Complex dynamics in a Pasinetti-Solow model of Growth and distribution

Abstract

In this paper, we study some of the properties of a discrete-time version of the two-class model of growth and distribution proposed by Pasinetti (1962) and Samuelson and Modigliani (1966) with a convex production technology of the CES type. We assume two distinct groups of agents, workers and capitalists. The first group saves out of wages and profits by applying to these income sources propensities to save *which are not necessarily equal* (this is a generalisation firstly proposed by Chiang, 1973). Capitalists' saving originates only from capital income. The resulting model is two-dimensional. Differently from Böhm and Kaas (2000), distributive processes occur not only between factor shares but also between the two groups existing in the economy. We explore through simulations the large variety of dynamic behaviours that emerge from this formulation.

Preliminary version do not quote

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1. Introduction

As it is well known, the standard one-sector neoclassical growth model (Solow, 1956, Swan, 1956) is able to generate only simple dynamics, that is, monotonic convergence to a steady state. The dynamic properties of the Solow model follows from the assumptions on the saving behaviour, the average propensity to save is constant, and on the neoclassical technology, represented usually by a Cobb-Douglas production function. As shown by Day (1982, 1999), when average savings are allowed to vary with the capital/labour ratio, under specific assumptions the discrete-time version of the Solow model is able to generate chaotic dynamics. More recently, Böhm and Kaas (2000) investigate the dynamics of a discrete-time Solow growth model modified by introducing different but constant saving propensities attached to factor shares, wages and profits, and a concave production function with more general properties than the standard neoclassical ones. Their assumption on saving behaviour corresponds to that proposed by Kaldor's (1956) in his model of equilibrium growth and distribution. The Solow model so revised is able to generate dynamic behaviour of capital and income per worker which is not limited to monotonic convergence to a steady state but it may also involve instability and chaos.

The hypothesis of constant saving propensities attached to income shares, which characterises Kaldor's (1956) model of growth and distribution, differs from that proposed in Pasinetti's (1962) and Samuelson and Modigliani's (1966) analyses according to which different saving propensities characterise two separate groups (or classes), workers and capitalists or pure shareholders. Following Pasinetti (1962), the former assumption cannot be used to interpret saving behaviour of separate groups in the economy since it implies that workers do not receive any revenue out of their savings. Other authors, considered that Kaldor's assumption on saving is logically coherent if one assumes that workers, in their quality of shareholders, behave like capitalists, applying to profits the same propensity to save as the latter group. However, it can be shown (see i.e. Maneschi, 1974, Fazi and Salvadori, 1981) that, following the latter interpretation, as long as workers' saving behaviour is the one postulated by Kaldor (1956), in equilibrium growth the two types of agents cannot both own a positive share of capital.[1]

We study some of the properties of a discrete-time version of the two-class model of growth and distribution proposed by Pasinetti (1962) and Samuelson and Modigliani (1966) with a convex production technology of the CES type. We assume two distinct groups of agents, workers and capitalists.[2] The first group saves out of wages and profits by applying to these income sources propensities to save *which are not necessarily equal* (this is a generalisation of Pasinetti's, 1962, and Samuelson and Modigliani's, 1966, analyses firstly proposed by Chiang, 1973; see also Faria and Teixeira, 1999, and Faria, 2000, for a dynamical analysis framed in continuous time). Capitalists' saving originates only from capital income. The resulting model is two-dimensional. Differently from Böhm and Kaas (2000), distributive processes occur not only between factor shares but also between the two groups existing in the economy. We explore through simulations the large variety of dynamic behaviours that emerge from this formulation.

2. The model

The economy

Consider a single good economy. Production involves only two factors, capital and labour, and a CES production function of the form:

$$f(k) = \left(a + bk^{\rho}\right)^{\frac{1}{\rho}} \tag{1}$$

[1] Fazi and Salvadori (1981) show that a steady growth equilibrium, in which workers' and pure capitalists' propensities to save out of profit are equal and they both own positive shares of capital, is possible if workers and pure capitalists earn different rates of return on their assets.

[2] Woodford's (1986, 1989) model of optimal growth cycles also involves two groups of agents, workers and capitalists, with differentiated saving behaviour. However, due to the presence of a financial constraint, workers do not save and, consequently, do not accumulate wealth.

where k is the capital/labour ratio and where $0 < b \leq 1 - a < 1$ and $-\infty < \rho < 1$ (with $\rho \neq 0$).[3]

The only sources of income in the economy are wages and profits. Perfectly competitive labour and capital markets ensure for each short run equilibrium equality between wage rate and marginal product of labour and between profit rate and marginal product of capital. The wage share and the profit share are respectively $f(k) - f'(k)k$ and $f'(k)k$.

The economy is also characterised by the existence of two distinct groups of agents, workers and capitalists. Both groups may save and accumulate capital, k_w and k_c representing respectively workers' and capitalists' capital per worker, where $0 \leq k_c \leq k$, $0 \leq k_w \leq k$ and $k = k_w + k_c$. The only income source of capitalists is profits out of which they save $s_c f'(k)k_c$, where $0 < s_c \leq 1$ is capitalists' invariant propensity to save.

Workers' income is composed of wages and capital revenues

$$f\left(k\right) - f'(k)k + f'(k)k_w = f\left(k\right) - f'(k)k_c \tag{2}$$

Workers principal source of income is wages, $f\left(k\right) - f'(k)k$, from which they save the constant proportion $0 \leq s_{ww} \leq 1$. Workers' savings generate capital revenues $f'(k)k_w$ from which the constant fraction $0 \leq s_{wp} \leq 1$ is saved. We assume that workers' saving propensities are not necessarily equal, that is, $s_{ww} \gtreqless s_{wp}$.[4]

$$s_{ww}\left(f(k) - f'(k)k\right) + s_{wp} f'(k)k_w$$

Aggregate savings corresponds to

$$s(k_c, k_w) = s_{ww}\left(f(k) - f'(k)k\right) + s_{wp} f'(k)k_w + s_c f'(k)k_c \tag{3}$$

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[3] It is easy to show that the production function (1) has the properties: $f(0) > 0$, $f'(0) > 0$, and $f''(0) < 0$

[4] It is not possible to establish a priori if s_{wp} is larger, smaller or equal to s_{ww}. s_{wp} could exceed s_{ww} because of the more uncertain nature of capital income. However, when savings mainly originates from wages, capital revenues may be treated as windfall income out of which only a comparatively smaller proportion is saved.

where $s_c > \max(s_{ww}, s_{wp})$.[5]

Assuming a constant growth rate of the labour force, n,[6] capital per worker accumulates according to the rule:

$$G(k) = \frac{1}{1+n}\left[(1-\delta)k + s(k)\right] \tag{4}$$

where δ is the (constant) depreciation rate of capital, with $0 < \delta \leq 1$.

Short-run equilibrium

Using (3), the accumulation rule (4) becomes

$$G(k_c, k_w) = \frac{1}{1+n}\left[(1-\delta)k + s_{ww}\left(f(k) - f'(k)k\right) + s_{wp}f'(k)k_w + s_c f'(k)k_c\right] \tag{5}$$

By disaggregating equation (5), we are able to describe separately capitalists' and workers' process of capital accumulation:[7]

$$G_w(k_c, k_w) = \frac{1}{1+n}\left[(1-\delta)k_w + s_{ww}\left(f(k) - f'(k)k\right) + s_{wp}f'(k)k_w\right] \tag{6}$$

$$G_c(k_c, k_w) = \frac{1}{1+n}\left[(1-\delta)k_c + s_c f'(k)k_c\right] \tag{7}$$

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[5] The assumption $s_c > \max(s_{ww}, s_{wp})$ follows from the fact that capitalists are typically more concerned than workers to control production.

[6] In our analysis we assume that the relative size of the two groups in the population does not vary through time. It follows that n is also the rate of population growth.

[7] Setting $s_{wp} = s_c$ expression (5) is equivalent to the one presented in Böhm and Kass (2000, p. 968):

$$G(k) = \frac{1}{1+n}\left[(1-\delta)k + s_{ww}\left(f(k) - f'(k)k\right) + s_{wp}f'(k)k\right]$$

Böhm and Kass (2000) confine their analysis to the above one-dimensional map, which describes the accumulation of the overall capital in a Kaldor type model. What follows, instead, is concerned with the two-dimensional system (6) and (7), which allows for a separate description of capitalists' and workers' capital accumulation.

