Asset price and wealth dynamics in a financial market with heterogeneous agents

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Abstract

This paper considers a discrete-time model of a financial market with one risky asset and one risk-free asset, where the asset price and wealth dynamics is determined by the interaction of two groups of agents, fundamentalists and chartists. In each period each group allocates its wealth between the risky asset and the safe asset according to myopic expected utility maximization, but the two groups have heterogeneous beliefs about the price change over the next period: the chartists are trend extrapolators, while the fundamentalists expect that the price will return to the fundamental. We assume that investors have CRRA utility, so that their optimal demand for each asset depends on wealth. A market maker is assumed to adjust the price at the end of each trading period, on the basis of the excess demand and according to particular stabilizing policies. The model results in a three-dimensional nonlinear discrete-time dynamical system, with growing price and wealth processes, but it is reduced to a stationary system in terms of asset returns and wealth shares of the two groups. It is shown that the long-run market dynamics is highly dependent on the parameters which characterize agents' behavior (in particular the risk aversion coefficient and the chartist extrapolation parameter) as well as on the initial condition (in particular the initial wealth shares of fundamentalists and chartists). It is also shown that for wide ranges of the parameters a (locally) stable fundamental steady state may coexist with a stable "nonfundamental" steady state, where price grows faster than fundamental and only chartists survive in the long-run. In such cases, the role played by the initial condition is analyzed by means of numerical investigation and graphical representation of the basins of attraction. Other dynamic scenarios include limit cycles, periodic orbits or more complex attractors, where in general both types of agents survive in the long run, with time varying wealth fractions.

1 Introduction

In recent years several models of asset price dynamics based on the interaction of *heterogeneous agents* have been proposed (Day and Huang (1990), Kirman (1991), Brock and Hommes (1998), Lux (1998), Gaunersdorfer (2000), Chiarella and He (2001a), (2003). These models assume a one risky/one riskless asset market, and focus on the effect of heterogeneous beliefs and trading rules on the dynamics of the price of the risky asset. Most of these models, some of which allow the size of the different groups of agents to vary according to the relative profitability of the adopted trading rules, are of necessity not very mathematically tractable. In Chiarella *et al* (2002), whose antecedents are Chiarella (1992), Beja and Goldman (1980), and Zeeman (1974), a two-dimensional discrete time model of asset price dynamics has been developed, which contains the essential elements of the heterogeneous agents paradigm whilst still remaining mathematically tractable. In that paper, a financial market with a risky asset and an alternative riskless asset has been assumed, consisting of two types of traders, *fundamentalists* and *chartists*, and of a *market maker*, who adjusts prices in each period depending on excess demand. In Chiarella *et al* (2002), as well as in most studies on heterogeneous agents' interaction, the evolution of agents' wealth and its effect on price dynamics is left in the background; indeed, in those papers the underlying assumptions about agents' portfolio allocation follow the framework of Brock and Hommes (1998), where optimal demand for the risky asset is independent on agents' wealth, as a result of the underlying CARA utility functions.

In general these assumptions are unrealistic: a more realistic framework, where investors' optimal decisions depend on their wealth, has been proposed and analyzed through numerical simulation by Levy *et al* (1994, 1995). This framework is consistent with the assumption of CRRA utility functions. More recently, Chiarella and He (2001b), (2002) have proved analytically the existence of multiple steady states, as well as of more complex dynamic scenarios, in heterogeneous agents models with wealth dynamics.

The present paper aims to contribute to the development and analysis of such models, by analyzing the dynamics of asset price and agents' wealth within a fundamentalists/chartists framework similar to the one developed in Chiarella *et al* 2002. In addition, we allow for a trend in the fundamental price of the risky asset, due to an assumed growing dividend process. As a consequence, the model that we develop results in price and wealth being determined simultaneously over time, as in real markets, which gives rise to interdependent growing wealth and price processes. In order to obtain analytical results about the dynamics of the growing system, to fully understand the range of dynamic scenarios generated by the model, and to discuss the role played by the key parameters and by the initial conditions in the long-run evolution, the nonstationary model is reformulated in terms of returns and wealth shares and reduced to a stationary system.

The structure of the paper is as follows. Section (2) presents the general framework of the model. In particular Section (2.1) derives the optimal agents' demand for the risky asset in a general setting, as a function of agents' beliefs about the risky return, under the assumption of myopic expected utility maximization with CRRA (power) utility of wealth function. Section (2.2) derives a benchmark notion of fundamental solution, which plays a role in fundamentalist expectations formation. Section (2.3) describes the schemes used by fundamentalists and chartists to revise their expectations.Section (2.4) describes how demands are aggregated by a market maker, who sets the price depending on excess demand and with a view to long-run market stability. Section (3) presents the resulting (growing) nonlinear dynamical system for the dynamic evolution of fundamental value and price, agents' expected returns, and wealth of the two groups; this is reduced to a stationary map (Section (3.1)) in terms of actual and expected capital gain of the risky asset, fundamental to price ratio, and wealth shares of the two groups. The steady states are determined and their properties are discussed in Section (3.2). Numerical simulation of the global behavior and discussion of the main dynamic scenarios is contained in Section (4). Section (5) contains some conclusions and discussion of future research.

2 The model

We consider a discrete-time model of a financial market with one risky asset and one riskless asset, two types of interacting agents, *fundamentalists* and *chartists* (denoted by $j \in \{f, c\}$), and a *market maker*. Each group has CRRA utility of wealth function.

The starting point is Chiarella, Dieci and Gardini (2002), whose antecedents are Chiarella (1992), Beja and Goldman (1980), and Zeeman (1974).

We denote, at time t, by P_t and Y_t the market price and the *fundamental* price of the risky asset, respectively, by $\Omega_t^{(j)}$ and $Z_t^{(j)}$ the wealth of agent type j and the fraction of wealth invested in the risky asset, by r the (constant) risk-free rate, by D_t the (random) dividend, while D_{t+1}/P_t is the dividend yield in $(t, t+1)$. The fundamental price is assumed to be known to the fundamentalists and to the market maker.

We denote by $E_t^{(j)}$, $Var_t^{(j)}$ the "beliefs" of investor type j about expectation and variance.

Wealth of agent j evolves according to

$$\begin{aligned} \Omega_{t+1}^{(j)} &= \Omega_t^{(j)} + \Omega_t^{(j)} Z_t^{(j)} \left(\frac{P_{t+1} + D_{t+1} - P_t}{P_t} \right) + \Omega_t^{(j)} (1 - Z_t^{(j)}) r = \\ &= \Omega_t^{(j)} \left[1 + r + Z_t^{(j)} \left(\frac{P_{t+1} + D_{t+1} - (1+r)P_t}{P_t} \right) \right] \end{aligned} \tag{1}$$

where $(P_{t+1} + D_{t+1} - (1+r)P_t)/P_t$ represents the excess return in $(t, t+1)$.

2.1 Asset demand

Each agent is assumed to have a CRRA power utility of wealth function of the type

$$u^{(j)}(x) = \begin{cases} \frac{1}{1-\lambda^{(j)}} x^{1-\lambda^{(j)}} & (\lambda^{(j)} \neq 1) \\ \ln(x) & (\lambda^{(j)} = 1) \end{cases}$$

where $x > 0$ and the parameter $\lambda^{(j)} > 0$ represents the relative risk aversion coefficient.

Each agent seeks the investment fraction $Z_t^{(j)}$ maximizing the expected utility of wealth at time $t+1$:

$$\max_{Z_t^{(j)}} E_t^{(j)}[u(\Omega_{t+1}^{(j)})]$$

Under simplifying assumptions (see e.g. Chiarella and He (2001b) the optimal investment fraction in the risky asset is approximately given by:

$$Z_t^{(j)} = \frac{E_t^{(j)}[(P_{t+1} + D_{t+1} - P_t)/P_t - r]}{\lambda^{(j)} Var_t^{(j)}[(P_{t+1} + D_{t+1} - P_t)/P_t]}$$

or

$$Z_t^{(j)} = \frac{E_t^{(j)}[\rho_{t+1} + \delta_{t+1} - r]}{\lambda^{(j)} Var_t^{(j)}[\rho_{t+1} + \delta_{t+1} - r]}$$

where $\rho_{t+1} \equiv (P_{t+1} - P_t)/P_t$ and $\delta_{t+1} \equiv D_{t+1}/P_t$ denote the capital gain and the dividend yield, respectively. Therefore, $Z_t^{(j)}$ is proportional to agent j's "risk-adjusted" expected excess return

2.2 Market clearing price and fundamental solution

Following the framework of Brock and Hommes (1998), the concept of fundamental solution which we use refers to the price that would be obtained if the agents were homogeneous with regard to their expectation of the excess return. Furthermore this price is assumed to satisfy a long-run stability condition, namely the "no bubbles" condition.

To get a benchmark notion of fundamental solution in this framework, denote by $N_t^{(j)} \equiv \Omega_t^{(j)} Z_t^{(j)}/P_t$ the number of shares demanded by agent type j at time t and by N_t^s the supply of shares at t. Let us consider the market equilibrium condition at time t

$$N_t = N_t^s$$

where $N_t = \sum_j N_t^{(j)}$. Rewrite the above equilibrium condition as

$$\sum_j \Omega_t^{(j)} \frac{E_t^{(j)}[P_{t+1} + D_{t+1} - (1+r)P_t]}{\lambda^{(j)} Var_t^{(j)}[\rho_{t+1} + \delta_{t+1} - r]} = N_t^s P_t^2 \tag{2}$$

Now assume for simplicity that agents have constant (not necessarily homogeneous) beliefs about the variance of the excess return in $(t, t+1)$ and denote by $\sigma^{2(j)} \equiv Var_t^{(j)}[\rho_{t+1} + \delta_{t+1} - r]$ these beliefs ($j = f, c$). Denote also by $\Omega_t = \sum_j \Omega_t^{(j)}$ the total wealth and by $w_t^{(j)} = \Omega_t^{(j)}/\Omega_t$ the wealth proportion of agent j, with $w_t^{(c)} = 1 - w_t^{(f)}$. Assume that all agents have homogeneous beliefs about the expected excess return. Then eq. (2) can be rewritten as

$$E_t[P_{t+1} + D_{t+1} - (1+r)P_t] \sum_j w_t^{(j)} \frac{1}{\lambda^{(j)}\sigma^{2(j)}} = Q_t P_t \tag{3}$$

where $Q_t \equiv N_t^s P_t/\Omega_t$ represents the value of the supply of shares over total agents' wealth. Finally

$$E_t[P_{t+1} + D_{t+1} - (1+r)P_t] = Q_t \xi_t P_t \tag{4}$$

where

$$\xi_t = \left(\sum_j w_t^{(j)} \frac{1}{\lambda^{(j)}\sigma^{2(j)}} \right)^{-1} = \left[w_t^{(f)} \frac{1}{\lambda^{(f)}\sigma^{2(f)}} + (1 - w_t^{(f)}) \frac{1}{\lambda^{(c)}\sigma^{2(c)}} \right]^{-1}$$

Therefore eq. (4) can be rewritten as

$$E_t[P_{t+1} + D_{t+1}] = (1 + r_t^*) P_t \tag{5}$$

where $r_t^* \equiv r + Q_t \xi_t$ represents the expected return that would be required in $(t, t+1)$ under homogeneous beliefs about tomorrow's expected price and dividend. In other words the quantity

$$\pi_t \equiv Q_t \xi_t = \frac{N_t^s P_t}{\Omega_t} \left[w_t^{(f)} \frac{1}{\lambda^{(f)}\sigma^{2(f)}} + (1 - w_t^{(f)}) \frac{1}{\lambda^{(c)}\sigma^{2(c)}} \right]^{-1}$$

represents the required risk-premium in $(t, t+1)$, under the same assumption.

