Auction Prices and Asset Allocations of the Electronic Security Trading System *Xetra*

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Preliminary and incomplete version: June 28, 2004

Abstract

In this paper we develop a theoretical framework for auctions of the *Xetra* System, the Electronic Security Trading System operated by Deutsche Börse on the German stock exchange. We formalize the price and the allocation mechanism of limit orders and investigate the fundamental trading principles of Xetra. We show that transactions are carried out using a rationing mechanism such that asset allocations will most likely be of a non-Walrasian type.

1 Introduction

In the past decades, the amount of worldwide security transactions that were processed by electronic trading platforms increased significantly. In Germany, for example, over 90% of security transactions are executed by the *Xetra* System operated by Deutsche Börse, cf. Gruppe Deutsche Börse (01.07.2003). Other well-established European trading platforms are the Pan European stock exchange, Euronext, which connects the stock exchanges of Amsterdam, Brussels, and Paris, the Portuguese stock exchange BVLP (Bolsa de Valores de Lisboa e Porto), and the London International Financial Futures and Options Exchange (LIFFE). Many countries including China are currently in the process of establishing domestic electronic trading platforms.

There are at least four advantages of using electronic trading platforms instead of traditional markets. First, electronic platforms provide more information during the process of trading. Second, electronic trading platforms are more transparent than conventional markets. Security prices are stipulated according to well-specified rules while market makers in conventional markets have a considerable influence on the price determination. This 'black-box' argument applies in particular for prices which are negotiated among dealers. Third, transaction costs of electronic trading platforms are lower than those of conventional floor markets. Moreover, they usually provide more liquidity as their transaction volume is usually higher than that of conventional markets.

Despite the popularity of electronic trading systems, little is known about a microeconomic foundation of investment strategies that are adapted to these markets, e.g., see Harris (1990) and Huang & Stoll (1991). Electronic security markets have attracted only relatively little attention in the theory of financial markets. The classical approach of the literature derives asset prices from intertemporal equilibrium conditions assuming that asset markets clear and expectations are always rational (e.g., see Ingersoll (1987), Pliska (1997), or LeRoy & Werner 2001). Böhm, Deutscher & Wenzelburger (2000) pointed out that this classical approach involves two implicit conditions: One for the assumption of market clearing in each trading period and the other for the assumption of rational expectations. The latter condition may be replaced by introducing the notion of a forecasting rule along with the concept of a *perfect* forecasting rule as an operational concept for rational expectations. Instead of reducing the expectations feedback to a consistency assumption between expectations and realizations, this concept leaves enough explanatory room for diverse and non-rational as well as rational beliefs of traders.

The market-clearing condition, however, still remains an unresolved conceptional problem as it is easy to construct an asset market for which market-clearing prices do not exist generically, e.g., see Böhm & Chiarella (2000). This theoretical insight provides the motivation to study the price and transaction mechanisms of ‘real’ financial markets which handle a great diversity of traders every day. A prominent example for such markets is an electronic market in which buyers and sellers interact through a computer system.

One of the well-established electronic markets is operated by the German stock exchange (Deutsche Börse) in Frankfurt, Germany. Deutsche Börse operates an electronic trading platform called *Xetra*. *Xetra* is an order-driven system in which agents can trade securities by entering certain order specifications through a computer interface. A description of this interface along with the trading rules may be found in a brochure distributed by Gruppe Deutsche Börse (01.07.2003). In Xetra *ask orders* to buy and *bid orders* to sell securities are either traded continuously or by multi-unit double auctions which take place several times during a trading day. Despite the clarity of the Xetra’s trading rules, financial markets literature so far has provided only little understanding of the nature of price formation in electronic markets and its implication for possible investment strategies. The price mechanism of electronic stock markets has intuitively been described in Sharpe, Alexander & Bailey (1999), however without formal rigor.

This paper provides a first formalization of the price and allocation mechanism of limit orders processed by auctions in Xetra. A primary goal of the present paper is to provide a microeconomic foundation of investment strategies for Xetra auctions. An auction in Xetra is composed of three phases: a call phase, a price determination phase, and an order book balancing phase. During the call phase, traders may enter ask orders and bid orders into the Xetra System. Orders will be tagged with a time priority index and collected in an order book. There is one order book for each security. The call phase has a random end after a fixed minimum time period. It is followed by the price determination phase in which the auction price is determined. As soon as the auction price has been determined, orders are matched and transactions are carried out. If not all of the orders can be fully executed, the surplus is offered again to traders in the order book balancing phase. At the end of the auction process, all orders which were not or only partially executed are forwarded to the next possible trading.

2 Call phase

We describe the auction of a single security by the Xetra System. As mentioned above, a Xetra auction consists of three phases, a call phase, a price determination phase and order book balancing phase. During the call phase, Xetra collects all asks and bids for a security quoted by traders in an order book, labeled with a time-priority index. Assume that there are I traders, indexed by $i \in \{1, \dots, I\}$, who submit bids and J traders indexed by $j \in \{1, \dots, J\}$ who submit asks. For simplicity, assume also that each trader submits only one order such that $\{1, \dots, I\}$ is also the index set for bids and $\{1, \dots, J\}$ the index set for asks.

Orders in Xetra will be executed according to price/time priority, such that the time index attached to an order determines its execution priority in the order book. To formulate the model, we first focus on a convenient presentation of individual bids and asks or, in other words, on individual demand and supply schedules.