Steady growth equilibrium

The stationary growth solutions are obtained by solving the following equations

$$(n+\delta)k_c = s_c f'(k)k_c \tag{8}$$

$$(n+\delta)k_w = s_{ww}\left(f(k)-f'(k)k\right)+s_{wp}f'(k)k_w \tag{9}$$

It is possible to envisage two different types of non-trivial equilibria. A Pasinetti equilibrium involves capitalists owning a positive share of capital. A dual equilibrium allows only workers to own capital.[8] The two types of equilibria, which may coexist, are defined as follows:

Pasinetti equilibrium

$$f'(k^P)=\frac{n+\delta}{s_c} \quad k_c^P=\left(1-\frac{s_{ww}}{s_c-s_{wp}}\frac{1-e_f(k^P)}{e_f(k^P)}\right)k^P \quad k_w^P=\frac{s_{ww}}{s_c-s_{wp}}\frac{1-e_f(k^P)}{e_f(k^P)}k^P$$

Dual equilibrium

$$\frac{f(k^D)}{k^D}=\frac{n+\delta}{s_{ww}(1-e_f(k^D))+s_{wp}e_f(k^D)} \quad k_c^D=0 \quad k_w^D=k^D$$

where $e_f(k)=f'(k)k/f(k)=b(ak^{-\rho}+b)^{-1}$ denotes the output elasticity of capital. From the properties of $f(k)$, it follows that $0<e_f(k)\le 1$.

The existence of Pasinetti or dual equilibria is verified as follows. We first define a function that relates the inverse capital/output ratio and $e_f(k)$:

$$\frac{f(k)}{k}=\varphi(e_f(k))$$

where $\varphi(x)=\left(\frac{b}{x}\right)^{\frac{1}{\rho}}$. The function $\varphi(\bullet)$ is plotted in Figure 1 for a) $\rho>0$, b) $-1<\rho<0$ and c) $\rho<-1$.

A Pasinetti equilibrium in which both workers and capitalists own a positive share of capital, $0<k_c^P<k^P$, exists if and only if:[9]

[8] The existence of a 'trivial' equilibrium, $k^0=0$, is excluded as long as $s_{ww}\neq 0$.

$$0 < \tilde{e}_f < e_f(k^P) < 1 \qquad (10)$$

where $\tilde{e}_f \equiv \dfrac{s_{ww}}{s_c - (s_{wp} - s_{ww})}$ and $e_f(k^P) = b^{\frac{1}{1-\rho}} \left(\dfrac{n+\delta}{s_c} \right)^{\frac{\rho}{\rho-1}}$.

In Figure 1, the intersection between the straight line at $e_f(k^P)$ and the curve $\varphi(e_f(k))$ identifies the unique Pasinetti equilibrium. Such an equilibrium is characterised by a positive share of capitalists' capital ($e_f(k^P)$ is on the right of the dotted line plotted at $\tilde{e}_f$) and by a positive share of workers' capital ($e_f(k^P) < 1$ and $s_{ww} > 0$).

As far as the dual equilibrium is concerned, we define also the following relationship

$$\frac{f(k)}{k} = \theta(e_f(k))$$

where $\theta(x) = \dfrac{n+\delta}{s_{ww}(1-x) + s_{wp}x}$.

As shown in Figure 2, which has been plotted for $\rho < -1$ and a) $s_{ww} = s_{wp}$, b) $s_{ww} < s_{wp}$ and c) $s_{ww} > s_{wp}$, an unique Dual equilibrium exists if

$$\left[\lim_{e_f(k)\to 0} \varphi(e_f(k)) - \theta(0) \right] \left[\theta(1) - \lim_{e_f(k)\to 1} \varphi(e_f(k)) \right] > 0 \qquad (11)$$

otherwise two or none of such equilibria exists, where $\theta(0) = \dfrac{n+\delta}{s_{ww}}$ and $\theta(1) = \dfrac{n+\delta}{s_{wp}}$.[10] Note that condition (11) can be written as:

$$\rho \left(\frac{n+\delta}{s_{wp}} - b^{\frac{1}{\rho}} \right) > 0 \qquad (12)$$

Local stability analysis

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[9] A special case of a Pasinetti equilibrium, the 'anti-dual' case, occurs when $s_{ww}(1 - e_f(k)) = 0$ and $k_w^P = 0$.

[10] A dual equilibrium collapses to a trivial equilibrium, $k^D = 0$, when $s_{ww} = 0$.

The Jacobian of the system (6) and (7) evaluated at the Pasinetti equilibrium (k^P, k_c^P, k_w^P) is

$$J(k_c^P, k_w^P) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{13}$$

where

$$J_{11} = \frac{1}{1+n}\left[1-\delta+(s_{wp}-s_{ww})f''(k^P)k^P + s_{wp}\left(f'(k^P)-f''(k^P)k_c^P\right)\right]$$

$$J_{12} = \frac{1}{1+n}\left[(s_{wp}-s_{ww})f''(k^P)k^P - s_{wp}f''(k^P)k_c^P\right]$$

$$J_{21} = \frac{1}{1+n}s_c f''(k^P)k_c^P$$

$$J_{22} = \frac{1}{1+n}\left[1-\delta+s_c\left(f'(k^P)+f''(k^P)k_c^P\right)\right]$$

The corresponding trace is:

$$T(k_c^P, k_w^P) = \frac{n+\delta}{1+n}\left[\frac{2(1-\delta)}{n+\delta}+1+e_{f'}(k^P)+\left(\frac{s_{wp}e_f(k^P)-s_{ww}e_{f'}(k^P)}{s_c e_f(k^P)}\right)\right] \tag{14}$$

and the corresponding determinant is:

$$D(k_c^P, k_w^P) = T(k_c^P, k_w^P)\left(\frac{1-\delta}{1+n}\right)-\left(\frac{1-\delta}{1+n}\right)^2+\frac{e_{f'}(k^P)(s_{wp}-s_{ww})+s_{wp}}{s_c}\left(\frac{n+\delta}{1+n}\right)^2 \tag{15}$$

where $e_{f'}(k) = f''(k)k/f'(k) = -a(1-\rho)(a+bk^\rho)^{-1}$.

The stability conditions for the Pasinetti equilibrium are the following:

(i) $1+T(k_c^P, k_w^P)+D(k_c^P, k_w^P)>0$;

(ii) $1-T(k_c^P, k_w^P)+D(k_c^P, k_w^P)>0$;

(iii) $1-D(k_c^P, k_w^P)>0$.

Conditions (i) to (iii) ensure that the eigenvalues of the Jacobian $J(k_c^P, k_w^P)$ are both confined within the unit circle. An unstable fixed point (k_c^P, k_w^P) corresponds to a violation of at least one of these conditions.[11] Each violation gives rise to a different bifurcation process. That is, if condition (i) does not hold stability is lost trough a Flip Bifurcation process; the failing of condition (ii) determines a Saddle Node bifurcation process; and the failing of condition (iii) generates a Hopf bifurcation process.

Condition (i) corresponds to

$$e_{f'}(k^P) > (<) -2e_f(k^P)\left(\frac{1+n}{n+\delta}\right)\frac{(n+2-\delta)s_c+(n+\delta)s_{ww}}{(n+2-\delta)(s_c e_f(k^P)-s_{ww})+(n+\delta)(s_{wp}-s_{ww})} \equiv e_{f'}^F$$

The direction of the inequality sign depending on:

$$e_f(k^P) > (<) \frac{s_{ww}(n+2-\delta)-(s_{wp}-s_{ww})(n+\delta)}{s_c(n+2-\delta)} \equiv \overline{e}_f$$

Note that

$$if \quad e_f(k^P) > \overline{e}_f \quad then \quad e_{f'}^F < -1 \text{ and } if \quad e_f(k^P) < \overline{e}_f \quad then \quad e_{f'}^F > 0$$

Condition (ii) can be reduced to condition (10).

