}$ with density function of the form (2) described by the first two moments*

$$\mu_t^{(ij)} := \begin{pmatrix} \mu_{t,t+1}^{(i)} \\ \vdots \\ \mu_{t,t+j}^{(i)} \end{pmatrix} \in \mathbb{R}^{Kj}, \qquad \Sigma_t^{(ij)} := \begin{bmatrix} \Sigma_{t,11}^{(i)} & \cdots & \Sigma_{t,1j}^{(i)} \\ \vdots & \ddots & \vdots \\ \Sigma_{t,j1}^{(i)} & \cdots & \Sigma_{t,jj}^{(i)} \end{bmatrix} \in \mathcal{M}_{Kj}^{\star}. \tag{4}$$

Here, $\mu_{t,t+s}^{(ij)} := \mathbb{E}_t^{(ij)}[p_{t+s}]$ denotes investor (i, j)'s subjective mean value for prices p_{t+s}, $s = 1, \ldots, j$ conditional on information available at time t corresponding to the density $f_{Kj}(\cdot, \mu_t^{(ij)}, \Sigma_t^{(ij)})$. The matrix

$$\Sigma_{t,ss'}^{(ij)} := \mathbb{E}_t^{(ij)}\left[\left(p_{t+s} - \mathbb{E}_t^{(ij)}[p_{t+s}]\right)\left(p_{t+s'} - \mathbb{E}_t^{(ij)}[p_{t+s'}]\right)^{\top}\right]$$

denotes investor (i, j)'s subjective conditional variance-covariance matrix between the prices p_{t+s}, $p_{t+s'}$, $s, s' = 1, \ldots, j$ corresponding to $f_{Kj}(\cdot, \mu_t^{(ij)}, \Sigma_t^{(ij)})$.

Assumption 1 states that all investors of the same type are characterized by the same risk aversion. The subjective beliefs of investor $(i, j) \in \mathbb{I}$ at time t are parameterized by subjective means $\mu_t^{(ij)} \in \mathbb{R}^{Kj}$ and subjective variance-covariance-matrices $\Sigma_t^{(ij)} \in \mathcal{M}_{Kj}^{\star}$ for future prices. Assumption 3, Appendix A shows that the restriction to $\mathcal{M}_{Kj}^{\star}$ is trivially satisfied if $\Sigma_{ss'}^{(i)} = 0$ for $s \neq s'$ and hence if correlations between prices of different periods are sufficiently small. For simplicity, we have assumed in Assumption 1 that any non-young investor $(i, j) \in \mathbb{I}$ with a planning horizon $j < J$ holds the same expectations for prices $p_{t+1}, \ldots, p_{t+j}$ as the young investor (i, J). Formally, this means that her beliefs are given by the marginal distributions[2] of the respective young investor (i, J). Economically, this assumption can be justified by the presumption that all investors of type i employ the same financial mediator. As a consequence, expectations of all investors at time t are completely described by the moments of the respective young generations.

[2] Mathematically, the subjective probability distribution of a non-young investor (i, j) is the projection of the the probability distribution of the corresponding young investor (i, J). By the properties of the multivariate normal distribution (e.g., see Tong 1990) this distribution is again normal with corresponding projected moments.

Consider the portfolio choice problem faced by an arbitrary investor $(i,j) \in \mathbb{I}$ in an arbitrary but fixed period, say $t = 0$. For simplicity of notation, we set

$$\mu := \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_j \end{pmatrix} \in \mathbb{R}^{Kj} \qquad \Sigma := \begin{bmatrix} \Sigma_{11} & \dots & \Sigma_{1j} \\ \vdots & \ddots & \vdots \\ \Sigma_{j1} & \dots & \Sigma_{jj} \end{bmatrix} \in \mathcal{M}^{\star}_{Kj} \tag{5}$$

and suppress indices referring to the decision period $t = 0$ for a moment. Given parametric prices p and initial wealth w defined by (1) assume that the investor chooses a self-financing trading strategy

$$H = (x_0, y_0, \dots, x_j, y_j)$$

consisting of a list of portfolios $(x_0, y_0) \in \mathbb{R}^K \times \mathbb{R}$ and planned portfolios

$$x_s = x_s(p_1, \dots, p_s) \in \mathbb{R}^K, \quad y_s = y_s(p_1, \dots, p_s) \in \mathbb{R}, \quad s = 1, \dots, j-1,$$

such $w = y_0 + p^\top x_0$ and for $s = 1, \dots, j-1$ and each possible realization of prices $p_1, \dots, p_s$

$$\begin{array}{rcl} y_s + p_s^\top x_s & = & R y_{s-1} + p_s^\top x_{s-1}, \quad s = 1, \dots, j-1, \\ W_j & = & R y_{j-1} + p_j^\top x_{j-1}. \end{array} \tag{6}$$

Observe that for each $s = 1, \cdots, j-1$, planned portfolios (x_s, y_s) are mappings that depend on prices $p_1, \dots, p_s$.[3]

Let $\mathscr{H}(p,w)$ denote the set of all self-financing strategies satisfying (6) with parametric prices p and initial wealth w at time $t = 0$. Setting $p_1^s := (p_1, \dots, p_s)$, the choice of a particular strategy $H \in \mathscr{H}(p,w)$ induces a random variable $W_j(H, p_1^j) := R y_{j-1}(p_1^{j-1}) + p_j^\top x_{j-1}(p_1^{j-1})$ which describes terminal wealth attained at the end of period j. Assuming that given his beliefs $(\mu, \Sigma) \in \mathbb{R}^{Kj} \times \mathcal{M}^{\star}_{Kj}$, parametric prices $p \in \mathbb{R}^K$ and wealth $w \in \mathbb{R}$ the investor maximizes the expected utility of terminal wealth his optimization problem at $t = 0$ reads

$$\max \left\{ \int_{\mathbb{R}^{Kj}} u\big(W_j(H, p_1^j);\, a^{(i)}\big)\, f_{Kj}(p_1^j, \mu, \Sigma)\, dp_1^j \text{ s.t. } H \in \mathscr{H}(p,w) \right\}. \tag{7}$$

Note that the investor is allowed to update beliefs and reoptimize planned portfolio decisions in any subsequent period. A solution to the optimization problem (7) will determine the investors' individual asset demand functions at time $t = 0$ as a function of prices, wealth and beliefs. Using a dynamic programming approach we show in Appendix A that the restriction of the subjective covariance-variance matrices Σ to the set $\mathcal{M}^{\star}_{Kj} \subset \mathcal{M}_{Kj}$ suffices to obtain well-defined asset demand functions. These demand functions are given in the following Theorem.

[3] This definition of a self-financing trading strategy is consistent with Pliska (1997) who defines a trading strategy as an adapted stochastic process.

Theorem 1

Let Assumption 1 be satisfied. Then for each planning horizon $j = 1, \ldots, J$ the investor's asset demand function derived from (7) given her beliefs $(\mu, \Sigma) \in \mathbb{R}^{Kj} \times \mathcal{M}^{\star}_{Kj}$ and risk aversion risk aversion $a^{(i)}$ takes the form:

$$\varphi^{(ij)}(p, \mu, \Sigma) := \frac{1}{R^{j-1} a^{(i)}} \Pi_j^{\top} \Sigma^{-1} \big(\mu - \Pi_j R p\big), \quad p \in \mathbb{R}^K. \tag{8}$$

where $\Pi_j := \big[I_K, \ldots, R^{j-1} I_K\big]^{\top} \in \mathbb{R}^{Kj \times K}$, $j = 1, \ldots, J$.

The proof of Theorem 1 is given in Appendix A. Observe that the demand for risky assets (8) is independent of the investor's initial wealth. For $j = 1$ and $(\mu, \Sigma) \in \mathbb{R}^K \times \mathcal{M}_K$ we obtain from (8)

$$\varphi^{(i1)}(p, \mu, \Sigma) = \frac{1}{a^{(i)}} \Sigma^{-1} \left(\mu - Rp\right). \tag{9}$$

Hence, for a 1-period planning-horizon, we obtain the classical demand function of an investor with linear mean-variance preferences (e.g., see Böhm & Chiarella 2000).
For $j = 2$ and moments $(\mu, \Sigma) \in \mathbb{R}^{2K} \times \mathcal{M}^{\star}_{2K}$ given by (5) the demand function (8) can be written as

$$\varphi^{(i2)}(p, \mu, \Sigma) = \frac{1}{R a^{(i)}} \begin{bmatrix} I_K & R I_K \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}^{-1} \begin{pmatrix} \mu_1 - Rp \\ \mu_2 - R^2 p \end{pmatrix}. \tag{10}$$

The restriction $\Sigma \in \mathcal{M}^{\star}_{2K} \subset \mathcal{M}_{2K}$ is in this case equivalent to the condition that the matrix $A_1 := R I_K - \Sigma_{21} \Sigma_{11}^{-1}$ be invertible (see Assumption 3 in Appendix A). In particular, this holds if investors assume future prices to be uncorrelated, i.e. $\Sigma_{12} = \Sigma_{21}^{\top} = 0$. In this special case the demand function (10) takes the form

$$\varphi^{(i2)}(p, \mu, \Sigma) = \frac{1}{R a^{(i)}} \left(\Sigma_{11}^{-1}(\mu_1 - Rp) + \left[\frac{1}{R^2} \Sigma_{22}\right]^{-1} \left(\frac{1}{R}\mu_2 - Rp\right)\right).$$

In this case the the asset demand function is the sum of two asset demand functions of the form (9) with adjusted risk aversion $a^{(i)} R$, moments (μ_1, Σ_{11}), and discounted moments $\big(\frac{1}{R}\mu_2, \frac{1}{R^2}\Sigma_{22}\big)$.

In the sequel we assume that in each period $t \in \mathbb{N}$ each investor $(i, j) \in \mathbb{I}$, $j > 0$, solves an optimization problem of the form (7) given her beliefs $(\mu_t^{(ij)}, \Sigma_t^{(ij)}) \in \mathbb{R}^{(Kj)} \times \mathcal{M}^{\star}_{Kj}$, her wealth defined by (1) and risk aversion $a^{(i)} > 0$. Utilizing Theorem 1 her asset demand function at time t can be written as

$$\varphi^{(ij)}(p, \mu_t^{(ij)}, \Sigma_t^{(ij)}) := \frac{1}{a^{(ij)}} \, \Theta_t^{(ij)-1} \big[\theta_t^{(ij)} - Rp\big], \quad p \in \mathbb{R}^K, \tag{11}$$

where we use the abbreviations

$$\begin{aligned} \Theta_t^{(ij)} &:= \Big[\Pi_j^{\top} \Sigma_t^{(ij)-1} \Pi_j\Big]^{-1} \\ \theta_t^{(ij)} &:= \Theta_t^{(ij)} \Pi_j^{\top} \Sigma_t^{(ij)-1} \mu_t^{(ij)} \\ a^{(ij)} &:= R^{j-1} a^{(i)}. \end{aligned} \tag{12}$$

Observe that each $\Theta_t^{(ij)}$ is symmetric and positive definite.

Corollary 1
Under the hypothesis of Theorem 1, the asset demand function (11) takes the form

$$\varphi^{(ij)}(p,\mu_t^{(ij)},\Sigma_t^{(ij)}) = \sum_{s=1}^{j} D_{t,s}^{(ij)}\mu_{t,t+s}^{(ij)} - R\Theta_t^{(ij)-1}p, \quad p\in\mathbb{R}^K, \tag{13}$$

where $D_{t,s}^{(ij)} \in \mathbb{R}^{K\times K}$ *and* $D_t^{(ij)} = \left[D_{t,1}^{(ij)},\dots,D_{t,j}^{(ij)}\right] := \Pi_j^\top \Sigma_t^{(ij)-1} \in \mathbb{R}^{K\times Kj}$.

The consequences of Theorem 1 seem quite remarkable. It establishes a structural equivalence between the demand functions of investors with a one period planning horizon and investors with a multiperiod planning horizon of arbitrary finite length. In principle, subjective expectations of an investor with a multiperiod planning horizon can always be transformed in such a way that his demand behavior is indistinguishable from the behavior of an investor with a one-period planning horizon. Despite this fact, it turns out that many interesting implications arise due to the intrinsic heterogeneity of different planning horizons some of which are addressed in the present paper.