2.1 Demand-to-buy schedule (bids)

Each bid i consists of a price-quantity pair (a_i, d_i), where d_i is the amount that trader i wants to buy and a_i is the highest price per unit of security that she is willing to buy. In other words, a_i is the highest possible price at which the bit (a_i, d_i) may be executed. A bid may be represented as an individual demand function as follows. If $1_{A_i^D}(p)$ denotes a characteristic function of the compact interval $A_i^D = [0, a_i]$ such that

$$1_{A_i^D}(p) := \begin{cases} 1 & \text{when} \quad p \in A_i^D, \\ 0 & \text{when} \quad p \in \mathbb{R}_+ \setminus A_i^D, \end{cases}$$

we define the individual demand function that represents a bid (a_i, d_i), $i = 1, \dots, I$ by the step function

$$\mathcal{L}_i^D : \mathbb{R}_+ \to \mathbb{R}_+, \quad p \mapsto d_i 1_{A_i^D}(p). \tag{1}$$

The aggregate demand function is defined as the sum of the individual demand functions:

$$\Phi_D : \mathbb{R}_+ \to \mathbb{R}_+, \quad p \mapsto \sum_{i=1}^{I} \mathcal{L}_i^D(p). \tag{2}$$

After a suitable renumbering we may assume without loss of generality that $a_I > \ldots > a_2 > a_1 > 0$. Then we obtain the following lemma.

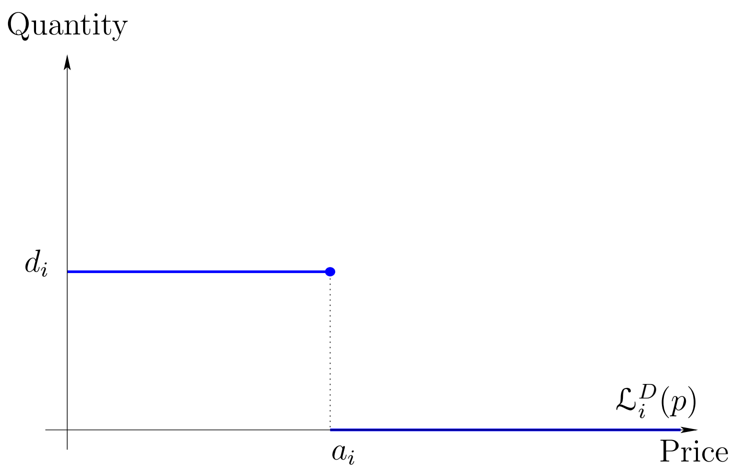

Figure 1: Individual demand function for bid (a_i, d_i).

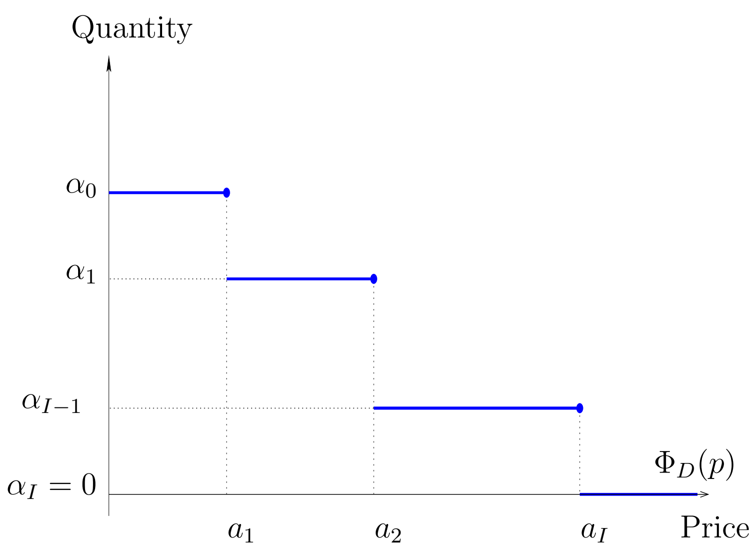

Figure 2: Aggregate demand function.

Lemma 1. *Let $a_I > \ldots > a_2 > a_1 > 0$. The aggregate demand function $\Phi_D(p)$ is non-increasing and takes the form:*

$$\Phi_D(p) = \sum_{i=0}^{I} \alpha_i 1_{A_i}(p), \quad p \in \mathbb{R}_+, \tag{3}$$

where $\alpha_i := \sum_{k=i+1}^{I} d_k$, $i = 0, 1, \ldots, I-1$, $\alpha_I := 0$ and $A_0 := [0, a_1]$, $A_i := (a_i, a_{i+1}]$, $i = 1, \ldots, I-1$, $A_I := (a_I, +\infty)$.

Proof. $\{A_0, \ldots, A_I\}$ is by construction a partition of $\mathbb{R}_+$. Let $i_* \in \{0, \ldots, I-1\}$

be arbitrary but fixed. Then $p \in A_{i_*}$ implies that all bids $i = i_* + 1, \ldots I$ are executable. The corresponding aggregate bids volume is $\alpha_{i_*} = \sum_{k=i_*+1}^{I} d_k$. $p \in A_I$ implies that no bids can be executed because $p > a_I$. The corresponding aggregate bids volume is $\alpha_I = 0$. This establishes the specific presentation of the aggregate demand function. Since $\alpha_0 > \alpha_1 > \ldots > \alpha_I$, Φ_D is non-increasing. $\square$

2.2 Supply-to-sell schedule (asks)

Each ask j consists of a price-quantity pair (b_j, s_j), where s_j is the amount that trader j wants to sell and b_j is the lowest price per unit of the security that she is willing to sell. In other words b_j is the lowest possible price at which the ask (b_j, s_j) may be executed. Analogous the bids, any ask may be represented as an individual supply function as follows. If $1_{B_j^S}(p)$ denotes a characteristic function of the interval $B_j^S = [b_j, +\infty)$, the individual supply function that represents an ask (b_j, s_j), $j = 1, \ldots, J$ is given by the step function:

$$\mathfrak{L}_j^S : \mathbb{R}_+ \to \mathbb{R}_+, \quad p \mapsto s_j 1_{B_j^S}(p). \tag{4}$$