Finally, condition (iii) corresponds to

$$e_{f'}(k^P) < (>) e_{f'}^H \quad for \quad e_{f'}^H > (<)\, 0 \tag{16}$$

where $e_{f'}^H \equiv \dfrac{e_f(k^P)(s_c-s_{wp})(1+n)}{e_f(k^P)(s_{wp}-s_{ww})(n+\delta)+(1-\delta)(s_c e_f(k^P)-s_{ww})}$.

If condition (10) holds, $e_{f'}^H > 0$ for $s_{ww} \le s_{wp}$ and $e_{f'}^H > -1$ for $s_{ww} > s_{wp}$.

The Jacobian of the system (6) and (7) evaluated at a dual equilibrium (k^D, k_c^D, k_w^D) is

$$J(k_c^D, k_w^D) = \begin{bmatrix} \frac{1}{1+n}\left[1-\delta+(s_{wp}-s_{ww})f''(k^D)k^D+s_{wp}f'(k^D)\right] & \frac{1}{1+n}\left[(s_{wp}-s_{ww})f''(k^D)k^D\right] \\ 0 & \frac{1}{1+n}\left[1-\delta+s_c f'(k^D)\right] \end{bmatrix} \tag{17}$$

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[11] For a detailed analysis on the stability conditions of two and higher dimensional difference equations systems, see Gandolfo (1997).

The two eigenvalues of the system, λ_1 and λ_2, lie along the principal diagonal of the triangular matrix (17), with

$$\lambda_2 = \frac{1}{1+n}\left[1-\delta + s_c f'(k^D)\right] \quad \text{and} \quad \lambda_1 = \frac{1}{1+n}\left[1-\delta + (s_{wp}-s_{ww}) f''(k^D)k^D + s_{wp} f'(k^D)\right].$$

The occurrence of a Hopf bifurcation is excluded. Only a Flip bifurcation (one of the roots passing through –1) or a Saddle node bifurcation (one of the two roots passing through 1) may occur. Stability requires $-1 < \lambda_1 < 1$ and $-1 < \lambda_2 < 1$.

We may distinguish three cases

1st case $s_{ww} = s_{wp}$. We have $0 < \lambda_1 < \lambda_2$ and $\lambda_2 < 1$ for $k^D > k^P$.[12]

2nd case $s_{ww} < s_{wp}$. We have $\lambda_2 > \max(\lambda_1, 0)$, $\lambda_2 < 1$ for $k^D > k^P$ and $\lambda_1 > -1$ for $e_{f'}(k^D) > -\dfrac{s_{wp} + \left(\dfrac{2-\delta}{f'(k^D)}\right)}{s_{wp} - s_{ww}} < -1$.

3rd case $s_{ww} > s_{wp}$. We have $\lambda_2 > 0$, $\lambda_1 > 0$; and, when $k^D > k^P$, $\lambda_2 < 1$ and $\lambda_1 < 1$ for $e_{f'}(k^D) > -\dfrac{\dfrac{n+\delta}{f'(k^D)} - s_{wp}}{s_{ww} - s_{wp}} < -1$.[13] It follows that, for this case, stability can be lost only through a Saddle node bifurcation process.

According to the above analysis, , as long as $k^P > k^D$, the condition $e_{f'}(k) \geq -1$ ensures stability to the Pasinetti equilibrium and, as long as $k^P < k^D$, it also ensures stability to a Dual equilibrium, where the elasticity $e_{f'}(k)$ measures the curvature of the production function. The inequality $e_{f'}(k) \geq -1$ follows from the Inada conditions. For example, it holds for the well-known Cobb-Douglas production function, $f(k) = Ak^{\alpha}$, where $0 < \alpha < 1$: for any k, $e_{f'}(k) = \alpha - 1 > -1$.

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[12] In the appendix we prove formally that when condition (10) fails to hold and $s_{ww} \leq s_{wp}$, it must be $k^D > k^P$.

[13] It can be shown that a trivial equilibrium k^0 is never stable for $k^P \geq k^0 = 0$.

However, it does not hold for a larger class of concave production functions.[14] For example it does not necessarily hold for the equally widely-used CES technology.[15] Specifically, for the production (1), the condition

$$e_{f'}(k) = \frac{f''(k)k}{f'(k)} = a(\rho - 1)(a + bk^{\rho})^{-1} > -1$$

is always true only for $0 < \rho < 1$. In the next section, we turn to the numerical analysis of the associated dynamical system (6) and (7) when such a technology is assumed.[16]

3. Numerical explorations

The significance of workers' propensities to save

We choose the constellation of parameters: $a = 0.7$, $b = 0.3$, $\rho = -50$, $n = 0.05$, $\delta = 0.2$, $s_c = 0.75$ and initial values: $k_{c,\,0} = 0.5$, $k_{\mathrm{w},\,0}$, and $k_{c,\,0} = k_{c,\,0} + k_{w,\,0} = 1$. The Pasinetti equilibrium value of the capital/labour ratio is given by

$$k^P = a^{\frac{1}{\rho}} \left[\left(\frac{n+\delta}{s_c b} \right)^{\frac{\rho}{1-\rho}} - b \right]^{-\frac{1}{\rho}} \cong 0.997 \tag{18}$$

From condition (10), $k_c^P > 0$, if and only if $s_{ww} < \tilde{s}_{ww} \equiv (s_c - s_{wp}) \dfrac{e(k^P)}{1 - e(k^P)}$. $\tilde{s}_{ww}$ represents, therefore, a critical value above which a Pasinetti equilibrium cannot exist.

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[14] The inequality $e_{f'}(k) > -1$ is not always satisfied even by concave production functions with slightly more general properties than the Cobb-Douglas production function (see Böhm and Kaas, 2000, p. 969).

[15] As it is well known (See Varian, 1992, pp. 19-20), the CES production function approximates a Cobb-Douglas production function for $\rho \to 0$.

[16] Böhm and Wenzelburger (2000) explore some of the dynamic properties of a Pasinetti type model with neoclassical features but their analysis is confined to some of the local stability properties of the case $s_{ww} = s_{wp}$.

It is not always possible to derive an explicit form for a dual equilibrium, depending on the workers' propensities to save out of wages and profits. From condition (12), a dual equilibrium exists and it is unique if $s_{wp} > \overline{s}_{wp} \cong 0.244$, where $\overline{s}_{wp} \equiv (n+\delta)b^{-\frac{1}{\rho}}$.

We consider first the case $s_{ww} = s_{wp}$. Figure 3 presents bifurcation diagrams which study the long term behaviour of k, k_c/k and k_w/k with respect to s_{ww} for $0 \leq s_{ww} \leq 0.3$.[17] Figure 4(a), instead, will help us to follow, for this case, the creation and the destruction of equilibria as s_{ww} increases. For $0 < s_{ww} < \tilde{s}_{ww}$, a Pasinetti equilibrium k^P exists, involving a positive share of capitalists' capital, $k_c / k > 0$, with $\tilde{s}_{ww} = s_c e_f(k^P) \cong 0.249$. Moreover, for $\sigma_{ww}^P < s_{ww} < \tilde{s}_{ww}$, k^P is stable with both eigenvalues of the Jacobian $J(k_c^P, k_w^P)$ lying inside the unit circle, where

$$\sigma_{ww}^P = -s_c e_f(k^P) \frac{e_{f'}(k^P) + 2\dfrac{1+n}{n+\delta}}{2e_f(k^P)\dfrac{1+n}{2+n-\delta} - e_{f'}(k^P)} \cong 0.186 .$$

As s_{ww} is lowered below σ_{ww}^P, the fixed point k^P loses stability. The stability condition (i) $1+T(k_c^P, k_w^P) + D(k_c^P, k_w^P) > 0$ (see above) is violated involving the smallest eigenvalue of $J(k_c^P, k_w^P)$ becoming less than –1. The system follows a period-doubling (or Flip) bifurcation route to chaos.[18] At $s_{ww} = \tilde{s}_{ww}$,[19] the share of capitalists' capital is zero. For $s_{ww} > \tilde{s}_{ww}$, as condition (10) is violated, the dual equilibrium

$$k^D = a^{\frac{1}{\rho}} \left[\left(\frac{n+\delta}{s_{ww}} \right)^{\rho} - b \right]^{-\frac{1}{\rho}}$$

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[17] To generate all our diagrams, we discard the first 1500 periods and employ the subsequent 2500 periods.

