Hölder continuity of the policy function approximation in the value function approximation

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Abstract

We show that given a value function approximation V of a strongly concave stochastic dynamic programming problem (SDDP), the associated policy function approximation is Hölder continuous in V.

Using this result, we obtain explicit error bounds for the approximations of the optimal policy function. The error bounds only depend on the primitive data of the problem. Neither differentiability of the return function nor interiority of solutions is required. Furthermore, similar error bounds are obtained when the maximization in the Bellman equation and the computation of the associated policy function are performed inexactly.

A stopping criterium for computational implementations is found using these error bounds and the contraction mapping defined from the SDPP.

Keywords: Stochastic dynamic programming problem, estimation of the policy function.

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1 Introduction

In stochastic dynamic programming problems (SDPP), Bellman's Principle of Optimality allows to define an estimated optimal policy function using a value function approximation. It also defines a contraction mapping which provides an efficient algorithm to approximate the value function with high precision. However, until now, only asymptotic convergence results (without numerical error bounds) were obtained for the sequence of policy function estimates found with this algorithm, (Christiano [4], Tauchen [14]). For example, Stokey and Lucas with Prescott [13](Theorem 9.9) established pointwise convergence (and uniform convergence if the domain is compact) to the optimal policy function under the assumption of strict concavity of the return function.

In this paper we prove that the policy function estimate obtained from the Bellman Principle of Optimality is a Hölder continuous function on the approximated value function. This Hölder continuity property allows us to obtain an explicit error bound for the estimated policy function.

There exist other approaches for obtaining good estimates for the optimal policy function. For example, the Euler equation grid method (Baxter *et al.* [1], Coleman [2, 3]), the parameterized expectations method (Marcet and Marshall [7]) and projection methods (Judd [5]). Again, only asymptotic convergence to the optimal policy function was proved for these methods.

Bounds for the distance between the optimal policy function (of the original problem) and the *exact* optimal policy function of a discretized (piecewise linear) version of the problem were obtained by Santos and Vigo-Aguiar [12]. In Santos [11], Euler equation residuals were used to obtain error bounds for an approximated policy function. Either an assumption on the repeated iterations of the approximated policy function (condition NDIV) or a bound on the second derivative of the return function evaluated at the optimal policy function was necessary. In both papers, [11] and [12], existence of interior solutions, twice differentiablity of the return function and strong concavity of this function with respect to the second variable were also required.

The error bounds presented in this paper only require boundedness of the return function and its strong concavity in *either* the first *or* the second variable. Neither differentiability of the return function nor existence of interior solutions is required to obtain our results.

We also prove the robustness of error bounds formulae by considering the use of an inexact operator defining the contractive method or inexact solutions in each maximization process.

Our result has a practical consequence: if the contraction method (defined by the value function iterations) is used to approximate the policy function then the number of iterations required to attain a given precision may be computed in advance, using only some primitive data of the problem.

This paper is organized into five sections. Section 2 describes the framework and the hypotheses that we will consider. Section 3 states the Hölder continuity of the policy function approximation and as a consequence, an error bound for the approximated policy function is provided. Section 4 shows the robustness of the error bounds under small numerical errors. Conclusions are given in Section 5 and the proofs are in the appendix.

2 The framework

The stochastic dynamic programming problem (SDPP) is defined using the following elements: the set of values for the endogenous state variables $X \subset \mathbb{R}^l$ (which is a convex Borel set), the set of values for the exogenous shocks $Z \subset \mathbb{R}^k$ (which is a compact set); both are measurable spaces with their σ-algebras denoted by $\mathcal{X}$ and $\mathcal{Z}$ respectively. The evolution of the stochastic shocks is given by the transition function Q defined on $(Z, \mathcal{Z})$ with the Feller property. A (measurable) set $\Omega \subset X \times X \times Z$ describing the feasibility of decisions, *i.e.* if $(x, z) \in X \times Z$ are the current values of the state variable and the shock then $y \in X$ is feasible for the next period if and only if $(x, y, z) \in \Omega$. From this we can define the correspondence $\Gamma : X \times Z \rightarrow \mathbb{R}^l$ by $\Gamma(x, z) = \{y \in X;\ (x, y, z) \in \Omega\}$. The one-period return function $F : \Omega \rightarrow \mathbb{R}$ is such that $F(x, y, z)$ is the current return if y is chosen for the next period from (x, z). The discount factor is $\beta \in (0, 1)$. With all these elements, the SDPP is to find a sequence of contingent plans $(\hat{x}_t)_{t\geq 1}$ (where for all $t \geq 1$, $\hat{x}_t : Z^t \rightarrow X$ is a measurable function) such that it solves the following maximization:

$$\begin{aligned} v(x_0, z_0) = \quad & \text{Max} \sum_{t=0}^{\infty} \int_{Z^t} \beta^t F(x_t, x_{t+1}, z_t)\, Q^t(z_0, dz^t) \\ & \text{subject to } (x_t, x_{t+1}, z_t) \in \Omega \text{ for all } t \geq 0 \\ & (x_0, z_0) \in X \times Z \text{ given} \end{aligned}$$