Here we focus on the particular case of zero supply of shares (similarly to Brock and Hommes (1998), Chiarella and He (2001a)), and we leave the general case to the Appendix. For $N_t^s = 0$, $\forall\, t$, eq. (4) becomes

$$E_t[P_{t+1} + D_{t+1}] = (1 + r) P_t \tag{6}$$

where the required expected return is equal to the risk-free rate. As it is well known, the unique "fundamental" solution to the expectational equation (6) which satisfies the "no-bubbles" transversality condition

$$\lim_{k \to +\infty} \frac{E_t[P_{t+k}]}{(1+r)^k} = 0$$

is given by

$$P_t = Y_t \equiv \sum_{k=1}^{\infty} \frac{E_t[D_{t+k}]}{(1+r)^k} \tag{7}$$

In particular, in the case of an i.i.d. dividend process $\{D_t\}$ with $E_t[D_{t+k}] = \overline{D}$, $k = 1, 2, ...,$ the fundamental solution (7) is constant, given by $Y_t = Y = \overline{D}/r$, while in the case of a dividend process described by $E_t[D_{t+k}] = (1+\phi)^k D_t$ $k = 1, 2, ..., \phi \geq 0$ (Gordon growth model) the fundamental solution is given by

$$Y_t = (1+\phi)D_t/(r-\phi) \tag{8}$$

We will use the latter specification of the dividend process: as one can easily check this implies that the fundamental evolves over time according to

$$E_t[Y_{t+1}] = (1+\phi)Y_t \tag{9}$$

and that along the fundamental path the expected dividend yield and the capital gain are given respectively by

$$E_t[\delta_{t+1}] \equiv E_t\left[\frac{D_{t+1}}{Y_t}\right] = r - \phi \equiv \overline{\delta}$$

$$E_t[\rho_{t+1}] \equiv E_t\left[\frac{Y_{t+1} - Y_t}{Y_t}\right] = \phi$$

while the expected return is the risk free rate $r = E_t[\rho_{t+1}] + E_t[\delta_{t+1}]$.

Throughout this paper it will be assumed that agents share the same beliefs about the dividend process, while they form different beliefs about the "price" component of the return.

2.3 Expectation formation

The two groups differ in the way they update their "beliefs" about the price change over the next period.

The *fundamentalists* believe that the price will return to the (known) fundamental in the future, so that their expected price change is given by

$$\begin{aligned} E_t^{(f)}[P_{t+1} - P_t] &= \eta(Y_t - P_t) + E_t^{(f)}[Y_{t+1} - Y_t] = \\ &= \eta(Y_t - P_t) + \phi Y_t \end{aligned}$$

The fundamentalist rule is based on the expected change in the underlying fundamental and includes a correction term, proportional to the difference between fundamental and current price, which depends on their beliefs about the speed of mean reversion (captured by the parameter η, $0 \leq \eta \leq 1$). We also assume that fundamentalist conditional variance is constant over time, $Var_t^{(f)}[\rho_{t+1} + \delta_{t+1}] = \sigma^{2(f)}$

The fundamentalist demand function becomes

$$Z_t^{(f)} = \frac{1}{P_t} \frac{\eta(Y_t - P_t) + \phi Y_t + (1+\phi)D_t - rP_t}{\lambda^{(f)}\sigma^{2(f)}} \tag{10}$$

The *chartists'* conditional expected price change evolves over time according to a weighted average (with geometrically declining weights) of past capital gains, which results in the adaptive rule

$$\begin{aligned} m_t^{(c)} &\equiv E_t^{(c)}\left[\rho_{t+1}\right] = E_t^{(c)}\left[\frac{P_{t+1} - P_t}{P_t}\right] = \\ &= (1-c)m_{t-1}^{(c)} + c\left(\frac{P_t - P_{t-1}}{P_{t-1}}\right) = \\ &= (1-c)m_{t-1}^{(c)} + c\rho_t \end{aligned}$$

where the parameter c, $0 \leq c \leq 1$, represents the weight given to the most recent price change.

In order to set an upper bound to chartists' trend extrapolation, we assume (similarly to Chiarella *et al* (2002)) that chartists also increase their estimate of the variance according to the magnitude of the expected excess return, $\sigma_t^{2(c)} = v^{(c)}(|x_t|)$ where $x_t \equiv E_t^{(c)}[\rho_{t+1} + \delta_{t+1} - r] = E_t^{(c)}[(P_{t+1} + D_{t+1} - (1+r)P_t)/P_t]$, so that their demand for the risky asset results in a nonlinear sigmoid function of the expected risk premium.

In our simulations we use the following specification, similar to Chiarella *et al* (2002).

$$\begin{aligned} Z_t^{(c)} &= \frac{\gamma}{\theta} Tanh\left\{\theta E_t^{(c)}[\rho_{t+1} + \delta_{t+1} - r]\right\} \\ &= \frac{\gamma}{\theta} Tanh\left\{\theta[m_t^{(c)} + (1+\phi)D_t/P_t - r]\right\} \end{aligned} \tag{11}$$

where the parameter $\gamma \equiv 1/(\lambda^{(c)} v^{(c)}(0))$ represents the *strength* of chartist demand when the expected excess return is zero (slope of the chartist demand function computed at the origin).

Remark

Using eq. (8) we get $(1+\phi)D_t = (r-\phi)Y_t$, and the demand functions (10) and (11) can be rewritten as

$$Z_t^{(f)} = \frac{(\eta + r)(Y_t - P_t)/P_t}{\lambda^{(f)}\sigma^{2(f)}} \tag{12}$$

$$Z_t^{(c)} = \frac{\gamma}{\theta} Tanh\left\{\theta[m_t^{(c)} + (r-\phi)Y_t/P_t - r]\right\} \tag{13}$$

2.4 Price setting rules

Price adjustment are operated by a *market maker*, who is assumed to know the underlying fundamental.

The market maker clears the market at the end of period t by taking an offsetting long or short position and announces the next period price depending on agents' *excess demand*. We assume that market maker price setting rule also includes a correction term aimed at ensuring long-run market stability.

The assumed price setting rule is given in general by

$$P_{t+1} - P_t = \alpha(Y_t - P_t) + E_t^{(m)}[Y_{t+1} - Y_t] + P_t\ H_t(N_t^{(d)} - N_t^{(s)}) \tag{14}$$

where $E_t^{(m)}[Y_{t+1} - Y_t] = E_t^{(f)}[Y_{t+1} - Y_t] = \phi Y_t$ is the market maker's expected change in the underlying fundamental, $N_t^{(d)} = (\Omega_t^{(f)} Z_t^{(f)} + \Omega_t^{(c)} Z_t^{(c)})/P_t$ is the total agents' demand at time t (number of shares), $N_t^{(s)}$ is the supply of shares at time t, and $H_t(\cdot)$ is a strictly increasing function, with $H_t(0) = 0$. In eq. (14) the term $P_t H_t(N_t^{(d)} - N_t^{(s)})$ represents the price change due to excess demand, while $\alpha(Y_t - P_t) + E_t^{(m)}[Y_{t+1} - Y_t]$ is a corrective term to prevent the price from moving too far away from the fundamental path.

Notice that total agents' demand $N_t^{(d)}$ (number of shares) can be rewritten as $N_t^{(d)} = Z_t \Omega_t / P_t$, where $\Omega_t = \Omega_t^{(f)} + \Omega_t^{(c)}$ is the total wealth and $Z_t \equiv (\Omega_t^{(f)} Z_t^{(f)} + \Omega_t^{(c)} Z_t^{(c)})/\Omega_t$ is the fraction of total wealth invested in the risky asset at time t. Denoting by $Q_t \equiv N_t^{(s)} P_t / \Omega_t$ the value of the supply of shares as a fraction of total agents' wealth, we also obtain $N_t^{(s)} = Q_t \Omega_t / P_t$, so that

$$N_t^{(d)} - N_t^{(s)} = (Z_t - Q_t)\frac{\Omega_t}{P_t}$$

We assume that the market maker reaction to the excess demand is invariant under changes in the level of Ω_t / P_t ("real" wealth), i.e. we assume

$$H_t\ (N_t^{(d)} - N_t^{(s)}) = H\left[(N_t^{(d)} - N_t^{(s)})/(\Omega_t/P_t)\right] = H(Z_t - Q_t)$$

where H is strictly increasing with $H(0) = 0$. We will assume a linear specification in our numerical simulations

$$H(Z_t - Q_t) = \beta(Z_t - Q_t) \quad (\beta > 0)$$

In particular, in the case of zero supply we get $H(Z_t - Q_t) = H(Z_t) = \beta Z_t$.

3 The dynamical system

Under the assumption of noisy dividend and fundamental processes, the dynamics of the model will be given in general by a random nonlinear dynamical system. In this paper we explore the dynamics of the "deterministic skeleton" of the model, i.e. we assume that dividends evolve in a deterministic way according to their (commonly shared) expected value. The dynamics can be summarized as:

$$\begin{aligned} P_{t+1} &= P_t + \alpha(Y_t - P_t) + \phi Y_t + P_t \beta Z_t \\ m_{t+1}^{(c)} &= (1-c)m_t^{(c)} + c[(P_{t+1} - P_t)/P_t] \\ Y_{t+1} &= (1+\phi)Y_t \\ \Omega_{t+1}^{(j)} &= \Omega_t^{(j)} \left[1 + r + Z_t^{(j)} \left(\frac{P_{t+1} + D_{t+1} - (1+r)P_t}{P_t}\right)\right] \qquad j \in \{f, c\} \end{aligned}$$

where:

$$\begin{aligned} \Omega_t &= \Omega_t^{(f)} + \Omega_t^{(c)} \\ Z_t &= (\Omega_t^{(f)} Z_t^{(f)} + \Omega_t^{(c)} Z_t^{(c)})/\Omega_t \\ Z_t^{(f)} &= \frac{(\eta + r)(Y_t - P_t)/P_t}{\lambda^{(f)} \sigma^{2(f)}} \\ Z_t^{(c)} &= \frac{\gamma}{\theta} Tanh\left\{\theta[m_t^{(c)} + (r-\phi)Y_t/P_t - r]\right\} \end{aligned}$$

Although the system results in growing price and wealth processes, it is possible to obtain a stationary system in terms of *capital gain* $\rho_{t+1} \equiv (P_{t+1} - P_t)/P_t$, *fundamental/price ratio* $y_t \equiv Y_t/P_t$, and *wealth shares* of fundamentalists and chartists $w_t^{(j)} \equiv \Omega_t^{(j)}/\Omega_t$, $j \in \{f, c\}$, with $w_t^{(c)} = (1 - w_t^{(f)})$.

