Aggregation of Dependent Risk with Specific Marginals by the Family of Koehler-Symanowski Distributions

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Preliminary Version

Abstract

This paper studies the family of Koehler and Symanovski multivariate distributions with specific marginals, as skewed Student t, generalized secant hyperbolic and generalized exponential power distributions, in order to model financial returns and measure dependent risks. This family of distributions can be specified using the cumulative distribution function adding interaction terms to the case of independence. Moreover, it can be also derived using a particular transformation of independent gamma functions. The advantage of using this distributions respect to others lies in the opportunity to model complex dependence structures among subsets of marginals, as shown with a Monte Carlo study, and to aggregate dependent risks of some market indices.

Keywords: Asset Returns, Risk Management, Skew Marginals, Monte Carlo Simulation, IFM method.

1 Introduction

Many problems in Finance, including risk management, optimal asset allocation and derivative pricing, require an understanding of the volatility and correlations of asset returns. In these cases it can be necessary to represent empricial data with a parametric distribution. In literature many distributions can be found able to model univariate data, but they cannot be easily extended to represent multivariate populations. In this context, the most used multivariate distribution in the aggregation of dependent risks is the normal distribution, which nevertheless has the drawback to be not very flexible and in many cases not appropriate to model returns.

An important tool to account of individual risks, the copula function, has been introduced in finance by Embrechts, McNeil and Straumann (1999, 2002), who have explained some essential concepts of dependence which have affected the construction of methods for the risk management industry (Embrechts, Lindskog, McNeil, 2003; Rosenberg, Schuermann, 2004). According to these specifications, this paper studies a method to obtain an analytical form of the joint distribution of asset returns in a portfolio based on the family of distributions introduced by Koehler and Symanowski (KS) (1995). This family of distributions is defined by the cdf adding interaction terms to the case of independence and it permits to specify arbitrary marginals. Moreover, it can be also derived using a particular transformation of independent exponential and gamma random variables. The advantage of using this distribution respect to others lies in the opportunity to model complex dependence structures among subsets of marginals, as shown with a Monte Carlo study, and to aggregate dependent risks.

The paper is organized as follows. In the next section we introduce the KS distribution family, while in third section we present the results of a Monte Carlo study on the dependence structure of the distribution when the marginals are skewed Student t, generalized secant hyperbolic and exponential power. Then, in section 4 we show the results of an application to four market indices and conclusions are left to section 5.

2 KS formulation

Koehler and Symanowski (1995) introduce a class of multivariate distribution families which can be built for almost any given univariate marginal distribution and can be viewed as a generalization of the Cook-Johnson family of distributions (Johnson, 1987; Cook and Johnson, 1981, 1986). Therefore, consider the p-dimensional random variable $\boldsymbol{U} = (U_1, U_2, \ldots, U_p)'$ with support on the unit hypercube $(0, 1]^p$ and cdf

$$F(u_1, u_2, \ldots, u_p) = \prod_{i=1}^{p} u_i \prod_{j=i+1}^{p} C_{ij}^{-\alpha_{ij}}, \tag{1}$$

where

$$C_{ij} = u_i^{1/\alpha_{i+}} + u_j^{1/\alpha_{j+}} - u_i^{1/\alpha_{i+}} u_j^{1/\alpha_{j+}}, \tag{2}$$

with $\alpha_{ij} = \alpha_{ji} \geq 0$ for all (i, j) and $\alpha_{i+} = \alpha_{i1} + \alpha_{i2} + \cdots + \alpha_{ip} > 0$ for all $i = 1, 2, \ldots, p$. Deriving the cdf respect to $u_1, u_2, \ldots, u_p$ we obtain the probability density function (pdf) of $\boldsymbol{U}$ which can be written as

$$\begin{aligned} f(u_1, u_2, \ldots, u_p) = & \prod_{i=1}^{p} \left(D_i \prod_{j=i+1}^{p} C^{-\alpha_{ij}} \right) \\ & \times \left[1 + \sum_{i=1}^{p} \sum_{j=i+1}^{p} \left(\frac{\alpha_{ij}}{\alpha_{i+}\alpha_{j+}} D_i^{-1} D_j^{-1} C_{ij}^{-2} u_i^{1/\alpha_{i+}} u_j^{1/\alpha_{j+}} \right) \right], \end{aligned} \tag{3}$$

where

$$D_i = \alpha_{i+}^{-1} \left[\alpha_{ii} + \sum_{k \neq i}^{p} (\alpha_{ik} U_k^{1/\alpha_{k+}} C_{ik}^{-1}) \right],$$

and $C_{ij} = C_{ji}$ is defined by (2).

