Games and Queues

Li Zhang Fang Wu[*] Bernardo A. Huberman

Information Dynamics Laboratory
HP Laboratories
Palo Alto, CA 94301, USA

Abstract

We consider scheduling in distributed systems from a game theoretic point view while taking into account queuing theory methodologies. In this approach no one knows the global state of the system while users try to maximize their utility. Since the performance of such a blind scheduler is worse than the optimal, it induces users to employ strategies to improve their own utilization of the system. One such strategy is that of restarting a request if it is not satisfied in a given time. Since we assume users as non-cooperative and selfish, the problem is that of studying the characteristic of the Nash equilibrium in a large distributed system with no omniscient controls. We study the problem through computer experiments and analytical approaches. We obtain exact solutions in situations delimited by two extremes: one in which users never restart an initial request, and another one in which the user's requests are restarted infinitely often. Users can switch between these two behaviors. When the system load is below certain threshold, it is always better off to be impatient, and when the system load is higher than some threshold, it is always better to be patient. Between them there exists a homogeneous Nash equilibrium with non-trivial properties.

[*]Department of Applied Physics, Stanford University

1 Introduction

Scheduling is one of the most well-studied topics in Computer Science. As such, it has been traditionally approached within the setting of an omniscient scheduler and passive users, where the scheduler has full knowledge of the state of the system and the users only place requests and wait passively for the requests being processed. Unfortunately, both of those assumptions are often invalid in distributed computer systems. The Internet is probably the most prominent counter-example to these assumptions since the scheduling (or routing) in the Internet is done as a collective effort among numerous routers which each only know their own state, with users that are far from passive in expressing their preferences. Actually, most users play various strategies to try to maximize their own utility, such as restarting their browsers.

In this paper, we consider scheduling in distributed systems from a game theoretic point view while taking into account queuing theory methodologies. In this approach no one knows the global state of the system while users try to maximize their utility, which is measured in terms of getting their needs served as fast as possible. The system thus acts very much like a "blind" scheduler, i.e., one that has virtually no knowledge about the system's state. Since one would expect the performance of such a blind scheduler to be worse than the optimal, it would induce users to employ strategies to improve their own utilization of the system. One common such strategy is that of restarting a request if it is not satisfied in a satisfactory interval which depends on the needs of the user.

While such restart strategies are commonly used in, they were first studied systematically a few years ago [5] in the context of reducing the average and variance of the waiting time in Internet transactions. In that study the only tractable scenarios corresponded to situations where congestion is light, allowing for the construction of restart strategies that minimize the response times and their variance. That methodology cannot easily be extended to more congested scenarios, thus motivating a present approach that can yield predictions at all levels of congestion.

In this paper we consider users that resort to a restart strategy in order to gain access to a particular resource. Since we assume users as non-cooperative and selfish, we cast the problem as that of studying the characteristic of the Nash equilibrium in a large distributed system with no omniscient controls within a game-theoretic sense. The approach of treating users in a distributed system as players in a non-cooperative game is fairly common and has already led to a number of interesting results [2, 6, 4, 8, 7].

We first study the mixed game theoretical queueing theory problem through computer experiments which reveal the possible regimes to be expected. Next, we proceed to obtain analytical solutions in situations delimited by two extremes: one in which *patient* users never restart an initial request, and another one in which the *impatient* user's requests are restarted infinitely often. Users can then switch between these two behaviors. Intuitively one expects that when the system load is below certain threshold, it will always be better off to be impatient, and when the system load is higher than some threshold, it will always be better to be patient. As we show in this paper, those thresholds can be derived analytically and have the property that between them there exists a homogeneous Nash equilibrium with non-trivial properties .

Thus, in spite of its simplicity a restart strategy turns out to be very powerful when dealing with a distributed system lacking an omniscient scheduler. This has particular appeal when designing a system with a blind or no scheduler such as the Internet. As is presently the case in the Ethernet, there is no scheduler to determine who may use the publicly shared media. Instead, each user uses an exponential backoff protocol to decide when they may send a message. Other examples include the routing on the Internet and scheduling in highly distributed server farms.

2 Modeling restart in blind scheduling

Consider a typical distributed system that consists of a set of users and machines, with the users generating requests, and machines processing them. When a request is assigned to a machine, it is placed in an internal first-in-first-out (FIFO) queue in the machine. An idle machine always takes the first request from its queue, whenever the queue is not empty, and processes the request. The time needed to process a request is S/V where S denotes the size of the request, and V the speed of the machine. Each machine also prescribes a maximum queue size. A machine only accepts requests when its queue size is under the limit. The response time of a request is defined to be the time between when a request is generated and when it has been processed.