2.3 Price formation

In order to determine market clearing prices, let $\bar{x} \in \mathbb{R}_+^K$ denote the total stock of risky assets. Assume that in each trading period t there is another group of so-called noise-traders who purchase or sell the random quantity $\xi_t \in \mathbb{R}^K$ in the market.[4]

Assumption 2
The portfolios of noise traders are given by an $\mathbb{R}^K$*-valued stationary ergodic stochastic process* $\{\xi_t\}_{t\in\mathbb{N}}$ *on a the probability space* $(\Omega,\mathcal{F},\mathbb{P})$, *which is adapted to the filtration* $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ *such that each* ξ_t *is* $\mathcal{F}_t$ *measurable.*

Market clearing in period t requires the existence of a price vector $p_t \in \mathbb{R}^K$ such that aggregate demand including noise traders equals the total stock of risky assets. Given the individual demand functions (11) for risky assets and the quantity of noise traders ξ_t, the market-clearing condition of period t reads

$$\sum_{(i,j)\in\mathbb{I}} \frac{1}{a^{(ij)}}\Theta_t^{(ij)-1}\left[\theta_t^{(ij)} - Rp_t\right] + \xi_t \stackrel{!}{=} \bar{x}. \tag{14}$$

Solving for p_t, given any list of subjective beliefs $\left(\mu_t^{(ij)},\Sigma_t^{(ij)}\right)_{(i,j)\in\mathbb{I}}$ and ξ_t, the market clearing prices are defined by a map

$$p_t = S\Big((\mu_t^{(ij)},\Sigma_t^{(ij)})_{(i,j)\in\mathbb{I}},\xi_t\Big) := \sum_{(i,j)\in\mathbb{I}} \Gamma_t^{(ij)}\theta_t^{(ij)} - \Gamma_t[\bar{x}-\xi_t], \tag{15}$$

[4] Noise traders will be thought of as traders whose portfolio decisions are not captured by a standard microeconomic decision model. Alternative interpretations as given in (De Long, Shleifer, Summers & Waldmann 1990), p. 709 apply as well.

where[5]

$$\Gamma_t := \frac{1}{R}\left[\sum_{(i,j)\in\mathbb{I}} \frac{1}{a^{(ij)}}\Theta_t^{(ij)-1}\right]^{-1}, \qquad \Gamma_t^{(ij)} := \frac{1}{a^{(ij)}}\Gamma_t\,\Theta_t^{(ij)-1}. \tag{16}$$

The temporary equilibrium map (15) defines the economic law S for our multiperiod version of the CAPM which determines market-clearing prices in each trading period as a function of agents' expectations for future prices. The mapping S is of *Cobweb-type* since it contains essentially expectations for prices as arguments. Since these expectations refer to future periods $t+1,\dots,t+J$, the law contains an *expectational lead* of length J. Apart from these expectational leads, the structure (15) is the same as in the case of investors with heterogeneous expectations and a one-period planning horizon studied in Wenzelburger (2004). In the present setup, heterogeneity may enter through possibly diverse beliefs *as well as* through the different planning horizons of investors belonging to different generations.

Realized portfolios of investors $(i,j)\in\mathbb{I}$ after trading in period t are given by

$$\begin{aligned} x_t^{(ij)} &= \varphi^{(ij)}(p_t,\mu_t^{(ij)},\Sigma^{(ij)}) \\ y_t^{(ij)} &= \begin{cases} e^{(i)} - p_t^\top x_t^{(ij)}, & \text{if } j=J \\ Ry_{t-1}^{(i,j+1)} + p_t^\top[x_{t-1}^{(i,j+1)} - x_t^{(ij)}], & \text{if } j=1,\cdots,J-1. \end{cases} \end{aligned} \tag{17}$$

3 Homogeneous expectations

In the following section we will show that the heterogeneous structure of the price law (15) is maintained even if expectations of investors are homogeneous. To study the impact of different planning horizons on portfolios and prices, the section focuses on the case of homogeneous expectations. This means that the beliefs of investor $(i,j)\in\mathbb{I}$ are independent of her type i and may depend only on the length j of his planning horizon. This implies in particular that all investors within one generation hold identical expectations. In the sequel we will therefore write $(\mu_t^{(j)},\Sigma_t^{(j)})\in\mathbb{R}^{Kj}\times\mathcal{M}^\star_{Kj}$, for the beliefs of generation j, $j=1,\dots,J$ in period t instead of $(\mu_t^{(ij)},\Sigma_t^{(ij)})$. As a consequence, the parameters (12) of the individual demand function (11) of investor $(i,j)\in\mathbb{I}$ in period t may be rewritten as

$$\Theta_t^{(j)} := \left[\Pi_j^\top\Sigma_t^{(j)-1}\Pi_j\right]^{-1}, \quad \text{and} \quad \theta_t^{(j)} := \Theta_t^{(j)}\Pi_j^\top\Sigma_t^{(j)-1}\mu_t^{(j)}, \quad i=1,\dots,I. \tag{18}$$

More precisely, the demand functions of any two investors $(i,j),(i',j)\in\mathbb{I}$ with homogeneous expectations satisfy the relation

$$\varphi^{(ij)}(p,\mu_t^{(j)},\Sigma_t^{(j)}) = \frac{1}{a^{(i)}R^{j-1}}\Theta_t^{(j)-1}[\theta_t^{(j)} - Rp] = \frac{a^{(i')}}{a^{(i)}R^{j-1}}\varphi^{(i'j)}(p,\mu_t^{(j)},\Sigma_t^{(j)}).$$

The demand functions for risky shares of investors belonging to the same generation are thus collinear by a factor determined by the possibly different risk aversions $a^{(i)}$ and

[5] Note that by (12), all matrices $\Theta_t^{(ij)}$ are positive-definite and hence invertible. Since the sum of positive definite matrices is again positive definite, Γ_t and the $\Gamma_t^{(ij)}$ are well-defined.

$a^{(i')}$. This implies that the proportions of shares held by investors of the same generation are identical. Let $\alpha := \left(\frac{1}{a^{(1)}} + \cdots + \frac{1}{a^{(I)}}\right) > 0$ denote the aggregate risk tolerance, p_t be the market-clearing price in period t, and

$$x_t^{(j)} := \sum_{i=1}^{I} \varphi^{(ij)}(p_t, \mu_t^{(j)}, \Sigma_t^{(j)}) = \frac{\alpha}{R^{j-1}} \Theta_t^{(j)-1}[\theta_t^{(j)} - Rp_t] \tag{19}$$

denote the *aggregate generational portfolio* held by generation j after trading in period t. Then we have proven the following result.

Theorem 2
Under homogeneous expectations, the risky portfolio $x_t^{(ij)}$ held by an investor $(i,j) \in \mathbb{I}$ after trading in period $t \in \mathbb{N}$ is given by a constant share of the aggregate generational portfolio (19) of generation j, such that

$$x_t^{(ij)} = \frac{\frac{1}{a^{(i)}}}{\alpha} x_t^{(j)}.$$

This share is determined by the individual risk tolerance $\frac{1}{a^{(i)}}$ relative to the aggregate risk tolerance α.

Theorem 2 is a generalization of the famous capital market line result from classical CAPM theory to the case with multiperiod planning horizons. Under homogeneous expectations, investors will hold a multiple of a generational portfolio rather than the market portfolio $\bar{x}$. In view of (19), generational portfolios corresponding to different planning horizons will, in general, not be collinear. Therefore, under homogeneous expectations, planning horizons of distinct lengths will lead to structurally distinct portfolio holdings.

4 Unbiased forecasting rules

The price law (15) determines market clearing prices in each period given the beliefs of all investors and the demand of noise traders. To obtain a complete description of the dynamic evolution of prices and portfolios we need to specify how investors form their expectations based on the available information. In this regard, the existence of forecasting rules generating expectations which are rational in some sense is of particular importance which will be studies in this section.

Following Wenzelburger (2004), we develop an unbiased forecasting rule that generates rational expectations for investors of type I. The notion of rational expectations used here requires that forecast must be unbiased in the sense that in each trading period the subjective expected values for future prices coincide with the respective true conditional expectations.

4.1 The general case

Assume that investors use a *no-updating* forecasting rule of the following form. The idea of such a forecasting rule is that in any period t, the first $J-1$ forecasts will not be updated (see Wenzelburger (2001)) such that

$$\mu^{(I)}_{t,t+j} = \mu^{(I)}_{t-1,t+j}, \quad j = 1, \ldots, J-1. \tag{20}$$

Using the price law (15), then $\mu^{(I)}_{t,t+J}$ will be chosen such that

$$\mathbb{E}_{t-1}[p_t] = \sum_{(i,j)\in\mathbb{I}} \Gamma^{(ij)}_t \theta^{(ij)}_t - \Gamma_t\big[\bar{x} - \mathbb{E}_{t-1}[\xi_t]\big] \overset{!}{=} \mu^{(I)}_{t-1,t}. \tag{21}$$

Suppose for a moment that the forecasts $\mu^{(I)}_{t,t+J}$ can be chosen such that (21) holds. Then $\mathbb{E}_{t-1}[p_t - \mu^{(I)}_{t-1,t}] = 0$ and the no-updating condition implies that the conditional forecast errors of all forecasts $\mu^{(I)}_{t-j,t}$, $j = 1, \ldots, J$ for p_t vanish, that is,

$$\mathbb{E}_{t-1}[p_t - \mu^{(I)}_{t-j,t}] = 0, \quad j = 1, \ldots, J.$$

Moreover, by the law of iterated expectations

$$\mathbb{E}_{t-j}[p_t - \mu^{(I)}_{t-j,t}] = \mathbb{E}_{t-j}\big[\mathbb{E}_{t-1}[p_t - \mu^{(I)}_{t-j,t}]\big] = 0, \quad j = 1, \ldots, J.$$

The problem of obtaining unbiased forecasts is therefore reduced to solving (21). This can be achieved as follows. Taking conditional expectations of the market-clearing condition (14), we have

$$\varphi^{(IJ)}\big(\mathbb{E}_{t-1}[p_t], \mu^{(I)}_t, \Sigma^{(I)}_t\big) + \sum_{(i,j)\neq(I,J)} \frac{1}{a^{(ij)}} \Theta^{(ij)-1}_t \big[\theta^{(ij)}_t - R\mathbb{E}_{t-1}[p_t]\big] + \mathbb{E}_{t-1}[\xi_t] - \bar{x} = 0.$$

Therefore, Condition (21) is equivalent to

$$\varphi^{(IJ)}(\mu^{(I)}_{t-1,t}, \mu^{(I)}_t, \Sigma^{(I)}_t) + \sum_{(i,j)\neq(I,J)} \frac{1}{a^{(ij)}} \Theta^{(ij)-1}_t \big[\theta^{(ij)}_t - R\mu^{(I)}_{t-1,t}\big] + \mathbb{E}_{t-1}[\xi_t] - \bar{x} = 0. \tag{22}$$

In view of the individual demand functions (11), let

$$\zeta_t := \sum_{(i,j)\neq(I,J)} \frac{1}{a^{(ij)}} \Theta^{(ij)-1}_t \big[\theta^{(ij)}_t - R\mu^{(I)}_{t-1,t}\big] + \mathbb{E}_{t-1}[\xi_t] \tag{23}$$

denote the expected aggregate portfolio of all investors $(i,j) \in \mathbb{I}$ except investor (I,J). Replacing (23) in Condition (22), we see that the condition

$$\varphi^{(IJ)}(\mu^{(I)}_{t-1,t}, \mu^{(I)}_t, \Sigma^{(I)}_t) + \zeta_t - \bar{x} = 0 \tag{24}$$

is equivalent to the original Condition (21). Using Corollary 1 we may solve (24) for $\mu^{(I)}_{t,t+J}$ to get

$$\mu^{(I)}_{t,t+J} = D^{(IJ)-1}_{t,J}\Big[\bar{x} - \zeta_t - \sum_{s=1}^{J-1} D^{(IJ)}_{t,s}\mu^{(I)}_{t,t+s} + R\Theta^{(IJ)-1}_t \mu^{(I)}_{t-1,t}\Big].$$

An unbiased forecasting rule von investors of type I are thus given by

$$\begin{cases} \mu^{(I)}_{t,t+j} &= \mu^{(I)}_{t-1,t+j}, \quad j = 1, \dots, J-1. \\ \mu^{(I)}_{t,t+J} &= \psi^{(I)}_{*}\left(\mu^{(I)}_{t-1}, \Sigma^{(IJ)}_t, \zeta_t\right) \\ &:= D^{(IJ)-1}_{t,J}\Big[\bar{x} - \zeta_t - \sum_{s=1}^{J-1} D^{(IJ)}_{t,s}\mu^{(I)}_{t-1,t+s} + R\Theta^{(IJ)-1}_t \mu^{(I)}_{t-1,t}\Big]. \end{cases} \tag{25}$$

Inserting the unbiased forecasting rule into the price law (15), we obtain the system of equations

$$\begin{cases} p_t &= \mu^{(IJ)}_{t-1,t} + \Gamma_t\big[\xi_t - \mathbb{E}_{t-1}[\xi_t]\big] \\ \zeta_t &= \displaystyle\sum_{(i,j)\neq(I,J)} \frac{1}{a^{(ij)}}\Theta^{(ij)-1}_t\big[\theta^{(ij)}_t - R\mu^{(I)}_{t-1,t}\big] + \mathbb{E}_{t-1}[\xi_t] \\ \mu^{(I)}_{t,t+j} &= \mu^{(I)}_{t-1,t+1}, \quad j = 1, \dots, J-1 \\ \mu_{t,t+J} &= D^{(IJ)-1}_{t,J}\Big[\bar{x} - \zeta_t - \sum_{s=1}^{J-1} D^{(IJ)}_{t,s}\mu^{(I)}_{t-1,t+s} + R\Theta^{(IJ)-1}_t \mu^{(I)}_{t-1,t}\Big] \end{cases} \tag{26}$$

that determine the asset prices of period t under rational expectations for investors of type I, given the beliefs of all investors $(i,j) \in \mathbb{I}'$.

