A double-auction artificial market with time-irregularly spaced orders

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Abstract

In this paper, a simulation of high-frequency market data is performed with the Genoa Artificial Stock Market. In the market model, heterogeneous agents trade a risky asset in exchange for cash. Agents have zero intelligence and issue random limit or market orders depending on their budget constraints. The price is cleared by means of a limit order book. The order generation process is a renewal process where the waiting-time distribution between two consecutive orders follows a Weibull law. This hypothesis is motivated by recent theoretical and empirical studies on high-frequency financial data. According to simulation results, the mechanism of the limit order book can reproduce fat-tailed distributions of returns without ad-hoc behavioral assumptions on agents. As for the simulated trade process, in the case of exponentially distributed order waiting times, also trade waiting times are exponentially distributed. Conversely, if order waiting times follow a Weibull law, the same does not hold true for trade waiting times. These findings are interpreted in terms of a random thinning of the order renewal process.

Key words: Agent-based simulation; artificial financial market; limit order book; renewal processes; Weibull distribution.
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1 Introduction

In recent years, thanks to the availability of large databases of financial data, the statistical properties of high-frequency financial data and market microstructural properties have been studied by means of different tools, including phenomenological models of price dynamics and agent-based market simulations [1–10]. Various studies on high-frequency econometrics appeared in the literature including the autoregressive conditional duration models [11–14].

Among these approaches, agent-based based simulations [7,8] are particularly flexible as they allow the study of both the behaviour of agents and the influence of market structures in a well-controlled way. Here, the focus is on the influence of the double auction clearing mechanism where the price is fixed by the order book.

An important empirical variable is the waiting time between two consecutive transactions. Due to the double auction mechanism, waiting times between two trades are themselves a stochastic variable. They may also be correlated to returns [15] as well as to traded volumes. Indeed, trading via the order book is asynchronous and a transaction occurs only if a trader issues a market order. For liquid stocks, waiting times can vary in a range between some seconds and a few minutes, depending on the specific stock and on the market considered. In ref. [15], the reader can find a study on General Electric stocks traded in October 1999. Waiting times between consecutive prices exhibit 1-day periodicity, typical of variable intraday market activity. Moreover, the survival probability (the complementary cumulative distribution function) of waiting times is well fitted by a Weibull function.

In this paper, we simulate different distributions of waiting times between consecutive limit orders, namely the Weibull distribution and the exponential distribution. Orders are then *selected* by means of the limit order book mechanism implemented in the Genoa Artificial Stock Market (GASM) and described in Section 2.2 The resulting distribution of waiting times between consecutive trades is then compared to a *zero-order* theory of order selection described in Section 2.3. Section 3 is devoted to a description of numerical simulation results Finally, Section 4 is devoted to discussion and conclusions.

2 Computational experiments

In the implemented simulation, agents in the GASM trade one single stock in exchange for cash. They are liquidity traders and, therefore, the decision making process is nearly random and depends on the finite amount of cash + stock available. At the beginning of the simulation, cash and stocks are uniformly distributed among agents.

2.1 Order generation

Trading is divided into M daily sections. Each trading day is divided into T elementary time steps of size one second. During the day, at given time steps t_h a trader k is randomly chosen for issuing an order. Order waiting times $\tau_h^o = t_h - t_{h-1}$ are extracted according to a Weibull distribution or to a power-law distribution. The Weibull probability density function is:

$$\varphi(\tau^o) = \frac{\beta}{\eta} \left(\frac{\tau^o}{\eta} \right)^{\beta-1} \exp[-(\tau^o/\eta)^\beta], \tag{1}$$

where η is the scale parameter and β is the shape parameter, also known as slope, as it is the slope of the regression line in a probability plot. The Weibull distribution reduces to the exponential distribution for $\beta = 1$. In these computational experiments we considered values of β less of equal to one.

The order generation process is then described as a general renewal process where the waiting times between two consecutive orders, τ^o, are indipendent and identically distributed (i.i.d.) random variables following the densities in eq. (1). In the case $\beta = 1$, the order generation process is a Poisson process with an exponential waiting-time distribution. For further information on renewal processes the reder is referred to ref. [16].

2.2 Order selection and trading

A trader issues a buy or sell order with probability 1/2. Let $a(t_{h-1})$ and $d(t_{h-1})$ be the values of the ask and bid prices stored in the book at time step t_{h-1}.