The performance of a system is evaluated primarily by the mean and by the variance of the response time [5]. If an omniscient scheduler is available, i.e. if the scheduler knows the speed and the queue length of each machine, a simple greedy strategy works well.[1] In this paper, we are interested in systems where such information is unavailable. Conceptually, such a system can be modelled as to have a blind scheduler: a scheduler that has no knowledge about the machine speed and queue length and assigns the jobs in a random manner. We should emphasize that the blind scheduler is only conceptual to capture the

[1]Computing the optimum is NP-hard even for an omniscient scheduler.

```
while(true) {
  generate(R);
  submit(R);
  restart_times = 0;
  while(!processed(R)) {
    if((waiting_time > T)&&(restart_times < k)) {
      restart(R);
      request_times++;
      reset(waiting_time);
    }
  }
}
```

Figure 1: The restart algorithm. T is the restart time, and k the maximum allowed restarts for each request.

behavior of a system where the global state or coordination is unavailable, such as routing on the Internet.

We assume that the users may have at most one pending request at any time, and they may restart a pending request as described by the pseudo-code in Figure 1. A restart request is ignored if the request is being processed. Otherwise, the request is taken out from the queue it is in and re-assigned to a randomly chosen machine (and therefore placed at the end of the queue in that machine).

We should note that in the above model, there is no cost to restarts. This assumption might not be true in practice as it takes scheduler and machine's effort to locate, cancel, and reschedule a request. To avoid the system being overloaded by the restart requests, the system may discourage restart requests by adding delays to the restart requests. On the other hand, if by using restart the system can have better load balance, it may encourage the use of restart strategy. To focus on our main point, in our experiments, we assume that each restart is free but there is a limit of the number of restarts per request. This is according to the intuition that we can allow small number of restarts but infinitely many restarts are costly for both the user and the system. For our experiments, we assume that the users use a fixed restart time, which they cannot adaptively or probabilistically change.

Whether to use restart strategy largely depends on the system load. Intuitively, when the system is less congested, one should use shorter restart time because the chance of reallocation to a lightly loaded machine is higher. But when the system load is high, using restart strategy is probably not beneficial because the user will lose their position in the queue and will have to line up in the tail of some other queue which is likely to be full. In Section 4, we will make these points clearer. For our experiments, we therefore focus on the interesting case when the load of the system is fair. We note that the analysis in [5] is related to the situation when the load of the network is light.

In what follows, we first show that for cooperative users, by using a restart strategy properly, they may achieve nearly optimal response time with a blind scheduler. Further, we study the more interesting case when the users are selfish and only care about the response time of their own requests. Central to games with selfish players is the notion of Nash equilibrium, the combination of strategies in which no user has incentive to change his strategy unilaterally. We study the characteristics of Nash equilibria in such a system with the focus on the homogeneous Nash equilibrium, in which all the users employ the same strategy. Since we forbid the users from probabilistically choosing the restart time, we can view that the users only play pure strategies, in the game theory terminology.

3 Experiment results

It is difficult to calculate analytically the mean response time as a function of the restart time. People have studied similar models and only have success with very simple model [1]. We instead study the problem by experiments and provide analytical analysis for a simplified model in the next section.

3.1 Setup

In our simulation setup, we assume that each user generates requests according to a Poisson process. The size of requests is from a log-normal distribution. In both processes, we use the same parameters for all the users. We fix the total number (2000) of users and the maximum number ($k = 10$) of times a request may be restarted. The mean of the request time is 1500. The mean of the request size is 300 and the standard deviation is 100.

We fix the number (50) of machines. The maximum queue size of each machine is 50. The base speed of each machine is fixed and uniformly distributed in the interval $[2.5, 9.5]$. To simulate different congestion condition, we can apply a multiplicative rate r to the machine speed, i.e. the real speed of a machine is r times its base speed. As explained before, we will focus on the cases when the load of the system is fair (when $r = 1.0$).

With the above fixed parameters, we vary two parameters: the number N_r of users that use restart strategies and the restart time T

for those users who use restart strategies. All of the simulations run fairly long with millions requests generated in total.

For comparison, we run an omniscient greedy scheduling algorithm and obtain the mean response time of 614.28 and standard deviation of response time of 39.38. In the omniscient greedy algorithm, we calculate the response time of a request on each machine by the assumption that we know all the requests in each machine and the size of each request. We then assign the request to the machine with the minimum response time. While the greedy algorithm may not find the optimum value, its solution should be fairly close to the optimum, and it is the equilibrium when all the users are omniscient and can decide which machine to assign theirs requests.