Koheler and Symanowski (1995) have shown that it is possible to obtain a scheme to generate this distribution using gamma distributions. In this sense, let $Y_1, Y_2, \ldots, Y_p$ be i.i.d. $Gamma(1, 1)$ and, independently, $G_{11}, G_{12}, \ldots, G_{pp}$ be $Gamma(\alpha_{ij}, 1)$ with $G_{i+} = \sum_{j=1}^{p} G_{ij}$. Then, the joint pdf of

$$U_i = (1 + Y_i/G_{i+})^{-\alpha_{i+}}, \quad i = 1, 2, \ldots, p, \tag{4}$$

has cdf given by (1).

The conditional means of U are not linear functions of the values of the conditioning variables. Consequently, it is more reasonable to measure dependence between variables using concordance coefficients as the Kendall's tau or the Spearman's rank correlation than using the correlation coefficient. However, the level of association between pairs of variables in (3) depends on the level of the parameters α_{ij}. In particular, the cdf of any bivariate marginal (U_i, U_j) approximates the Frechét upper bound as $\alpha_{ij} \to 0$ and at the same time $(\alpha_{i+} - \alpha_{ij})/\alpha_{i+} \to 0$ and $(\alpha_{j+} - \alpha_{ij})/\alpha_{j+} \to 0$; vice versa, it approximates $(U_i U_j)$, paired independence, if $\alpha_{ij} \to 0$ when α_{i+} and α_{j+} are greater than zero and at the same time $\alpha_{i+} \to \infty$ and $\alpha_{j+} \to \infty$. Moreover, the standard form of U determines that all the correlations are positive. Variations in the standard form that also take into account negative correlations can be obtained by applying the transformation $V_i = 1 - U_i$ to some, but not all, variables (other characteristics of the KS formulation can be found in Caputo, 1998; Manomaiphiboon, K. and Russel, A.G., 2003).

We can obtain also specific marginal distributions of (3) by applying the inverse probability transforms to U_i. Let the random variables X_i, $i = 1, 2, \ldots, p$, have cdf F_i with corresponding inverse F_i^{-1} and pdf f_i. Then, the $X_i = F_i^{-1}(U_i)$ have joint cdf

$$F(x_1, x_2, \ldots, x_p) = \prod_{i=1}^{p} F_i(x_i) \prod_{j=i+1}^{p} C_{ij}^{-\alpha_{ij}}, \tag{5}$$

and pdf

$$\begin{aligned} f(x_1, x_2, \ldots, x_p) = & \prod_{i=1}^{p} \left(f_i(x_i) D_i \prod_{j=i+1}^{p} C^{-\alpha_{ij}} \right) \\ & \times \left[1 + \sum_{i=1}^{p} \sum_{j=i+1}^{p} \left(\frac{\alpha_{ij}}{\alpha_{i+}\alpha_{j+}} D_i^{-1} D_j^{-1} C_{ij}^{-2} F_i(x_i)^{1/\alpha_{i+}} F_j(x_j)^{1/\alpha_{j+}} \right) \right], \end{aligned} \tag{6}$$

where

$$D_i = \alpha_{i+}^{-1} \left[\alpha_{ii} + \sum_{k \neq i}^{p} (\alpha_{ik} F_k(x_k)^{1/\alpha_{k+}} C_{ik}^{-1}) \right],$$

and, in this case,

$$C_{ij} = F_i(x_i)^{1/\alpha_{i+}} + F_j(x_j)^{1/\alpha_{j+}} - F_i(x_i)^{1/\alpha_{i+}} F_j(x_j)^{1/\alpha_{j+}},$$

where the α_{ij} parameters assumes the meaning seen before. It is immediate to verify that, in order to simulate the joint distribution of $X_1, X_2, \ldots, X_p$ with the Monte Carlo method, it is sufficient to generate some variates using (4) and then apply the inverse transformation to each marginal.