Moreover, we denote by

$$\omega_{t+1}^{(j)} = r + Z_t^{(j)} \left(\frac{P_{t+1} + D_{t+1} - (1+r)P_t}{P_t}\right) \qquad j \in \{f, c\}$$

the growth rate of wealth of agent-type j over $(t, t+1)$, and by

$$\omega_{t+1} = r + Z_t \left(\frac{P_{t+1} + D_{t+1} - (1+r)P_t}{P_t}\right) = w_t^{(f)} \omega_{t+1}^{(f)} + (1 - w_t^{(f)}) \omega_{t+1}^{(c)}$$

the rate of growth of total agents' wealth. Notice also that (in the deterministic skeleton of the model) the actual return on the risky asset in $(t, t+1)$ can be rewritten as

$$\begin{aligned} \frac{P_{t+1} + D_{t+1} - (1+r)P_t}{P_t} &= \frac{P_{t+1} - P_t}{P_t} + \frac{(r-\phi)Y_t}{P_t} - r = \\ &= \rho_{t+1} + (r-\phi)y_t - r \end{aligned}$$

In particular, a dynamic equation for the wealth shares is obtained by rewriting the wealth recurrent equations (1) for $j = f, c$ as

$$w_{t+1}^{(j)} \Omega_{t+1} = w_t^{(j)} \Omega_t (1 + \omega_{t+1}^{(j)})$$

i.e.

$$
\begin{aligned}
w_{t+1}^{(f)} &= w_t^{(f)}(1+\omega_{t+1}^{(f)})\frac{\Omega_t}{\Omega_{t+1}} \\
w_{t+1}^{(c)} &= w_t^{(c)}(1+\omega_{t+1}^{(c)})\frac{\Omega_t}{\Omega_{t+1}}
\end{aligned}
$$

By summing up the above equations and recalling that $w_{t+1}^{(f)} + w_{t+1}^{(c)} = 1$ we obtain

$$
\frac{\Omega_{t+1}}{\Omega_t} = w_t^{(f)}(1+\omega_{t+1}^{(f)}) + w_t^{(c)}(1+\omega_{t+1}^{(c)})
$$

and therefore

$$
\begin{aligned}
w_{t+1}^{(f)} &= w_t^{(f)}(1+\omega_{t+1}^{(f)})\frac{\Omega_t}{\Omega_{t+1}} = \\
&= \frac{w_t^{(f)}(1+\omega_{t+1}^{(f)})}{w_t^{(f)}(1+\omega_{t+1}^{(f)}) + (1-w_t^{(f)})(1+\omega_{t+1}^{(c)})} = \frac{w_t^{(f)}(1+\omega_{t+1}^{(f)})}{(1+\omega_{t+1})}
\end{aligned}
$$

The stationary system is thus given by

$$
\begin{aligned}
\rho_{t+1} &= \alpha(y_t - 1) + \phi y_t + \beta Z_t \quad \text{(capital gain)} \\
y_{t+1} &= \frac{(1+\phi)}{(1+\rho_{t+1})} y_t \quad \text{(fundamental/price ratio)} \\
m_{t+1}^{(c)} &= (1-c)m_t^{(c)} + c\rho_{t+1} \quad \text{(chartist expected cap. gain)} \\
w_{t+1}^{(f)} &= \frac{w_t^{(f)}(1+\omega_{t+1}^{(f)})}{(1+\omega_{t+1})} \quad \text{(wealth shares)}
\end{aligned}
$$

where:

$$
\begin{aligned}
Z_t &= w_t^{(f)} Z_t^{(f)} + (1-w_t^{(f)}) Z_t^{(c)} \\
Z_t^{(f)} &= \frac{(\eta + r)(y_t - 1)}{\lambda^{(f)}\sigma^{2(f)}} \\
Z_t^{(c)} &= \frac{\gamma}{\theta} Tanh[\theta(m_t^{(c)} + (r-\phi)y_t - r)] \\
\omega_{t+1}^{(f)} &= r + Z_t^{(f)}(\rho_{t+1} + (r-\phi)y_t - r) \\
\omega_{t+1} &= r + Z_t(\rho_{t+1} + (r-\phi)y_t - r)
\end{aligned}
$$

3.1 The map

The time evolution of the stationary system is given by the iteration of the following *nonlinear* map $T : (\rho, y, m^{(c)}, w^{(f)}) \longmapsto (\rho', y', m^{(c)\prime}, w^{(f)\prime})$, where the symbol $'$ denotes the unit time advancement operator:

$$T:\begin{cases}\rho' = \alpha(y-1) + \phi y + \beta Z \\ y' = y(1+\phi)/(1+\rho') \\ m^{(c)\prime} = (1-c)m^{(c)} + c\rho' \\ w^{(f)\prime} = w^{(f)}[1 + r + Z^{(f)}(\rho' + (r-\phi)y - r)]/[1 + r + Z(\rho' + (r-\phi)y - r)]\end{cases} \tag{15}$$

where

$$\begin{aligned} Z^{(f)} &= \frac{(\eta + r)(y-1)}{\lambda^{(f)}\sigma^{2(f)}} \\ Z^{(c)} &= \frac{\gamma}{\theta} Tanh[\theta(m^{(c)} + (r-\phi)y - r)] \\ Z &= w^{(f)}Z^{(f)} + (1-w^{(f)})Z^{(c)} \end{aligned}$$

Although in (15) we have 4 dynamic variables, the map T is in fact a 3-D map, being ρ' a function of y, $m^{(c)}$, and $w^{(f)}$.

3.2 Steady states

As one can check, the map (15) has two types of steady states, that we denote by "fundamental steady states" and by "nonfundamental steady states", respectively. The map also presents other important "invariant" subsets of the phase-space.

"Fundamental" steady states

Fundamental steady states are characterized by

$$\begin{aligned} y &= 1; \quad \rho = m^{(c)} = \phi \\ w^{(f)} &= \overline{w}^{(f)}, \quad \overline{w}^{(f)} \in [0,1] \end{aligned}$$

i.e. by the price being at the fundamental ($P_t = Y_t$, for any t) and growing at the fundamental rate ϕ, and by zero excess demand, $Z = 0$. The long-run wealth distribution $(\overline{w}^{(f)}, 1-\overline{w}^{(f)})$ at a fundamental steady state may be given, in general, by any $\overline{w}^{(f)} \in [0,1]$ (numerical simulation of the dynamical system will confirm that the steady state wealth distribution depends on the initial condition).

"Non fundamental steady states"

It is easy to check that the "fundamental steady states" are not the only steady states of the model formulated in terms of returns and wealth shares.

Depending on the parameters, the map may have other "steady states", coexisting with the fundamental ones, characterized by

$$
\begin{aligned}
y &= 0; \quad w^{(f)} = 0; \quad \rho = m^{(c)} = \overline{\rho} > \phi \\
\text{where } \overline{\rho} \text{ solves} \quad &: \quad \frac{\overline{\rho} + \alpha}{\beta} = \frac{\gamma}{\theta} Tanh[\theta(\overline{\rho} - r)]
\end{aligned}
$$

It can be easily shown that the "nonfundamental" growth rates $\overline{\rho}$ which come out as (positive) solutions of the above equation are higher than the risk free rate r (and than the fundamental growth rate ϕ). Nonfundamental steady states are thus characterized by price growing faster than fundamental, $\rho = m^{(c)} = \overline{\rho} > r > \phi$, fundamental/price ratio converging to 0, $y = 0$, i.e. $\lim_{t\to\infty} P_t/Y_t = +\infty$, market dominated by chartists, $w^{(f)} = 0$, and permanent positive excess demand $Z = Z^{(c)} = (\gamma/\theta)Tanh[\theta(\overline{\rho} - r)] > 0$.

The map also presents other important "invariant" subsets of the phase-space, that are represented by the cases where only fundamentalists ($w^{(f)} = 1$) or only chartists ($w^{(f)} = 0$) populate the market.

In particular, in the case $w^{(f)} = 0$ (market dominated by chartists) the dynamics is driven by the lower dimensional map

$$
T^{(c)} : \begin{cases} \rho' = \alpha(y-1) + \phi y + \beta Z^{(c)} \\ y' = y(1+\phi)/(1+\rho') \\ m^{(c)\prime} = (1-c)m^{(c)} + c\rho' \end{cases}
$$

In this particular case it can be proved that

- the "fundamental equilibrium" $y = 1$, $\rho = m^{(c)} = \phi$, is locally asymptotically stable for low values of c (*chartists extrapolation rate*), β (*price reaction parameter*) and γ (i.e. for high agents' *risk aversion*).
- for higher values of c, β and γ the system converges to an attracting limit cycle (with long-run fluctuations of the price around the fundamental) or to a "non fundamental" equilibrium (with permanent and increasing deviation of the price away from the fundamental).
- numerical simulations show that the attractors of the map $T^{(c)}$ (limit cycle, nonfundamental steady state) are in general attractors also for the map T, i.e. they can be reached also starting from initial points with $w_0^{(f)} > 0$. Thus the analysis of the lower dimensional case with no fundamentalists helps to understand the dynamics of the system in the general case.

4 Numerical simulation of the global dynamics

This section contains numerical experiments aimed at gaining some insight into the global dynamics of the model. Overall, these simulations show the range of dynamic scenarios that the dynamic model is able to generate. In particular the analysis will focus on the situations of coexistence of fundamental and non-fundamental steady states, with an analysis of the role of initial conditions and basins of attraction of coexisting equilibria.

Although the model is described by a 3-D system, all the phase-space representations will be obtained by means of projections in the $\rho - y$ plane, except for the basins of attraction of *Fig. 6* (where the initial condition are taken in the $w^{(f)} - y$ plane).

Fig. 1 is devoted to the dynamic behavior of the model restricted to the lower dimensional invariant manifold $w^{(f)} = 0$ (i.e. when market is dominated by chartists). As remarked in the previous Section, when the (chartist) risk aversion coefficient $\lambda^{(c)}$ is decreased (the parameter γ is increased) the fundamental steady state F changes from stable (*Fig. 1a*) to unstable focus (*Fig. 1b*) through a (supercritical) Neimark-Hopf bifurcation, which creates a stable limit cycle. The size of fluctuations becomes wider for lower risk aversion (*Fig. 1c*) until a stable non fundamental steady state NF appears, to which the system converges with a persistent deviation of the price away from the fundamental (*Fig. 1d*). Similar phase-space transitions can be obtained by increasing the extrapolation parameter c or the price adjustment parameter β. The attractors (limit cycle and non fundamental steady state) which exist for the subcase $w^{(f)} = 0$ may be reached also starting from initial conditions with $w^{(f)} > 0$. This is shown in the following *Fig. 2* (where starting from $w_0^{(f)} = 0.45$ the market ends up in the same limit cycle as in *Fig. 1c*) and *Fig. 3* (which shows convergence to the nonfundamental equilibrium represented in *Fig. 1d*, starting with $w_0^{(f)} = 0.56$).

While at a fundamental steady states typically both types of agents survive in the long-run, with constant stationary wealth shares, *Figs. 2,3* represent situations where fundamentalists are out of the market in the long-run, because their average profits are lower than chartists' profits. This is not the only possible outcome associated with disequilibrium dynamics: depending on the parameters, more complex attractors exist, where both types of agents survive in the long-run, with fluctuations of wealth shares. An example is the strange attractor whose projection is represented in *Fig. 4a*. *Figs. 4b,c* show that the motion on the strange attractor has alternating phases, with price much higher than fundamental when the market is dominated by chartists, whereas the fundamentalist wealth proportion increases when the price returns close to the fundamental.