4.2 The case $J = 2$

For simplicity of exposition, consider to the case $J = 2$ in which the population consists of investors who trade for three consecutive periods in the market. In the sequel, generation $j = 2$ will therefore be referred to as the young, $j = 1$ as the middle-aged and $j = 0$ as the old generation, respectively. Furthermore, we shall confine the analysis to the case with homogeneous expectations and assume that second moment beliefs of all agents are constant over time. In each period $t \in \mathbb{N}$ beliefs are therefore completely described by the expected values of both generations $j = 1, 2$, given by

$$\mu^{(2)}_t = \begin{pmatrix} \mu_{t,t+1} \\ \mu_{t,t+2} \end{pmatrix} \in \mathbb{R}^{2K}, \quad \text{and} \quad \mu^{(1)}_t = \mu_{t,t+1} \in \mathbb{R}^K, \tag{27}$$

and by constant variance-covariance matrices

$$\Sigma^{(2)}_t = \Sigma^{(2)} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \in \mathcal{M}^{\star}_{2K} \quad \text{and} \quad \Sigma^{(1)}_t = \Sigma^{(1)} = \Sigma_{11} \in \mathcal{M}^{\star}_K. \tag{28}$$

In the sequel, the expected values $\mu^{(j)}_t \in \mathbb{R}^{Kj}$, $j = 1, 2$ will be referred to as the *forecast* made by generation j. Defining from (28) the parameters $A_1 := RI_K - \Sigma_{21}\Sigma_{11}^{-1}$ and $\Sigma_2 := \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$ the price law (15) for $J = 2$ becomes

$$p_t = \tfrac{\alpha}{R}\Gamma A_1^{\top}\Sigma_2^{-1}\mu_{t,t+2} + \left[\tfrac{1}{R}I_K - \alpha\Gamma A_1^{\top}\Sigma_2^{-1}\right]\mu_{t,t+1} - \Gamma[\bar{x} - \xi_t], \tag{29}$$

where the matrix $\Gamma_t \equiv \Gamma$ given in (16) is constant over time and reads

$$\Gamma = \tfrac{1}{\alpha}\left[R\Theta^{(1)-1} + \Theta^{(2)-1}\right]^{-1}.$$

Given the price law (29), unbiased expectations require that in each period t the forecast $\mu_{t-1,t}$ made at time $t-1$ for prices in t must coincide with the true conditional expectation, i.e.,

$$\begin{aligned}\mathbb{E}_{t-1}[p_t] &= \tfrac{\alpha}{R}\Gamma A_1^\top \Sigma_2^{-1}\mu_{t,t+2} + \left[\tfrac{1}{R}I_K - \alpha\Gamma A_1^\top\Sigma_2^{-1}\right]\mu_{t,t+1} - \Gamma\left[\bar{x} - \mathbb{E}_{t-1}[\xi_t]\right] \\ &\stackrel{!}{=} \mu_{t-1,t}, \end{aligned} \tag{30}$$

where again $\mathbb{E}_{t-t}[\cdot]$ denotes the expectations operator taken with respect to the σ-field $\mathcal{F}_{t-1}$. The *no-updating-condition* reads

$$\mu_{t,t+1} = \mu_{t-1,t+1} \quad \forall t \in \mathbb{N}. \tag{31}$$

Inserting (31) into (30) and solving for $\mu_{t,t+2}$ yields an unbiased forecasting rule

$$\begin{aligned}\mu_{t,t+2} &= \psi_\star(\mu_{t-1,t}, \mu_{t-1,t+1}, \mathbb{E}_{t-1}[\xi_t]) \\ &:= \left[RI_K - \tfrac{1}{\alpha}\Sigma_2 A_1^\top \Gamma^{-1}\right]\mu_{t-1,t+1} + \tfrac{R}{\alpha}\Sigma_2 A_1^{-\top}\Gamma^{-1}\mu_{t-1,t} \\ &\quad + \tfrac{R}{\alpha}\Sigma_2 A_1^{-\top}\Big[\bar{x} - \mathbb{E}_{t-1}[\xi_t]\Big]. \end{aligned} \tag{32}$$

The unbiased forecasting rule (32) is a linear function of the previous forecast $\mu_{t-1}^{(2)}$ as well as of the conditional expectation $\mathbb{E}_{t-1}[\xi_t]$ and is independent of previous realizations of prices. Condition (30) is therefore satisfied for all times t if young agents determine their forecast according to (31) and (32). Observe that applying the unbiased forecasting rule (32) requires knowledge not only of the previous forecast $\mu_{t-1}^{(2)}$ but also of the true conditional expectation $\mathbb{E}_{t-1}[\xi_t]$ of the random variable ξ_t. In addition to that, knowledge of some of the market fundamentals (stock of assets, aggregate risk-tolerance) is required to apply (32).

Inserting (32) and (31) into the price law (29), one obtains a random difference equation (Arnold 1998)

$$\begin{cases} p_t &= \mu_{t-1,t} + \Gamma\left[\xi_t - \mathbb{E}_{t-1}[\xi_t]\right] \\ \mu_{t,t+1} &= \mu_{t-1,t+1} \\ \mu_{t,t+2} &= \tfrac{R}{\alpha}\Sigma_2 A_1^{-\top}\mu_{t-1,t} + \left[RI_K - \tfrac{1}{\alpha}\Sigma_2 A_1^{-\top}\Gamma^{-1}\right]\mu_{t-1,t+1} \\ & \quad + \tfrac{R}{\alpha}\Sigma_2 A_1^{-\top}\left[\bar{x} - \mathbb{E}_{t-1}[\xi_t]\right] \end{cases} \tag{33}$$

which describes the evolution of expectations and prices in the three-period CAPM under unbiased homogeneous expectations and constant second moment beliefs

Observe that prices at time t are determined from the pervious forecast $\mu_{t-1,t}$ and an additive deviation from the noise traders' transactions. Using (17), the aggregate generational portfolios $x_t^{(j)}$, $j = 1,2$ are

$$x_t^{(1)} := \sum_{i=1}^{I} \varphi^{(i1)}(\mu_{t,t+1}, p_t) = \alpha\Theta^{(1)-1}(\mu_{t,t+1} - Rp_t). \tag{34}$$

and, using the market clearing condition

$$x_t^{(2)} := \sum_{i=1}^{I} \varphi^{(i2)}(\mu_{t,t+1}, \psi_\star(\mu_{t-1,t}, \mu_{t,t+1}, \mathbb{E}_{t-1}[\xi_t]), p_t) = \bar{x} - \xi_t - x_t^{(1)}. \tag{35}$$

The portfolios (34) and (35) confirm above's observation, that under homogeneous unbiased expectations the portfolios held by different generations $j = 1, 2$ will in general not be collinear.

5 Evolution of asset prices under unbiased expectations

Since the unbiased forecasting rule (32) contains no past prices as arguments the evolution of expectations in (33) decouples completely from the evolution of the price process. The qualitative behavior of the expectations process is crucially influenced by the noise traders' portfolios process $\{\xi_t\}_{t\in\mathbb{N}}$. In the particular case in which the noise $\{\xi_t\}_{t\in\mathbb{N}}$ is iid one has $\mathbb{E}_{t-1}[\xi_t] = \mathbb{E}[\xi_t] = \bar{\xi}$ for all $t \in \mathbb{N}$. In this case the dynamic evolution of beliefs will be described by a deterministic dynamical system and only the price process will be stochastic.

To analyze the stability of the dynamical system (33), it suffices to consider the process $\{\mu_t^{(2)}\}_{t\in\mathbb{N}}$ of forecasts generated by the second two equations in (33). These form an affine-linear random difference equation with a (block) coefficient matrix

$$\Lambda := \begin{bmatrix} 0 & I_K \\ \frac{R}{\alpha}\Sigma_2 A_1^{-T} & \left[RI_K - \frac{1}{\alpha}\Sigma_2 A_1^{-T}\Gamma^{-1}\right] \end{bmatrix} \in \mathbb{R}^{2K\times 2K}. \tag{36}$$

The long-run behavior of the random difference equation (33) is described by random attractors which is the random analogue of an attractor of a deterministic system (see Arnold 1998, p. 483). Each path starting from the corresponding domain of attraction will then eventually end up on such an attractor. Typical candidates for these special orbits are generated by asymptotically stable *random fixed points*.[6] A random fixed point may be seen as a special solution to the difference equation (33) that induces a stationary and ergodic process. Loosely speaking, asymptotic stability of a random fixed point means that for almost all noise paths $\{\xi_t\}_{t\in\mathbb{N}}$, all paths starting from sufficiently close initial conditions eventually converge to paths of the random fixed point and after sufficiently long time be indistinguishable from the paths of the random fixed point. The following theorem provides conditions under which a globally asymptotically stable random fixed point obtains.

Theorem 3
Let Assumption 2 on the noise process $\{\xi_t\}_{t\in\mathbb{N}}$ be satisfied. In addition, assume that all eigenvalues of the coefficient matrix $\Lambda \in \mathbb{R}^{2K\times 2K}$ defined in (36) are smaller than one in modulus, i.e. $\chi(\lambda) := \det(\Lambda - \lambda I_{2K}) \neq 0 \ \forall \lambda \in \mathbb{C}$ with $|\lambda| \geq 1$. Then the expectations process defined by (33) possesses a unique random fixed point $\{\mu_t^\star\}_{t\in\mathbb{N}}$ which is globally asymptotically stable.

The proof of Theorem 3 follows directly from Arnold (1998), Corollary 5.6.6. From the definition of the coefficient matrix (36) it follows that stability of the random fixed point

[6] In classical terminology, a random fixed point of (33) corresponds to a *'steady state solution'* of a linear stochastic system, see Hannan & Deistler (1988, Chap. 1).

is essentially determined by the entries of the subjective variance-covariance matrix (28) and the safe rate $R > 0$. This result imposes further restrictions on the class of admissible second moment beliefs to ensure stability under homogeneous unbiased expectations. Assuming that the conditions given in Theorem3 are satisfied, the long run behavior of (33) will be described by a stationary stochastic process with constant (unconditional) moments. Moreover, exploiting ergodicity, the realizations observed along a particular sample path can be used to estimate the moments of the invariant distribution associated with the random fixed point.

To obtain further insight into the nature of the price process in (33), consider the case with a single risky asset $K = 1$. In this case, the block matrix entries of the variance-covariance matrices in (28) reduce to scalars. Using lowercase letters for these entries, (28) is written as

$$\Sigma^{(2)} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \in \mathcal{M}_2, \tag{37}$$

and the corresponding parameters as

$$a_1 = R - \frac{\sigma_{12}}{\sigma_{11}} \qquad \text{and} \qquad \sigma_2 = \sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}. \tag{38}$$

The coefficient matrix (36) reads $\Lambda = \begin{bmatrix} 0 & 1 \\ -R\lambda_2 & R + \lambda_2 \end{bmatrix}$ with

$$-\lambda_2 \;:=\; \frac{(1+R)\sigma_2}{R\sigma_{11} - \sigma_{12}} + a_1 = \frac{(1+R)(\sigma_{22}\sigma_{11} - \sigma_{12}^2) + (R\sigma_{11} - \sigma_{12})^2)}{(R\sigma_{11} - \sigma_{12})\sigma_{11}}$$

The eigenvalues of Λ are $\lambda_1 = R$ and λ_2. Due to the assumption $\Sigma^{(2)} \in \mathcal{M}_2^\star$ we have $a_1 \neq 0$ and therefore $\lambda_2 < 0 \iff a_1 > 0$. This yields the following corollary to Theorem 3.

Corollary 2
For $K = 1$ the random difference equation (33) possesses a unique globally stable random fixed if and only if

$$(i)\ R < 1, \quad (ii)\ \frac{(1+R)(\sigma_{22}\sigma_{11} - \sigma_{12}^2) + (R\sigma_{11} - \sigma_{12})^2}{(R\sigma_{11} - \sigma_{12})\sigma_{11}} \begin{cases} < 1, & \text{if } R > \frac{\sigma_{12}}{\sigma_{11}} \\ > -1, & \text{if } R < \frac{\sigma_{12}}{\sigma_{11}} \end{cases}.$$

Condition (i) in Corollary 2 parallels Theorem 3.2 in (Böhm & Chiarella 2000) for the case with a one-period planning horizon ($J = 1$). In addition to that, Condition (ii) imposes further assumptions on the subjective variance-covariance matrices to ensure asymptotic stability of (33) in the multiperiod case. Namely, σ_{22} must be small in relation to σ_{11}.[7] Stability of the system (33), requires moreover a safe rate R which is

[7] It may seem reasonable to choose $\sigma_{22} = \sigma_{11}$ such that investors hypothesize a constant variance of the price process over time. This, however, would lead to an unstable price process.

smaller than one. At first sight this may seem quite unrealistic, although it should be stressed that R denotes a *real* rather than a *nominal* rate of return.

For our numerical investigation we choose the following parameters. For the process $\{\xi_t\}_{t\in\mathbb{N}}$ describing noise traders' portfolios we assume a stable $AR(1)$-process of the form

$$\xi_t \quad = \quad \gamma^{(0)} + \gamma^{(1)}\xi_{t-1} + \gamma^{(2)}\varepsilon_t. \tag{39}$$

The process $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of innovations consists of uncorrelated random variables drawn from a standard normal distribution such that $\mathbb{E}[\varepsilon_t] = 0$, $\mathbb{V}[\varepsilon_t] = 1$ for all $t \in \mathbb{N}$ and $\mathbb{E}[\varepsilon_t\varepsilon_s] = 0$, $t \neq s$. Since the stock of risky assets was set to $\bar{x} = 100$, the parameters $\gamma^{(h)}$, $h = 0, 1, 2$ were chose such that the portfolios of noise traders take positive values between 0 and 100. All parameters are displayed in the following Table 1.

Parameter	Value	Description	Parameter	Value	Description
α	50	Aggr.risk tolerance	$\gamma^{(1)}$	0.9	AR-parameter
σ_{11}	5	Variance p_{t+1}	$\gamma^{(2)}$	2.5	AR-parameter
σ_{22}	1.875	Variance p_{t+2}	R	0.99	Safe rate
σ_{12}	2.5	Covariance p_{t+1}, p_{t+2}	$\bar{x}$	100	Stock of assets
$\gamma^{(0)}$	5	AR-parameter	p_0	60	Initial price

Table 1: Parameter set.