3 Monte Carlo study

In this section we present the results of a Monte Carlo study on the dependence structure of the KS distribution with skewed, fatted tails risks. The results of simulations and of the application shown in the next section have been obtained using a FORTRAN 90 code implemented on a 2300 MHz PC Intel on Windows 2000. The code uses the random number generator and the optimization routines of the NAG Fortran library.

3.1 KS Marginals

The Monte Carlo experiment has been run using as data generator process the skewed Student t, generalized secant hyperbolic and exponential power (*generalized error*) distributions, which are useful in financial risk analysis because have heavier tails than the normal distribution.

Since in their principal definition these distributions are symmetric, to skew them we have followed Fernandéz and Steel (1998), who have introduced skewness in a symmetric distribution adding inverse scale factors in positive and negative orthants. More precisely, if g is a symmetric pdf on zero with support $\mathbb{R}$, then for any $\kappa > 0$ we obtain a skew density

$$f(x) = \left(\frac{2}{1 + \frac{1}{\kappa}} \right) g(\kappa^{-\mathrm{sgn}(x)} x).$$

We can write the cdf F and the quantile function F^{-1} corresponding to a skew density f of an X distribution using the cdf G and the quantile function G^{-1} of the symmetric density. We have

$$F(x) = \begin{cases} \left(\frac{2}{1+\kappa^2}\right) G(\kappa x) & \text{for } x \leq 0, \\ 1 - \left(\frac{2}{1+\kappa^{-2}}\right) G(-\frac{x}{\kappa}) & \text{for } x > 0. \end{cases}$$

for the cdf and

$$F^{-1}(x) = \begin{cases} \frac{1}{\kappa} G^{-1}\left[(1+\kappa^2)\frac{p}{2}\right] & \text{for } x \leq 0, \\ -\kappa G^{-1}\left[(1+\kappa^{-2})\frac{1-p}{2}\right] & \text{for } x > 0, \end{cases}$$

for the quantile function. Moreover, the moments of order r, $r = 1, 2, \ldots$, can be written as

$$E(X^r) = 2E^+(X^r)\frac{\kappa^{r+1} + \frac{(-1)^r}{\kappa^{-(r+1)}}}{\kappa + \frac{1}{\kappa}},$$

where

$$E^+(X^r) = \int_0^\infty x^r g(x) dx$$

is the r-th moment of the symmetric distribution truncated to positive values. Clearly $E^+(X^r)$ assumes the value of the r-th moment of the symmetric distribution divided by 2 when r is even.

On the basis of these formulations, we present the main characteristics of the distributions considered (for details see, among others, Ayebo and Kozubowski, 2003; Lambert and Laurent, 2001; Palmitesta and Provasi, 2004).

Skew Student-t. A random variable X has a skew Student t distribution (SST) if the parameters $\kappa > 0$ and $v > 2$ exist such that the pdf of X is

$$f_{\text{SST}}(x) = \left(\frac{2}{1+\frac{1}{\kappa}}\right) \frac{\Gamma(\frac{v+1}{2})}{\sqrt{\pi(v-2)}\Gamma(\frac{v}{2})} \left[1 + \frac{\kappa^{-2\text{sgn}(x)} x^2}{v-2}\right]^{-(v+1)/2},$$

where $\Gamma(\cdot)$ indicates the gamma fucntion. If $\kappa = 1$, the distribution is symmetric on zero and has zero mean and unit variance. The cdf of X is given by

$$F_{\text{SST}}(x) = \begin{cases} \left(\frac{2}{1+\kappa^2}\right) t_v \left[\kappa \sqrt{\frac{v}{v-2}} x\right] & \text{for } x \leq 0, \\ 1 - \left(\frac{2}{1+\kappa^{-2}}\right) t_v \left[-\frac{1}{\kappa}\sqrt{\frac{v}{v-2}} x\right] & \text{for } x > 0, \end{cases}$$

where

$$t_v(z) = \int_{-\infty}^{z} \frac{\Gamma(\frac{v+1}{2})}{\sqrt{\pi v}\Gamma(\frac{v}{2})} \left(1 + \frac{w^2}{v}\right)^{-(v+1)/2} dw$$

is the cdf of the unscaled Student t with v degrees of freedom.