Figs. 5, 6 explore the dynamic behaviour under coexistence of fundamental and non fundamental steady states. Small differences in the initial condition (e.g. initial wealth shares) may change the asymptotic dynamics of the system (compare *Figs. 5a,c*, where $w_0^{(f)} = 0.56$, with *Figs. 5b,d*, where $w_0^{(f)} = 0.57$). *Fig. 6a* represents the basins of attraction associated with the numerical example of *Fig. 5*: the basins are obtained by letting the initial values of fundamentalist wealth share $w_0^{(f)}$ and fundamental/price ratio y_0 vary in the $w^{(f)} - y$ plane, for fixed initial values of the other dynamic variables, and by representing in blue (red) the region of initial points which generate trajectories converging to the fundamental (nonfundamental) equilibrium. Of course, the basins' structure depends on the particular parameter set used in the simulation. For instance, higher values of the chartist parameter c determine an increase of the size of the basin of the nonfundamental steady state (compare *Fig. 6a*, where $c = 0.25$, with Fig. *6b*, where $c = 0.75$).

Figs. 6a,b show that for sufficiently high values of the initial fundamentalist wealth share, the system converges to the fundamental (no matter how far is the initial price from the fundamental). However, for low initial fundamentalist wealth share, the system may converge to the non fundamental steady state: surprisingly, this occurs when the initial price is close to the fundamental, while when the initial price is far enough from fundamental, the price is capable to return to the fundamental. A possible explanation of this phenomenon is that when the price is far from the fundamental, a higher fundamentalist demand (proportional to the deviation from fundamental price) acts as a stronger mean reverting force.

Figs. 7, 8 show that increasing values of the chartist extrapolation rate c can destabilize the price and produce a negative effect on fundamentalist profits and wealth. The numerical experiment is obtained by starting with the same initial condition, with $w_0^{(f)} = 40\%$, and simulating the system under increasing values of c. In the cases of convergence to the fundamental steady state (*Fig. 7* and *Figs. 8a,b*) the higher is the chartist extrapolation rate, the lower is the stationary fundamentalist wealth share which is reached by the system in the long-run, as shown by *Figs. 7b,d,f* and *8b*; this happens because the higher is c, the longer is the transient characterized by price fluctuations around the fundamental, with fundamentalists' average profits lower than chartists' (*Figs. 7a,c,e* and *Fig. 8a*). When c becomes higher than a certain threshold, then the system is completely destabilized and no longer converges to the fundamental price but ends up in a limit cycle, with zero fundamentalist wealth share and market dominated by chartists (*Figs. 8c,d*)

Fig. 9 shows another example of an attractor which "allows" both groups to survive in the long-run, with time varying wealth shares. This is a periodic orbit (*Fig. 9c*) on which the fundamentalist wealth fraction fluctuates approximately in the range $[35\%, 43\%]$ (*Fig. 9d*). For slightly different initial wealth shares, the same figure shows a different trajectory converging to a (coexisting) fundamental equilibrium, with much higher stationary fundamentalist wealth proportion (see also *Figs. 9a,b*).

5 Conclusions and further research

Following the framework of Chiarella (1992), Chiarella et al (2002), Chiarella and He (2001a, 2003) and Brock and Hommes (1998), the interaction of fundamentalists and chartists has been incorporated in a market maker model of asset price and wealth dynamics. The resulting dynamical system for asset price and wealth turns out to be nonstationary, and a stationary system is developed by expressing the laws of motion in terms of capital gain, fundamental/price ratio and wealth proportions (among the two types of agents). It is found that the presence of fundamentalists and chartists leads the stationary model to have two kinds of steady states, which often coexist in the phase-space, with different long-run stationary returns and wealth distributions: *fundamental* steady states, where the price is at the fundamental level, and *nonfundamental* steady states, where price grows faster than fundamental, while fundamentalist wealth proportion ultimately converges to zero.

The chartists' extrapolation parameter c, together with the chartists' risk aversion $\lambda^{(c)} = 1/\gamma$ and the market reaction coefficient β, play an important role in the local asymptotic stability of the fundamental steady states, and for sufficiently high values of c, β, γ the price and return dynamics become unstable due to a Neimark-Hopf bifurcation of the fundamental steady states.

The main impression gained from the numerical simulation of the global dynamics (Section 4) is that the model is able to generate a wide range of different dynamic scenarios, with a strong dependence on small changes of the parameters and of the initial conditions: limit cycles, periodic orbits, strange attractors, cases of coexistence of multiple steady states, or coexistence of a steady state with a cyclic attractor. In particular, in the case of coexistence of fundamental and nonfundamental steady states, the initial wealth distribution and the initial distance of the price from the fundamental play an important role in determining the long-run evolution.

Another important feature of this model is that it considers explicitly the dynamic interdependence between price and wealth distribution among agent-types: it is found that in general fundamentalists' average profits are lower than chartists' profits (and thus fundamentalist wealth proportion tends to vanish) when the system moves on a limit cycles or is at a nonfundamental steady state; on the other hand both types of agents survive in the long-run when the market is at a fundamental equilibrium, or when it fluctuates on periodic orbits or strange attractors.

Our analysis in this paper is based on a simplified model, and some extensions are needed in order to develop a more realistic one. First the analysis here has focused on a deterministic dynamic model which can be interpreted as the deterministic skeleton of a stochastic model with a noisy growing dividend process: particularly interesting seems the interaction of a noisy dividend process with the basins' structure of the underlying deterministic scenarios with coexisting attractors. Second, although the dynamic modelling of the wealth proportions "keeps track" of realized profits of the two types of agents and determines endogenously time varying "weights" of fundamentalists and chartists, this model is one with fixed agents' proportions, in the sense that agents do not "switch" amongst different strategies on the basis on their realized profits or wealth (according to the adaptive belief system introduced by Brock and Hommes (1997, 1998)). The introduction of "switching" mechanisms and time varying proportion (similar to Chiarella and He (2002)) would be a very interesting extension of this model. Third, the introduction of a more flexible and realistic price setting rule, where the market maker inventory position is also taken into account, is likely to lead to more realistic dynamics of returns and wealth fractions.

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Appendix

Derivation of the dynamic model under the assumption of positive supply of shares

In this Appendix we show how the model can be extended to the case of positive supply of shares.

Again we derive a benchmark notion of fundamental solution, which refers to the price that would be obtained if agents were homogeneous with regard to their expectation of the excess return, and which satisfies a long-run "no bubbles" condition.

Denote by $N_t^{(j)} = \Omega_t^{(j)} Z_t^{(j)} / P_t$ the number of shares demanded by agent type j at time t and by N_t^s the supply of shares at t, and consider the market equilibrium condition at time t (see Section 2.2)

$$\sum_j \Omega_t^{(j)} \frac{E_t^{(j)}[P_{t+1} + D_{t+1} - (1+r)P_t]}{\lambda^{(j)} Var_t^{(j)}[\rho_{t+1} + \delta_{t+1} - r]} = N_t^s P_t^2$$

Now denote by $\sigma_t^{2(j)} \equiv Var_t^{(j)}[\rho_{t+1} + \delta_{t+1} - r]$ the belief of agent j about the variance of the excess return in $(t, t{+}1)$, by $\Omega_t = \sum_j \Omega_t^{(j)}$ the total wealth and by $w_t^{(j)} = \Omega_t^{(j)}/\Omega_t$ the wealth share of agent j, with $w_t^{(c)} = 1 - w_t^{(f)}$. Assume that all agents have homogeneous beliefs about the expected excess return. Then the above market clearing condition can be rewritten as

$$E_t[P_{t+1} + D_{t+1} - (1+r)P_t] = Q_t \xi_t P_t$$

where $Q_t \equiv N_t^s P_t / \Omega_t$ represents the value of the supply of shares as a fraction of total agents' wealth and

$$\xi_t = \left(\sum_j w_t^{(j)} \frac{1}{\lambda^{(j)} \sigma_t^{2(j)}} \right)^{-1} = \left[w_t^{(f)} \frac{1}{\lambda^{(f)} \sigma_t^{2(f)}} + (1 - w_t^{(f)}) \frac{1}{\lambda^{(c)} \sigma_t^{2(c)}} \right]^{-1} \tag{16}$$

is a (time varying) weighted harmonic mean of the "risk-adjustment coefficients" $\lambda^{(j)} \sigma_t^{2(j)}$, $j = f, c$. Finally

$$E_t[P_{t+1} + D_{t+1}] = (1 + r_t^*) P_t \tag{17}$$

where $r_t^* \equiv r + Q_t \xi_t$ may be interpreted within this framework as the required expected return on the risky asset in $(t, t+1)$, while

$$\pi_t \equiv Q_t \xi_t = \frac{N_t^s P_t}{\Omega_t} \left[w_t^{(f)} \frac{1}{\lambda^{(f)} \sigma_t^{2(f)}} + (1 - w_t^{(f)}) \frac{1}{\lambda^{(c)} \sigma_t^{2(c)}} \right]^{-1} \tag{18}$$

represents the required risk-premium.

The case of zero supply ($N_t^s = 0$) has already been considered in the paper. Here we consider the case of positive supply, and we make the additional assumption that in this market (characterized by growing wealth process) the value of the supply of shares as a fraction of total agents' wealth is constant over time, $Q_t = N_t^s P_t / \Omega_t = Q$, $\forall\, t$.

Same risk aversion and homogeneous beliefs about variance

First assume that agents have the same risk aversion and share the same (constant) beliefs about the variance, $\lambda^{(f)}\sigma^{2(f)} = \lambda^{(c)}\sigma^{2(c)} = \lambda\sigma^2$. In this case the (constant) required risk premium $\pi_t = \pi$ and the required expected return $r_t^* = r^*$ turn out to be

$$\pi \equiv Q\lambda\sigma^2$$

$$r^* = r + \pi = r + Q\lambda\sigma^2$$

and the market clearing condition yields

$$E_t[P_{t+1} + D_{t+1}] = (1 + r^*)P_t$$

Assuming homogeneous beliefs about dividends and a dividend process which evolves according to $E_t[D_{t+k}] = (1 + \phi)^k D_t,\ k = 1, 2, ...,\ \phi \geq 0$, the unique fundamental solution Y_t is given by

$$Y_t = \frac{(1+\phi)D_t}{(r^* - \phi)} = \frac{(1+\phi)D_t}{(r + Q\lambda\sigma^2 - \phi)} \tag{19}$$

where Y_t evolves over time according to

$$E_t[Y_{t+1}] = (1 + \phi)Y_t$$

At the fundamental solution the expected dividend yield and capital gain are given respectively by

$$E_t[\delta_{t+1}] = E_t\left[\frac{D_{t+1}}{Y_t}\right] = r^* - \phi \equiv \overline{\delta}$$

$$E_t[\rho_{t+1}] = E_t\left[\frac{Y_{t+1} - Y_t}{Y_t}\right] = \phi$$

while the expected return is $r^* = r + Q\lambda\sigma^2 = E_t[\rho_{t+1}] + E_t[\delta_{t+1}]$.