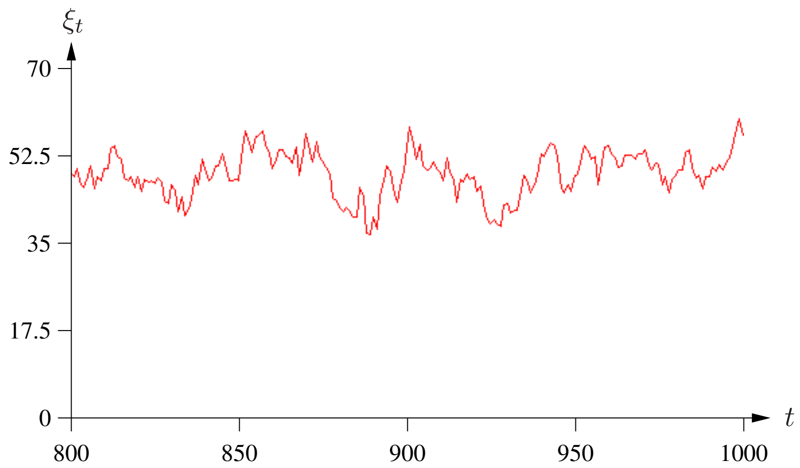

Figure 2: Time series window with noise trader portfolios

Figure 2 depicts a typical realization of the noise process $\{\xi_t\}_{t\in\mathbb{N}}$, which describes the noise traders' portfolios while Figure 3 illustrates an asymptotically stable random fixed point as follows. Using the same realization of the noise process displayed in Figure 2, Figure 3 displays three different time series corresponding to three different initial values $\mu_0^1 = 50$, $\mu_0^2 = 60$ and $\mu_0^3 = 70$ of the expectations process $\{\mu_{t-1,t}\}_{t\in\mathbb{N}}$. Independently of

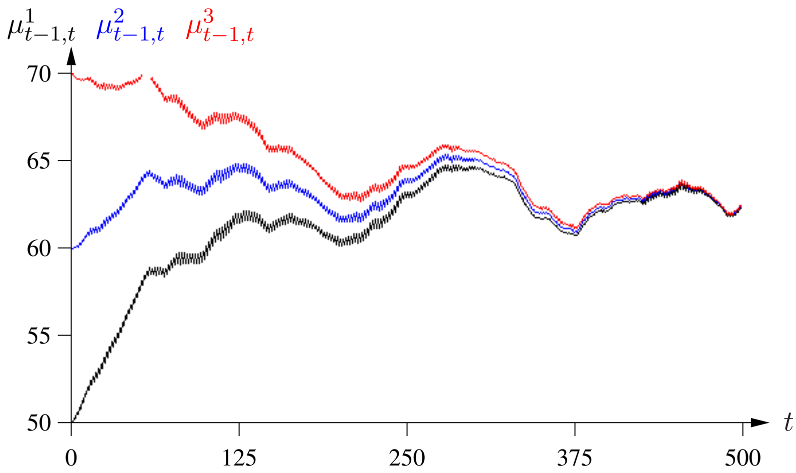

Figure 3: Convergence of expectations for alternative initial values.

these initial conditions, all three time series converge to the same path within the first 500 iterations.

The next two figures portray the dynamic behavior of the price process of the risky asset. Figure 4 shows the induced sample path, the corresponding empirical density defined by relative frequencies is depicted in Figure 5. The latter is based on a length of iteration of $T = 10^6$. The associated empirical moments are summarized in the table below.

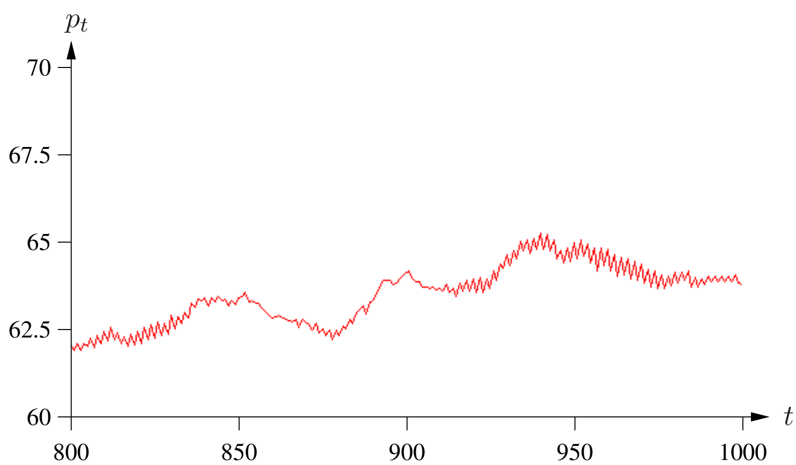

Figure 4: Time series window with the price process

Prices are strictly positive taking values in the interval $[60, 70]$. As an important observation, note that the volatility of the price process appears not to be constant over time but in some regions ($t \in [800, 840]$, $t \in [920, 1000]$) appears to be significantly greater than in other regions ($t \in [840, 870]$,$t \in [890, 910]$). This feature was observed throughout all simulations carried out and is very much in line with empirical observations suggesting that the conditional variance of the price process is time-dependent and the process undergoes phases of high volatility followed by phases with low volatility

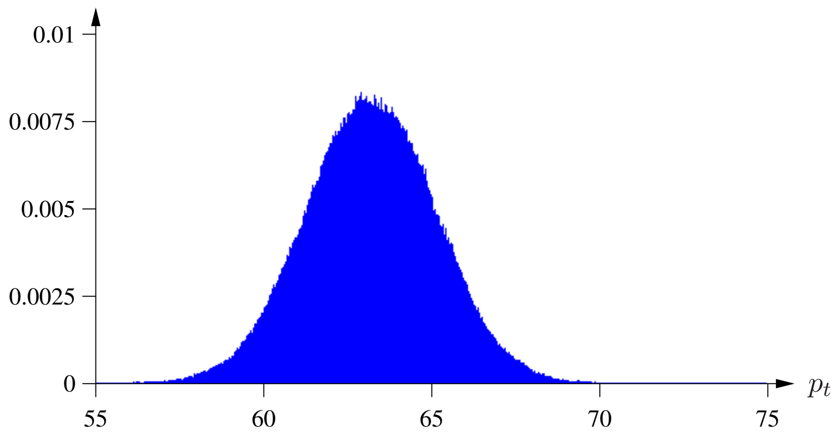

Figure 5: Empirical density/frequencies of prices

Statistic	Estimate	Statistic	Estimate
Mean	63.1962	Variance	3.78956
Std.deviation	1.94668	Skewness	-0.0189265

(*volatility-clustering*). Hence it seems that our simple multiperiod model is capable of replicating a stylized fact of empirically observed in financial data.

The following Figures 6–8 show the evolution of portfolios of the two generations $j = 1, 2$ and of noise traders. First observe that young investors hold throughout a larger quantity of the risky asset than the middle-aged generation, confirming the result established above. In addition to that, one observes that volatility in the portfolios of noise traders is fully absorbed by a movement in the portfolios of young agents into the opposite direction. However, portfolios of young investors seem to fluctuate significantly more than the noise process. Furthermore, one again observes the phenomenon of clustered volatility in the portfolios of the two generations. This holds in particular for the portfolios of middle-aged investors belonging to generation $j = 1$.

We finish this section with Figure 9 by displaying a time series of risky returns $\{r_t\}_{t\in\mathbb{N}}$ with $r_t := \frac{p_t - p_{t-1}}{p_{t-1}}$ induced by the price process. As in empirical return series, we again find strong volatility clustering.

6 Conclusions

We developed a fully explicit and dynamic model of a financial market in which investors with different planning horizons of *arbitrary finite length* interact. An innovative feature of the models is that investors are allowed to reoptimize their portfolios through the

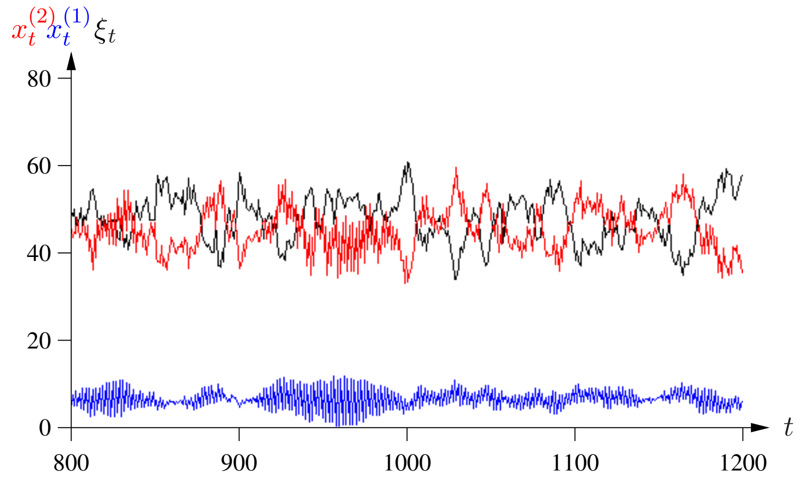

Figure 6: Portfolios of young and middle-aged investors and noise traders.

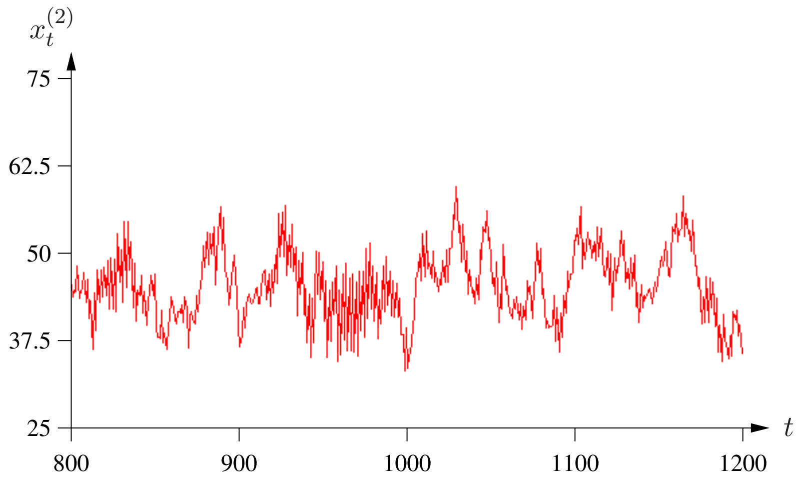

Figure 7: Time series window with portfolios of young investors.

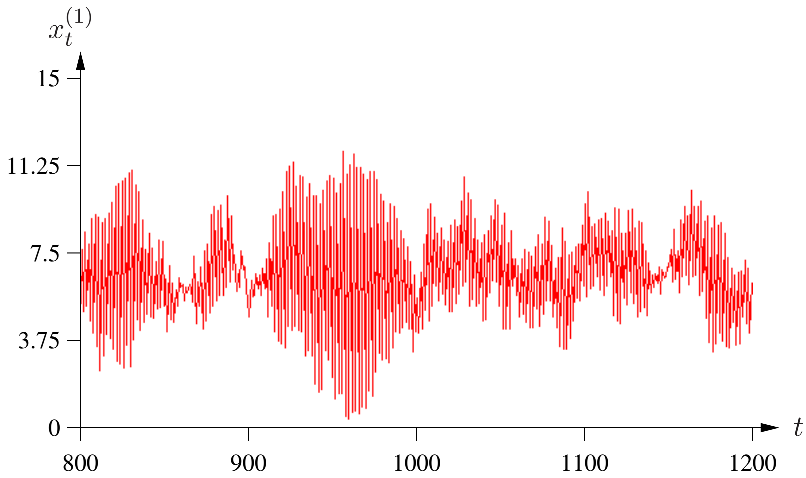

Figure 8: Time series window with portfolios of middle-aged investors.

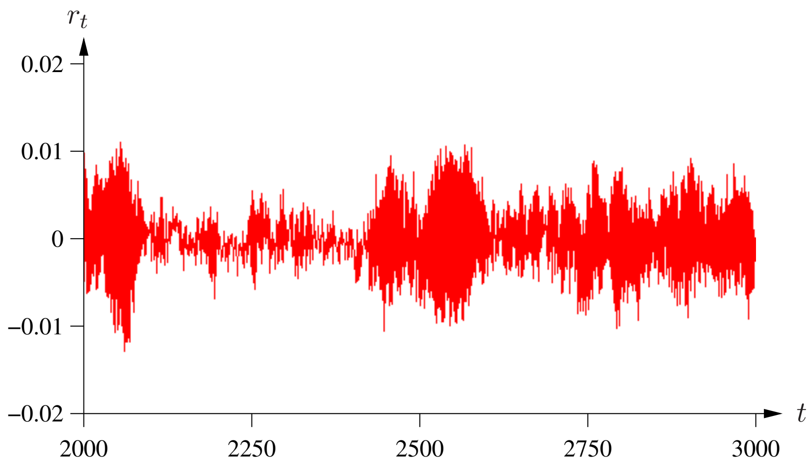

Figure 9: Time series window with asset returns.

course of time until they leave the market. In addition to possibly diverse beliefs, this model allows to investigate the impact of different planning horizons on asset prices, returns, and portfolios. Introducing the notion of a *generational portfolio*, we extended the notion of a market portfolio and showed that traders with different planning horizons will generally hold different portfolios even if expectations are homogeneous. Moreover, we provided numerical evidence that different planning horizons account for volatility clustering in time series of asset prices, returns and portfolios. It would be interesting to now what *mean-variance efficient* portfolios look alike in the presence of multiperiod planning horizons and whether the results on their performance Böhm & Wenzelburger (2002) can be generalized.

A Appendix

A.1 Proof of Theorem 1.