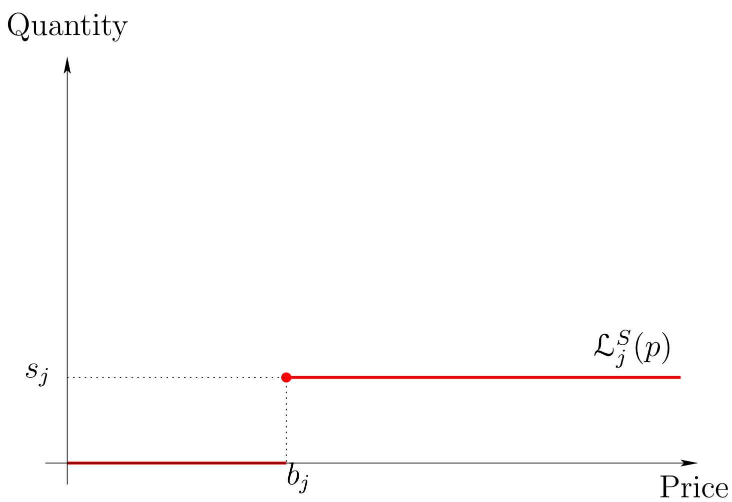

Figure 3: Individual supply function for (b_j, s_j).

The aggregate supply function is defined as the sum of the individual supply functions

$$\Phi_S : \mathbb{R}_+ \to \mathbb{R}_+, \quad p \mapsto \sum_{j=1}^{J} \mathfrak{L}_j^S(p). \tag{5}$$

Without loss of generality, let $b_J > \ldots > b_2 > b_1 > 0$. Then we obtain the following lemma:

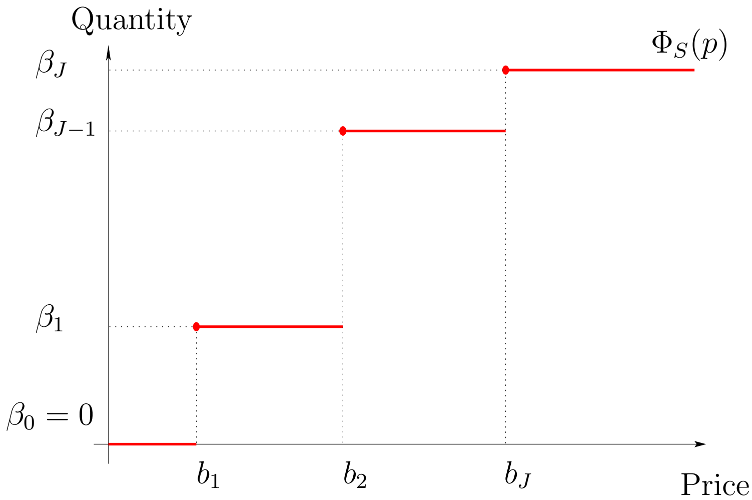

Figure 4: Aggregate supply function.

Lemma 2. *Let $b_J > \ldots > b_2 > b_1 > 0$. The aggregate supply function $\Phi_S(p)$ is non-decreasing and takes the form:*

$$\Phi_S(p) = \sum_{j=0}^{J} \beta_j 1_{B_j}(p) \tag{6}$$

where $\beta_0 := 0$, $\beta_j := \sum_{k=1}^{j} s_k$, for $j = 1, \ldots, J$;
and $B_0 := [0, b_1)$, $B_j := [b_j, b_{j+1})$, for $j = 1, \ldots, J-1$, $B_J := [b_J, +\infty)$.

Proof. $\{B_0, \ldots, B_J\}$ is by construction a partition of $\mathbb{R}_+$. $p \in B_0$ implies that no asks can be executable because $p < b_1$. The corresponding aggregate asks volume is $\beta_0 = 0$. Let $j_* \in \{1, \ldots, J\}$ be arbitrary but fixed. Then $p \in B_{j_*}$ implies that all bids $j = 1, \ldots, j_*$ are executable. The corresponding aggregate asks volume is $\beta_{j_*} = \sum_{k=1}^{j_*} s_k$. This establishes the specific presentation of the aggregate supply function. Since $\beta_J > \ldots > \beta_1 > \beta_0$, Φ_S is non-decreasing. $\square$

3 Price determination phase

The call phase stops with a random end and is followed by the price determination phase during which the auction price and the transaction are determined. In this phase the order book is closed and no new orders will be accepted. The status of the order book is then given by a collection of bids (a_i, d_i), $i = 1, \ldots, I$ and asks (b_j, s_j), $j = 1, \ldots, J$ including the corresponding time-priority indices which will be introduced later. In order to describe how he auction price determined in the price determination phase, it is useful to represent the order book by its associated aggregate demand and aggregate supply functions. An auction price determined by Xetra has to obey two principles. First, it has to allow for the highest order volume that can possibly be executed. Second, it has to be such that the surplus of non-executable orders is minimal. The rules according to which this auction price is stipulated are found on on page 32 in the brochure by Gruppe Deutsche Börse (01.07.2003):

> A price which allows for the highest executable order volume and the lowest surplus is called a candidate price.
>
> *Rule 1.* The auction price is the candidate price if there is only one candidate price.
>
> *Rule 2.* If there is more than one candidate price, then there are two cases:
>
>> *Rule 2.1.* If the surplus for prices satisfying Rule 1 is on the demand side, then the auction price is stipulated as the highest candidate price.
>>
>> *Rule 2.2.* If the surplus for the prices satisfying Rule 1 is on the supply side, then the auction price is stipulated as the lowest candidate price.
>
> *Rule 3.* If Rule 1 and Rule 2 can not determine a unique auction price, a certain reference price P_{ref} designated by *Xetra* is included as an additional criterion. There are three cases with the reference price included.
>
>> *Rule 3.1.* The auction price is the highest candidate price if the reference price is higher than the highest candidate price.
>>
>> *Rule 3.2.* The auction price is the lowest candidate price if the reference price is lower than the lowest candidate price.
>>
>> *Rule 3.3.* The auction price is equal to the reference price if the reference price lies between the highest candidate price and the lowest candidate price.
>
> *Rule 4.* If Rule 1 to Rule 3 fail, there exists no auction price.