The moments of order r, $r = 1, 2, \ldots$, of the SST distribution are:

$$\begin{aligned} E(X^r) =& \frac{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{v-r}{2}\right)}{\sqrt{\pi(v-2)}\Gamma\left(\frac{v}{2}\right)} \left(\frac{1}{1+\kappa^{-2}}\right) \\ & \times \left[\left(\frac{1}{\kappa^2(v-2)}\right)^{-(r+1)/2} + (-1)^r \left(\frac{\kappa^2}{v-2}\right)^{-(r+1)/2}\right] \end{aligned}$$

and exist when $v > r$.

Skew Generalized Secant Hyperbolic. A random variable X has a skew generalized secant hyperbolic distribution with parameters $\kappa > 0$ and $\lambda > -\pi$ is its density is

$$f_{\mathrm{SGSH}}(x) = \left(\frac{1}{1+\frac{1}{\kappa}}\right) \frac{c_1}{a + \cosh\left(c_2\, \kappa^{-\operatorname{sgn}(x)} x\right)},$$

where

$$\begin{array}{llll} a = \cos(\lambda), & c_2 = \sqrt{\dfrac{\pi^2 - \lambda^2}{3}}, & \text{and} \quad c_1 = \dfrac{\sin(\lambda)}{\lambda} c_2 & \text{if } \lambda \in (-\pi, 0), \\ a = 1, & c_2 = \sqrt{\dfrac{\pi^2}{3}}, & \text{and} \quad c_1 = c_2 & \text{if } \lambda = 0, \\ a = \cosh(\lambda) & c_2 = \sqrt{\dfrac{\pi^2 + \lambda^2}{3}} & \text{and} \quad c_1 = \dfrac{\sinh(\lambda)}{\lambda} c_2 & \text{if } \lambda > 0. \end{array}$$

The skew logistic distribution is a special case of this distribution when $\kappa \neq 1$ and $\lambda = 0$.

The cdf of X is given by

$$F_{\mathrm{SGSH}}(x) = \begin{cases} \left(\frac{2}{1+\kappa^2}\right) G(\kappa x) & \text{for } x \leq 0, \\ 1 - \left(\frac{2}{1+\kappa^{-2}}\right) G(\frac{x}{\kappa}) & \text{for } x > 0, \end{cases}$$

where

$$G(x) = \begin{cases} \frac{1}{2} + \frac{1}{\lambda} \tan^{-1}\left[\tan\left(\frac{\lambda}{2}\right) \tanh\left(\frac{c_2}{2} x\right)\right] & \text{for } \lambda \in (-\pi, 0), \\ \frac{1}{2}\left[1 + \tanh\left(\frac{c_2}{2} x\right)\right] & \text{for } \lambda = 0, \\ \frac{1}{2} + \frac{1}{\lambda} \tanh^{-1}\left[\tanh\left(\frac{\lambda}{2}\right) \tanh\left(\frac{c_2}{2} x\right)\right] & \text{for } \lambda > 0, \end{cases}$$

is the cdf of the symmetric generalized secant hyperbolic distribution with zero mean and unit variance obtained when $\kappa = 1$. Moreover, the moments of order r, $r = 1, 2, \ldots$, are:

$$\begin{aligned} E(X^r) = & \frac{c_1\, \Gamma(r+1)}{2\, c_2^{r+1} \sqrt{a^2 - 1}} \left[L_{r+1}\left(-\frac{1}{\sqrt{a^2-1}+a}\right) - L_{r+1}\left(\frac{1}{\sqrt{a^2-1}-a}\right)\right] \\ & \times \frac{\kappa^{r+1} + (-1)^r \kappa^{-(r+1)}}{\kappa + \frac{1}{\kappa}}, \end{aligned}$$

where $L_r(\cdot)$ indicates the polylogarithmic function whose primary definition is[1]

$$L_r(w) = \sum_{k=1}^{\infty} \frac{w^k}{k^r}, \qquad |w| < 1,\ w \in \mathbb{C}.$$