Following similar steps as in the case of zero supply, we obtain the following dynamical system

$$\begin{aligned}
P_{t+1} &= P_t + \alpha(Y_t - P_t) + \phi Y_t + P_t\beta(Z_t - Q) \\
m_{t+1}^{(c)} &= (1-c)m_t^{(c)} + c[(P_{t+1} - P_t)/P_t] \\
Y_{t+1} &= (1+\phi)Y_t \\
\Omega_{t+1}^{(j)} &= \Omega_t^{(j)}\left[1 + r + Z_t^{(j)}\left(\frac{P_{t+1} + D_{t+1} - (1+r)P_t}{P_t}\right)\right] \qquad j \in \{f, c\}
\end{aligned}$$

where:

$$\begin{aligned}
\Omega_t &= \Omega_t^{(f)} + \Omega_t^{(c)} \\
Z_t &= (\Omega_t^{(f)} Z_t^{(f)} + \Omega_t^{(c)} Z_t^{(c)})/\Omega_t \\
Z_t^{(f)} &= \frac{1}{P_t}\frac{\eta(Y_t - P_t) + \phi Y_t + (1+\phi)D_t - rP_t}{\lambda\sigma^2} \\
Z_t^{(c)} &= \frac{m_t^{(c)} + (1+\phi)D_t/P_t - r}{\lambda\sigma_t^{2(c)}}
\end{aligned}$$

and we may again assume that chartists increase their estimate of the variance according to the magnitude of the expected excess return, $\sigma_t^{2(c)} = v^{(c)}(|x_t|)$ where $x_t \equiv E_t^{(c)}[\rho_{t+1} + \delta_{t+1} - r] = m_t^{(c)} + (1+\phi)D_t/P_t - r$ (so that their demand for the risky asset results in a nonlinear sigmoid function of the expected risk premium). Consistently with our assumption that agents have homogeneous beliefs about variances in equilibrium (where $x_t = r^* - r$) we also assume $v^{(c)}(r^* - r) = \sigma^2 = \sigma^{2(f)}$.

Notice that from (19) we get

$$(1+\phi)D_t = (r^* - \phi)Y_t$$

so that agents' demand functions can be rewritten as (recall also that $(r^* - r) = Q\lambda\sigma^2$)

$$\begin{aligned}
Z_t^{(f)} &= \frac{(\eta + r)(Y_t - P_t)/P_t + (r^* - r)Y_t/P_t}{\lambda\sigma^2} = \frac{(\eta + r)(Y_t - P_t)/P_t}{\lambda\sigma^2} + Q\frac{Y_t}{P_t} \\
Z_t^{(c)} &= \frac{m_t^{(c)} + (r^* - \phi)Y_t/P_t - r}{\lambda\sigma_t^{2(c)}} = \frac{m_t^{(c)} + (r - \phi)Y_t/P_t - r}{\lambda\sigma_t^{2(c)}} + \frac{Q\sigma^2}{\sigma_t^{2(c)}}\frac{Y_t}{P_t}
\end{aligned}$$

A stationary system can be obtained through the same changes of variables used in the simplified zero-supply case, and similar results about the steady states hold. Notice that, at the fundamental steady states, the total agents demand is exactly equal to the supply, $\overline{Z}^{(c)} = \overline{Z}^{(f)} = \overline{Z} = Q = (r^* - r)/\lambda\sigma^2$.

Different risk aversion and heterogeneous beliefs about variance
Consider the market clearing condition, rewritten in the form (17). In this case it is not immediate to obtain the fundamental solution in closed form, unless we make simplifying assumptions about future risk premia π_{t+1}, π_{t+2}, ... In the derivation of the fundamental price we assume, for simplicity, that the risk premium computed at time t according to (18) is believed to hold at all future times $t+1$, $t+2$, ..., but the estimate of π_t and r_t^* is revised at each point in time.

The fundamental price at time t turns out to be

$$Y_t = \frac{(1+\phi)D_t}{(r_t^* - \phi)} \tag{20}$$

and the expectation (taken at time t) of the fundamental in $(t+1)$ will be again given by.

$$E_t[Y_{t+1}] = (1+\phi)Y_t$$

Since the dividend growth rate ϕ is assumed constant, but r^*_{t+1} will be different from r^*_t (unless the system is at a steady state), the fundamental at $(t+1)$ will be

$$Y_{t+1} = \frac{(1+\phi)D_{t+1}}{(r^*_{t+1} - \phi)}$$

and this implies that the law of motion of the fundamental (in the deterministic skeleton of the model) is now given by

$$Y_{t+1} = (1+\phi)Y_t \frac{r^*_t - \phi}{r^*_{t+1} - \phi}$$

It is convenient to consider in the general model a dynamic equation which accounts for the time evolution of the dividend, instead of the fundamental. We obtain the following dynamical system

$$\begin{aligned}
P_{t+1} &= P_t + \alpha(Y_t - P_t) + \phi Y_t + P_t \beta(Z_t - Q) \\
m^{(c)}_{t+1} &= (1-c)m^{(c)}_t + c[(P_{t+1} - P_t)/P_t] \\
D_{t+1} &= (1+\phi)D_t \\
\Omega^{(j)}_{t+1} &= \Omega^{(j)}_t \left[1 + r + Z^{(j)}_t \left(\frac{P_{t+1} + D_{t+1} - (1+r)P_t}{P_t}\right)\right] \qquad j \in \{f, c\}
\end{aligned}$$

where:

$$\begin{aligned}
\Omega_t &= \Omega^{(f)}_t + \Omega^{(c)}_t \\
Z_t &= (\Omega^{(f)}_t Z^{(f)}_t + \Omega^{(c)}_t Z^{(c)}_t)/\Omega_t \\
Z^{(f)}_t &= \frac{1}{P_t} \frac{\eta(Y_t - P_t) + \phi Y_t + (1+\phi)D_t - rP_t}{\lambda^{(f)} \sigma^{2(f)}_t} \\
Z^{(c)}_t &= \frac{m^{(c)}_t + (1+\phi)D_t/P_t - r}{\lambda^{(c)} \sigma^{2(c)}_t} \\
Y_t &= \frac{(1+\phi)D_t}{(r^*_t - \phi)} \\
r^*_t &= r + Q\left[\frac{\Omega^{(f)}_t}{\Omega_t} \frac{1}{\lambda^{(f)} \sigma^{2(f)}_t} + \frac{\Omega^{(c)}_t}{\Omega_t} \frac{1}{\lambda^{(c)} \sigma^{2(c)}_t}\right]^{-1}
\end{aligned}$$

and the fundamentalist and chartist estimates of the variance $\sigma^{2(f)}_t$ and $\sigma^{2(c)}_t$ are allowed in general to depend on the state of the system at time t.

Also in the general case a stationary system can be obtained in terms of *capital gain* $\rho_{t+1} \equiv (P_{t+1}-P_t)/P_t$, *fundamental/price ratio* $y_t \equiv Y_t/P_t$, *dividend to price ratio* $\psi_t \equiv D_t/P_t$, and *wealth shares* $w_t^{(j)} \equiv \Omega_t^{(j)}/\Omega_t$, $j \in \{f, c\}$, with $w_t^{(c)} = (1 - w_t^{(f)})$. The stationary system is the following

$$
\begin{aligned}
\rho_{t+1} &= \alpha(y_t - 1) + \phi y_t + \beta(Z_t - Q) \quad \text{(capital gain)} \\
\psi_{t+1} &= \frac{(1+\phi)\psi_t}{(1+\rho_{t+1})} \qquad \text{(dividend to price ratio)} \\
m_{t+1}^{(c)} &= (1-c)m_t^{(c)} + c\rho_{t+1} \quad \text{(chartists expected cap. gain)} \\
w_{t+1}^{(f)} &= \frac{w_t^{(f)}(1+\omega_{t+1}^{(f)})}{(1+\omega_{t+1})} \quad \text{(wealth shares)}
\end{aligned}
$$

where:

$$
\begin{aligned}
Z_t &= w_t^{(f)} Z_t^{(f)} + (1 - w_t^{(f)}) Z_t^{(c)} \\
Z_t^{(f)} &= \frac{\eta(y_t - 1) + \phi y_t + (1+\phi)\psi_t - r}{\lambda^{(f)} \sigma_t^{2(f)}} \\
Z_t^{(c)} &= \frac{m_t^{(c)} + (1+\phi)\psi_t - r}{\lambda^{(c)} \sigma_t^{2(c)}} \\
y_t &= \frac{(1+\phi)\psi_t}{(r_t^* - \phi)} \quad \text{(fundamental/price ratio)} \\
r_t^* &= r + Q\left[w_t^{(f)} \frac{1}{\lambda^{(f)}\sigma_t^{2(f)}} + (1 - w_t^{(f)}) \frac{1}{\lambda^{(c)}\sigma_t^{2(c)}} \right]^{-1} \\
\omega_{t+1}^{(f)} &= r + Z_t^{(f)}(\rho_{t+1} + (1+\phi)\psi_t - r) \\
\omega_{t+1} &= r + Z_t(\rho_{t+1} + (1+\phi)\psi_t - r)
\end{aligned}
$$

The dynamic behavior of this dynamical system will be different, in general, from the one of the simplified cases considered above. The main difference is due to the time varying risk premium (which depends on time varying wealth shares and risk perceptions) that causes in general the fundamental to grow at a rate different from the dividend growth rate.

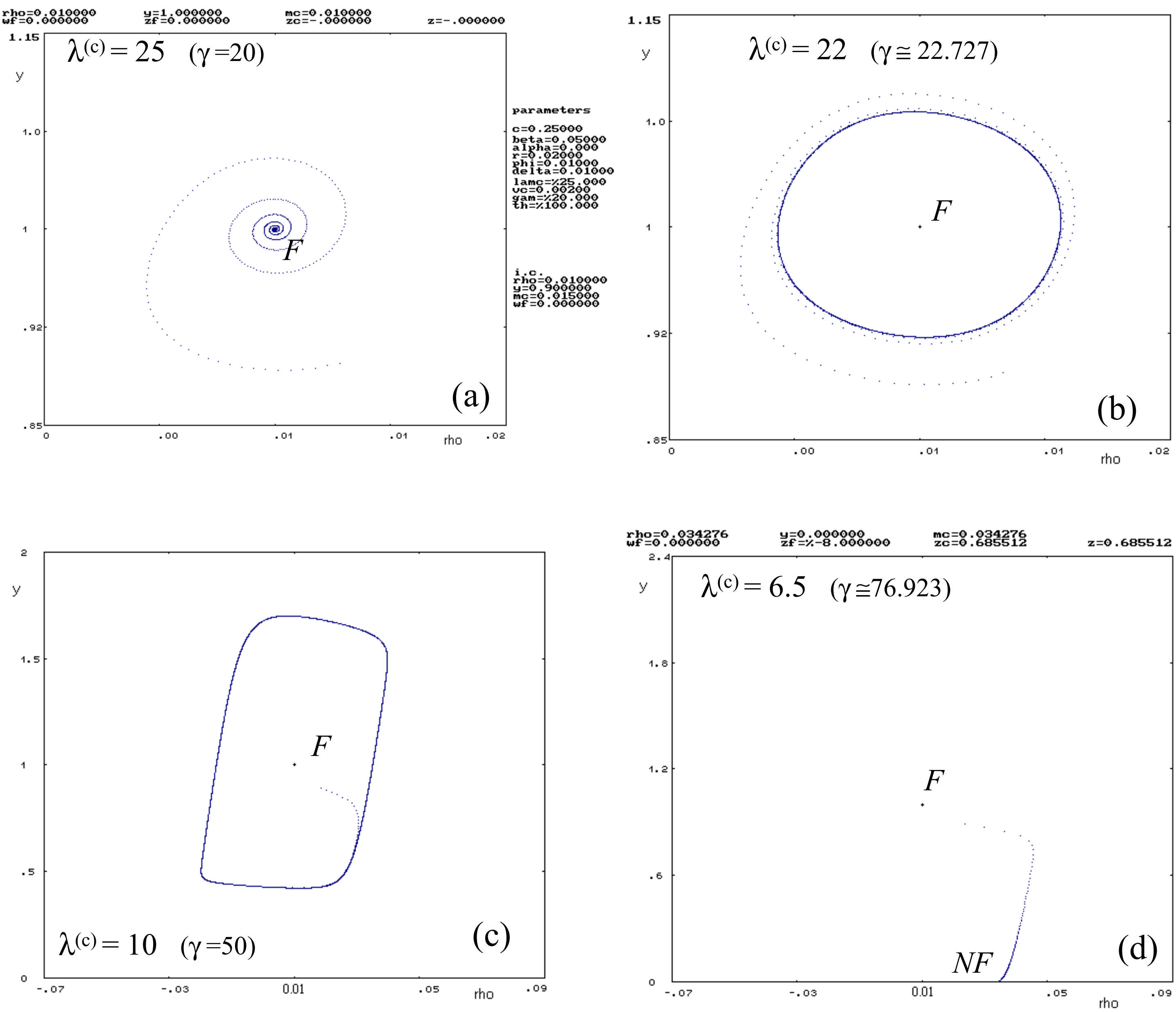
We leave to future research a detailed analysis of the general case.