[18] Complex behaviour can be found for many constellations of the parameters. Our paper is however confined in its scope. We are not going to proof rigorously the existence of chaos.

[19] Note that for $s_{ww} = s_{wp}$, the switch between a Pasinetti and dual equilibrium necessarily occurs when the Pasinetti equilibrium is stable since $\sigma_{ww}^P < \tilde{s}_{ww}$ always.

is created through a saddle node bifurcation (see Figure 4(a)). The steady growth capital/labour ratio, k^D, goes from $k^D = k^P$ to $k^D \cong 1.209$ as s_{ww} is increased from $s_{ww} = \tilde{s}_{ww}$ to $s_{ww} = 0.3$. As shown in the previous section, when $s_{ww} = s_{wp}$ and $s_{ww} > \tilde{s}_{ww}$ the dual equilibrium is always stable.

We turn now to the case $s_{ww} < s_{wp}$. Figure 5 presents bifurcation diagrams for the capital/labour ratio with respect to s_{ww} for $0 \leq s_{ww} \leq 0.3$ and for (a) $s_{wp} = 0.3$, (b) $s_{wp} = 0.45$, (c) $s_{wp} = 0.475$ and (d) $s_{wp} = 0.5$. Figure 4(b), instead, shows, the creation and the destruction of equilibria for $0 \leq s_{ww} \leq 0.3$ and $s_{wp} > \overline{s}_{wp}$. As shown in figure 5(a), for $s_{wp} = 0$, the qualitative behaviour of the system is analogous to the case $s_{ww} = s_{wp}$: for $0 \leq s_{ww} \leq \tilde{s}_{ww}$, a Pasinetti equilibrium exists, where $\tilde{s}_{ww} \cong 0.224$, and for $\sigma^P_{ww} < s_{ww} < \tilde{s}_{ww}$ it is stable. At the bifurcation value $\sigma^P_{ww} \cong 0.199$, the Pasinetti equilibrium loses stability through a Flip bifurcation, where

$$\sigma^P_{ww} = -\frac{1}{2(e_f(k^P) - e_{f'}(k^P))(1+n)}\left\{ s_c e_f(k^P)(n+\delta-2)\left[e_{f'}(k^P) + 2\frac{1+n}{n+\delta}\right] + s_{wp} e_{f'}(k^P)(n+\delta)\right\}$$

For $s_{wp} > 0.341$, $\sigma^P_{ww} > \tilde{s}_{ww}$, that is, the Pasinetti equilibrium is never stable. As Figures 5(b), 5(c), and 5(d) show there is a threshold value of the workers' propensity to save, s^{tsd}_{ww}, above which the system behaviour changes. In particular, capitalists' capital becomes zero after a sufficiently long number of iterations. Capitalists' share of capital vanishes as $t \to \infty$ before condition (10) is violated, that is, $s^{tsd}_{ww} < \tilde{s}_{ww}$ ($s^{tsd}_{ww} = \tilde{s}_{ww}$ *for* $\tilde{s}_{ww} > \sigma^D_{ww}$). Above s^{tsd}_{ww} the two-dimensional system (6) and (7) collapses to a one-dimensional system in which all the capital is owned by workers.

As Figure 5(b) shows, when s_{ww} passes through $\tilde{s}_{ww}$, a dual equilibrium k^D is created via a Flip bifurcation. k^D is unstable for $s_{ww} < \sigma^D_{ww}$, where σ^D_{ww} represents the Flip bifurcation value of s_{ww}. As shown in Figs 5(b)-4(d), σ^D_{ww} increases with s_{wp} ($\sigma^D_{ww} \cong 0.206$ for $s_{wp} \cong 0.45$, $\sigma^D_{ww} \cong 0.208$ for $s_{wp} \cong 0.475$ and $\sigma^D_{ww} \cong 0.209$ for $s_{wp} \cong 0.5$). Above σ^D_{ww} the dual equilibrium is stable if $\sigma^D_{ww} > \tilde{s}_{ww}$.

Finally, we consider the case $s_{ww} > s_{wp}$. If $s_{ww} > s_{wp}$, a Pasinetti equilibrium, if it exists, can lose stability only through a Hopf bifurcation. Figure 6 presents bifurcation diagrams for k, k_c/k and k_w/k with respect to s_{ww} for $0.25 \leq s_{ww} \leq 0.265$ and $s_{wp} = 0.3$. Figure 4(c) shows, for this case, the creation and the destruction of equilibria as s_{ww} increases. As shown in Figure 6, the Pasinetti equilibrium is stable for $s_{ww} < \sigma^H$, where

$$\sigma^H = -\frac{e_f(k^P)}{e_{f'}(k^P)}\left\{\frac{(s_c - s_{wp})(1+n) - E_f[(1-\delta)s_c + (\delta+n)s_{wp}]}{e_f(k^P)(n+\delta)+1-\delta}\right\} \cong 0.251$$

As shown in Figure 4(c), in correspondence of σ^F two dual equilibria are created as the system undergoes a Saddle node bifurcation. For $s_{ww} > \sigma^F$, the larger of these equilibria becomes the attractor of the system in correspondence of which, as shown by Figure 6, the capitalists' share of capital vanishes. At $s_{ww} = \tilde{s}_{ww}$ the system goes through another Saddle node bifurcation as the Pasinetti equilibrium ceases to exist (see Fig. 4(c))

Changes in the workers' propensity to save have also an impact on the distribution of wealth between workers and capitalists. Figure 7, shows the behaviour of the average capital/labour ratio and of the average shares of capitalists' and workers' capital for $0 \le s_{ww} \le \tilde{s}_{ww}$ and $s_{wp} = 0.15$, and compare it with the Pasinetti equilibrium (the dotted line). As shown in Figure 6, for $s_{ww} < \sigma^F$, when the Pasinetti equilibrium is unstable, the share of capitalists' capital is above the equilibrium value. Conversely, the average share of workers' capital is below the equilibrium value. The behaviour of the wealth distribution between workers and capitalists follows the behaviour of the income distribution between profits and wages. According to Figure 7, when the Pasinetti equilibrium is unstable and $s_{ww} < \sigma^F$, the share of profits (wages) in output is larger (smaller) than equilibrium value. These results are reversed for $s_{ww} > \sigma^F$.

4. Final remarks

In this paper, we have presented a discrete-time Solovian growth model that allows for equilibrium growth with two types of agents, workers and capitalists. In line with Pasinetti's (1962), Samuelson and Modigliani's (1966) and Chiang (1973) analyses, we have assumed that workers, in their quality of shareholders, save out of profit in a lower proportion than pure capitalists do. We have confirmed the validity of Böhm and Kaas's (2000) proposition according to which the properties of the equilibrium depend crucially on the characteristics of the technology and on differential saving rates. Finally, we have presented some numerical explorations showing the long run behaviour of the distributive processes between the two groups existing in the economy.

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Appendix

The condition $\lambda_2 < 1$, which corresponds to $f'(k^D) < \frac{n+\delta}{s_c} = f'(k^P)$, is satisfied for $k^D > k^P$. To prove that when condition (10) holds and $s_{ww} \le s_{wp}$, it must be $k^D > k^P$, we proceed generalizing Samuelson and Modigliani, 1966, and Miyazaki, 1991:

Assume that $k^D > k^P$. It follows from the strict monotonicity of $f(k)$ that

$$\frac{f(k^P)}{k^P} > \frac{f(k^D)}{k^D}$$

Considering that

$$\frac{f(k^D)}{k^D} = \frac{n+\delta}{s_{ww}} - \frac{s_{wp} - s_{ww}}{s_{ww}} f'(k^D) \text{ and } f'(k^P) = \frac{n+\delta}{s_c},$$

through substitution one obtains

$$s_{ww} \frac{f(k^P)}{k^P} > s_c f'(k^P) - (s_{wp} - s_{ww}) f'(k^D).$$

Moreover, when $k^D > k^P$, it must be that $f'(k^P) > f'(k^D)$. It follows that, as long as $s_{ww} \le s_{wp}$,

$$s_{ww} \frac{f(k^P)}{k^P} > [s_c - (s_{wp} - s_{ww})] f'(k^P),$$

which is true by condition (10).