Consider the decision problem (7) of investor $(i,j) \in \mathbb{I}$ in period $t = 0$, with planning horizon $j > 0$. Since the case with a one-period planning horizon ($j = 1$) is well-understood (e.g., see Ingersoll 1987) resulting in the demand function (9), we assume $j > 1$. For notational convenience, the index (i,j) as well as the time index $t = 0$ will be omitted in the sequel. Expectations at time $t = 0$ for prices $p_1, \ldots, p_j$ are given by a joint multivariate normal distribution characterized by the first two moments given in (5). In order to solve the decision problem, we need to impose certain restrictions on variance-covariance matrices (5). Setting $\Sigma_1^j := \Sigma$, for each $s = 2, \ldots, j$ we partition the matrix Σ_1^s into $\Sigma_1^s = \begin{bmatrix} \Sigma_1^{s-1} & C_s^\top \\ C_s & \Sigma_{ss} \end{bmatrix}$ with $C_s := [\Sigma_{s,1} \ldots \Sigma_{s,s-1}] \in \mathbb{R}^{K \times K(s-1)}$ and define the following parameters

$$\begin{aligned} \Sigma_s &:= \Sigma_{ss} - C_s\, [\Sigma_1^{s-1}]^{-1} C_s^\top \in \mathcal{M}_K \\ B_s &= \left[B_s^{(s-1)} \ldots B_s^{(1)}\right] := C_s \left[\Sigma_1^{s-1}\right]^{-1} \in \mathbb{R}^{K \times K(s-1)}, \end{aligned} \tag{40}$$

where $B_s^{(h)} \in \mathbb{R}^{K \times K}$, $h = 1, \ldots, s-1$. Note that each Σ_s in (40) is well defined, symmetric, and positive definite, cf. Ouellette (1981, Corollary 3.1, p. 208). With the above definitions the following Lemma describes a factorization of the joint probability distribution into marginal and conditional distributions.

Lemma 1
Let the joint distribution of the random variables $p_1, \cdots, p_j$ be a normal distribution with moments $(\mu, \Sigma) \in \mathbb{R}^{Kj} \times \mathcal{M}_{Kj}$ given in (5). Then the following holds:

(1) *For each $s = 2, \ldots, j$ the conditional distribution of the random variable p_s given previous observations $p_1, \ldots, p_{s-1}$ is given by a non-singular normal distribution on $\mathbb{R}^K$ with moments $(\mu_{s|s-1}, \Sigma_s)$, where*

$$\mu_{s|s-1} := \mu_s + B_s^{(1)}(p_{s-1} - \mu_{s-1}) + \ldots + B_s^{(s-1)}(p_1 - \mu_1). \tag{41}$$

and B_s, Σ_s are defined in (40).

(2) *The marginal distribution of the random variable p_1 is given by a non-singular normal distribution on $\mathbb{R}^K$ with moments $(\mu_1, \Sigma_{11}) \in \mathbb{R}^K \times \mathcal{M}_K$ given in (5).*

Proof: The assertion follows from a repeated application of Theorems 2.4.3 and 2.5.1 given in Anderson (1984), p. 37, see also Tong (1990), Theorem 3.3.4, p. 35. ■

Letting $0_K, I_K \in \mathbb{R}^{K \times K}$ denote the $K \times K$ zero and identity matrix, respectively, define for each $s = 0, 1, \ldots, j-1$ the following parameters

$$\begin{aligned} \Pi_{s,n} &:= \Big[\underbrace{0_K, \ldots, 0_K}_{s \text{ terms}}, \underbrace{I_K, RI_K, \ldots, R^{n-s-1} I_K}_{n-s \text{ terms}}\Big]^\top \in \mathbb{R}^{nK \times K},\ n = s+1, \ldots, j \\ \pi_{s,m} &:= \Big(\underbrace{\mu_1^\top - p_1^\top, \ldots, \mu_s^\top - p_s^\top}_{s \text{ terms}}, \underbrace{\mu_{s+1}^\top, \ldots, \mu_m^\top}_{m-s \text{ terms}}\Big)^\top \in \mathbb{R}^{mK},\ m = s, \ldots, j. \end{aligned} \tag{42}$$

In particular, $\Pi_{0,n} \equiv \Pi_n = \left[I_K, \dots, R^{n-1} I_K\right]^\top$ and $\pi_{0,m} \equiv \pi_m = \left(\mu_1^\top, \dots, \mu_m^\top\right)^\top$. Given the parameter definitions (40) and (42) some important relations which will be exploited in the sequel are collected in Lemma 2.
The next assumption imposes further restrictions on the covariance matrix Σ from (5).

Assumption 3
(Hillebrand 2003, Annahme 3.3, p. 66) All variance covariance matrices (5) satisfy the following condition. Given the parameters (40) and (42) the matrices

$$A_s \quad := \quad [-B_s, I_K]\,\Pi_{s-2,s} = RI_K - B_s^{(1)} \in \mathbb{R}^{K\times K}, \quad s = 2, \cdots, j, \tag{43}$$

are non-singular and hence invertible. For each $j = 1, \dots, J$, the subset of all variance covariance matrices satisfying (43) are denoted by $\mathcal{M}_{Kj}^\star$.

Observe that each set $\mathcal{M}_{Kj}^\star$ contains all covariance matrices (5) with $\Sigma_{ss'} = 0$ for $s \neq s'$. Hence the set is non-empty for each j. In particular, $\mathcal{M}_K^\star = \mathcal{M}_K$ for $j = 1$.

Consider now the investor's decision problem (7). To derive a solution, we will employ a dynamic programming approach. This amounts to solving a sequence of j one-stage problems. For each stage $s = 0, 1, \dots, j-1$ a one period-problem is solved given wealth[8]

$$w_s = Ry_{s-1} + x_{s-1}^\top p_s$$

and realizations of prices $p_1, \dots, p_s$ up to time s. For the following derivations we set $\mu_{s|s-1} := \mu_1$ and $\Sigma_s := \Sigma_{11}$ for $s = 1$ and write $p_1^s := (p_1, \dots, p_s)$. For each $s = 1, \dots, j$ the conditional distribution of prices p_s given pervious observations $p_1, \dots, p_{s-1}$ is given by a normal distribution with moments $(\mu_{s+1|s}, \Sigma_{s+1})$ given in Lemma 1(1). Setting $V_j(w_j, p_1^j) \equiv u(w_j; a)$ we employ for each $s = 1, \dots, j-1$ the following recursive relation, referred to as Bellmann's equation (e.g., see Pliska 1997):

$$V_s(w_s, p_1^s) = \max_{y + x^\top p_s = w_s} \left\{ \int_{\mathbb{R}^K} V_{s+1}(Ry + x^\top p, p_1^s, p)\; f(p; \mu_{s+1|s}, \Sigma_{s+1}) dp \right\}. \tag{44}$$

The function V_s is called *value function* of period s. The **principle of optimality** reads

$$\begin{aligned} V_0(w, p) \quad &:= \quad \max_{y + x^\top p = w} \left\{ \int_{\mathbb{R}^K} V_1(Ry + x^\top p, p)\; f(p; \mu_1, \Sigma_{11}) dp \right\} \\ &= \quad \max_{H \in \mathscr{H}(p,w)} \left\{ \int_{\mathbb{R}^{Kj}} u(W_j(H, p_1^j); a)\; f_{Kj}(p_1^j; \mu, \Sigma)\, dp_1^j \right\}. \end{aligned} \tag{45}$$

Equation (45) states that we can derive an optimal portfolio decision for $t = 0$ as a solution to a one-stage problem involving the value function V_1 together with the marginal distribution of p_1. In order to utilize this fact we first need to show that the sequence of value functions defined in (44) is well-defined and then compute an explicit functional form for the value function V_1.

[8] Since the investor is assumed to choose a self-financing strategy, no wealth is added from outside and no consumption takes place prior to period j. Hence wealth accumulated at stage s is equal to the return on the investment made in previous period

For this purpose we introduce the notion of a Gaussian function. A *Gaussian function* in $x \in \mathbb{R}^N$ with parameters $(c, \mu, \Sigma) \in \mathbb{R}_{++} \times \mathbb{R}^N \times \mathcal{M}_N$ is a real-valued function

$$g_N(x; c, \mu, \Sigma) := c \exp\left\{ -\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu) \right\}, \quad x \in \mathbb{R}^N. \tag{46}$$

Setting, $c_N(\Sigma) := (2\pi)^{-\frac{N}{2}} [\det \Sigma]^{-\frac{1}{2}}$, we have the relation

$$f_N(x; \mu, \Sigma) \quad = \quad g_N(x; c_N(\Sigma), \mu, \Sigma). \tag{47}$$

To simplify notation, we shall frequently suppress the dimension index N. Furthermore, we will omit parameter c as an argument of a Gaussian function if $c = 1$ and write $g(x; \mu, \Sigma) \equiv g(x; 1, \mu, \Sigma)$. Important properties of Gaussian functions are collected in Appendix A.2.

The next proposition provides the desired representation of the value functions.

Proposition 1
Let Assumption 3 be satisfied. Then for each $s = 1, \cdots, j-1$ the value functions V_s defined in (44) are well defined and take the form

$$V_s(w_s, p_1^s) = u\left(w_s; aR^{j-s}\right)\ g\left(p_s; c_s, \vartheta_s, \Omega_s\right), \tag{48}$$

where for each $s = 1, \ldots, j-1$

$$\begin{aligned}
\Omega_s &:= \left[\Pi_{s-1,j}^\top \Sigma^{-1} \Pi_{s-1,j} - \Sigma_s^{-1}\right]^{-1} \\
\vartheta_s &:= \Omega_s \left(\Pi_{s-1,j} \Sigma^{-1} \pi_{s-1,j} - \Sigma_s^{-1} \mu_{s|s-1}\right) \\
\tilde{c}_s &:= \begin{cases} 1 & s = j-1 \\ \displaystyle\prod_{n=1}^{j-s-1} \frac{c\left(\Sigma_{s+n}^{-1} + \Omega_{s+n}^{-1}\right)}{c\left(\Sigma_{s+n}^{-1}\right)} & s = 1, \ldots, j-2, \quad j > 2 \end{cases} \\
c_s &:= \frac{g(0; \tilde{c}_s, \pi_{s-1,j}, \Sigma)}{g(0; \pi_{s-1,s}, \Sigma_1^s)\ g(0; \vartheta_s, \Omega_s)}
\end{aligned} \tag{49}$$

We preface the proof of Proposition 1 by showing that the matrices Ω_s appearing in (49) are well-defined. From Lemma 2 (2c) we obtain

$$\Pi_{s-1,s}^\top \Sigma^{-1} \Pi_{s-1,s} - \Sigma_s^{-1} \quad = \quad \sum_{n=1}^{j-s} \Pi_{s-1,s+n}^\top \begin{bmatrix} B_{s+n}^\top \\ -I_K \end{bmatrix} \Sigma_{s+n}^{-1} \begin{bmatrix} B_{s+n} - I_K \end{bmatrix} \Pi_{s-1,s+n} \tag{50}$$

As noted earlier, each Σ_{s+n} $(n = 1, \ldots, j-s)$ is a symmetric, positive definite matrix. Furthermore, the first term appearing in the sum in (50) is given by $A_{s+1}^\top \Sigma_{s+1}^{-1} A_{s+1}$ where $A_{s+1} = -[B_{s+1}, -I_K]\Pi_{s-1,s+1} = RI_K - B_{s+1}^{(1)}$ is invertible by Assumption 3. Hence this term is again a positive definite, symmetric matrix while all the other terms appearing in the sum in (50) are positive-semi-definite matrices. This implies that each Ω_s is a symmetric, positive definite matrix and the scalars $\tilde{c}_s$ and c_s defined in (49) are strictly positive real number (the latter for each realization of previous prices $p_1, \ldots, p_{s-1}$).

Proof of Proposition 1: The proof consists of an induction argument and proceeds in two steps. In step one, we verify directly that the claim is true for $s = j-1$. For $j = 2$, this is already sufficient. In a second step we assume that $j > 2$ and prove the claim for arbitrary s given that it is true for $s+1$.