Notice that Rule 1 and Rule 2 do no apply, if there exists an excess supply for one set of candidate prices and an excess demand for another set of candidate prices or if there is zero surplus. Rule 4 implies that there could be no executable order volume in *Xetra* such that no auction price exists. Only after an auction price has been determined can the allocation mechanism be formulated. Thus, we first formalize the price mechanism and then the allocation mechanism. In doing so we first introduce the concept of an *executable order volume* and a *surplus* in Xetra.

3.1 Executable order volume and surplus

Let $p \in \mathbb{R}_+$ be some arbitrary price such that aggregate demand $\Phi_D(p)$ may be unequal to aggregate supply $\Phi_S(p)$. Then only the minimum of $\Phi_D(p)$ and $\Phi_S(p)$ could possibly be traded. The quantity which can be traded will henceforth be called *executable order volume* and is defined by

$$\Phi_V : \mathbb{R}_+ \to \mathbb{R}_+, \quad p \mapsto \min\{\Phi_D(p), \ \Phi_S(p)\}. \tag{7}$$

The function (7) will also be referred as the trading-volume function. The highest executable order volume V_{max} is the maximum value of the trading-volume function and given by

$$V_{max} := \max \{\Phi_V(p) \mid p \in \mathbb{R}_+\}.$$

Notice that V_{max} exists and is finite: the image of the trading-volume function Φ_V is a finite set because the images of Φ_D and Φ_S have finitely many values. The set of *volume-maximizing prices* is defined by

$$\Omega := \{p \in \mathbb{R}_+ \mid \Phi_V(p) = V_{max}\}.$$

In other words, each price $p \in \Omega$ allows the executable order volume be maximal. The excess demand function is, as usual, defined by

$$\Phi_Z(p) : \mathbb{R}_+ \to \mathbb{R}, \quad p \mapsto \Phi_D(p) - \Phi_S(p). \tag{8}$$

The absolute value of the excess demand $|\Phi_Z(p)|$ is called *surplus* in Xetra.

3.2 Price mechanism

Since V_{max} is well defined, we may define the best bid price $\overline{p}$ and the best ask price $\underline{p}$ conditional on V_{max} by

$$\overline{p} := \max\{p \in \mathbb{R}_+ \mid \Phi_D(p) \geq V_{max}\}, \tag{9}$$

$$\underline{p} := \min\{p \in \mathbb{R}_+ \mid \Phi_S(p) \geq V_{max}\}. \tag{10}$$

Notice that $\overline{p}$ and $\underline{p}$ are well defined since $A_i, \ i = 1, \ldots, I$ are right closed and $B_j, j = 1, \ldots, J$ are left closed intervals. If the best bid price $\overline{p}$ is greater than the best ask price $\underline{p}$, then we say that the order book is crossed implying that an executable transactions exist. On the other hand, if $\overline{p} < \underline{p}$, then the order book is uncrossed and no transactions are executable. We have the following lemma:

Lemma 3. *If $V_{max} > 0$, then $\overline{p} \geq \underline{p}$ and the order book is crossed.*

The proof of Lemma 3 is provided in the appendix. Lemma 3 shows that executable order volume can be maximized, if the order book is crossed. The set of volume-maximizing prices Ω takes the following form:

Proposition 1. *If $V_{max} > 0$, then $\Omega = [\underline{p}, \overline{p}]$.*

Proof. Since $\Phi_S(p)$ is a non-decreasing function and $\Phi_D(p)$ a non-increasing function, we have

$$\Phi_S(p) \;\geq\; \Phi_S(\underline{p}) \;\geq\; V_{max} \quad \text{for } p \;\geq\; \underline{p},$$

and

$$\Phi_D(p) \;\geq\; \Phi_D(\overline{p}) \;\geq\; V_{max} \quad \text{for } p \;\leq\; \overline{p}.$$

By definition of V_{max}, this implies

$$\Phi_V(p) = \min\{\Phi_D(p), \Phi_S(p)\} \geq V_{max} \quad \text{for } p \in [\underline{p}, \overline{p}].$$

Thus we have $\Phi_V(p) = V_{max}$ for $p \in [\underline{p}, \overline{p}]$ thus $[\underline{p}, \overline{p}] \subset \Omega$.

Now let $p \in \Omega$ be arbitrary. By definition of Ω and $\Phi_V(p)$, we have $\Phi_S(p) \geq V_{max}$ implying $p \geq \underline{p}$ and $\Phi_D(p) \geq V_{max}$ implying $p \leq \overline{p}$. Thus $\Omega \subset [\underline{p}, \overline{p}]$ and hence $\Omega = [\underline{p}, \overline{p}]$. □

As can be seen from Proposition 1, there could be more than one volume-maximizing price, if $V_{max} > 0$. Therefore, additional selection criteria have to be applied, in order to determine a unique auction price from the set of volume-maximizing prices $[\underline{p}, \overline{p}]$. The following theorem formalizes the determination of an auction price in *Xetra*, applying the above cited matching rules.