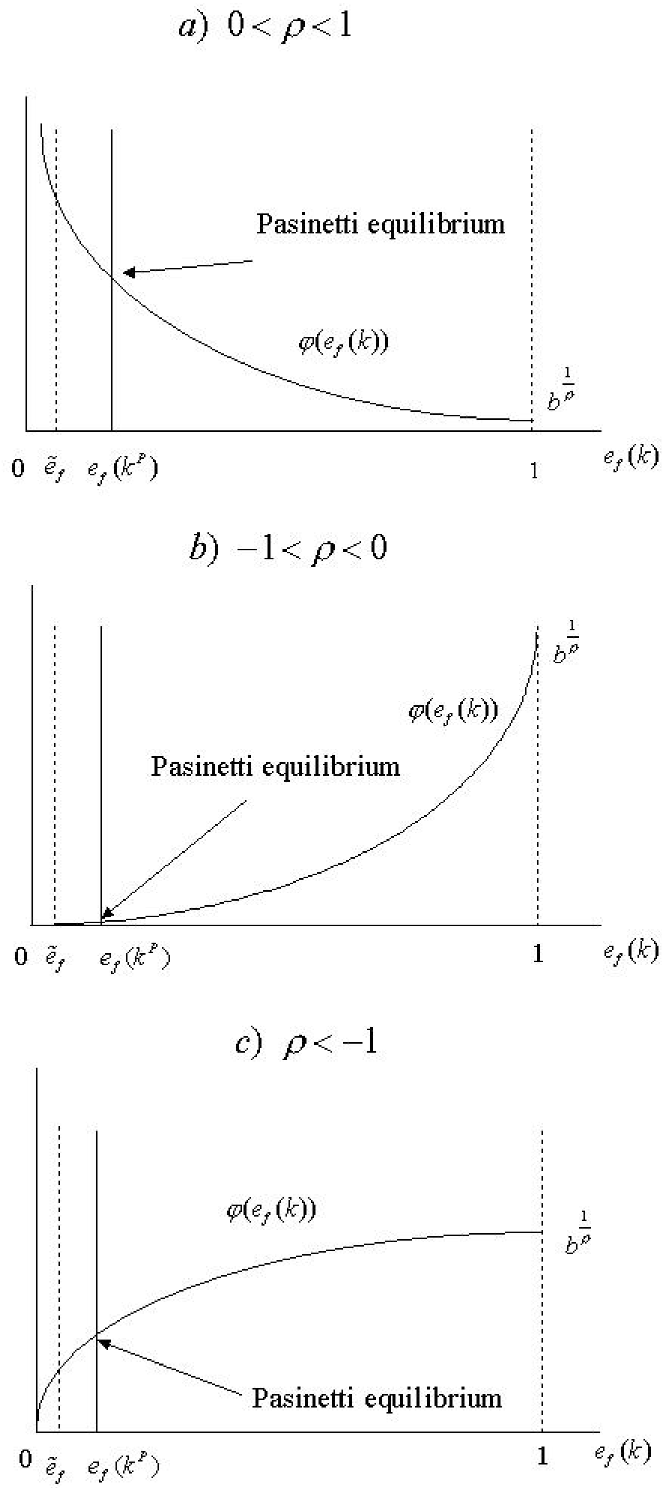
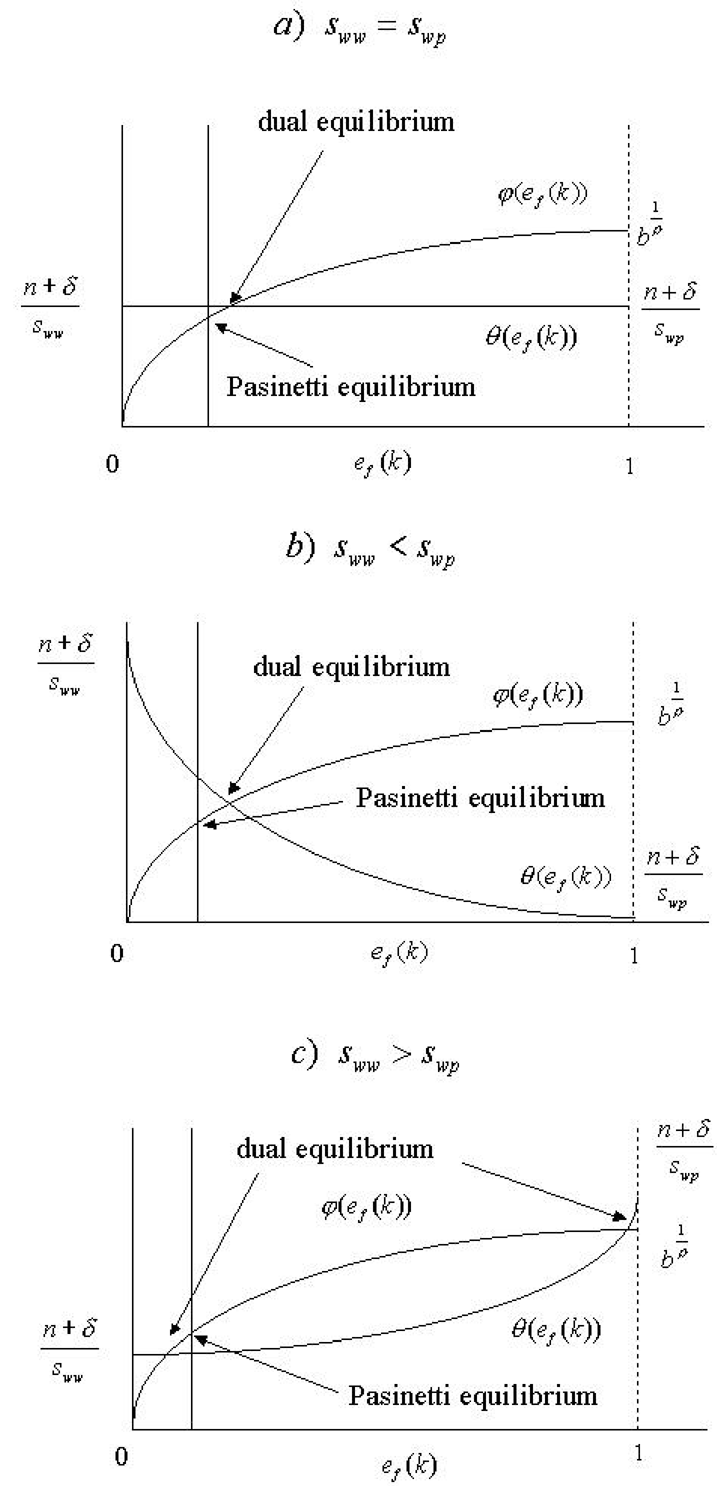

a) $0<\rho<1$
Pasinetti equilibrium
$\varphi(e_f(k))$
$b^{\frac{1}{\rho}}$
0
$\tilde{e}_f$
$e_f(k^P)$
1
$e_f(k)$
b) $-1<\rho<0$
$b^{\frac{1}{\rho}}$
$\varphi(e_f(k))$
Pasinetti equilibrium
0
$\tilde{e}_f$
$e_f(k^P)$
1
$e_f(k)$
c) $\rho<-1$
$\varphi(e_f(k))$
$b^{\frac{1}{\rho}}$
Pasinetti equilibrium
0
$\tilde{e}_f$
$e_f(k^P)$
1
$e_f(k)$

Figure 1



a) $s_{ww} = s_{wp}$
dual equilibrium
$\varphi(e_f(k))$
$b^{\frac{1}{\rho}}$
$\frac{n+\delta}{s_{ww}}$
$\frac{n+\delta}{s_{wp}}$
$\theta(e_f(k))$
Pasinetti equilibrium
0
$e_f(k)$
1
b) $s_{ww} < s_{wp}$
$\frac{n+\delta}{s_{ww}}$
dual equilibrium
$\varphi(e_f(k))$
$b^{\frac{1}{\rho}}$
Pasinetti equilibrium
$\theta(e_f(k))$
$\frac{n+\delta}{s_{wp}}$
0
$e_f(k)$
1
c) $s_{ww} > s_{wp}$
$\frac{n+\delta}{s_{wp}}$
dual equilibrium
$\varphi(e_f(k))$
$b^{\frac{1}{\rho}}$
$\frac{n+\delta}{s_{ww}}$
$\theta(e_f(k))$
Pasinetti equilibrium
0
$e_f(k)$
1