Step 1: Let $s = j-1$ and let wealth w_{j-1} and realizations of prices $p_1, \ldots, p_{j-1}$ be given. Consider the following maximization problem:

$$U_{j-1}(x, y, p_1^{j-1}) := \int_{\mathbb{R}^K} u(Ry + x^\top p; a)\; f(p; \mu_{j|j-1}, \Sigma_j)\; dp \longrightarrow \max_{y + x^\top p_{j-1} = w_{j-1}}. \tag{51}$$

Using (47) and Lemma 4 (setting $\alpha = a$, $c = c(\Sigma_j)$, $\theta = \mu_{j|j-1}$ and $\Theta = \Sigma_j$) the objective function $U_{j-1}(\cdot)$ in (51) can be rewritten to:

$$U_{j-1}(x, y; p_1^{j-1}) \;\; = \;\; u\left(Ry + x^\top \mu_{j|j-1} - \frac{a}{2}\, x^\top \Sigma_j\; x;\; a\right). \tag{52}$$

Using (52) in (51) and applying Lemma 5 (setting $\alpha = a$, $c = c(\Sigma_j)$, $\theta = \mu_{j|j-1}$, $\Theta = \Sigma_j$, $w = w_{j-1}$ and $p = p_{j-1}$) the maximum of problem (51) can be written as

$$\begin{aligned} V_{j-1}(w_{j-1}, p_1^{j-1}) \;\; &:= \;\; \max_{x,y}\left\{ U_{j-1}(x, y; p_1^{j-1}) \,\middle|\, y + x^\top p_{j-1} = w_{j-1} \right\} \\ &= \;\; u\big(w_{j-1}; aR\big)\;\; g\big(Rp_{j-1}; \mu_{j|j-1}, \Sigma_j\big) \end{aligned} \tag{53}$$

The assertion will thus follow if we show that

$$g(Rp_{j-1}; \mu_{j|j-1}, \Sigma_j) = g(p_{j-1}, \vartheta_{j-1}, \Omega_{j-1}) \quad \text{and} \quad c_{j-1} = 1.$$

Note from (41) and (42) that $(\mu_{j|j-1} - Rp_{j-1}) = [B_j, -I_K]\,(\Pi_{j-2,j}\; p_{j-1} - \pi_{j-2,j})$. Applying Lemma 2(1) we can rewrite the quadratic form in the exponent of $g(Rp_{j-1}; \mu_{j|j-1}, \Sigma_j)$ as follows:

$$\begin{aligned} & (Rp_{j-1} - \mu_{j|j-1})^\top \Sigma_j^{-1} (Rp_{j-1} - \mu_{j|j-1}) \\ = \;\; & (\Pi_{j-2,j} p_{j-1} - \pi_{j-2,j})^\top\, [B_j, -I_K]^\top \Sigma_j^{-1}\, [B_j, -I_K]\, (\Pi_{j-2,j} p_{j-1} - \pi_{j-2,j}) \\ = \;\; & (\Pi_{j-2,j} p_{j-1} - \pi_{j-2,j})^\top \left(\Sigma^{-1} - \begin{bmatrix} [\Sigma_1^{j-1}]^{-1} & 0_{K(j-1)\times K} \\ 0_{K\times K(j-1)} & 0_{K\times K} \end{bmatrix} \right) (\Pi_{j-2,j} p_{j-1} - \pi_{j-2,j}) \\ = \;\; & p_{j-1}^\top \left(\Pi_{j-2,j}^\top \Sigma^{-1} \Pi_{j-2,j} - \Pi_{j-2,j-1}^\top \left[\Sigma_1^{j-1}\right]^{-1} \Pi_{j-2,j-1} \right) p_{j-1} \\ & -2 p_{j-1}^\top \left(\Pi_{j-2,j}^\top \Sigma^{-1} \pi_{j-2,j} - \Pi_{j-2,j-1}^\top \left[\Sigma_1^{j-1}\right]^{-1} \Pi_{j-2,j} \mu_{j-1|j-2} \right) \\ & + \pi_{j-2,j}^\top \Sigma^{-1} \pi_{j-2,j} - \pi_{j-2,j-1}^\top \left[\Sigma_1^{j-1}\right]^{-1} \pi_{j-2,j-1}. \end{aligned} \tag{54}$$

Note that by virtue of Lemma 2(2a) $\Pi_{j-2,j-1}^\top [\Sigma_1^{j-1}]^{-1} \Pi_{j-2,j-1} = \Sigma_{j-1}^{-1}$. Defining the parameters

$$\begin{aligned} \Omega_{j-1} \;\; &:= \;\; \left[\Pi_{j-2,j}^\top \Sigma^{-1} \Pi_{j-2,j} - \Sigma_{j-1}^{-1} \right]^{-1} \\ \vartheta_{j-1} \;\; &:= \;\; \Omega_{j-1} \left(\Pi_{j-2,j}^\top \Sigma^{-1} \pi_{j-2,j} - \Sigma_{j-1}^{-1} \mu_{j-1|j-2} \right). \end{aligned} \tag{55}$$

we obtain from (54)

$$(Rp_{j-1}-\mu_{j|j-1})^\top\Sigma_j^{-1}(Rp_{j-1}-\mu_{j|j-1}) = p_{j-1}^\top\Omega_{j-1}^{-1}p_{j-1} - 2p_{j-1}\Omega_{j-1}^{-1}\vartheta_{j-1} + \pi_{j-2,j}^\top\Sigma^{-1}\pi_{j-2,j} - \pi_{j-2,j-1}^\top\left[\Sigma_1^{j-1}\right]^{-1}\pi_{j-2,j-1} \tag{56}$$

Observing that the matrix $A_j := -[B_j, -I_K]\Pi_{j-2,j}$ is invertible by Assumption 3 and using Lemma 2(1) the last two terms in (56) can be rearranged as follows:

$$\begin{aligned}
&\pi_{j-2,j}^\top\Sigma^{-1}\pi_{j-2,j} - \pi_{j-2,j-1}^\top\left[\Sigma_1^{j-1}\right]^{-1}\pi_{j-2,j-1}\\
=\;& \pi_{j-2,j}^\top\left(\Sigma^{-1} - \begin{bmatrix} [\Sigma_1^{j-1}]^{-1} & 0_{K(j-1)\times K}\\ 0_{K\times K(j-1)} & 0_{K\times K}\end{bmatrix}\right)\pi_{j-2,j}\\
=\;& \vartheta_{j-1}^\top\Omega_{j-1}^{-1}\vartheta_{j-1}.
\end{aligned} \tag{57}$$

Using the result (57) in (56) we therefore obtain

$$(Rp_{j-1}-\mu_{j|j-1})^\top\Sigma_j^{-1}(Rp_{j-1}-\mu_{j|j-1}) = (p_{j-1}-\vartheta_{j-1})^\top\Omega_{j-1}^{-1}(p_{j-1}-\vartheta_{j-1}). \tag{58}$$

It remains to show that the parameter definition (49) implies that $c_{j-1}=1$. Exploiting (57) we can write

$$g(0;\vartheta_{j-1},\Omega_{j-1}) = \frac{g(0;\pi_{j-2,j},\Sigma)}{g(0;\pi_{j-2,j-1},(\Sigma_1^{j-1}))} \tag{59}$$

From (49) we have $\tilde{c}_{j-1}=1$. Using this and (59) in (49) we therefore obtain

$$\begin{aligned}
c_{j-1} &= \frac{g(0;\tilde{c}_{j-1},\pi_{j-2,j},\Sigma)}{g(0;\pi_{j-2,j-1},(\Sigma_1^{j-1}))\;g(0;\vartheta_{j-1},\Omega_{j-1})}\\
&= \frac{g(0;\pi_{j-2,j},\Sigma)}{g(0;\pi_{j-2,j-1},(\Sigma_1^{j-1}))}\cdot\frac{g(0;\pi_{j-2,j-1},(\Sigma_1^{j-1}))}{g(0;\pi_{j-2,j},\Sigma)} = 1.
\end{aligned}$$

Given these results we can write the value function from (53) as

$$V_{j-1}(w_{j-1},p_1^{j-1}) = u\Big(w_{j-1};aR\Big)\,g\Big(p_{j-1};c_{j-1},\vartheta_{j-1},\Omega_{j-1}\Big) \tag{60}$$

where $c_{j-1}=1$ and the parameters $\vartheta_{j-1},\Omega_{j-1}$ are defined by (55). This proves the assertion for $s=j-1$. In particular, we have verified Theorem 1 for the case $j=2$.

Step 2: In the second step we assume that $j > 2$ and prove that the claim is true for arbitrary $s = 1, \ldots, j-2$ given that it is true for $s+1$. The induction hypothesis is thus

$$\begin{aligned} V_{s+1}(w_{s+1}, p_1^{s+1}) &= u\left(w_{s+1}; aR^{j-(s+1)}\right) g\left(p_{s+1}; c_{s+1}, \vartheta_{s+1}, \Omega_{s+1}\right) \end{aligned} \quad (61)$$

$$\begin{aligned} \text{where} \quad \Omega_{s+1} &:= \left[\Pi_{s,j}^\top \Sigma^{-1} \Pi_{s,j} - \Sigma_{s+1}^{-1}\right]^{-1} \\ \vartheta_{s+1} &:= \Omega_{s+1}\left(\Pi_{s,j}\Sigma^{-1}\pi_{s,j} - \Sigma_{s+1}^{-1}\mu_{s+1|s}\right) \\ \tilde{c}_{s+1} &:= \begin{cases} 1 & s+1 = j-1 \\ \prod_{n=2}^{j-s-1} \frac{c\left(\Sigma_{s+n}^{-1} + \Omega_{s+n}^{-1}\right)}{c\left(\Sigma_{s+n}^{-1}\right)} & \text{else} \end{cases} \\ c_{s+1} &:= \tilde{c}_{s+1} \frac{g(0; \pi_{s,j}, \Sigma)}{g(0; \pi_{s,s+1}, \Sigma_1^{s+1})\, g(0; \vartheta_{s+1}, \Omega_{s+1})} \end{aligned} \quad (62)$$

To show that this implies the form (48) of the value function V_s, consider the following optimization problem given wealth w_s and realizations of prices $p_1, \ldots, p_s$:

$$U_s(x, y; p_1^s) := \int_{\mathbb{R}^K} V_{s+1}(Ry + x^\top p, p_1^s, p)\, f(p; \mu_{s+1|s}, \Sigma_{s+1})\, dp \longrightarrow \max_{y + x^\top p_s = w_s}. \quad (63)$$

Given the form (61) of the value function V_{s+1} we may apply Lemma 6 (setting $c = c_{s+1}$, $\vartheta = \vartheta_{s+1}$, $\Omega = \Omega_{s+1}$, $\hat{\mu} = \mu_{s+1|s}$, $\hat{\Sigma} = \Sigma_{s+1}$, $\alpha = aR^{j-s-1}$) to write the objective function in (63) as

$$U_s(x, y; p_1^s) = \frac{\hat{c}_{s+1}}{c(\Theta_{s+1})}\, u\left(Ry + x^\top \theta_{s+1} - \frac{aR^{j-s-1}}{2} x^\top \Theta_{s+1} x; aR^{j-s-1}\right) \quad (64)$$

$$\begin{aligned} \text{where} \quad \Theta_{s+1} &:= \left[\Sigma_{s+1}^{-1} + \Omega_{s+1}^{-1}\right]^{-1} \\ \theta_{s+1} &:= \Theta_{s+1}\left[\Sigma_{s+1}^{-1}\mu_{s+1|s} + \Omega_{s+1}^{-1}\vartheta_{s+1}\right] \\ \hat{c}_{s+1} &:= \frac{g(0; c_{s+1}, \vartheta_{s+1}, \Omega_{s+1}) g(0; c(\Sigma_{s+1}), \mu_{s+1|s}, \Sigma_{s+1})}{g(0; \theta_{s+1}, \Theta_{s+1})}. \end{aligned} \quad (65)$$

Using (64) in (63) and applying Lemma 5(2) (setting $c = \frac{\hat{c}_{s+1}}{c(\Theta_{s+1})}$, $\Theta = \Theta_{s+1}$, $\theta = \theta_{s+1}$, $\alpha = aR^{j-s-1}$, $p = p_s$ and $w = w_s$) the value function associated to problem (63) takes the form

$$\begin{aligned} V_s(w_s, p_1^s) &:= \max_{x,y} \left\{ U_s(x, y; p_1^s) \middle| y + x^\top p_s = w_s \right\} \\ &= u(w_s; aR^{j-s})\, g(Rp_s; \frac{\hat{c}_{s+1}}{c(\Theta_{s+1})}, \theta_{s+1}, \Theta_{s+1}). \end{aligned}$$

The assertion will follow if we show that

$$g(Rp_s; \frac{\hat{c}_{s+1}}{c(\Theta_{s+1})}, \theta_{s+1}, \Theta_{s+1}) = g(p_s; c_s, \vartheta_s, \Omega_s). \quad (66)$$

From (46), (65) and Lemma 3(2) one observes the relation

$$g(Rp_s; \frac{\hat{c}_{s+1}}{c(\Theta_{s+1})}, \theta_{s+1}, \Theta_{s+1}) = \frac{c(\Sigma_{s+1})}{c(\Theta_{s+1})}\, g(Rp_s; \mu_{s+1|s}, \Sigma_{s+1})\, g(Rp_s; c_{s+1}, \vartheta_{s+1}, \Omega_{s+1}). (67)$$

Using equations (46) and the definition of parameter c_{s+1} given in (65) we can expand the second term on the r.h.s of (67) as

$$
\begin{aligned}
& g(Rp_s; \mu_{s+1|s}, \Sigma_{s+1})\, g(Rp_s; c_{s+1}, \vartheta_{s+1}, \Omega_{s+1}) \qquad (68)\\
= \quad & \underbrace{\tilde{c}_{s+1} \frac{g(0; \pi_{s,j}, \Sigma)}{g(0; \pi_{s,s+1}, \Sigma_1^{s+1})\, g(0; \vartheta_{s+1}, \Omega_{s+1})}}_{c_{s+1}}\, g(Rp_s; \mu_{s+1|s}, \Sigma_{s+1})\, g(Rp_s; , \vartheta_{s+1}, \Omega_{s+1})
\end{aligned}
$$

Using the definitions (65) the exponents of the Gaussian functions appearing on the r.h.s. of (68) can be summarized as

$$
\begin{aligned}
- \quad & \frac{1}{2} \Big[\pi_{s,j}^\top \Sigma^{-1} \pi_{s,j} - \pi_{s,s+1}^\top (\Sigma_1^{s+1})^{-1} \pi_{s,s+1} + (Rp_s - \vartheta_{s+1})^\top \Omega_{s+1}^{-1} (Rp_s - \vartheta_{s+1}) \\
& \quad - \vartheta_{s+1}^\top \Omega_{s+1}^{-1} \vartheta_{s+1} + (Rp_s - \mu_{s+1|s}^\top) \Sigma_{s+1}^{-1} (Rp_s - \mu_{s+1|s}) \Big] \\
= \quad & -\frac{1}{2} \Big[\mu_{s+1|s}^\top \Sigma_{s+1}^{-1} \mu_{s+1|s} + Rp_s^\top \Pi_{s,j}^\top \Sigma^{-1} \Pi_{s,j} Rp_s - 2Rp_s^\top \Pi_{s,j}^\top \Sigma^{-1} \pi_{s,j} \\
& \quad + \pi_{s,j}^\top \Sigma^{-1} \pi_{s,j} - \pi_{s,s+1}^\top (\Sigma_1^{s+1})^{-1} \pi_{s,s+1} \Big] \\
= \quad & -\frac{1}{2} \Big[(\Pi_{s,j} Rp_s - \pi_{s,j})^\top \Sigma^{-1} (\Pi_{s,j} Rp_s - \pi_{s,j}) \\
& \quad + \mu_{s+1|s}^\top \Sigma_{s+1}^{-1} \mu_{s+1|s} - \pi_{s,s+1}^\top (\Sigma_1^{s+1})^{-1} \pi_{s,s+1} \Big] \qquad (69)
\end{aligned}
$$