Theorem 1. *If $V_{max} > 0$, then*

$$P_{Xetra} = \begin{cases} \overline{p} & \text{if } \;\Phi_Z(\overline{p}) > 0, \\ \underline{p} & \text{if } \;\Phi_Z(\underline{p}) < 0, \\ \max\{\underline{p}, \min\{P_{ref}, \overline{p}\}\} & \text{otherwise.} \end{cases} \tag{11}$$

Proof. Since $V_{max} > 0$, only Rule 1 to Rule 3 need to be considered. By Proposition 1, the auction price in Xetra must lie in $\Omega = [\underline{p}, \overline{p}]$, the set of volume maximizing prices. Using excess demand function, Rule 2.1 states that $P_{Xetra} = \overline{p}$, if $\Phi_Z(p)$ for all $p \in [\underline{p}, \overline{p}]$. Since $\Phi_Z(\overline{p}) > 0$ implies

$$\Phi_D(p) \geq \Phi_D(\overline{p}) > \Phi_S(\overline{p}) \geq \Phi_S(p), \quad \text{for all } p \leq \overline{p},$$

Rule 2.1 is equivalent to $P_{Xetra} = \underline{p}$, if $\Phi_Z(\overline{p}) > 0$.

On the other hand, Rule 2.2 states that $P_{Xetra} = \underline{p}$ if $\Phi_Z(p) < 0$ for all $p \in [\underline{p}, \overline{p}]$. By an analogous reasoning, Rule 2.2 is equivalent to $P_{Xetra} = \underline{p}$, if $\Phi_Z(\underline{p}) < 0$.

If the surplus is neither on the demand nor on the supply side, Rule 2 cannot be satisfied and a reference price P_{ref} comes into play. According to Rule 3, we have $P_{Xetra} = \overline{p}$ if $P_{ref} \geq \overline{p}$ (Rule 3.1), $P_{Xetra} = \underline{p}$ if $\underline{p} \geq P_{ref}$ (Rule 3.2), or $P_{Xetra} = P_{ref}$ when $\overline{p} \geq P_{ref} \geq \underline{p}$ (Rule 3.3). This proves the theorem. □

Theorem 1 formalizes the price mechanism in Xetra. Given an order book with bids $(a_i, d_i)_{i\in\mathfrak{I}}$ and asks $(b_j, s_j)_{j\in\mathfrak{J}}$, a unique auction price P_{Xetra} is determined by Theorem 1. The price mechanism is illustrated in Figure 5. Notice that Ω is reduced to one point, if $\underline{p} = \overline{p}$. In this case there exists only one volume maximizing price $P_{Xetra} = \underline{p} = \overline{p}$ which is market clearing such that the surplus is zero. After determining P_{Xetra}, we formulate the allocation mechanism.

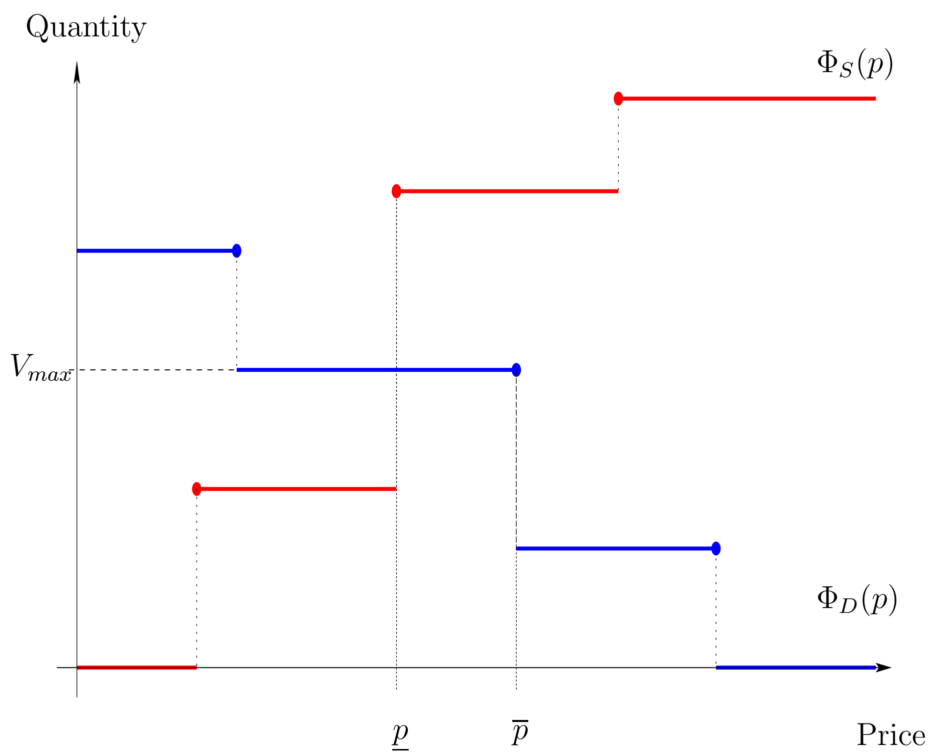

Figure 5: Price mechanism in *Xetra*.

3.3 Allocation Mechanism

When an order is submitted to the order book, it is labeled with a time tag which determines the time priority with which it is executed. The time tag attached to each order determines the ranking of execution in the order book. Given a Xetra Price, executable orders are executed by time priority.

Denote the execution priority of bid i by $\iota_d(i)$, and the execution priority of bid j by $\iota_s(j)$, respectively, where $\iota_d(i) \in \{1, \dots, I\}$ and $\iota_s(j) \in \{1, \dots, J\}$. The position in the execution sequence of trader (bid) i then is $\iota_d(i)$, which implies that there are $\iota_d(i) - 1$ bids which will be executed before i. Analogously, there are $\iota_s(j) - 1$ asks which will be executed before j.