In case the order issued at time step t_h is a sell order, the limit price s_k associated to the sell order is:

$$s_k(t_h) = n_k(t_h) \cdot a(t_{h-1}) \tag{2}$$

where $n_k(t_h)$ is a random draw by trader k at time step t_h from a Gaussian distribution with mean $\mu = 1$ and standard deviation σ. If $s_k(t_h) > d_k(t_{h-1})$ then the limit order is recorded in the book and no trade occurs, else the order becomes a market order and a transaction takes place at the price $S(t_h) =$

$d(t_{h-1})$. In the latter case, the sell order is partially or totally fulfilled and the bid price is updated. The quantity of stock offered for sale is a random fraction of the quantity owned by the trader.

In case the order is a buy oder, the limit price $b_k(t_h)$ is now:

$$b_k(t_h) = n_k(t_h) \cdot d(t_{h-1}), \tag{3}$$

where $n_k(t_h)$ is determined as above. If $b_k(t_h) < a_k(t_{h-1})$ then the limit order is recorded in the book and no trade occurs, else the order becomes a market order and a transaction takes place at the price $S(t_h) = a(t_{h-1})$. The quantity of stock ordered depends on the cash of trader k and on the value of $b_k(t_h)$.

As a final remark for this subsection, it is worth observing that, in this framework, agents compete for liquidity. If a buy order is issued by an agent, its benchmark is the best limit buy order given buy the bid price. As $\mu = 1$ for half of the times, the agent offers a more competitive buy order (if $b_k(t_h) > d(t_{h-1})$), that can result in a trade if $b_k(t_h) \geq a(t_{h-1})$. The same is valid for sell limit orders.

2.3 Random thinning for order selection and trading

As a *zero-order* model of order selection and trading, let us consider the random thinning [16] of the order generation process. This has been studied by Gnedenko and Kovalenko [17]. It means that given a random sequence of i.i.d. order waiting times τ_h^o, for each index h, a decision is taken: the event is deleted with probability p or kept with probability $q = 1-p$, with $0 < q < 1$. In order to compute the probability density function $(T_q\varphi)(\tau)$, the probability density function $f_k(t)$ of the sum of k waiting times is needed. As waiting times are i.i.d. variables, $f_k(t)$ is given by the k-fold convolution of φ:

$$f_1(t) = \varphi(t), \;\; f_k(t) = \int_0^t f_{k-1}(t-t')\varphi(t')dt'. \tag{4}$$

Now, $(T_q\varphi)(\tau)$ can be obtained by purely probabilistic arguments, by noting that, after a kept event, the next one of the original process is kept with probability q but dropped in favour of the second next with probability pq and, in general, $n-1$ events are dropped in favour of the n-th next with probability $p^{n-1}q$. Therefore one has:

$$(T_q\varphi)(\tau) = \sum_{n=1}^{\infty} qp^{n-1} f_n(\tau). \tag{5}$$

Let $\tilde{f}_n(s) = \int_0^\infty \mathrm{e}^{-st} f_n(t)\, dt$ be the Laplace transform of $f_n(t)$. From the behaviour of the Laplace tranform of a convolution, it turns out that the Laplace tranform of eq. (5) is:

$$(\tilde{T_q \varphi})(s) = \sum_{n=1}^{\infty} q p^{n-1} [\tilde{\varphi}(s)]^n = \frac{q\tilde{\varphi}(s)}{1-(1-q)\tilde{\varphi}(s)}, \qquad (6)$$

from which by Laplace inversion, in principle, we can reconstruct the probability density function of the thinned process.

Eq. (5) or eq. (6) can be used to estimate the density of waiting times between two consecutive transaction from the knowledge of the order waiting-time probability density function. Alternatively, a Monte Carlo simulation can be used, in which the thinning procedure is performed directly on a pseudo-random sequence of waiting times. Random thinning for order selection and trading is a rough approximation as it does not take into account many features which are present in a market, including price and volume feedback and partial order fulfillment. However the distributions obtained by random thinning can be easily generated and compared with those obtained with GASM by the procedure described in Section 2.2.