Note from (41) and (42) that $\mu_{s+1|s} = -[B_{s+1}, -I_K]\pi_{s,s+1}$ and therefore, applying Lemma 2(1)

$$
\begin{aligned}
& \mu_{s+1|s}^\top \Sigma_{s+1}^{-1} \mu_{s+1|s} - \pi_{s,s+1}^\top (\Sigma_1^{s+1})^{-1} \pi_{s,s+1} \\
= \quad & \pi_{s,s+1}^\top \left([B_{s+1}, -I_K]^\top \Sigma_{s+1}^{-1} [B_{s+1}, -I_K] - (\Sigma_1^{s+1})^{-1} \right) \pi_{s,s+1} \\
= \quad & \pi_{s,s+1}^\top \begin{bmatrix} (\Sigma_1^s)^{-1} & 0_{Ks \times K} \\ 0_{K \times Ks} & 0_{K \times K} \end{bmatrix} \pi_{s,s+1} = \pi_{s,s}^\top (\Sigma_1^s)^{-1} \pi_{s,s}.
\end{aligned}
$$

Note from the definitions (42) that $\pi_{s,s} = (\pi_{s-1,s} - \Pi_{s-1,s} p_s)$ and $(\Pi_{s,j} Rp_s - \pi_{s,j}) = (\Pi_{s-1,j} p_s - \pi_{s-1,j})$. Now define the parameters (Ω_s, ϑ_s) as in (49) and note that $\Sigma_s^{-1} = \Pi_{s-1,s}^\top (\Sigma_1^s)^{-1} \Pi_{s-1,s}$ by virtue of Lemma 2(2a). With these properties write (69) as

$$
\begin{aligned}
- \quad & \frac{1}{2} \Big[(\Pi_{s,j} Rp_s - \pi_{s,j})^\top \Sigma^{-1} (\Pi_{s,j} Rp_s - \pi_{s,j}) + \mu_{s+1|s}^\top \Sigma_{s+1}^{-1} \mu_{s+1|s} - \pi_{s,s+1}^\top (\Sigma_1^{s+1})^{-1} \pi_{s,s+1} \Big] \\
= \quad & -\frac{1}{2} \Big[(\Pi_{s-1,j} p_s - \pi_{s-1,j})^\top \Sigma^{-1} (\Pi_{s-1,j} p_s - \pi_{s-1,j}) \\
& + (\Pi_{s-1,s} p_s - \pi_{s-1,s})^\top (\Sigma_1^s)^{-1} (\Pi_{s-1,s} p_s - \pi_{s-1,s}) \Big] \\
= \quad & -\frac{1}{2} \Big[p_s^\top \left(\Sigma^{-1} - \Pi_{s-1,s}^\top (\Sigma_1^s)^{-1} \Pi_{s-1,s} \right) p_s + \pi_{s-1,j}^\top \Sigma^{-1} \pi_{s-1,j} \\
& - 2p_s^\top \left(\Pi_{s-1,j}^\top \Sigma^{-1} \pi_{s-1,j} - \Pi_{s-1,s}^\top \Sigma_1^s)^{-1} \pi_{s-1,s} \right) - \pi_{s-1,s}^\top (\Sigma_1^s)^{-1} \pi_{s-1,s} \Big] \\
= \quad & \frac{1}{2} \Big[p_s^\top \Omega_s^{-1} p_s - 2p_s^\top \Omega_s^{-1} \vartheta_s \pm \vartheta_s^\top \Omega_s^{-1} \vartheta_s + \pi_{s-1,j}^\top \Sigma^{-1} \pi_{s-1,j} - \pi_{s-1,s}^\top (\Sigma_1^s)^{-1} \pi_{s-1,s} \Big] \\
= \quad & \frac{1}{2} \Big[(p_s - \vartheta_s)^\top \Omega_s^{-1} (p_s - \vartheta_s) - \vartheta_s^\top \Omega_s^{-1} \vartheta_s + \pi_{s-1,j}^\top \Sigma^{-1} \pi_{s-1,j} - \pi_{s-1,s}^\top (\Sigma_1^s)^{-1} \pi_{s-1,s} \Big] \quad (70)
\end{aligned}
$$

Using the results from (69) and (70) the exponents of the Gaussian functions on the r.h.s. of (68) can thus be rewritten to

$$\begin{aligned} - & \frac{1}{2}\Big[\pi_{s,j}^\top\Sigma^{-1}\pi_{s,j} - \pi_{s,s+1}^\top(\Sigma_1^{s+1})^{-1}\pi_{s,s+1} + (Rp_s - \vartheta_{s+1})^\top\Omega_{s+1}^{-1}(Rp_s - \vartheta_{s+1}) \\ & \quad -\vartheta_{s+1}^\top\Omega_{s+1}^{-1}\vartheta_{s+1} + (Rp_s - \mu_{s+1|s}^\top)\Sigma_{s+1}^{-1}(Rp_s - \mu_{s+1|s})\Big] \\ = & -\frac{1}{2}\Big[(p_s - \vartheta_s)^\top\Omega_s^{-1}(p_s - \vartheta_s) + \pi_{s-1,j}^\top\Sigma^{-1}\pi_{s-1,j} - \pi_{s-1,s}^\top(\Sigma_1^s)^{-1}\pi_{s-1,s} - \vartheta_s^\top\Omega_s^{-1}\vartheta_s\Big] \end{aligned} \tag{71}$$

Using (71) the r.h.s of (68) can thus be written as

$$\begin{aligned} & \tilde{c}_{s+1}\,\frac{g(0;\pi_{s,j},\Sigma)}{g(0;\pi_{s,s+1},\Sigma_1^{s+1})\;g(0;\vartheta_{s+1},\Omega_{s+1})}\;g(Rp_s;\mu_{s+1|s},\Sigma_{s+1})\;g(Rp_s;,\vartheta_{s+1},\Omega_{s+1}) \\ = & \tilde{c}_{s+1}\,\frac{g(0;\pi_{s-1,j},\Sigma)}{g(0;\pi_{s-1,s},\Sigma_1^s)\;g(0;\vartheta_s,\Omega_s)}\;g(p_s;,\vartheta_s,\Omega_s) \end{aligned} \tag{72}$$

Observe from (49) and (65) the recursive relation $\tilde{c}_s = \frac{c(\Sigma_{s+1}^{-1}+\Omega_{s+1}^{-1})}{c(\Sigma_{s+1}^{-1})}\tilde{c}_{s+1}$ which yields

$$\frac{c(\Sigma_{s+1})}{c(\Theta_{s+1})}\tilde{c}_{s+1} = \frac{c(\Sigma_{s+1})}{c([\Sigma_{s+1}^{-1}+\Omega_{s+1}^{-1}]^{-1})}\tilde{c}_{s+1} = \frac{c(\Sigma_{s+1}^{-1}+\Omega_{s+1}^{-1})}{c(\Sigma_{s+1}^{-1})}\tilde{c}_{s+1} = \tilde{c}_s. \tag{73}$$

Using (68), (72) and (73) in (67) we obtain

$$\begin{aligned} & \frac{c(\Sigma_{s+1})}{c(\Theta_{s+1})}\;g(Rp_s;\mu_{s+1|s},\Sigma_{s+1})\;g(Rp_s;c_{s+1},\vartheta_{s+1},\Omega_{s+1}) \\ = & \frac{c(\Sigma_{s+1})}{c(\Theta_{s+1})}\tilde{c}_{s+1}\;\frac{g(0;\pi_{s-1,j},\Sigma)}{g(0;\pi_{s-1,s},\Sigma_1^s)\;g(0;\vartheta_s,\Omega_s)}\;g(p_s;,\vartheta_s,\Omega_s) \\ = & \frac{g(0;\tilde{c}_s,\pi_{s-1,j},\Sigma)}{g(0;\pi_{s-1,s},\Sigma_1^s)\;g(0;\vartheta_s,\Omega_s)}\;g(p_s;\vartheta_s,\Omega_s) = g(p_s;c_s,\vartheta_s,\Omega_s). \end{aligned}$$

Using (67) this proves equation (66) and hence Proposition 1. ∎

Given parametric prices $p \in \mathbb{R}^K$ and wealth w the value function V_1 obtained from Proposition 1 together with the marginal distribution of prices p_1 given by Lemma 1 allows us by virtue of the principle of optimality (45) to obtain an optimal portfolio decision for $t = 0$ as a solution to the following optimization problem

$$U_0(x,y) := \int_{\mathbb{R}^K} V_1\left(Ry + x^\top\hat{p},\hat{p}\right) f(\hat{p};\mu_1,\Sigma_{11})\,d\hat{p} \longrightarrow \max_{y+x^\top p=w}. \tag{74}$$

By Proposition 1 the value function V_1 in (74) takes the form

$$V_1(w_1,p_1) = u\left(w_1;aR^{j-1}\right)\;g\left(p_1;c_1,\vartheta_1,\Omega_1\right), \tag{75}$$

where (setting $\Pi_{0,n} \equiv \Pi_n$, $\pi_{0,n} \equiv \pi_n$)

$$\begin{aligned} \Omega_1 &:= \left[\Pi_j^\top\Sigma^{-1}\Pi_j - \Sigma_{11}^{-1}\right]^{-1} \\ \vartheta_1 &:= \Omega_1\left(\Pi_j\Sigma^{-1}\mu - \Sigma_{11}^{-1}\mu_1\right) \\ \tilde{c}_1 &:= \prod_{n=2}^{j-1}\frac{c\left(\Sigma_n^{-1}+\Omega_n^{-1}\right)}{c\left(\Sigma_n^{-1}\right)} \\ c_1 &:= \frac{g(0;\tilde{c}_1,\mu,\Sigma)}{g(0;\mu_1,\Sigma_{11})\;g(0;\vartheta_1,\Omega_1)} \end{aligned} \tag{76}$$

Given the form (75) of the value function V_1 we may apply Lemma 6 (setting $c = c_1$, $\vartheta = \vartheta_1$, $\Omega = \Omega_1$, $\hat{\mu} = \mu_1$, $\hat{\Sigma} = \Sigma_{11}$, $\alpha = aR^{j-1}$) to write the objective function in (74) as

$$U_0(x, y) = \frac{\hat{c}_1}{c(\Theta_1)}\, u\left(Ry + x^\top \theta_1 - \frac{aR^{j-1}}{2} x^\top \Theta_1\, x; aR^{j-1}\right).$$

where (using equation (76) the parameters $\hat{c}$, Θ_1 and θ_1 are given by

$$\begin{aligned}
\Theta_1 &:= \left[\Sigma_{11}^{-1} + \Omega_1^{-1}\right]^{-1} && (77)\\
&= \left[\Pi_j^\top \Sigma^{-1} \Pi_j\right]^{-1} \in \mathcal{M}_K \\
\theta_1 &:= \Theta_1 \left[\Sigma_{11}^{-1}\mu_1 + \Omega_1^{-1}\vartheta_1\right] && (78)\\
&= \Theta_1 \left[\Pi_j \Sigma^{-1} \mu\right] \in \mathbb{R}^K \\
\hat{c}_1 &:= \frac{g(0; c_1, \vartheta_1, \Omega_1)\; g(0; c(\Sigma_{11}), \mu_1, \Sigma_{11})}{g(0; \theta_1, \Theta_1)} > 0.
\end{aligned}$$

By Lemma 5(1), for each $p \in \mathbb{R}^K$ and $w \in \mathbb{R}$, the solutions to (74) are given by

$$\begin{aligned}
x_0^\star &= \frac{1}{aR^{j-1}} \Theta_1^{-1}(\theta_1 - Rp) && (79)\\
&= \frac{1}{aR^{j-1}} \Pi_j^\top \Sigma^{-1} (\mu - \Pi_j Rp) \\
y_0^\star &= w - p^\top x_0^\star.
\end{aligned}$$

Since the moments $(\mu, \Sigma) \in \mathbb{R}^{Kj} \times \mathcal{M}_{Kj}^\star$ were arbitrary, this proves Theorem 1. ■

A.2 Technical appendix

In this section we collect some well-known properties of Gaussian functions together with some important relations of parameters. Here for each $n, m \in \mathbb{N}$, $0_{n\times m}$ denotes the $n \times m$ zero matrix.