Let us recall the definitions of strong concavity (see [8]):

Definition 2.1 *Let $f(x)$ be defined on a convex set $X \subseteq \mathbb{R}^n$, and $\alpha > 0$.*

The function $f(x)$ is α-concave if $f(x) + (1/2)\alpha\|x\|^2$ is concave on X, or, equivalently, if for all $x_1, x_2 \in X$ and $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2) + (\alpha/2)t(1-t)\|x_1 - x_2\|^2.$$

Definition 2.2 *Let $U(x, y)$ be defined on a convex set $D \subseteq \mathbb{R}^n \times \mathbb{R}^n$, and $\alpha > 0$.*

The function $U(x,y)$ is α_x-concave on D if $U(x,y)+\alpha/2\,\|x\|^2$ is concave in D, or, equivalently, if for all $(x_1,y_1),(x_2,y_2)\in D$ and $t\in[0,1]$

$$\begin{aligned} U(tx_1+(1-t)x_2, ty_1+(1-t)y_2) \ &\geq \ tU(x_1,y_1)+(1-t)U(x_2,y_2) \\ &\quad +(\alpha/2)t(1-t)\|x_1-x_2\|^2 \end{aligned}$$

The function $U(x,y)$ is α_y-concave on D if $U(x,y)+\alpha/2\,\|y\|^2$ is concave in D, or, equivalently, if for all $(x_1,y_1),(x_2,y_2)\in D$ and $t\in[0,1]$

$$\begin{aligned} U(tx_1+(1-t)x_2, ty_1+(1-t)y_2) \ &\geq \ tU(x_1,y_1)+(1-t)U(x_2,y_2) \\ &\quad +(\alpha/2)t(1-t)\|y_1-y_2\|^2 \end{aligned}$$

An α-concave function is also called a strongly concave function, with modulus α. Analogous expressions are used for α_x and α_y concave functions. The following hypotheses will be used in this work.

Hypothesis 1. The correspondence Γ is nonempty, compact-valued, continuous and for all $x, x' \in X$, $z\in Z$ and $t\in[0,1]$ it satisfies:

$$t\Gamma(x,z)+(1-t)\Gamma(x',z)\subset\Gamma(tx+(1-t)x',z).$$

Hypothesis 2. The function F is bounded, continuous and there exists $\eta_1>0$ such that, for each $z\in Z$, $(x,y)\mapsto F(x,y,z)$ is $(\eta_1)_x$-concave.

As an alternative to Hypotheses 2, we will use

Hypothesis 3. The function F is bounded, continuous and there exists $\eta_2>0$ such that, for each $z\in Z$, $(x,y)\mapsto F(x,y,z)$ is $(\eta_2)_y$-concave.

Hypothesis 1 as well as boundedness and continuity of F are quite general in models with bounded returns and technologies with non-increasing returns. A discussion on the hypotheses of strong concavity of the return function is presented at the end of Section 3.

Under these assumptions, the value function v is well-defined and satisfies $\|v\|_\infty \leq \|F\|_\infty/(1-\beta)$. From now on, $\|\cdot\|$ stands for $\|\cdot\|_\infty$.

3 Hölder continuity of the policy function approximation

In this section we prove that the policy function approximation of the SDPP is Hölder continuous on the approximated value function. Let T be the operator on $C(X\times Z)$ (the set of continuous and bounded functions defined in $X\times Z$ with the topology induced by the supremum norm) defined by:

$$TV(x,z)=\operatorname*{Max}_{\{y\in X\,;\,(x,y,z)\in\Omega\}} F(x,y,z)+\beta\int_Z V(y,z')\,Q(z,dz').$$

It is well known (see Stokey and Lucas with Prescott [13]) that under hypotheses 1, boundedness and continuity of F, this operator is a contraction mapping with modulus β and fixed point v (the value function). Let us consider the following sets:

$\widehat{C}(X \times Z) = \{ f \in C(X \times Z) \,/\, f(\cdot\,, z)$ is a concave function for each $z \in Z \}$;

$\widehat{C}_{\eta_1}(X \times Z) = \{ f \in C(X \times Z) \,/\, f(\cdot\,, z)$ is a η_1–concave function for each $z \in Z \}$;

The following lemma claims that concave functions are mapped by T into strongly concave functions. It is an extension to the stochastic case of Montrucchio [8, Prop. 4.1].