Skew Power Exponential. A random variable X has a skew exponential power distribution (SEP) with parameters $\kappa > 0$ and $\beta > 0$ is its density is

$$f_{\mathrm{SEP}}(x) = \left(\frac{2}{\kappa + \frac{1}{\kappa}}\right) \Gamma^{\frac{1}{2}}\left(\frac{3}{2\beta}\right) \Gamma^{-\frac{3}{2}}\left(\frac{1}{2\beta}\right) \beta \exp\left[-c^{\beta} \kappa^{-2\beta \operatorname{sgn}(x)} |x|^{2\beta}\right],$$

with $c = \Gamma\left(\frac{3}{2\beta}\right) \Gamma^{-1}\left(\frac{1}{2\beta}\right)$. The skew Laplace distribution is a special case of the SEP when $\kappa \neq 1$ and $\beta = \frac{1}{2}$. Likewise, when $\beta = 1$ we obtain a normal distribution.

[1]See at the web page http://functions.wolfram.com/10.08.02.0001.01

The cdf of X is given by

$$F_{\mathrm{SEP}}(x) = \begin{cases} \left(\frac{2}{1+\kappa^2}\right) G(\kappa x) & \text{for } x \leq 0, \\ 1 - \left(\frac{2}{1+\kappa^{-2}}\right) G(\frac{x}{\kappa}) & \text{for } x > 0, \end{cases}$$

where

$$G(x) = \begin{cases} \frac{1}{2} G\left(\frac{1}{2\beta}, c^\beta (-x)^{2\beta}\right) & \text{for } x \leq 0, \\ \frac{1}{2}\left[1 - G\left(\frac{1}{2\beta}, c^\beta x^{2\beta}\right)\right] & \text{for } x > 0, \end{cases}$$

being

$$\Gamma(a, z) = \frac{1}{\Gamma(a)} \int_z^\infty u^{a-1} \mathrm{e}^{-u} du, \quad a > 0,\, z > 0,$$

the regularized gamma function. G is the cdf of the symmetric exponential power distribution with zero mean and unit variance obtained when $\kappa = 1$.

The momemts of order r, $r = 1, 2, \ldots$, of the SEP distribution are:

$$E(X^r) = \frac{\Gamma^{\frac{r}{2}-1}(\frac{1}{2\beta})\Gamma(\frac{r+1}{2\beta})}{\Gamma^{\frac{r}{2}}(\frac{3}{2\beta})} \frac{\left(\kappa^{r+1} + (-1)^r \kappa^{-(r+1)}\right)}{\kappa + \frac{1}{\kappa}}.$$

If $\theta \in \mathrm{I\!R}$ is a position parameter and $\varphi > 0$ is a scale parameter, then it is possible to bring back the three distributions to be characterized by four parameters using the transformation $Y = \theta + \varphi X$. Moreover, the skewness and kurtosis indices can be easily obtained relating moments from the origin and central moments.

3.2 The experiment

Here we show the results of the Monte Carlo experiment run to interpret the association parameters of the (5) when the marginals are the three distributions seen above. According to the instruction given in Koehler and Symanowski (1995) samples are taken from a three-dimensional KS distribution for different values of $\alpha_{12}, \alpha_{13}, \alpha_{23}$ and setting $\alpha_{11} = \alpha_{22} = \alpha_{33} = 0$. For each sample the correlation coefficient as a measure for the linear relation and the Kendall's coefficient for the monotone relation are calculated for all pairs of variables. Table 1 shows the results for samples of size $n = 50000$, where the marginals have zero mean, unit variance and, respectively, skewness equal to -0.5, 0, 0.5 and kurtosis equal to 5, 6, 8.

To comment the results, note that changing the values of the association parameters we obtain very different dependence measures, but no particular links among them can be pointed out. Moreover, it seems that marginal have not a big effect on the dependence structure of the KS distribution, because the values of the correlation coefficients and Kendall's taus do not vary a lot in the three cases we have studied.