3.2 Cooperative restarts

In our first set of experiments, we assume that each user uses the same preset restart time. A special case is when none of the users uses restart, or in the other words, when their restart time is infinity. Figure 2 shows the mean and the standard deviation of the response time as a function of the restart time. When no users uses restart strategy, the mean response time is about 1200, about twice as much as the mean response time of omniscient greedy algorithm. The standard deviation is 1800, which is significantly higher. When the users use the restart strategy with appropriate restart time, both the mean and the standard deviation of the response time become significantly lower. In particular, the mean response time is very close to that produced by the omnisicent greedy algorithm. The performance is fairly insensitive to the restart time — the minimal mean response time is achieved over a large interval ($[20, 400]$) of restart time. The standard deviation of response time has a sharp drop for small restart time and reaches the minimum when $T \approx 80$ and remains fairly stable at 400 until T reaches 500. Therefore, we can draw conclusion that when all the users use restart strategy with the same restart time, it is easy to achieve the nearly optimal mean response time with relatively small standard deviation in the response time.

3.3 Non-Cooperative homogeneous restarts

As shown in the previous section, when all the users use restart strategy with the same restart time, the best restart time is located in the interval $[80, 400]$ when both the mean and the standard deviation of the response time are low. But what if the users are non-cooperative, i.e. a user may defect from that restart time everyone else uses? To answer that question, we first experiment with the extreme cases: in one case, we let one user not to restart and all the othe users use the

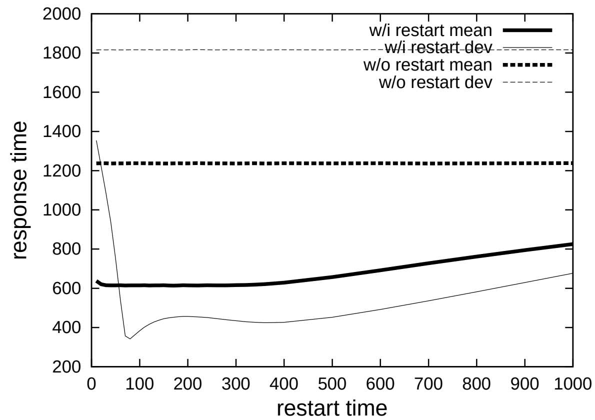


Figure 2: The mean and standard deviation of the response time for homogeneous cooperative users.

same restart time; in the other case, we let only one user restart and all the other users not to. The results are shown in Figure (3).

In Figure (3).(1), when there is only one restart user, it has much lower response time than those who do not restart. It is even significantly better than that by omniscient greedy algorithm. When all the users restart, the non-restart user has smaller mean response time except when the restart users use really small restart time.(Figure 3.(2)).

We further experiment with the cases when the restart users use small restart time. We then vary one user's restart time and find that it is still beneficial for the user to defect. Figure 3.(3) shows the result. In Figure 3.(3), we plot the response time when all the users are restart users, and one user uses different restart time from the other users. The x-axis in Figure 3.(3) represents the restart time of the defect user. The solid curve corresponds to the mean response time to the homogeneous group of users. Since the group restart time is chosen from the interval such that the mean response time is small, the mean response time remains the same. The dashed curves correspond to the response time to the defect user with different group restart time.

¿From the figure, we can see that when all the users use the same restart time, one is always better off to be a little bit more patient, i.e. using a slightly longer restart time. However, as we show earlier, when all the users are patient, it is always better off to restart. Therefore, there exists a critical value when it is no longer true that waiting a little bit more patient is better. In such cases, it is always better to restart really sooner than the other users.