Lemma 2
(1) Given the partition $\Sigma_1^s = \begin{bmatrix} \Sigma_1^{s-1} & C_s^\top \\ C_s & \Sigma_{ss} \end{bmatrix}$ *and the parameters* Σ_s *and* B_s *defined in (40) we have for each* $s = 2, \ldots, j$

$$[\Sigma_1^s]^{-1} = \begin{bmatrix} \left[\Sigma_1^{s-1}\right]^{-1} & 0_{Ks\times K} \\ 0_{K\times Ks} & 0_{K\times K} \end{bmatrix} + \begin{bmatrix} B_s^\top \\ -I_K \end{bmatrix} \Sigma_s^{-1} \begin{bmatrix} B_s & I_k \end{bmatrix}$$

(2) Given the parameters defined in (40) and (42) we have for each $s = 1, \ldots, j-1$ *the following relations:*

$$\begin{aligned}
&(a)\quad \Pi_{s-1,s}^\top \left[\Sigma_1^s\right]^{-1} \Pi_{s-1,s} &&= \Sigma_s^{-1} \\
&(b)\quad \Pi_{s-1,s}^\top \left[\Sigma_1^s\right]^{-1} \pi_{s-1,s} &&= \Sigma_s^{-1} \mu_{s|s-1} \\
&(c)\quad \Pi_{s-1,s}^\top \Sigma^{-1} \Pi_{s-1,s} - \Sigma_s^{-1} &&= \sum_{n=1}^{j-s} \Pi_{s-1,s+n}^\top \begin{bmatrix} B_{s+n}^\top \\ -I_K \end{bmatrix} \Sigma_{s+n}^{-1} \begin{bmatrix} B_{s+n} - I_K \end{bmatrix} \Pi_{s-1,s+n}
\end{aligned}$$

Proof: (1): By Assumption the matrix Σ given in (5) is positive definite implying that each submatrix Σ_1^s ($s = 1, \dots, j$) is also positive definite and hence invertible. Given the definitions (40) the assertion follows from Ouellette (1981) Theorem 2.7, p.201.

(2a): Using (1) we have

$$\begin{aligned}\Pi_{s-1,s}^\top \left[\Sigma_1^s\right]^{-1} \Pi_{s-1,s} &= \Pi_{s-1,s}^\top \begin{bmatrix} \left[\Sigma_1^{s-1}\right]^{-1} & 0_{Ks\times K} \\ 0_{K\times Ks} & 0_{K\times K} \end{bmatrix} \Pi_{s-1,s} \\ &+\Pi_{s-1,s}^\top \begin{bmatrix} B_s^\top \\ -I_K \end{bmatrix} \Sigma_s^{-1} \begin{bmatrix} B_s & -I_k \end{bmatrix} \Pi_{s-1,s}.\end{aligned}$$

Note that $[B_s, -I_K]\Pi_{s-1,s} = \Pi_{s-1,s}^\top [B_s, -I_K]^\top = -I_K$ and that the first term on the r.h.s. is equal to $0_{K\times K}$ to obtain (2a).

(2b): Use (1) again and note that $[B_s, -I_K]\pi_{s-1,s} = -\mu_{s|s-1}$.

(2c): Using (2a) and (1) we can write

$$\begin{aligned}\Sigma_s^{-1} &= \Pi_{s-1,s}^\top \left[\Sigma_1^s\right]^{-1} \Pi_{s-1,s} = \Pi_{s-1,s+1}^\top \begin{bmatrix} \left[\Sigma_1^s\right]^{-1} & 0_{Ks\times K} \\ 0_{K\times Ks} & 0_{K\times K} \end{bmatrix} \Pi_{s-1,s+1} \\ &= \Pi_{s-1,s+1}^\top \left[\Sigma_1^{s+1}\right]^{-1} \Pi_{s-1,s+1} - \Pi_{s-1,s+1}^\top \left[B_{s+1}, -I_k\right]^\top \Sigma_{s+1}^{-1} \left[B_{s+1}, -I_k\right] \Pi_{s-1,s+1}\end{aligned}$$

By simple induction one can verify that for each $n = 0, 1, \dots, j - s - 1$

$$\begin{aligned}\Pi_{s-1,s+n-1}^\top \left[\Sigma_1^{s+n-1}\right]^{-1} \Pi_{s-1,s+n-1} &= \Pi_{s-1,s+n}^\top \left[\Sigma_1^{s+n}\right]^{-1} \Pi_{s-1,s+n} \\ &- \Pi_{s-1,s+n}^\top \left[B_{s+n}, -I_k\right]^\top \Sigma_{s+n}^{-1} \left[B_{s+n}, -I_k\right] \Pi_{s-1,s+n}\end{aligned}$$

We thus obtain

$$\Sigma_s^{-1} = \Pi_{s-1,j}^\top \Sigma^{-1} \Pi_{s-1,j} - \sum_{n=1}^{j-s} \Pi_{s-1,s+n}^\top \begin{bmatrix} B_{s+n}^\top \\ -I_K \end{bmatrix} \Sigma_{s+n}^{-1} \begin{bmatrix} B_{s+n} - I_K \end{bmatrix} \Pi_{s-1,s+n}$$

and rearranging yields (2c). ∎

Lemma 3
Given the definition of a Gaussian function (46), the following properties hold:

(1) *The product of m Gaussian functions with parameters $(c^{(h)}, \vartheta^{(h)}, \Omega^{(h)}) \in \mathbb{R}_{++} \times \mathbb{R}^K \times \mathcal{M}_K$, $h = 1, \dots, m$ is again a Gaussian function, that is,*

$$\prod_{h=1}^{m} g(x; c^{(h)}, \vartheta^{(h)}, \Omega^{(h)}) = g(x; c, \vartheta, \Omega), \tag{80}$$

where the parameters $(c, \vartheta, \Omega) \in \mathbb{R}_{++} \times \mathbb{R}^K \times \mathcal{M}_K$ in (80) are given by

$$\Omega = \left[\Omega^{(1)-1} + \ldots + \Omega^{(m)-1}\right]^{-1}, \quad \vartheta = \Omega \left[\Omega^{(1)-1}\vartheta^{(1)} + \ldots + \Omega^{(m)-1}\vartheta^{(m)}\right],$$

$$\textit{and} \quad c = \frac{\prod_{h=1}^{m} g(0; c^{(h)}, \vartheta^{(h)}, \Omega^{(h)})}{g(0; \vartheta, \Omega)}.$$

(2) The integral of a Gaussian function $g(\cdot; c, \vartheta, \Omega)$ over $\mathbb{R}^K$ satisfies

$$\int_{\mathbb{R}^K} g(x; c, \vartheta, \Omega)\, dx = \frac{c}{c(\Omega)} > 0.$$

Proof: (1) follows by induction and straightforward calculations, (2) follows from the relation (47) and the properties of multivariate density functions, (3) can be verified from the definition given in (46) and direct calculations. ∎

Lemma 4
Let $(\theta, \Theta) \in \mathbb{R}^K \times \mathcal{M}_K$, $c > 0$ and $\alpha > 0$ be given and the functions $u(\cdot; \alpha)$ and $g(\cdot; c, \theta, \Theta)$ be as defined in (3) and (46). Then

$$\int_{\mathbb{R}^K} u(Ry + x^\top p; \alpha) g(p; c, \theta, \Theta)\, dp = \frac{c}{c(\Theta)}\, u\left(Ry + x^\top \theta - \frac{\alpha}{2} x^\top \Theta\, x; \alpha\right). \tag{81}$$

Proof of Lemma 4. We have

$$u(Ry + x^\top p; \alpha)\, g(p; c, \theta, \Theta) = -c \exp\left\{-\alpha(Ry + x^\top p) - \frac{1}{2}\big(p - \theta\big)^\top \Theta^{-1}\,(p - \theta)\right\}$$

and using symmetry of the matrix $\Theta^{-1} = \Theta^{-\top}$, the exponent can be rewritten as

$$\begin{aligned}
& -\alpha\left(Ry + x^\top p\right) - \frac{1}{2}\big(p - \theta\big)^\top \Theta^{-1}\,(p - \theta) \\
= \;& -\alpha Ry - \frac{1}{2}\Big[\big(p - (\theta - \alpha\Theta x)\big)^\top \Theta^{-1}\big(p - (\theta - \alpha\Theta x)\big) + 2\alpha x^\top \theta - \alpha^2 x^\top \Theta x\Big] \\
= \;& -\alpha(Ry + x^\top \theta - \frac{\alpha}{2} x^\top \Theta x) - \frac{1}{2}\Big[\big(p - (\theta - \alpha\Theta x)\big)^\top \Theta^{-1}\big(p - (\theta - \alpha\Theta x)\big)\Big].
\end{aligned}$$

Rearranging, the integrand in (81) becomes

$$u(Ry + x^\top p; \alpha) g(p; c, \theta, \Theta) = u\left(Ry + x^\top \theta - \frac{\alpha}{2} x^\top \Theta x; \alpha\right) g(p; c, \theta - \alpha\Theta x, \Theta).$$

Since the first factor is independent of the integration variable p, the assertion follows from Lemma 3 (2). ∎

Lemma 5
Let the parameters $(\theta, \Theta) \in \mathbb{R}^K \times \mathcal{M}_K$, $c > 0$, $\alpha > 0$, prices $p \in \mathbb{R}^K$, and wealth $w \in \mathbb{R}$ be given. Then the following holds true:

(1) The optimization problem

$$c\, u\left(Ry + x^\top \theta - \frac{\alpha}{2} x^\top \Theta\, x; \alpha\right) \longrightarrow \max_{x,y} \quad s.t. \quad y + x^\top p = w \tag{82}$$

has a unique solution $(x^\star, y^\star) \in \mathbb{R}^K \times \mathbb{R}$ of the form

$$x^\star = \frac{1}{\alpha}\Theta^{-1}(\theta - Rp), \qquad y^\star = w - p^\top x^\star. \tag{83}$$

(2) The maximum takes the form

$$u^\star := c\,u\left(Ry^\star + x^{\star\top}\theta - \frac{\alpha}{2}x^{\star\top}\Theta\, x^\star;\alpha\right) = u(w;\alpha R)\; g(Rp;c,\theta,\Theta). \tag{84}$$

Proof of Lemma 5. (1) Due to the strict monotonicity of the function $cu(\cdot;\alpha)$ for each fixed $c > 0$ and $\alpha > 0$ maximizing the function

$$c\,u\left(Ry + x^\top\theta - \frac{\alpha}{2}x^\top\Theta\, x;\alpha\right) = -c\exp\left\{-\alpha\left(Ry + x^\top\theta - \frac{\alpha}{2}x^\top\Theta\, x\right)\right\}.$$

is equivalent to maximizing the function $(x,y) \longmapsto Ry + x^\top\theta - \frac{\alpha}{2}x^\top\Theta\ x$. From the Lagrangian function

$$\mathcal{L}(x,y;\lambda) := Ry + x^\top\theta - \frac{\alpha}{2}x^\top\Theta\, x + \lambda(w - x^\top p - y)$$

one obtains the first order conditions

$$D_x\mathcal{L}(x^\star,y^\star;\lambda) = \theta - \alpha\Theta x^\star - \lambda p \overset{!}{=} 0 \quad \text{and} \quad D_y\mathcal{L}(x^\star,y^\star;\lambda) = R - \lambda \overset{!}{=} 0.$$

Combining this with the constraint $y + x^\top p = w$ yields the solution (83).

(2) Substituting the solutions (83) into the objective function in (84) and exploiting the symmetry of the matrix Θ the maximum $u^\star$ reads:

$$\begin{aligned} u^\star &= c\,u\left(Rw + x^{\star\top}(\theta - Rp) - \frac{\alpha}{2}x^{\star\top}\Theta\, x^\star;\alpha\right) \\ &= c\,u\left(Rw + \frac{1}{2\alpha}(\theta - Rp)^\top\Theta^{-1}(\theta - Rp);\alpha\right) \\ &= u(w;\alpha R)\; g(Rp;c,\theta,\Theta) \end{aligned}$$

which proves the second assertion. ■

Lemma 6
Let $(c,\vartheta,\Omega) \in \mathbb{R}_{++} \times \mathbb{R}^K \times \mathcal{M}_K$, $(\hat\mu,\hat\Sigma) \in \mathbb{R}^K \times \mathcal{M}_K$ and $\alpha > 0$ be given and the functions $u(\cdot;\alpha)$, $g(\cdot;c,\vartheta,\Omega)$ and $f(\cdot;\hat\mu,\hat\Sigma)$ be as defined in (2), (3) and (46). Then

$$\int_{\mathbb{R}^K} u(Ry + x^\top p;\alpha) g(p;c,\vartheta,\Omega)\; f(p;\hat\mu,\hat\Sigma)dp = \frac{\hat c}{c(\Theta)}\; u\left(Ry + x^\top\theta - \frac{\alpha}{2}x^\top\Theta\, x;\alpha\right) \tag{85}$$

where

$$\begin{aligned} \Theta &:= \left[\Omega^{-1} + \hat\Sigma^{-1}\right]^{-1} \\ \theta &:= \Theta\left[\Omega^{-1}\vartheta + \hat\Sigma^{-1}\hat\mu\right] \\ \hat c &:= \frac{g(0;c(\hat\Sigma),\hat\mu,\hat\Sigma)\; g(0;c,\vartheta,\Omega)}{g(0;\theta,\Theta)}. \end{aligned} \tag{86}$$

Proof of Lemma 6. Using (47) and Lemma 3 (1) the Gaussian functions in (85) can be written as

$$g(p;c,\vartheta,\Omega)\; g(p;c(\hat\Sigma),\hat\mu,\hat\Sigma)dp = g(p;\hat c,\theta,\Theta)$$

with parameters $(\hat c,\theta,\Theta)$ as defined in (86). Applying Lemma 4 yields

$$\int_{\mathbb{R}^K} u(Ry + x^\top p;\alpha) g(p;\hat c,\theta,\Theta)\; dp = \frac{\hat c}{c(\Theta)}\; u\left(Ry + x^\top\theta - \frac{\alpha}{2}x^\top\Theta\, x;\alpha\right)$$

which proves the assertion ■

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