4 Fitting the KS distributions to data

4.1 Data

The raw data used in this paper are weekly prices of four market indices: the S&P 500 Composite index (S&PCOMP), NASDAQ Composite index (NASCOMP), NIKKEI 500 index (JAPA500) and MSCI AC World index (MSACWFL). The observations have been obtained by Datastream for the period 1/1/1988 to 12/31/2003. Then, we have computed the returns as first differences of the natural logarithm of each series, $r_t =$

Skew Student-t Distribution

Association parameters			Kendall's coefficients			Correlation coefficients		
0.01	0.01	0.01	0.3293	0.3250	0.3299	0.3979	0.3887	0.4104
0.10	0.10	0.01	0.3933	0.3977	0.0451	0.4948	0.5028	0.0608
0.50	0.01	0.01	0.4805	0.0170	0.0164	0.6698	0.0412	0.0358
0.10	0.02	0.01	0.6430	0.1495	0.0760	0.7403	0.2261	0.1165
0.50	0.50	0.50	0.1473	0.1431	0.1471	0.2140	0.2009	0.1992
1.00	0.10	0.10	0.2785	0.0628	0.0617	0.4319	0.0939	0.0895
1.50	0.50	0.50	0.1455	0.0778	0.0761	0.2301	0.1127	0.1048
5.00	0.10	0.01	0.0893	0.0158	0.0054	0.1570	0.0298	0.0078
5.00	5.00	5.00	0.0235	0.0285	0.0252	0.0418	0.0447	0.0416

Skew Generalized Secant Hyperbolic Distribution

Association parameters			Kendall's coefficients			Correlation coefficients		
0.01	0.01	0.01	0.3231	0.3256	0.3248	0.3827	0.3824	0.3953
0.10	0.10	0.01	0.3965	0.3943	0.0447	0.4847	0.4737	0.0520
0.50	0.01	0.01	0.4806	0.0177	0.0159	0.6637	0.0422	0.0416
0.10	0.02	0.01	0.6416	0.1487	0.0755	0.7294	0.2234	0.1161
0.50	0.50	0.50	0.1429	0.1427	0.1443	0.2045	0.1944	0.1884
1.00	0.10	0.10	0.2791	0.0640	0.0611	0.4277	0.0921	0.0906
1.50	0.50	0.50	0.1431	0.0799	0.0786	0.2324	0.1142	0.1068
5.00	0.10	0.01	0.0925	0.0201	0.0048	0.1628	0.0354	0.0072
5.00	5.00	5.00	0.0219	0.0217	0.0209	0.0360	0.0356	0.0312

Skew Exponential Power Distribution

Association parameters			Kendall's coefficients			Correlation coefficients		
0.01	0.01	0.01	0.3231	0.3256	0.3248	0.3819	0.3820	0.3954
0.10	0.10	0.01	0.3965	0.3943	0.0447	0.4812	0.4709	0.0517
0.50	0.01	0.01	0.4806	0.0177	0.0159	0.6621	0.0421	0.0416
0.10	0.02	0.01	0.6416	0.1487	0.0755	0.7265	0.2229	0.1160
0.50	0.50	0.50	0.1429	0.1427	0.1443	0.2034	0.1935	0.1874
1.00	0.10	0.10	0.2791	0.0640	0.0611	0.4271	0.0917	0.0902
1.50	0.50	0.50	0.1431	0.0799	0.0786	0.2319	0.1137	0.1063
5.00	0.10	0.01	0.0925	0.0201	0.0048	0.1626	0.0354	0.0071
5.00	5.00	5.00	0.0219	0.0217	0.0209	0.0358	0.0355	0.0312

Table 1: Correlation and Kendall's coefficients for $\alpha_{11} = \alpha_{22} = \alpha_{33} = 0$ and marginals with zero mean, unit variance, skewness equal to -0.5, 0, 0.5 and kurtosis equal to 5, 6, 8.

$\ln I_t - \ln I_{t-1}$, where I_t indicates the price at time t, obtaining a sample of $T = 854$ returns. Table 2 reports preliminary statistics for the four return series, and gives the mean, median, minimum, maximum, standard deviation and skewness and kurtosis indices.

The nonparametric densities with normal kernel of the distributions of the returns is given in fig. 1[2]. From the latter figure, also if the distributions have fat tails, it is difficult to identify their shape. Moreover, since multivariate skewness and curtosis indices are equal, respectively, to 2.246 and 61.523, the data are not normally distributed.