Fig. 1

The case with $w^{(f)} = 0$ (market with no fundamentalists)
Effect of decreasing the chartist risk aversion

damped fluctuations and
convergence to fundamental (*F*)

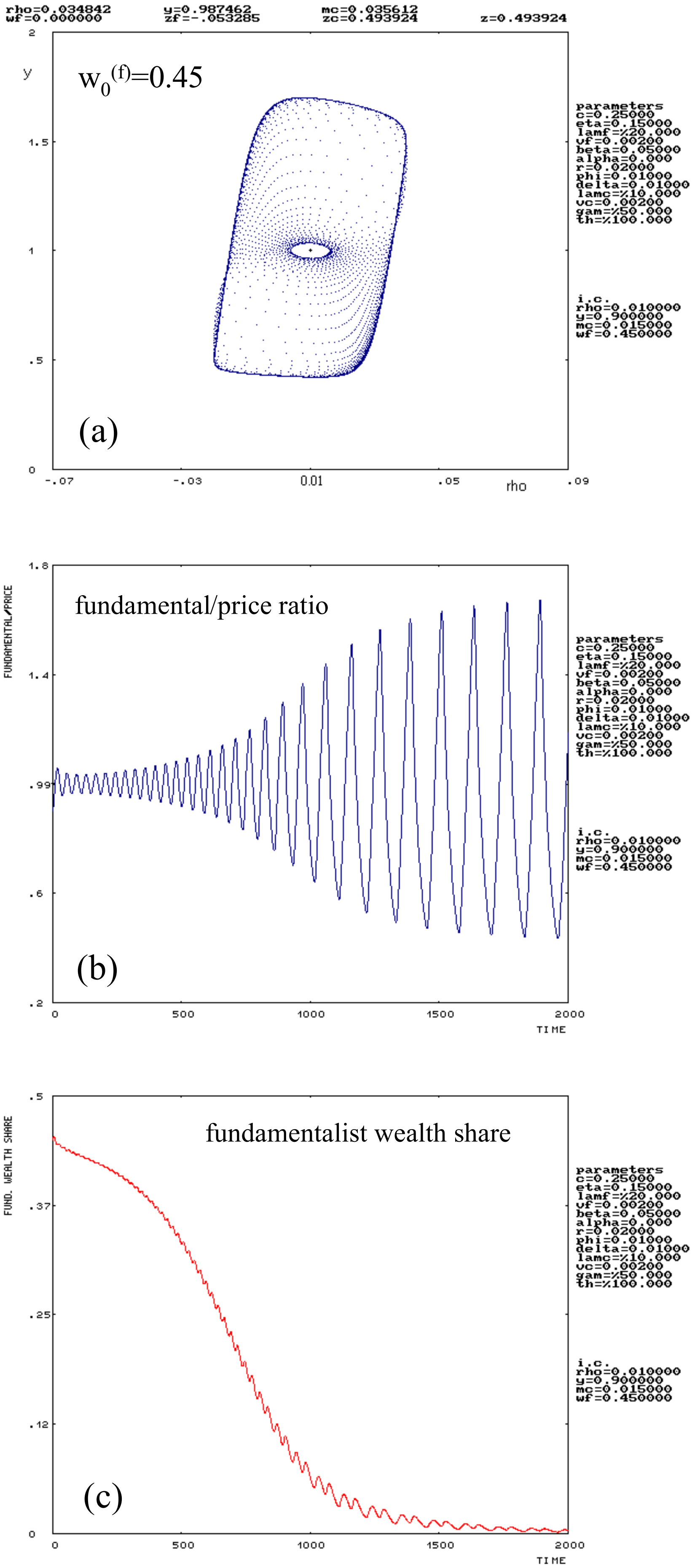
limit cycle



limit cycle with wide fluctuations
of fundamental/price ratio

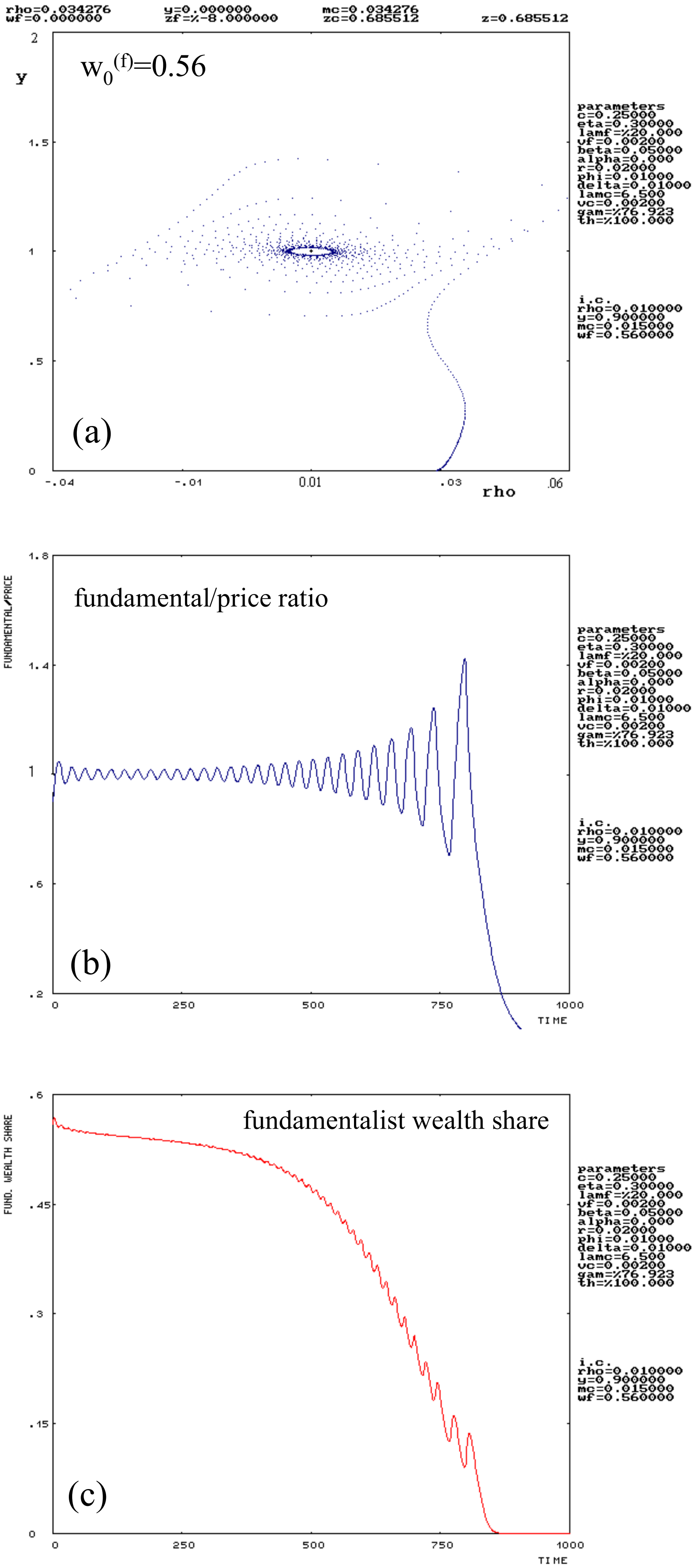
convergence to a “non fundamental”
steady state (*NF*)
(price growing faster than fundamental)

Fig. 2



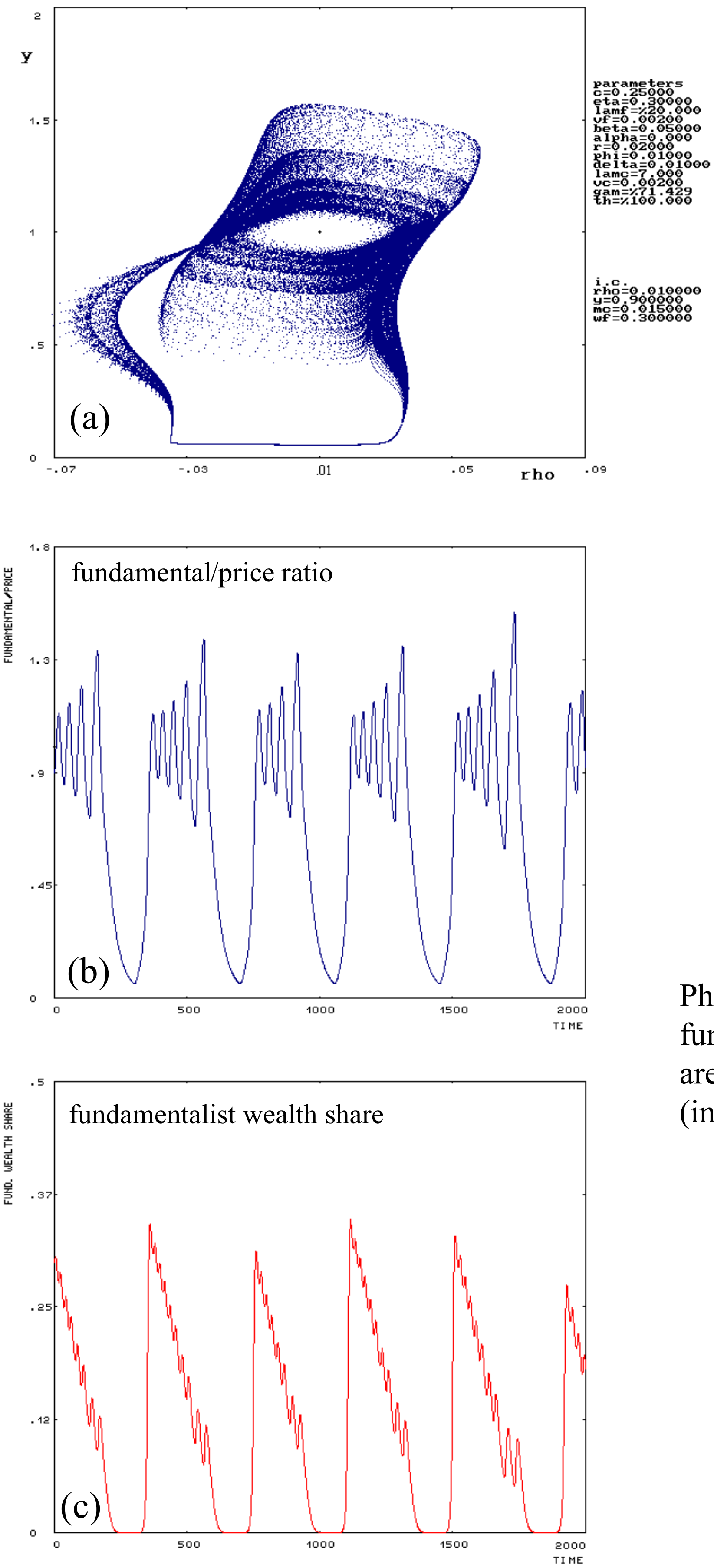
Convergence to
a limit cycle where
only chartists survive