The final transaction for each order is highly affected by its position in the execution sequence since Xetra applies the rule of ***First Come First Serve (FCFS)*** for the order execution.[1] Given the fixed ranking of the execution sequence, a bid i will not be executed until all higher ranked bids are executed. The maximum feasible quantity that trader i can get is therefore the quantity which higher ranked traders have left over, that is, the positive difference between the highest executable order volume $\Phi_V(P_{Xetra})$ and the aggregate executed order volume before bid i is handled. The maximum feasible quantity for trader i is given by

$$\bar{\mathfrak{L}}_i^D(P_{Xetra}) := \max\Big\{0, \Phi_V(P_{Xetra}) - \sum_{m=1}^{\iota_d(i)-1} \mathfrak{L}^D_{\iota_d^{-1}(m)}(P_{Xetra})\Big\}, \tag{12}$$

where $\iota_d^{-1}(m)$ denotes the bid in position m. If the individual demand $\mathfrak{L}_i^D(P_{Xetra})$ of trader i is less than $\bar{\mathfrak{L}}_i^D(P_{Xetra})$, then i is fully served and she receives

$$\mathfrak{L}_i^D(P_{Xetra}) = \begin{cases} d_i & \text{if } \ P_{Xetra} \in [0, a_i], \\ 0 & \text{otherwise.} \end{cases}$$

If $\bar{\mathfrak{L}}_i^D(P_{Xetra})$ is smaller than $\mathfrak{L}_i^D(P_{Xetra})$ trader i can only be partially executed. The final transaction is $\bar{\mathfrak{L}}_i^D(P_{Xetra})$ and trader i is rationed. Denoting the final transaction of trader i by X_i^d, we have

$$X_i^d(P_{Xetra}) := \min\Big\{\mathfrak{L}_i^D(P_{Xetra}), \bar{\mathfrak{L}}_i^D(P_{Xetra})\Big\}, \quad i = 1, \dots, I. \tag{13}$$

For the supply side, the maximum feasible quantity for any arbitrary trader j is the positive difference between the executable order volume $\Phi_V(P_{Xetra})$ and the aggregate executed order volume before ask j is handled. The maximum feasible quantity for trader j is given by

$$\bar{\mathfrak{L}}_j^S(P_{Xetra}) := \max\Big\{0, \Phi_V(P_{Xetra}) - \sum_{n=1}^{\iota_s(j)-1} \mathfrak{L}^S_{\iota_s^{-1}(n)}(P_{Xetra})\Big\}, \tag{14}$$

where $\iota_s^{-1}(n)$ denotes the ask in position n. Denoting the final transaction for trader j by X_j^s, by an analogous reasoning, we have

$$X_j^s(P_{Xetra}) := \min\Big\{\mathfrak{L}_j^S(P_{Xetra}), \bar{\mathfrak{L}}_j^S(P_{Xetra})\Big\}, \quad j = 1, \dots, J, \tag{15}$$

[1]FCFS is equivalent to the rule of *First In First Out (FIFO)*.

where

$$\mathcal{L}_j^S(P_{Xetra}) = \begin{cases} s_j & \text{if} \quad P_{Xetra} \in [b_j, +\infty) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the aggregate final transaction of bids is equal to aggregate final transaction of asks, that is,

$$\sum_{i=1}^{I} X_i^d(P_{Xetra}) = \sum_{j=1}^{J} X_j^s(P_{Xetra}) = \Phi_V(P_{Xetra}) = V_{max}.$$

Summarizing, the Xetra allocation mechanism for any given Xetra price P_{Xetra} is given by

$$\begin{aligned} X_i^d(P_{Xetra}) &:= \min\Big\{\mathcal{L}_i^D(P_{Xetra}), \bar{\mathcal{L}}_i^D(P_{Xetra})\Big\}, \quad i = 1, \dots, I \\ X_j^s(P_{Xetra}) &:= \min\Big\{\mathcal{L}_j^S(P_{Xetra}), \bar{\mathcal{L}}_j^S(P_{Xetra})\Big\}, \quad j = 1, \dots, J. \end{aligned} \tag{16}$$

Also notice that the market-clearing situation is included as a special case in which for all traders the individual demand $\mathcal{L}_i^D(P_{Xetra})$ happens to be equal to the final transaction $X_i^d(P_{Xetra})$ and the individual supply $\mathcal{L}_j^S(P_{Xetra})$ happens to be equal to the final transaction $X_j^s(P_{Xetra})$, that is:

$$\begin{aligned} \mathcal{L}_i^D(P_{Xetra}) &= X_i^d(P_{Xetra}), \quad i = 1, \dots, I \\ \mathcal{L}_j^S(P_{Xetra}) &= X_j^s(P_{Xetra}), \quad j = 1, \dots, J. \end{aligned} \tag{17}$$

3.4 Properties of the Xetra allocation mechanism

The Xetra allocation mechanism has some well-known properties of rationing mechanisms, found in Benassy (1982) and Böhm (1989).

Voluntary Exchange. The property of voluntary exchange states that no trader is forced to trade more than he claims. Intuitively, this property holds in Xetra since traders can never trade a quantity that she did not claim. More formally, (16) satisfies this property, because for all i, j,

$$\begin{aligned} X_i^d(P_{Xetra}) &\leq \mathcal{L}_i^D(P_{Xetra}), \\ X_j^s(P_{Xetra}) &\leq \mathcal{L}_j^S(P_{Xetra}). \end{aligned}$$

The Short-side Rule. An allocation mechanism is called *'efficient'*, or frictionless, if no mutually advantageous trade can be carried out from the transaction attained. This implies that traders on the short side of a market will realize their desired transactions. [2] Combining the property of voluntary exchange and market efficiency, we obtain the so-called *'short-side rule'* stating that traders on the short side will realize all of their effective demand (supply). Formally the Xetra allocation mechanism (16) satisfies the short-side rule, if

$$\Phi_D(P_{Xetra}) \geq \Phi_S(P_{Xetra}) \quad \Rightarrow \quad X_j^s(P_{Xetra}) = \mathcal{L}_j^S(P_{Xetra}), \quad \forall j; \tag{18}$$

$$\Phi_D(P_{Xetra}) \leq \Phi_S(P_{Xetra}) \quad \Rightarrow \quad X_i^d(P_{Xetra}) = \mathcal{L}_i^D(P_{Xetra}), \quad \forall i. \tag{19}$$