Finally, if we express the depedence structure of the four indices with the correlation index, we obtain the following matrix;

$$\hat{\boldsymbol{R}} = \begin{pmatrix} 1.0000 & 0.8196 & 0.3684 & 0.8877 \\ & 1.0000 & 0.3697 & 0.7694 \\ & & 1.0000 & 0.3697 \\ & & & 1.0000 \end{pmatrix},$$

while if we use the Kendall's tau we obtain:

$$\hat{\boldsymbol{\tau}} = \begin{pmatrix} 1.0000 & 0.6347 & 0.2499 & 0.6807 \\ & 1.0000 & 0.2421 & 0.5517 \\ & & 1.0000 & 0.4578 \\ & & & 1.0000 \end{pmatrix}.$$

Note the high dependence of S&PCOMP with NASCOMP and MSACWFL.

	S&PCOMP	NASCOMP	JAPA500	MSACWFL
Mean	0.0017	0.0020	-0.0004	0.0012
Median	0.0035	0.0052	0.0007	0.0028
Min	-0.1218	-0.1646	-0.1181	-0.0931
Max	0.1237	0.1272	0.1523	0.0898
StDev	0.0225	0.0341	0.0294	0.0199
Skewness	-0.1806	-0.6725	0.0330	-0.4371
Kurtosis	5.9210	6.0317	5.1286	5.6514

Table 2: Summary statistics for weekly returns.

4.2 Parameter estimation and results

We have adapted to the four series of returns the KS distribution with marginals skew Student t, generalized secant hyperbolic and exponential power seen in the previous section. According to the IFM (inference for margins) method (Joe and Xu, 1996), the parameters of the marginals have been estimated distinctly from the parameters of the KS function. In other words, the process of estimation has been divided in the following steps:

i) estimating the location, scale, skewness and kurtosis parameters of marginal distributions using the maximum likelihood method:

ii) estimating the KS parameter $\boldsymbol{\alpha}$, always with the maximum likelihood method, given the estimations performed in step (i).

[2]The bandwidth of the nonparametric densities has been obtained with the Sheather and Jones method (Sheater and Jones, 1991).

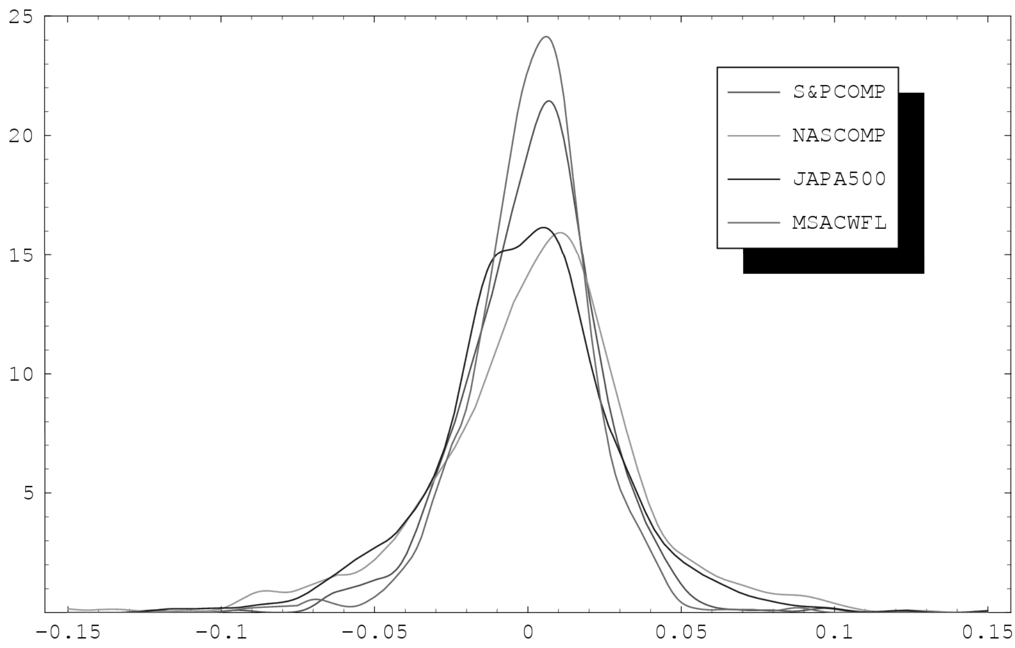

Figure 1: Kernel estimates of the distribution of weekly returns.