Figure 2

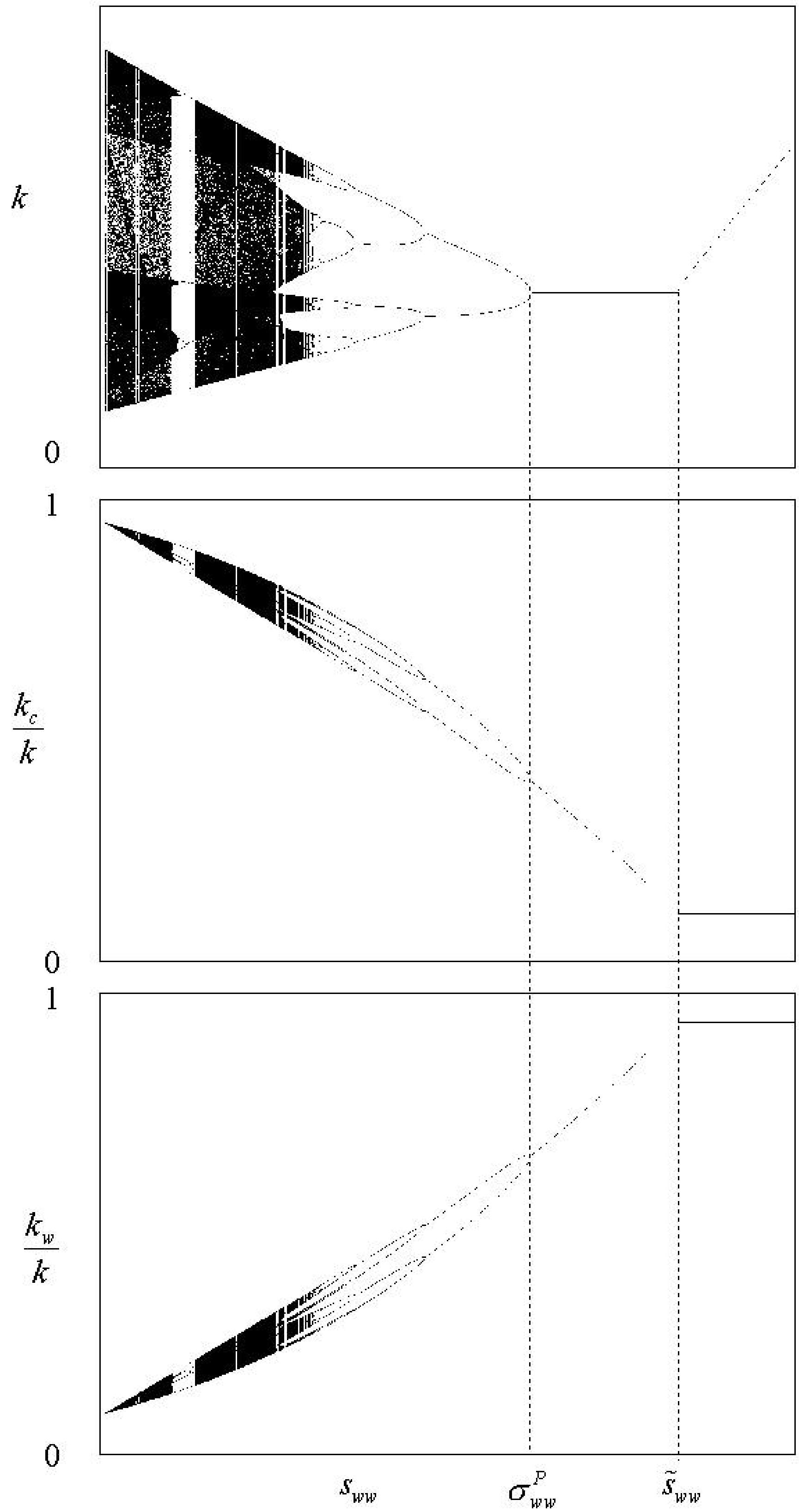


Figure 3

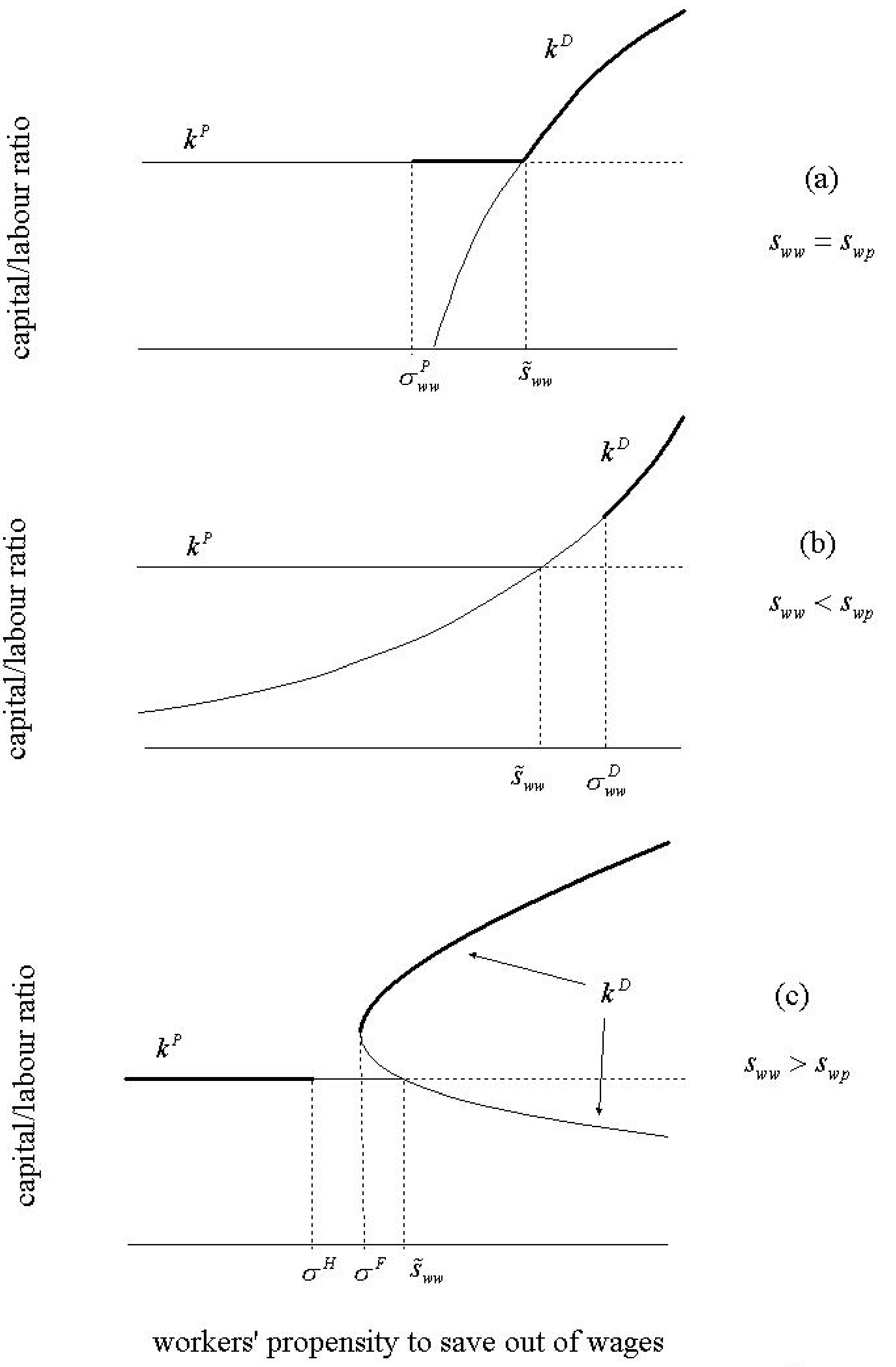

capital/labour ratio
k^D
k^P
(a)
$s_{ww} = s_{wp}$
σ^P_{ww}
$\tilde{s}_{ww}$
capital/labour ratio
k^D
k^P
(b)
$s_{ww} < s_{wp}$
$\tilde{s}_{ww}$
σ^D_{ww}
capital/labour ratio
k^D
(c)
k^P
$s_{ww} > s_{wp}$
σ^H
σ^F
$\tilde{s}_{ww}$
workers' propensity to save out of wages