By analogy, we only verify condition (18). Clearly, $\Phi_D(P_{Xetra}) \geq \Phi_S(P_{Xetra})$ implies $\Phi_V(P_{Xetra}) = \Phi_S(P_{Xetra})$ and hence

$$\Phi_S(P_{Xetra}) - \sum_{n=1}^{\iota_s(j)-1} \mathcal{L}_{\iota_s^{-1}(n)}^S(P_{Xetra}) \geq \mathcal{L}_j^S(P_{Xetra}), \quad j = 1, \ldots, J.$$

Therefore (18) holds.

Anonymity. Loosely speaking, a rationing mechanism is called anonymous, if any two traders with the same characteristics attain the same final transaction. In the Xetra case, for any two traders i and i' with the same time priority and with the same limit order $\mathcal{L}_i^D(P_{Xetra}) = \mathcal{L}_{i'}^D(P_{Xetra})$ attain the same final transaction $X_i^d(P_{Xetra}) = X_{i'}^d(P_{Xetra})$. The same holds true for the supply side. Hence, the Xetra allocation mechanism satisfies anonymity in that sense.

Notice, however, that the time priority concept of Xetra might by subject to various influences which are beyond the control of the system in the sense of queuing theory. In view of stochastic rationing mechanisms (Weinrich 1984), then anonymity would hold only, if traders with the same orders attain the same final transactions on average.

Manipulability. An allocation mechanism is called *non-manipulable* in quantity if the trader, when he is rationed, faces a bound to his transaction which depends solely on the quoted quantities of the other traders which he can not manipulate. It is called manipulable in quantity if the trader can, when he is rationed, increase his final transaction by increasing his quoted quantity. Intuitively, non-manipulability implies that the individual quantity quoted by a trader has no impact on his maximum feasible quantity and vice versa.

[2] Benassy (1982) states that the 'short' side of a market is that side where the aggregate transaction is smallest. It is thus the demand side if there is excess supply, the supply side if excess demand exists. The other side is called the 'long' side.

In Xetra, traders face upper bounds $\bar{\mathcal{L}}_i^D(P_{Xetra})$ and $\bar{\mathcal{L}}_j^S(P_{Xetra})$ for their final transactions, should they be rationed. In the case of excess demand $\Phi_D(P_{Xetra}) > \Phi_S(P_{Xetra})$, only traders on the demand side will be rationed. The maximum feasible quantity of trader i is

$$\bar{\mathcal{L}}_i^D(P_{Xetra}) = \max\{0, \Phi_S(P_{Xetra}) - \sum_{m=1}^{\iota_d(i)-1} \mathcal{L}_{\iota_d^{-1}(m)}^D(P_{Xetra})\}, \quad i = 1, \ldots, I$$

which is independent of his individual quantity $\mathcal{L}_i^D(P_{Xetra})$.

Analogously, in the of case excess supply $\Phi_S(P_{Xetra}) > \Phi_D(P_{Xetra})$, only traders on the supply are rationed. The maximum feasible quantity of trader j is

$$\bar{\mathcal{L}}_j^S(P_{Xetra}) = \max\{0, \Phi_D(P_{Xetra}) - \sum_{n=1}^{\iota_s(j)-1} \mathcal{L}_{\iota_s^{-1}(n)}^S(P_{Xetra})\}, \quad j = 1, \ldots, J$$

which is independent of her individual quantity $\mathcal{L}_j^S(P_{Xetra})$.

At first sight, this observation seems to imply that the Xetra mechanism is non-manipulable in the above sense of classical rationing theory. However, since traders do influence the price by submitted their limit orders, matters are more complicated then the classical case which prices are assumed to be fixed.

To be continued ...

4 Conclusions

This paper provides a first formalization of the price and allocation mechanism of limit orders processed by auctions in Xetra. This approach should be seen as a first step towards a better understanding of electronic systems. It provides a basis for the development and analysis of trading strategies in view of a more complete understanding of the properties and the role of electronic markets. A primary goal of the present paper **will be** to develop a microeconomic foundation of investment strategies for electronic trading platforms and to establish a theoretical framework for the dynamics of prices and allocations generated by these platforms.

To be continued ..

A Appendix

A.1 Proof of Lemma 3

We will now prove Lemma 3. Clearly, $V_{max} > 0$ must be equal to some α_{i_0} or to some β_{j_0}. Therefore, we have two cases.

CASE I: $V_{max} = \alpha_{i_0} > 0$.

We show that there exists some $j^* \in \{1, \ldots, J\}$ such that

$$\beta_{j^*-1} < \alpha_{i_0} \leq \beta_{j^*}. \tag{20}$$

Since $V_{max} = \alpha_{i_0}$, there exists some $\tilde{p}$, such that $\Phi_D(\tilde{p}) \leq \Phi_S(\tilde{p})$ and

$$\Phi_V(\tilde{p}) = \min\ \{\Phi_D(\tilde{p}),\ \Phi_S(\tilde{p})\} = \Phi_D(\tilde{p}) = \alpha_{i_0} > 0.$$

Let $\beta_{\tilde{j}} = \Phi_S(\tilde{p})$. Notice that $\tilde{j} > 0$ since $\beta_{\tilde{j}} = \Phi_S(\tilde{p}) \geq \Phi_D(\tilde{p}) = \alpha_{i_0} > 0 = \beta_0$. Set $j^* = \min\{\tilde{j} \in \{1, \ldots, J\} \mid \beta_{\tilde{j}} \ \geq \ \alpha_{i_0}\ \}$. Since $\beta_{j^*-1} < \beta_{j^*}$, we have $\beta_{j^*-1} < \alpha_{i_0} \leq \beta_{j^*}$. This shows (20).