Lemma 3.1 *Under hypotheses 1 and 2,* $T : \widehat{C}(X \times Z) \to \widehat{C}_{\eta_1}(X \times Z)$. *In particular* $v \in \widehat{C}_{\eta_1}(X \times Z)$

To state the main theorems of this section let us introduce the following notation. For $V \in \widehat{C}_{\eta_1}(X \times Z)$ define $h(V) : X \times Z \to X$ by:

$$h(V)(x,z) = \underset{\{y \in X;\, (x,y,z) \in \Omega\}}{\text{Argmax}} \quad F(x,y,z) + \beta \int_Z V(y,z')\, Q(z,dz'). \qquad (1)$$

Note that under hypotheses 1 and 2, $h(V) \in C(X \times Z)$. It is clear that the *optimal policy function* of the SDPP is $g(x,z) = h(v)(x,z)$.

Theorem 3.2 *Suppose that hipotheses 1 and 2 hold. If* $V, W \in \widehat{C}_{\eta_1}(X \times Z)$ *then:*

$$\|h(V) - h(W)\| \leq (2/\eta_1)^{1/2} \, \|V - W\|^{1/2}.$$

Now we are able to give an error bound for the policy function approximation obtained from a η_1-concave approximation of the value function. Note that in this setting, the *exact* value function is η_1-concave.

Proposition 3.3 *Suppose that hipotheses 1 and 2 hold. If* $V \in \widehat{C}_{\eta_1}(X \times Z)$ *then:*

$$\|g - h(V)\| \leq \left(\frac{2}{(1-\beta)\eta_1} \right)^{1/2} \|TV - V\|^{1/2}. \qquad (2)$$

The proof of proposition above is straightforward from Theorem 3.2, putting $W = v$ and using the fact that $\|v - V\| \leq \|TV - V\|/(1-\beta)$ (recall that T is a β-contraction mapping with fixed point v).

Remark If iterations of T are used to approximate the value function (i.e., $v_0 = 0$ and $v_{n+1} = Tv_n$ for $n \geq 0$) then the inequality (2) results:

$$\|g - h(v_n)\| \leq \left(\frac{2}{(1-\beta)\eta_1} \|F\| \right)^{1/2} \beta^{n/2}, \qquad (3)$$

which is similar to the error bound reported in Maldonado and Svaiter [6].

Similar results can be proved using hypothesis 3, instead of hypothesis 2. Under these hypotheses, $h(V) \in C(X \times Z)$ for any $V \in \widehat{C}(X \times Z)$.

Theorem 3.4 *Suppose that hipotheses 1 and 3 hold. If* $V, W \in \widehat{C}(X \times Z)$ *then:*

$$\|h(V) - h(W)\| \leq (2\beta/\eta_2)^{1/2}\ \|V - W\|^{1/2}.$$

Remarks: Note that under hypothesis 3, the functions V and W do not need to be strongly concave as in Theorem 3.2. Analogous formulae to (2) and (3) can be also obtained in this case.

It is important to note that our error bounds can be estimated if the return function $(x, y) \mapsto F(x, y, z)$ is strongly concave in either the first or the second group of variables. This is an improvement on Santos and Vigo-Aguiar [12] result, which depends on strong concavity in the second group of variables. Venditti [15] provided sufficient conditions to obtain strong concavity of $F(\cdot, \cdot, z)$ in multisector optimal growth models. He also showed that the conditions to get strong concavity in the second variable are much more restrictive than the ones to obtain strong concavity in the first variable.

Strong concavity of the return function was also used to prove differentiability of the policy function. Santos [10] used strong concavity in the second variable whereas Montrucchio [9] used strong concavity in the first variable to obtain such a differentiability.

4 Robustness of the error bounds

In this section we will show that the error bound formulae given in section 3 are robust by jointly considering errors in computing the T operator and errors in computing the maximizer in (1).

Suppose that T is performed using a numerical method and that $\widetilde{T}$, an "approximated" operator, is computed.