Figure 4

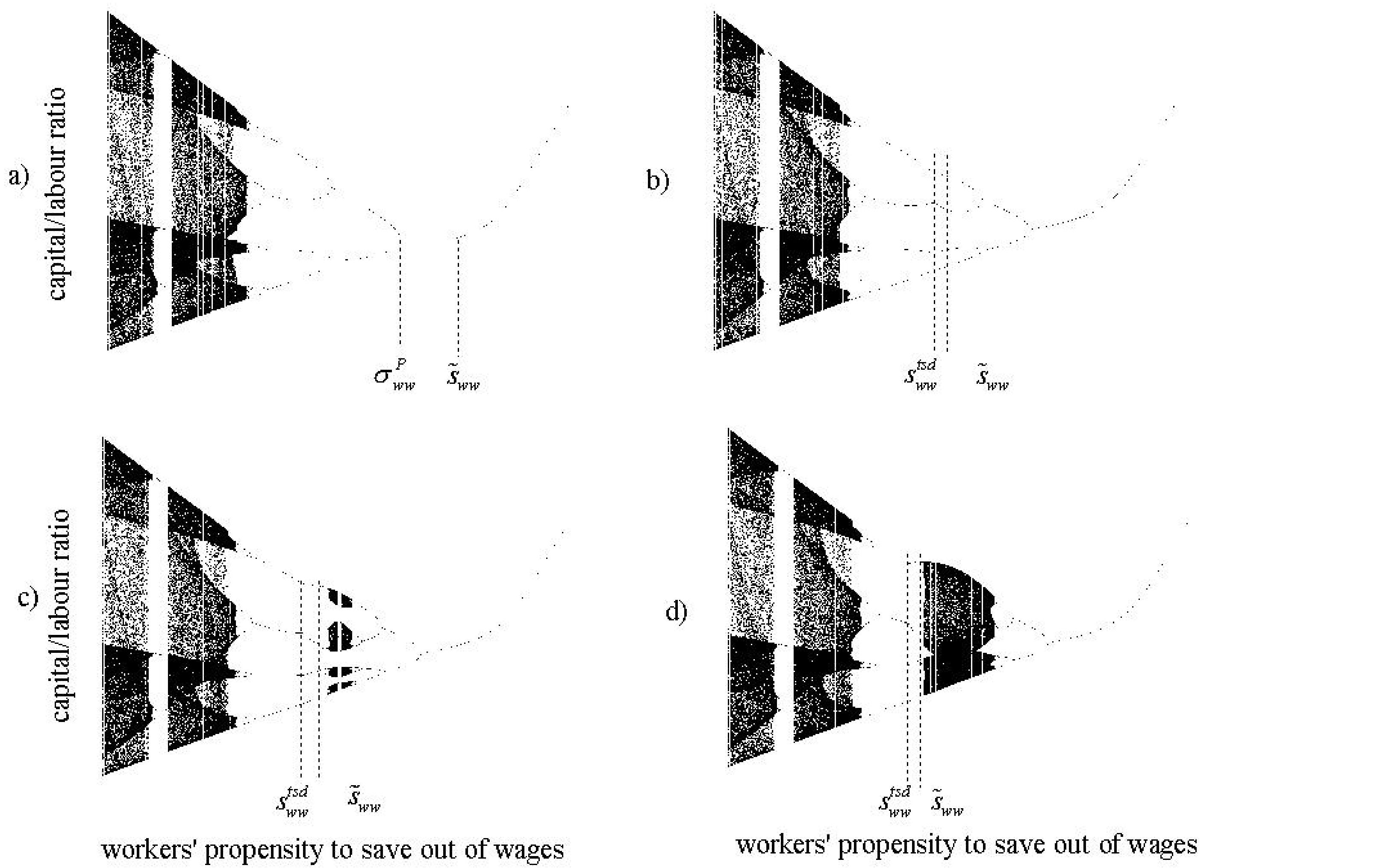

a)
b)
c)
d)
capital/labour ratio
capital/labour ratio
σ^{P}_{ww}
$\tilde{s}_{ww}$
s^{tsd}_{ww}
$\tilde{s}_{ww}$
s^{tsd}_{ww}
$\tilde{s}_{ww}$
s^{tsd}_{ww}
$\tilde{s}_{ww}$
workers' propensity to save out of wages
workers' propensity to save out of wages