Fig. 3



Convergence to a "*non fundamental* equilibrium" where only chartists survive (price grows faster than fundamental $\rho \cong 3.428\%$; $\phi = 1\%$)

Fig. 4



A strange attractor with cyclical movements of prices and wealth shares

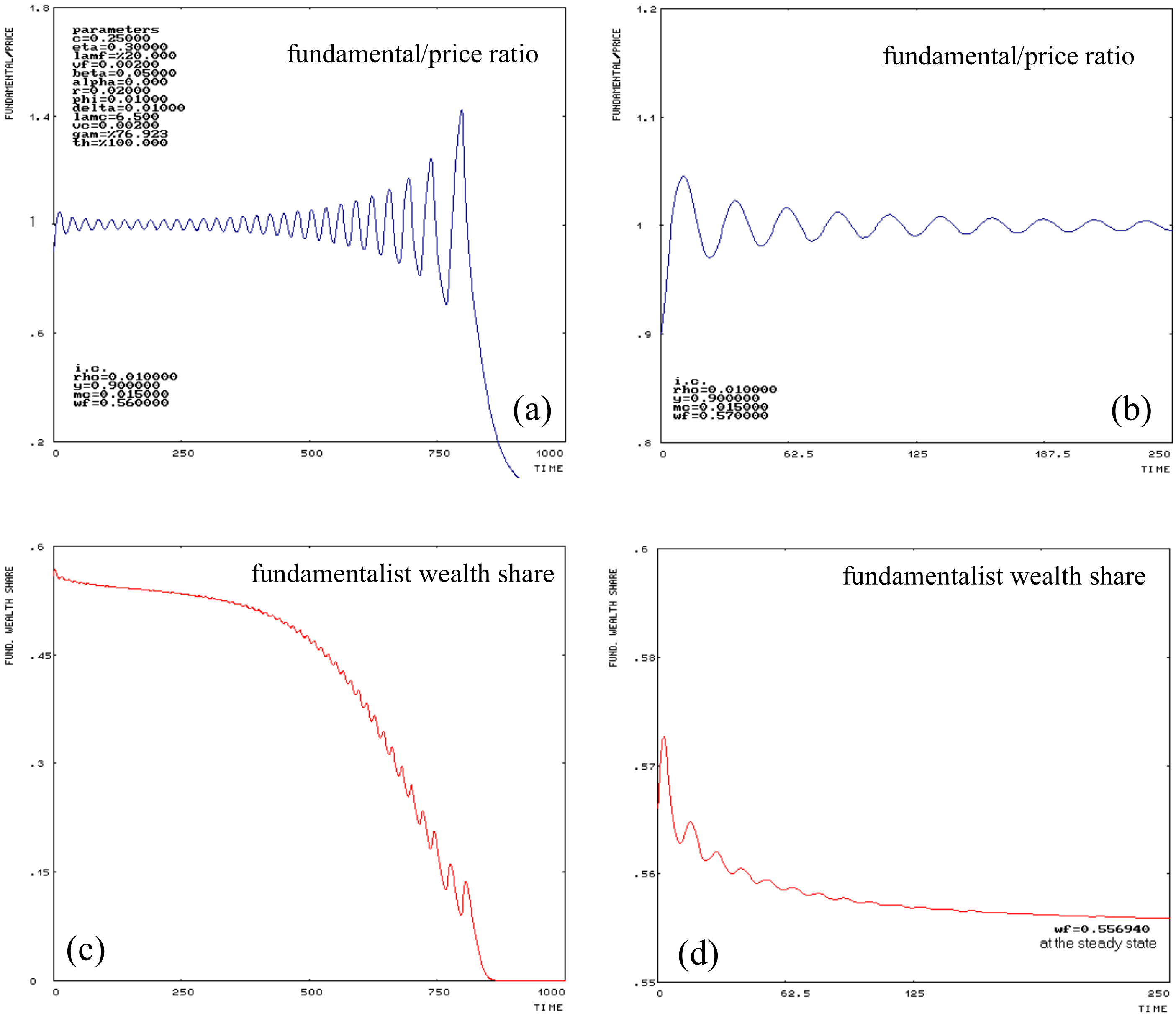
Phases where price is much higher than fundamental (low fundamental/price ratio) are dominated by chartists (in terms of wealth shares)

Role of initial condition (wealth shares) in long-run dynamics
(*crossing* the border of a *basin* of attraction)

Fig. 5

$w_0^{(f)}$=0.56: convergence to "non fundamental" steady state

$w_0^{(f)}$=0.57: convergence to fundamental steady state



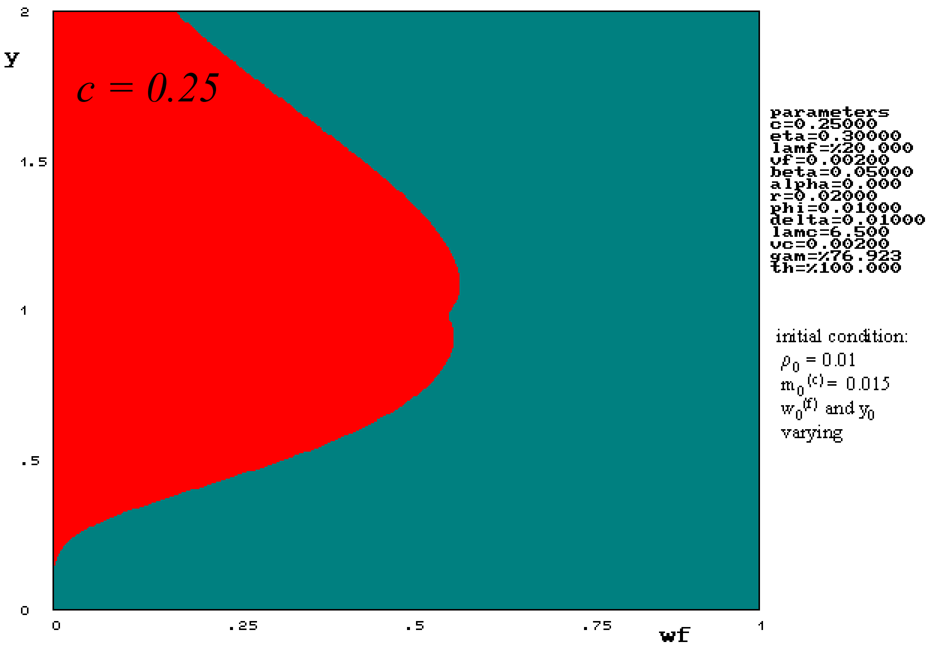
Role of initial condition (initial wealth shares and fundamental/price ratio) in the long-run dynamics
Basins of attraction of fundamental and nonfundamental steady states and their dependence on chartist extrapolation parameter(c)

Fig. 6

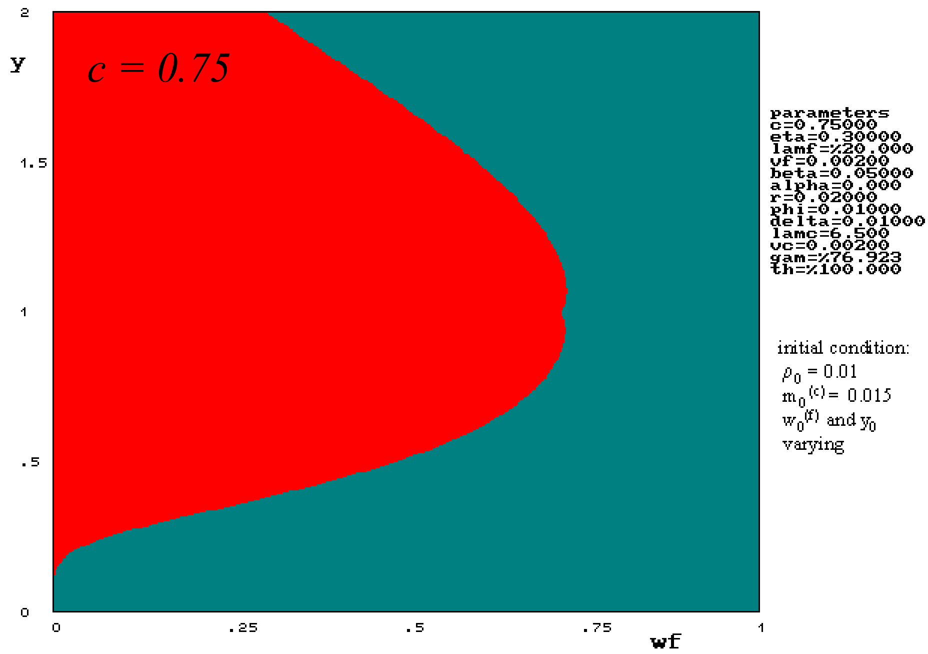
initial condition: $\rho_0 = 0.01$ $m_0^{(c)} = 0.015$
$w_0^{(f)}$ and y_0: varying in the $(w^{(f)}, y)$ plane

i.c. converging to fundamental steady state

i.c. converging to non fundamental steady state



(a)