Hypothesis 4 Let $\widetilde{T} : C(X \times Z) \to C(X \times Z)$. Assume that there exists $\varepsilon \geq 0$ such that for all $f \in C(X \times Z)$, it holds that: $\|\widetilde{T}(f) - T(f)\| \leq \varepsilon$.

Now let $(\tilde{v}_n)_{n \geq 0}$ be a sequence generated by the rule

$$\tilde{v}_{n+1} = \widetilde{T}(\tilde{v}_n).$$

Proposition 4.1 *If the correspondence* Γ *is nonempty, compact-valued, and continuous, the function* F *is bounded and continuous, and* $\widetilde{T}$ *satisfies Hypothesis 4, then the sequence* $(\tilde{v}_n)_{n \geq 0}$ *satisfies*

$$\|\tilde{v}_n - v\| \leq \frac{\varepsilon}{1 - \beta} + \beta^n \left(\frac{\|F\|}{1 - \beta} + \|\tilde{v}_0\| \right).$$

Remark The application $\widetilde{T}$ does not have to satisfy the usual assumptions of monotonicity ($f \leq g \Rightarrow \widetilde{T}f \leq \widetilde{T}g$) and discounting ($\widetilde{T}(f+a) = \widetilde{T}f + \beta a,\ a \in \mathbb{R}$) (see Santos and Vigo-Aguiar [12]). Since $\widetilde{T}$ represents the "inexact" operator T these assumptions are hard to check (and may not hold) when rounding and chopping errors are embedded into the analysis.

Proposition 4.1 also says that if

$$\beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right) \ll \frac{\varepsilon}{1-\beta}$$

then more than n iterations may not appreciably improve the accuracy of the value function approximation.

Now suppose that $\widetilde{V} \in C(X \times Z)$ is a value function approximation and an *inexact* maximization is used to compute the policy associated to $\widetilde{V}$. That is, take a tolerance $\tau \geq 0$ and define $\widetilde{G}(x,z)$ as those $\tilde{y} \in \Gamma(x,z)$ such that $\forall y \in \Gamma(x,z)$

$$F(x,\tilde{y},z) + \beta \int_Z \widetilde{V}(\tilde{y},z')Q(z,dz') \geq F(x,y,z) + \beta \int_Z \widetilde{V}(y,z')Q(z,dz') - \tau. \quad (4)$$

Observe that $\widetilde{G}$ can be a correspondence. In general, practical computation does not provide *the whole* set $\widetilde{G}(x,z)$. Instead, the inexact maximization will provide some $\tilde{y} \in \widetilde{G}(x,z)$. Even so, we have the following estimation.

Theorem 4.2 *Suppose that hypotheses 1, 2 and 4 are satisfied. Let $\tilde{g} : X \times Z \to X$ be a selection of $\widetilde{G}$, that is, $\tilde{g}(x,z) \in \widetilde{G}(x,z)$ for all (x,z). Then,*

$$\|g - \tilde{g}\| \leq \left[\frac{4}{\eta_1} \|v - \widetilde{V}\| + \frac{2}{\beta \eta_1} \tau \right]^{1/2}.$$

Remarks:

1) Using that T is a $\beta-$contraction mapping and hypothesis 4 it is easy to see that $\|v - \widetilde{V}\| \leq \left(\|\widetilde{V} - \widetilde{T}\widetilde{V}\| + \epsilon \right) / (1-\beta)$. Therefore we have the following error bound for the approximation of the optimal policy function:

$$\|g - \tilde{g}\| \leq \left(\frac{4}{\eta_1} \left[\frac{1}{1-\beta} \left(\|\widetilde{V} - \widetilde{T}\widetilde{V}\| + \epsilon \right) \right] + \frac{2}{\beta \eta_1} \tau \right)^{1/2}.$$

2) If $\widetilde{V}$ is found from the $n-$iterative of $\widetilde{T}$ (i.e. $\widetilde{V} = \tilde{v}_n$) then using $\|\tilde{v}_{n+1} - \tilde{v}_n\| \leq \|v_{n+1} - v\| + \|v - v_n\|$, Proposition 4.1 and inequality above it results:

$$\|g - \tilde{g}\| \leq \left[\frac{8}{\eta_1} \left(\frac{\varepsilon}{1-\beta} + \beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right) \right) + \frac{2}{\beta \eta_1} \tau \right]^{1/2}.$$

Therefore if n is such that

$$\beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right) \ll \frac{\varepsilon}{1-\beta} + \frac{\tau}{2\beta}$$

then more than n iterations may not appreciably improve the accuracy of the policy function approximation.