Table 3 presents the results of the maximum likelihood estimation of the parameters of the marginals. In the table we present also the values of the averaged loglikelihood function at maximum. Then, we have obtained the following maximum likelihood estimates of the parameters $\boldsymbol{\alpha}$ in the three cases considered:

$$\hat{\boldsymbol{\alpha}}_{\mathrm{SST}} = \begin{pmatrix} 0.0000 & 0.0702 & 0.0000 & 0.1072 \\ & 0.0476 & 0.0000 & 0.0357 \\ & & 0.0491 & 0.0342 \\ & & & 0.0000 \end{pmatrix},$$

$$\hat{\boldsymbol{\alpha}}_{\mathrm{SGSH}} = \begin{pmatrix} 0.0000 & 0.0730 & 0.0000 & 0.1095 \\ & 0.0467 & 0.0000 & 0.0363 \\ & & 0.0419 & 0.0338 \\ & & & 0.0000 \end{pmatrix}$$

and

$$\hat{\boldsymbol{\alpha}}_{\mathrm{SEP}} = \begin{pmatrix} 0.0000 & 0.0728 & 0.0000 & 0.1086 \\ & 0.0473 & 0.0000 & 0.0358 \\ & & 0.0385 & 0.0332 \\ & & & 0.0000 \end{pmatrix}.$$

with the values of the averaged loglikelihood function at maximum, respectively, equal to 12.9449, 12.9496 and 12.9415[3].

Note that the three marginals have a similar behaviour, also if the shape parameter of the skew Student-t for the NASCOMP index is lower than 4 and, as a consequence, it has not kurtosis index. Also the structure of $\boldsymbol{\alpha}$ is very similar in the three cases considered, confirming what we have writtem in the previous section.

[3]We have used a quasi-Newton algorithm to obtain the maximum of loglikelihood functions.

Skew Student-t Distribution

	S&PCOMP	NASCOMP	JAPA500	MSACWFL
Location	0.0062	0.0096	0.0013	0.0055
Scale	0.0222	0.0357	0.0300	0.0196
Skewness	0.8664	0.8506	0.9593	0.8593
Kurtosis	5.4329	3.4153	4.5528	4.8423
LogLik	*2.4179*	*2.0313*	*2.1460*	*2.5503*

Skew Generalized Secant Hyperbolic Distribution

	S&PCOMP	NASCOMP	JAPA500	MSACWFL
Location	0.0061	0.0010	0.0011	0.0055
Scale	0.0220	0.0334	0.0294	0.0193
Skewness	0.8738	0.8440	0.9645	0.8599
Kurtosis	-1.4014	-2.1752	-1.7483	-1.6308
LogLik	*2.4160*	*2.0360*	*2.1477*	*2.5497*

Skew Exponential Power Distribution

	S&PCOMP	NASCOMP	JAPA500	MSACWFL
Location	0.0065	0.0105	0.0020	0.0056
Scale	0.0221	0.0330	0.0292	0.0193
Skewness	0.8673	0.8416	0.9479	0.8571
Kurtosis	0.6278	0.5358	0.6151	0.6166
LogLik	*2.4135*	*2.0332*	*2.1446*	*2.5467*

5 Conclusions

In this paper we have studied the family of multivariate distributions by Koheler and Smanowski in order to model financial returns and measure dependent risks. Fitting the distribution to the returns of four market indices, the S&P 500 Composite index (S&PCOMP), the NASDAQ Composite index (NASCOMP), the NIKKEI 500 index (JAPA500) and the MSCI AC World index (MSACWFL), with skewed Student t, generalized secant hyperbolic and generalized exponential power marginals, we have seen that this distribution succeeds in properly interpreting the dependence structure of data, apart from the marginals, introducing therefore some measures which can couple the usual correlation indices in the interpretation of the strength of the link among dependent risks. However, other important aspects of the distribution must be studied, for example the analysis of sensitivity of the parameters in presence of dependence structures of higher order introduced by Koeheler and Symanovski (1995).

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