3 Simulation results

The simulations were performed with the following parameters. The number of daily sections M was set equal to 50. The length of the daily sections was $T = 25,200$ s (corresponding to 7 hours of trading activity). In Fig. 1, data are presented for Weibull-distributed orders with $\beta = 1$ (the exponential case), whereas in Fig. 2, the case $\beta = 0.4$ is discussed. The average waiting-time $\langle \tau^o \rangle$ between orders was set to 20 s for every simulation. The scale factor η is related to $\langle \tau^o \rangle$ according to the following equation:

$$\eta = \frac{\langle \tau^o \rangle}{\Gamma(1/\beta + 1)}, \qquad (7)$$

where $\Gamma(\cdot)$ is Euler's Gamma function. The survival probability $P_>(\tau)$ corresponding to the Weibull density in eq. (1) is:

$$P_>(\tau) = \exp(-\tau/\eta)^{\beta}. \qquad (8)$$

The lifespan of orders was 600 s, a time much larger than $\langle \tau^o \rangle$. Sell and buy limit prices were computed following eq. (2) and eq. (3), respectively. The random numbers $n_i(t_h)$ were drawn from a Gaussian distribution with $\mu = 1$ and $\sigma = 0.005$. The number of agents was 10,000. The initial stock price was 100.00 units of cash, say Euro, and each trader owned an equal amount of cash and of shares: 100,000 Euro and 1,000 shares. These simulations produce realistic intraday price paths [18,19].

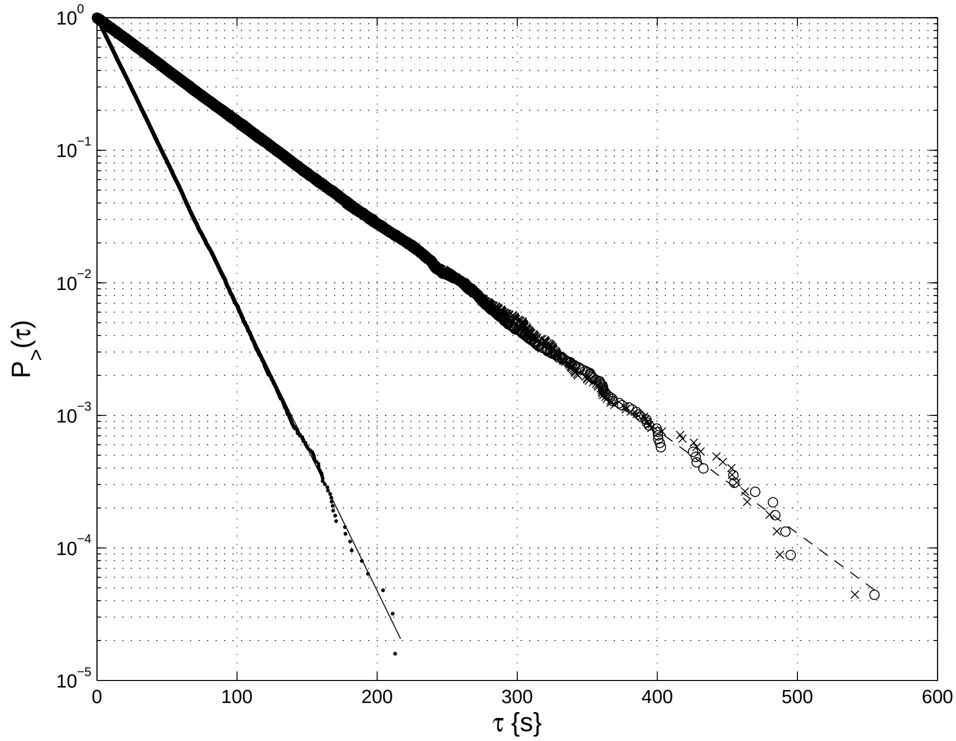

Fig. 1. Survival probability distribution of order waiting times (dots) and of trade waiting times (crosses) in the case $\beta = 1$ (exponential distribution). The two lines represent the corresponding exponential fits. The hollow circles represent the result of a Monte Carlo simulation of random thinning with probability $q = 0.36$

In Fig. 1, the survival probability distribution of order waiting times is compared with the one for trade waiting times, in the case $\beta = 1$. This case corresponds to exponentially distributed order waiting times. As a consequence of the GASM order selection procedure, the waiting time between trades, $\langle \tau^t \rangle$, is still exponentially distributed, with a larger average waiting time. The hollow circles in Fig. 1 correspond to a Monte Carlo simulation of random thinning with a probability $q = \langle \tau^o \rangle / \langle \tau^t \rangle = 0.36$. The agreement between the GASM oder selection and the random thinning procedure is good.