Alternatively, we can use hypothesis 3 instead of hypothesis 2, to obtain a similar error bound in the policy function approximation.

Theorem 4.3 *Suppose that hypotheses 1, 3 and 4 are satisfied. Let* $\tilde{g} : X \times Z \to X$ *be a selection of* $\widetilde{G}$*, that is,* $\tilde{g}(x,z) \in \widetilde{G}(x,z)$ *for all* (x,z)*. Then,*

$$\|g - \tilde{g}\| \leq \left[\frac{4\beta}{\eta_2} \|v - \widetilde{V}\| + \frac{2}{\eta_2}\tau \right]^{1/2}.$$

5 Conclusions

In this paper we proved that policy function approximations obtained using Bellman's equation depends Hölder continuously on the value function approximation. This property allows us to provide an error bound for the optimal policy function of the stochastic dynamic programming problem. Such an error bound only depends on the value function approximation, the norm and the modulus of strong concavity of the return function and the discount factor. Neither differentiability of the return function nor interiority of the solution are required. This can be useful when the model involves piece-wise linear taxation/subsidies or short-run Leontief technologies because in these cases the return function may not be differentiable. Also interiority of solutions can not be guaranteed if (for example) Inada's condition is not considered.

When inexact computations are performed in the calculation of T (making a discretization of the state space, for example) or in the maximizer defined from T we also provide an error bound for the approximation of the value and policy functions. The intuition is quite simple: Perturbations of a contraction mapping are stable even though a fixed point for the perturbed mapping may not exist.

If the iterations of the T operator are used to solve the SDPP then the error bounds presented in this paper can be used for evaluating *a priori* the number of iterations needed in practical computations. For example, following the notation of Section 4, let ε be the error on the T operator and τ be the error on the maximization procedure used to compute the associated policy. If n is such that

$$\beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right) \ll \frac{\varepsilon}{1-\beta} + \frac{\tau}{4\beta}$$

then more than n iterations may not appreciably improve the accuracy of the policy function approximation.

A Appendix

Proof of Lemma 3.1 Let $V \in \widehat{C}(X \times Z)$, $x_1, x_2 \in X$, $\alpha \in [0,1]$, $x^\alpha = \alpha x_1 + (1-\alpha)x_2$ and for $i = 1, 2$ let $y_i \in \Gamma(x_i, z)$ be such that:

$$TV(x_i, z) = F(x_i, y_i, z) + \beta \int_Z V(y_i, z') \, Q(z, dz').$$

Then, using hypotheses 1, 2 and Definition 2.2 we have that:

$$\begin{aligned} TV(x^\alpha) &\geq F(x^\alpha, \alpha y_1 + (1-\alpha)y_2, z) + \beta \int_Z V(\alpha y_1 + (1-\alpha)y_2, z')Q(z, dz') \\ &\geq \alpha F(x_1, y_1, z) + (1-\alpha)F(x_2, y_2, z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2 + \\ &\quad \beta \int_Z [\alpha V(y_1, z') + (1-\alpha)V(y_2, z')] \, Q(z, dz') \\ &= \alpha TV(x_1, z) + (1-\alpha)TV(x_2, z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2. \end{aligned}$$

Since the set of strongly concave functions is a closed set with the topology induced by the sup norm it follows that the fixed point of T is strongly concave. This proves the second part of the lemma.

To prove Theorem 3.2, we will need the following lemma. Although its proof is trivial, we will give it for the sake of completeness.

Lemma A.1 *Let $f : C \subset \mathbb{R}^n \to \mathbb{R}$ (C is a convex set) be a η-concave function. If $x^* = Argmax_{x \in C} f(x)$ then*

$$f(x) \leq f(x^*) - \frac{\eta}{2}|x - x^*|^2, \quad \forall x \in C.$$

Proof: Let $x \in C$ and $\alpha \in (0,1)$. By definition of x^* and Definition 2.1, we have:

$$f(x^*) \geq f(\alpha x^* + (1-\alpha)x) \geq \alpha f(x^*) + (1-\alpha)f(x) + \tfrac{\eta}{2}\alpha(1-\alpha)|x - x^*|^2$$

$$\Rightarrow f(x^*) \geq f(x) + \tfrac{\eta}{2}\alpha|x - x^*|^2,$$

making $\alpha \to 1$ we obtain the result.