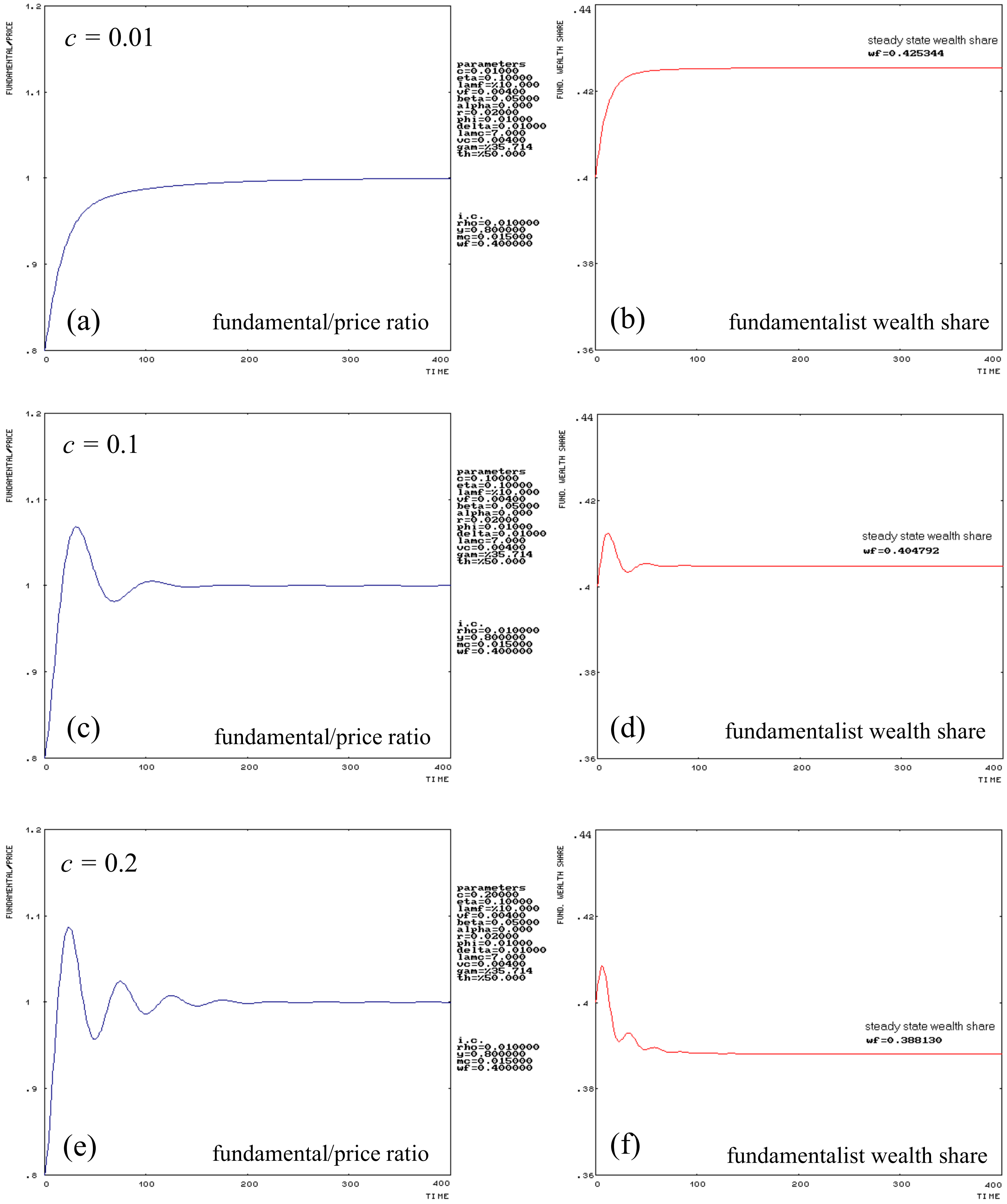


(b)

Effect of the chartist extrapolation rate (c) on transient dynamics and on long-run wealth shares, starting from the same initial condition

$\rho_0 = 0.01$; $m_0^{(c)} = 0.015$; $y_0 = 0.8$; $w_0^{(f)} = 40\%$

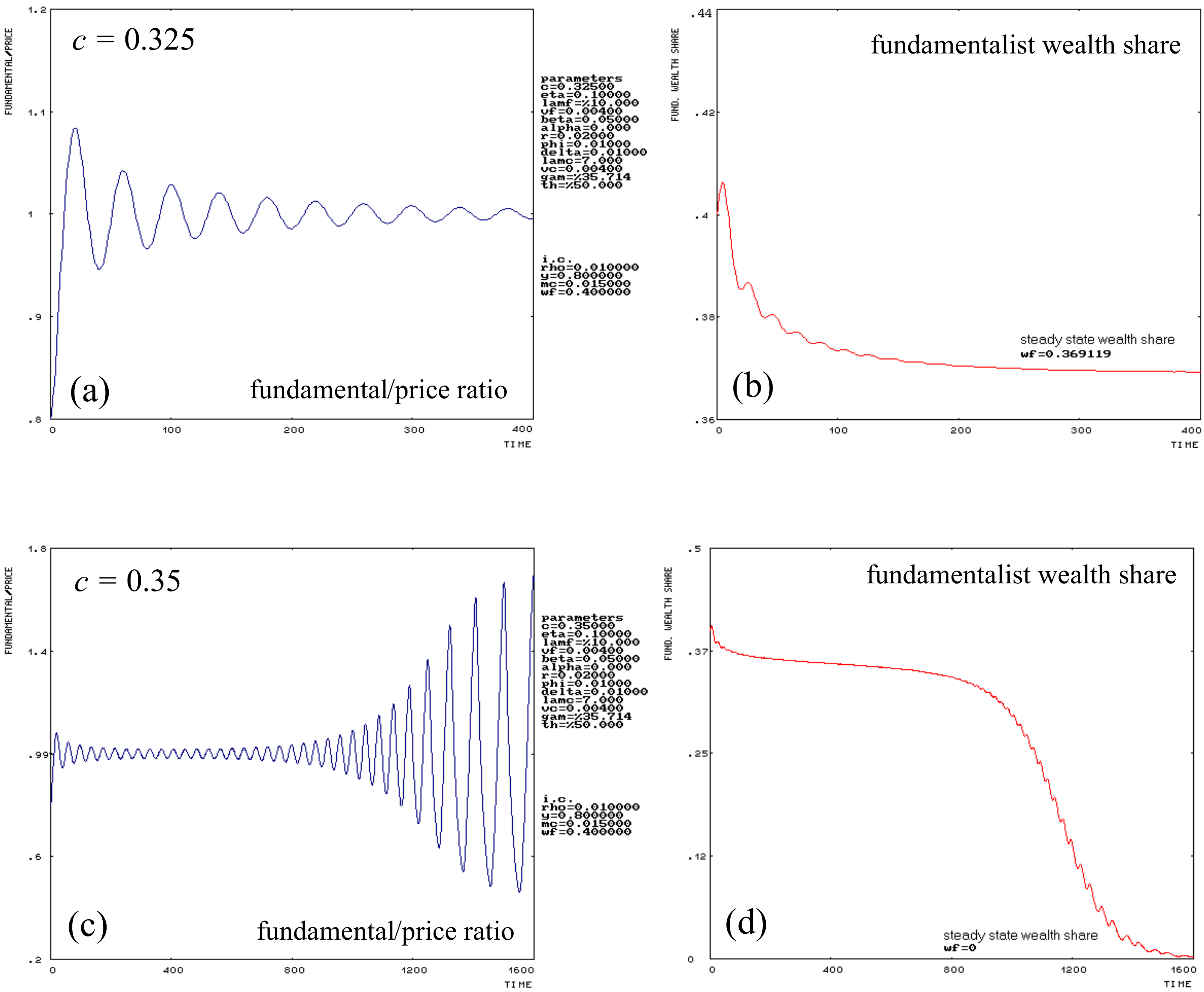
Fig. 7



Effect of the chartist extrapolation rate (c) on the asymptotic dynamics and on long-run wealth shares, starting from the same initial condition

Fig. 8

$\rho_0 = 0.01;\ m_0^{(c)} = 0.015;\ y_0 = 0.8;\ w_0^{(f)} = 40\%$

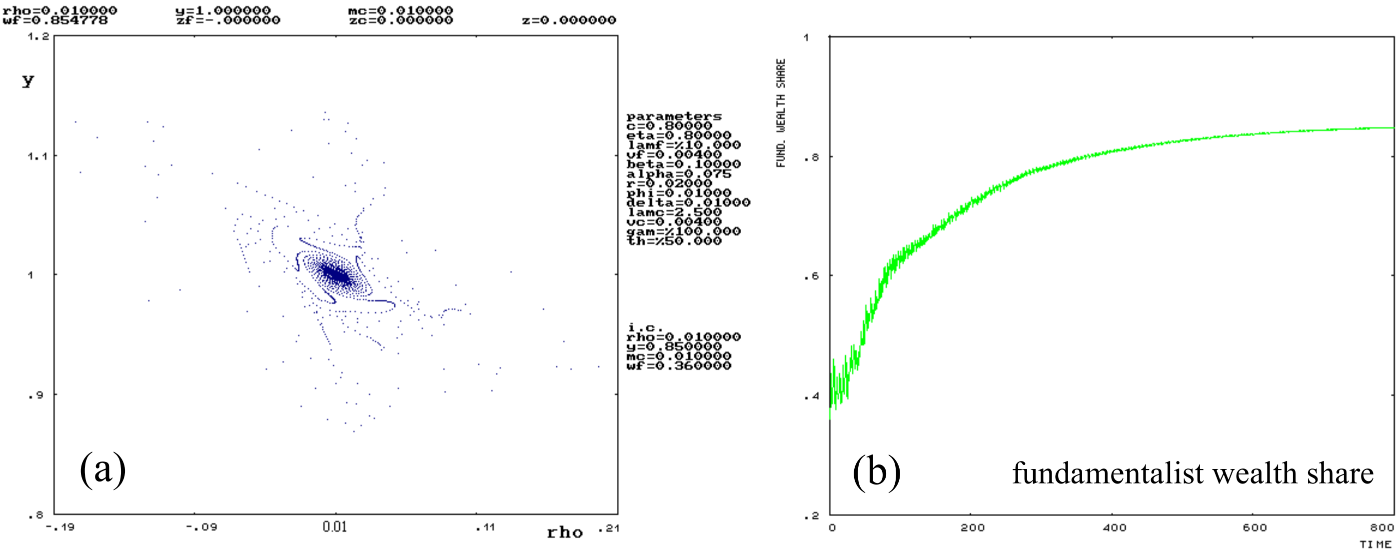


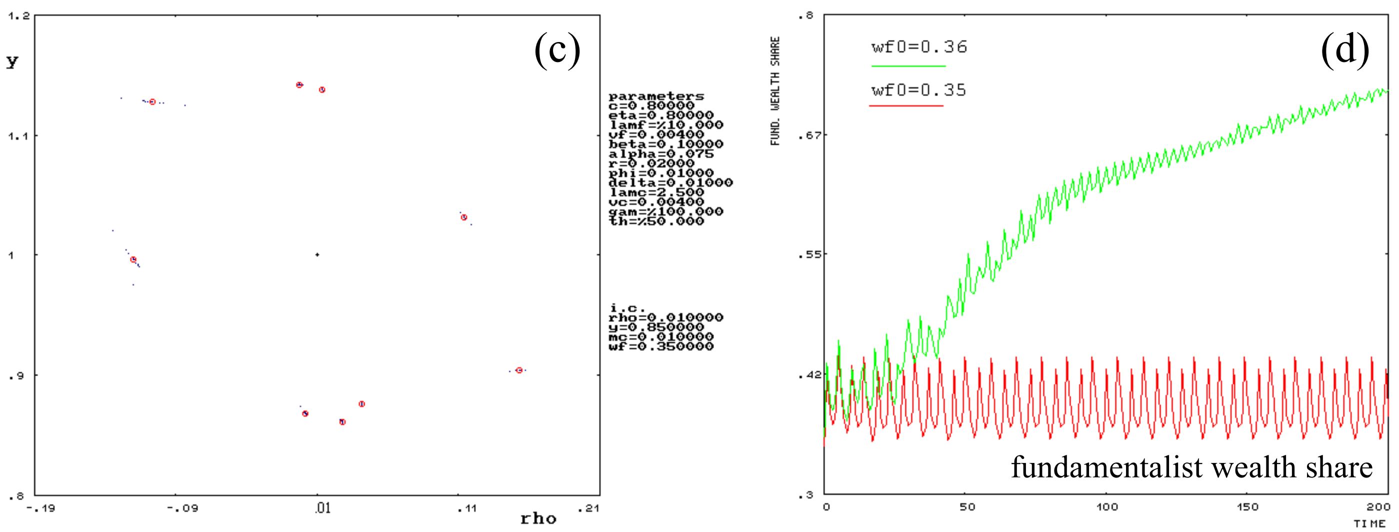
High values of agents' reaction coefficients ($\eta = 0.8$, $c = 0.8$)
Existence of periodic orbits with long-run fluctuations in wealth shares
Role of initial condition

Fig. 9

initial condition: $\rho_0 = m_0^{(c)} = 0.01$; $y_0 = 0.85$

$w_0^{(f)} = 36\%$: convergence to fundamental after a long transient





$w_0^{(f)} = 35\%$: convergence to a periodic orbit with long-run fluctuations in wealth shares