Figure 5

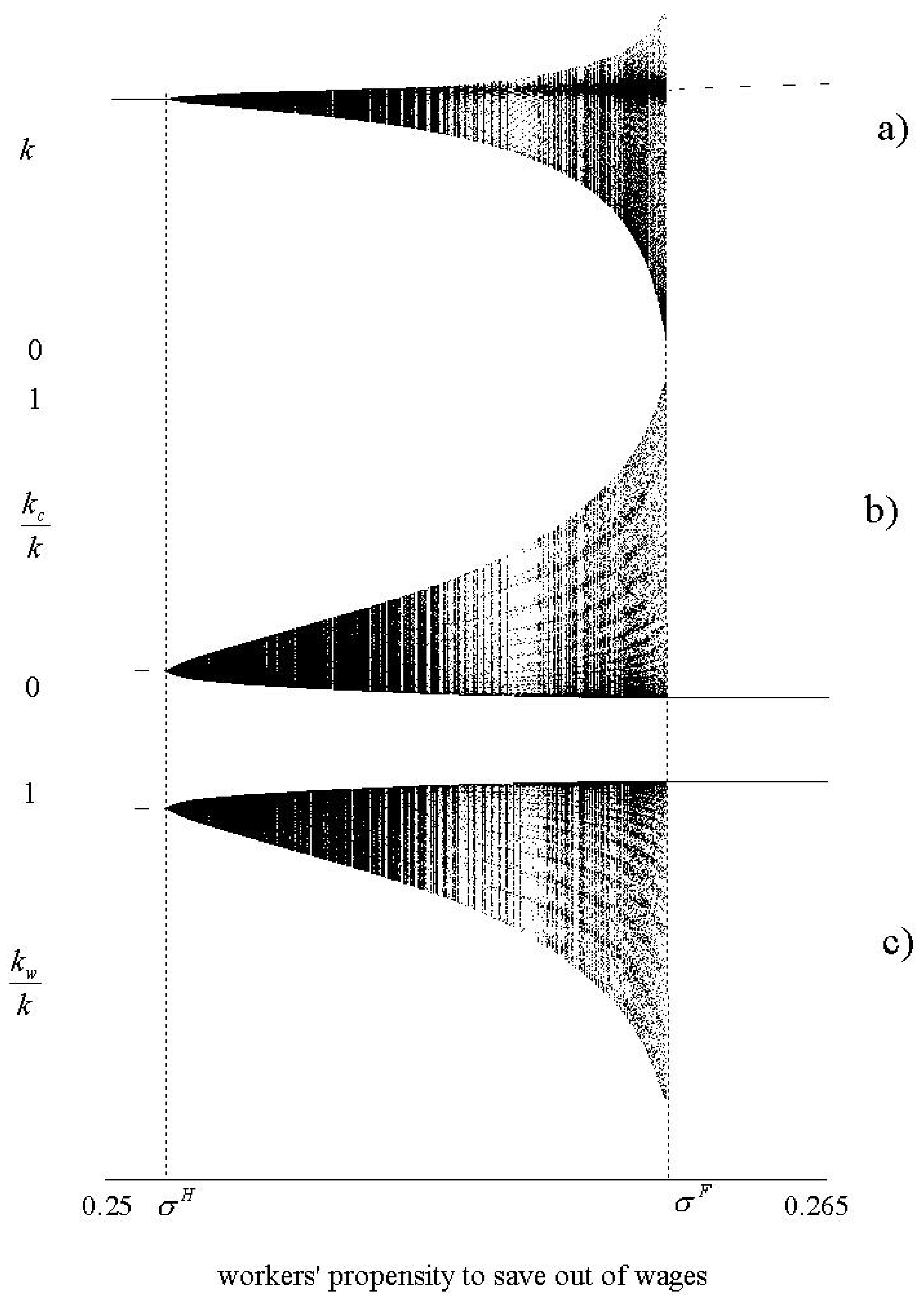
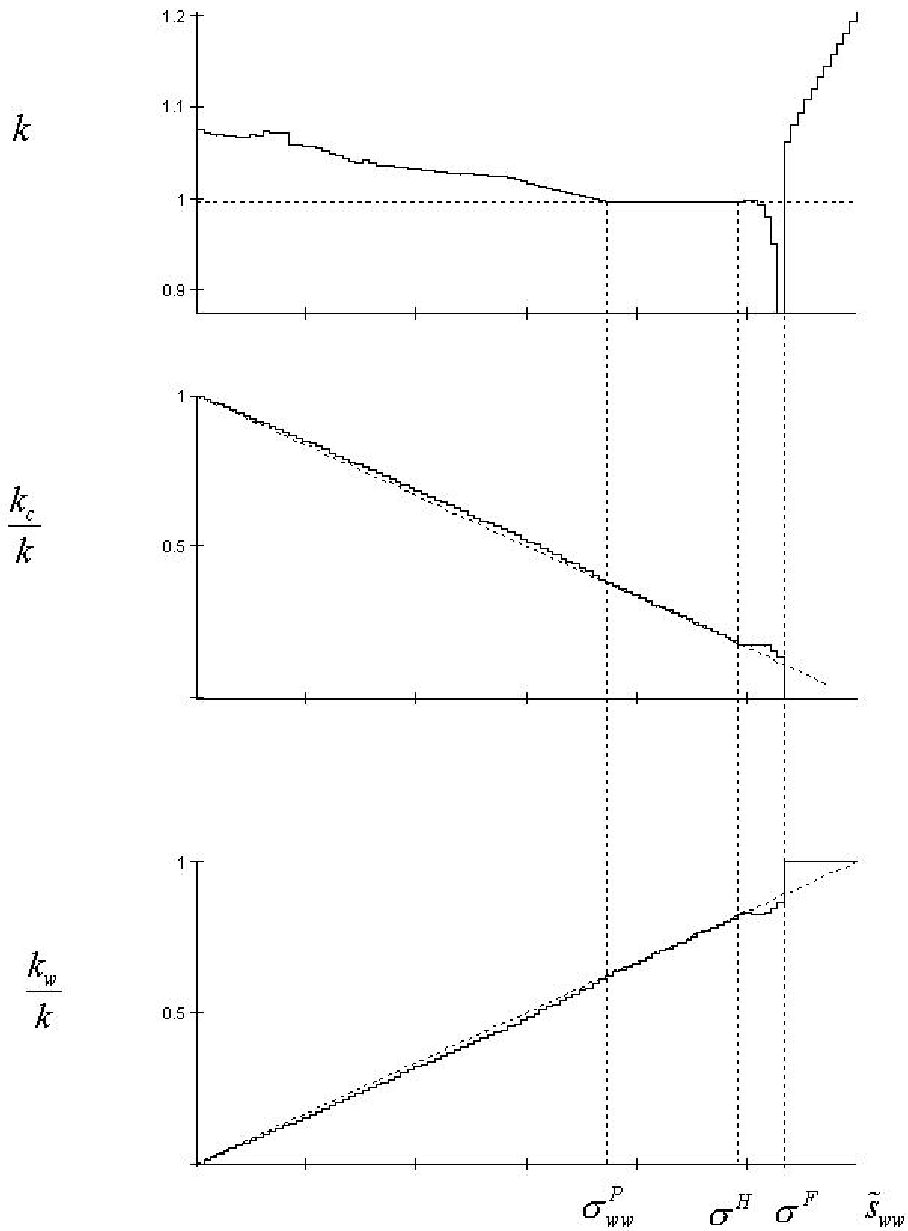
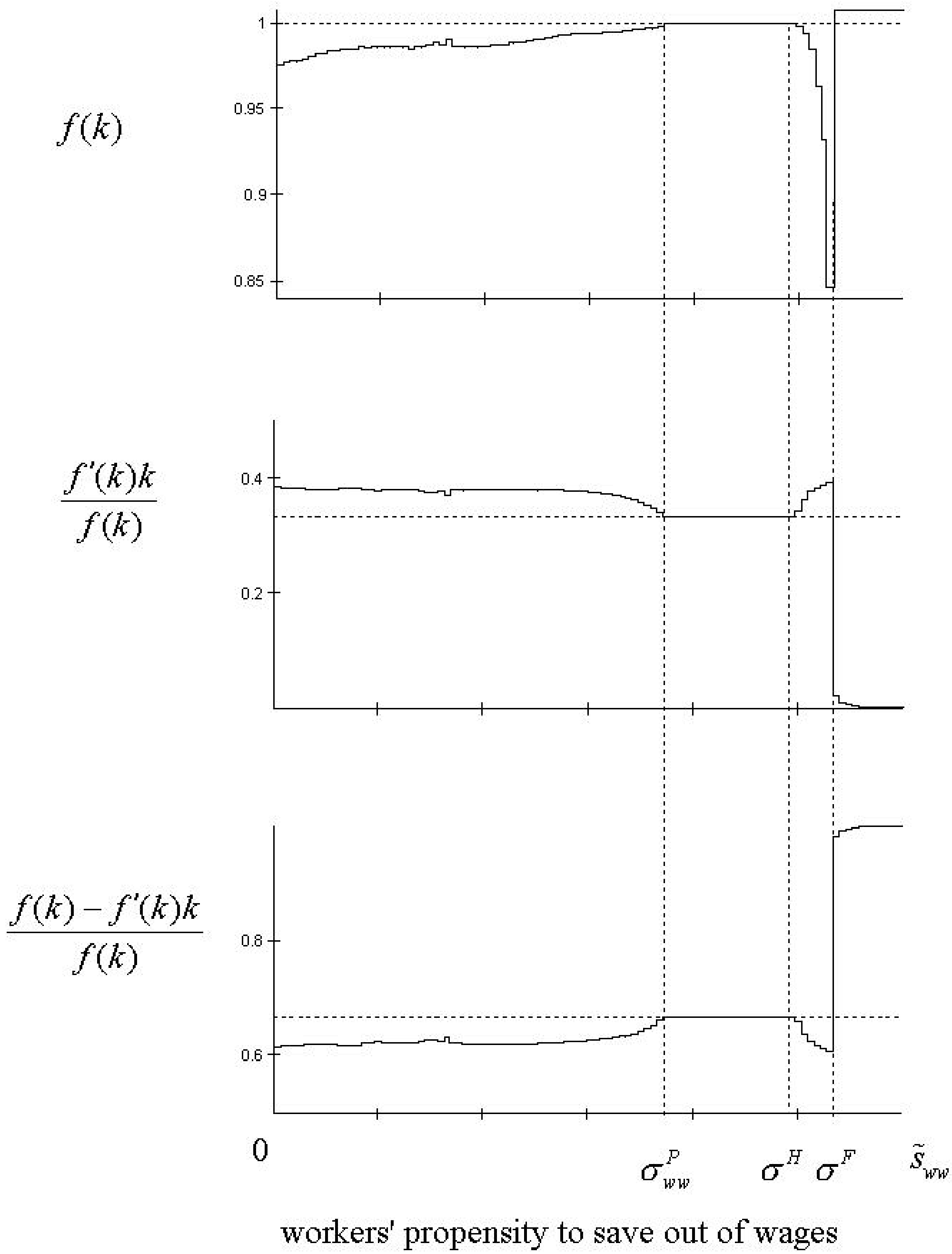

a)
k
0
1
b)
$\frac{k_c}{k}$
0
1
c)
$\frac{k_w}{k}$
0.25
σ^H
σ^F
0.265
workers' propensity to save out of wages

Figure 6



workers' propensity to save out of wages

Figure 7



f(k)
1
0.95
0.9
0.85
f'(k)k / f(k)
0.4
0.2
f(k) − f'(k)k / f(k)
0.8
0.6
0
σ_{ww}^{P}
σ^{H}
σ^{F}
$\tilde{s}_{ww}$
workers' propensity to save out of wages

Figure 8