Proof of Theorem 3.2. For $V, W \in \widetilde{C}_{\eta_1}(X \times Z)$ define:

$$\phi_V(x, y, z) = F(x, y, z) + \beta \int_Z V(y, z',) \, Q(z, dz'),$$

and an analogous expression for ϕ_W. The functions $\phi_V(x,.,z)$ and $\phi_W(x,.,z)$ are $\beta\eta_1$-concave. Then by lemma A.1 we have that:

$$\phi_W(x, h(W)(x,z), z) \geq \phi_W(x, h(V)(x,z), z) + \frac{\beta\eta_1}{2}|h(W)(x,z) - h(V)(x,z)|^2,$$

$$\phi_V(x, h(V)(x,z), z) \geq \phi_V(x, h(W)(x,z), z) + \frac{\beta\eta_1}{2}|h(W)(x,z) - h(V)(x,z)|^2.$$

Summing up the above inequalities we obtain that:

$$\beta\{\int_Z [(V-W)(h(V)(x,z), z') + (W-V)(h(W)(x,z), z')]\, Q(z, dz')\} \geq$$

$$\beta\eta_1 |h(W)(x,z) - h(V)(x,z)|^2$$

$$\Rightarrow 2\|W - V\| \geq \eta_1 |h(W)(x,z) - h(V)(x,z)|^2$$

this inequality holds for all $(x,z) \in X \times Z$, so we conclude:

$$\|h(V) - h(W)\| \leq [\frac{2}{\eta_1}\|V - W\|]^{1/2}.$$

Proof of Theorem 3.4 Under hypothesis 3, the functions $\phi_V(x,.,z)$ and $\phi_W(x,.,z)$ given above are η_2-concave. Then by lemma A.1 we have that:

$$\phi_W(x, h(W)(x,z), z) \geq \phi_W(x, h(V)(x,z), z) + \frac{\eta_2}{2}|h(W)(x,z) - h(V)(x,z)|^2,$$

$$\phi_V(x, h(V)(x,z), z) \geq \phi_V(x, h(W)(x,z), z) + \frac{\eta_2}{2}|h(W)(x,z) - h(V)(x,z)|^2.$$

Using the same reasoning as in the proof of Theorem 3.2 we conclude that:

$$\|h(V) - h(W)\| \leq [(2\beta/\eta_2)\|V - W\|]^{1/2}.$$

Proof of Proposition 4.1 By our assumptions, T is a β-contraction on $C(X \times Z)$ with fixed point v. Using also the definition of the sequence $(\tilde{v}_n)_{n\geq 0}$, the triangular inequality and Hypothesis 4 we get

$$\|\tilde{v}_{n+1} - v\| = \|\widetilde{T}(\tilde{v}_n) - v\| \leq \|\widetilde{T}(\tilde{v}_n) - T(\tilde{v}_n)\| + \|T(\tilde{v}_n) - v\| \leq \varepsilon + \beta\|\tilde{v}_n - v\|.$$

Hence

$$\|\tilde{v}_n - v\| \leq \sum_{j=0}^{n-1} \varepsilon\beta^j + \beta^n\|\tilde{v}_0 - v\| \leq \varepsilon/(1-\beta) + \beta^n\|\tilde{v}_0 - v\|.$$

Proof of Theorem 4.2 Let

$$\tilde{\phi}(x,y,z) = F(x,y,z) + \beta\int_Z \widetilde{V}(y,z')\, Q(z,dz').$$

Then

$$\tilde{\phi}(x, \tilde{g}(x, z), z) \geq \tilde{\phi}(x, g(x, z), z) - \tau.$$

As discussed in the proof of Theorem 3.2, under hypotheses 1 and 2, $\phi_v(x, \cdot, z)$ is $\beta\eta_1$-concave. So, using again lemma A.1 we have:

$$\phi_v(x, g(x, z), z) \geq \phi_v(x, \tilde{g}(x, z), z) + \frac{\beta\eta_1}{2}|g(x, z) - \tilde{g}(x, z)|^2.$$

Adding up these inequalities and following the same procedure as in the proof of Theorem 3.2 we will obtain

$$\|g - \tilde{g}\| \leq \left[\frac{4}{\eta_1}\|v - \widetilde{V}\| + \frac{2}{\beta\eta_1}\tau\right]^{1/2}.$$

Proof of Theorem 4.3 Analogous to Theorem 4.2, using the fact that $\phi_v(x, \cdot, z)$ is η_2-concave.

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