In Fig. 2, the survival probability distribution of order waiting times is compared with the one for trade waiting times, in the case $\beta = 0.4$. This case corresponds to Weibull distributed order waiting times. As a consequence of the GASM order selection procedure, the waiting time between trades, $\langle \tau^t \rangle$, no longer follows the Weibull distribution. In Fig. 2, the dashed line is the Weibull fit of the trade waiting-time survival function and a Kolmogorov-Smirnov test rejects the null hypothesis of Weibull-distributed trade waiting times at the 5% significance level. The hollow circles in Fig. 2 correspond to a Monte Carlo simulation of random thinning with a probability $q = \langle \tau^o \rangle / \langle \tau^t \rangle = 0.36$. Again, the agreement between the GASM oder selection and the random thinning procedure is good.

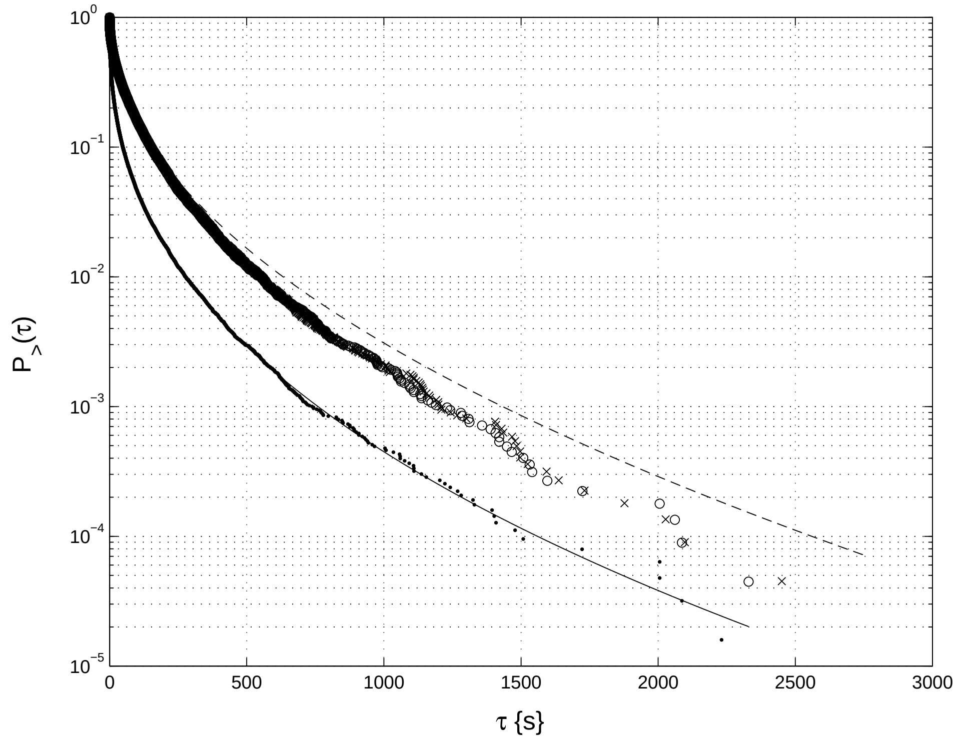

Fig. 2. Survival probability distribution of order waiting times (dots) and of trade waiting times (crosses) in the case $\beta = 0.4$. The two curves represent the corresponding Weibull fits. The hollow circles represent the result of a Monte Carlo simulation of random thinning with probability $q = 0.36$

4 Discussion and conclusions

The simulation results described in the previous section can be interpreted as follows. When the waiting time distribution between orders is exponential, then the GASM order selection procedure described in Section 2.2 leads to exponentially distributed waiting times between two consecutive trades. When the order waiting times follow a Weibull renewal process with $0 < \beta < 1$, then the trade waiting-time distribution is no longer ruled by the Weibull law. However, in both cases, for what concerns waiting times, the outcome of the order selection process can be well-mimicked by a simple random thinning with probability q given by the ratio between the average order waiting time and the average trade waiting time: $q = \langle \tau^o \rangle / \langle \tau^t \rangle$. In other words, the GASM selection process of the order book is equivalent to a random thinning for the simulation parameters investigated.

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