By the definition of $\Phi_D(p)$ and $\Phi_S(p)$, there exists $A_{i_0} = (a_{i_0},\ a_{i_0+1}]$ corresponding to α_{i_0} and $B_{j^*} = [b_{j^*}, b_{j^*+1})$ corresponding to β_{j^*}. Since $\Phi_D(p)$ is a non-increasing function, it follows from (9) that

$$\overline{p} = \max\{p \mid \Phi_D(p) \geq \alpha_{i_0}\} = \max\{\ p \mid p \in A_{i_0}\} = a_{i_0+1}.$$

Noticing that $\Phi_S(p)$ is a non-decreasing function and $\alpha_{i_0} \leq \beta_{j^*}$ from (20), (10) implies

$$\underline{p} = \min\{p \mid \Phi_S(p) \geq \alpha_{i_0}\} = \min\{p \mid \Phi_S(p) = \beta_{j^*}\} = \min\{p \mid p \in B_{j^*}\} = b_{j^*}.$$

This shows that $\overline{p}$ and $\underline{p}$ are well-defined.

We are now left to prove $\overline{p} \geq \underline{p}$, that is, to prove $a_{i_0+1} \geq b_{j^*}$. Assume on the contrary, that $a_{i_0+1} < b_{j^*}$. Since $0 < a_{i_0} < a_{i_0+1} < b_{j^*}$, we have $A_{i_0} = (a_{i_0}, a_{i_0+1}] \subset [0, b_{j^*})$ Since $\Phi_S(p) \in \{\beta_0, \beta_1, \ldots, \beta_{j^*-1}\}$ for $p \in [0, b_{j^*-1})$ and $\alpha_{i_0} > \beta_{j^*-1} > \ldots > \beta_0$, we have $\Phi_V(p) < \alpha_{i_0}$ for $p \in [0, b_{j^*-1})$. Since $A_{i_0} \subset [0, b_{j^*})$, this contradicts $V_{max} = \alpha_{i_0}$.

CASE II: $V_{max} = \beta_{j_0} > 0$.

We show that there exists some $i^* \in \{0, 1, \ldots, I-1\}$ such that

$$\alpha_{i^*+1} \ < \ \beta_{j_0} \ \leq \ \alpha_{i^*}. \tag{21}$$

Since $V_{max} = \beta_{j_0}$, there exists some $\tilde{p}$, such that $\Phi_S(\tilde{p}) \leq \Phi_D(\tilde{p})$ and

$$\Phi_V(\tilde{p}) = \min\ \{\Phi_D(\tilde{p}),\ \Phi_S(\tilde{p})\} = \Phi_S(\tilde{p}) = \beta_{j_0} > 0.$$

Let $\alpha_{\tilde{i}} = \Phi_D(\tilde{p})$. Notice that $\tilde{i} < I$ since $\alpha_{\tilde{i}} = \Phi_D(\tilde{p}) \geq \Phi_S(\tilde{p}) = \beta_{j_0} > 0 = \alpha_I$.

Let $i^* = \max\{\tilde{i} \in \{0, 1, \ldots, I-1\} \mid \alpha_{\tilde{i}} \ \geq \ \beta_{j_0}\ \}$. We have $\alpha_{i^*+1} \ < \ \beta_{j_0} \ \leq \ \alpha_{i^*}$ since $\alpha_{i^*} < \alpha_{i^*+1}$. This shows (21).

By the definition of $\Phi_D(p)$ and $\Phi_S(p)$, there exists $B_{j_0} = (b_{j_0},\ b_{j_0+1}]$ corresponding to β_{j_0} and $A_{i^*} = [a_{i^*}, a_{i^*+1})$ corresponding to α_{i^*}.

Since $\Phi_D(p)$ is a non-increasing function and $\alpha_{i^*+1} < \beta_{j_0} \leq \alpha_{i^*}$ from (21), (9) implies

$$\overline{p} = \max\ \{p \mid \Phi_D(p) \geq \beta_{j_0}\} = \max\{\ p \mid \Phi_D(p) = \alpha_{i^*}\} = \max\{\ p \mid p \in A_{i^*}\} = a_{i^*+1}.$$

Noticing that $\Phi_S(p)$ is a non-decreasing function, it follows from (10) that

$$\underline{p} = \min\ \{p \mid \Phi_S(p) \geq \beta_{j_0}\} = \min\{\ p \mid p \in B_{j_0}\} = b_{j_0}.$$

This shows that $\overline{p}$ and $\underline{p}$ are well-defined.

We are now left to prove $\overline{p} \geq \underline{p}$, that is, to prove $a_{i^*+1} \geq b_{j_0}$. Assume on the contrary, that $a_{i^*+1} < b_{j_0}$. Since $a_{i^*+1} < b_{j_0} < b_{j_0+1} < +\infty$, we have $B_{j_0} = (b_{j_0}, b_{j_0+1}] \subset (a_{i^*+1}, +\infty)$.

Since $\Phi_D(p) \in \{\alpha_{i^*+1}, \ldots, \alpha_I\}$ for $p \in (a_{i^*+1}, +\infty)$ and $\beta_{j_0} > \alpha_{i^*+1} > \ldots > \alpha_I$, we have $\Phi_V(p) < \beta_{j_0}$ for $p \in B_{j_0}$. Since $B_{j_0} \subset (a_{i^*+1}, +\infty)$, this contradicts $V_{max} = \beta_{j_0}$.

The proof of Lemma 3 follows from *CASE I* and *CASE II*. $\square$

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