¿From these experiments, we conclude that it is always better to

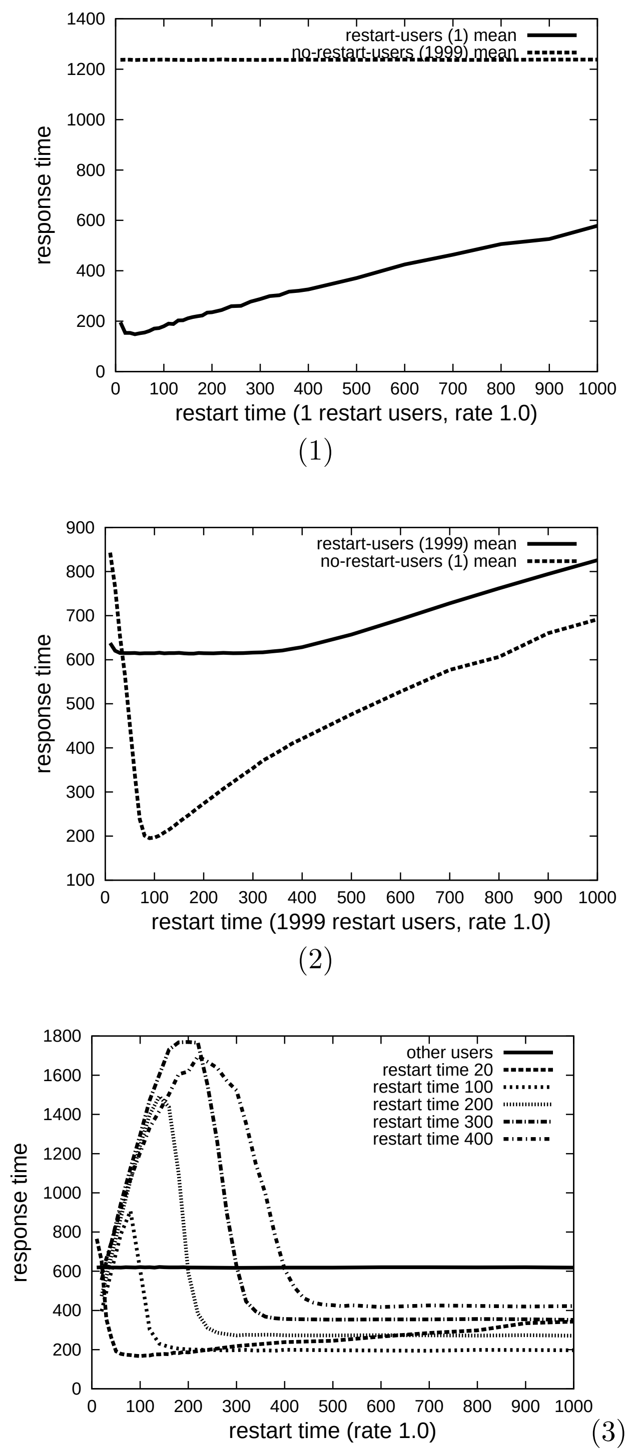


Figure 3: The mean and variance of the response time when there is one defect user

use a different restart time when all the other uses use the same restart time. In the game theory terminology, this means that there does not exist a pure strategy homogeneous Nash equilibrium.

3.4 Non-cooperative non-homogeneous restarts

In the previous section, we show, by experiments, that there does not exist a pure strategy homogeneous Nash equilibrium. It raises question of whether there exists a pure strategy Nash equilibrium at all. We experiment the situation when there are two groups of homogeneous users, and one group does not use restart strategy. Figure 4 shows three different settings where the number of restart users is 500, 1000, and 1800 respectively. In all the cases, the interval $[100, 400]$ seems the most appropriate restart time for the restart users as both the mean and the variance are low. From the figure, we can see that when N_r, the number of restart users, is 500, it is good for a non-restart user to switch to using restart. When $N_r = 1800$, it is however beneficial for a restart user to switch to non-restart. When $N_r = 1000$, the gap between the restart users and non-restart users diminishes, which indicates the possibility of the existence of a Nash equilibrium in the neighborhood.

4 Analysis

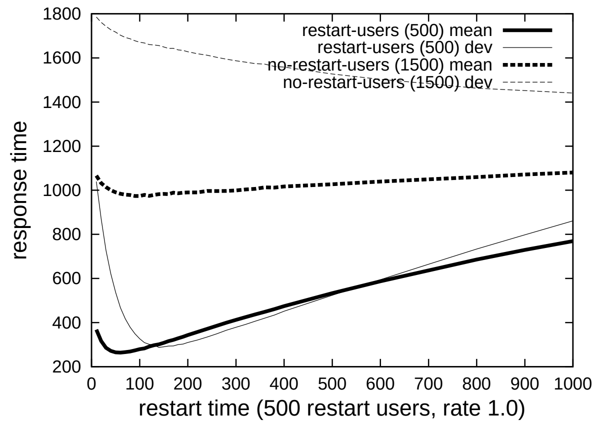
It is difficult to analytically analyze the mean response time in our experiment setup. In this section, we analyze a simpler model and show some characteristics of the restart strategy.

4.1 A simplified model

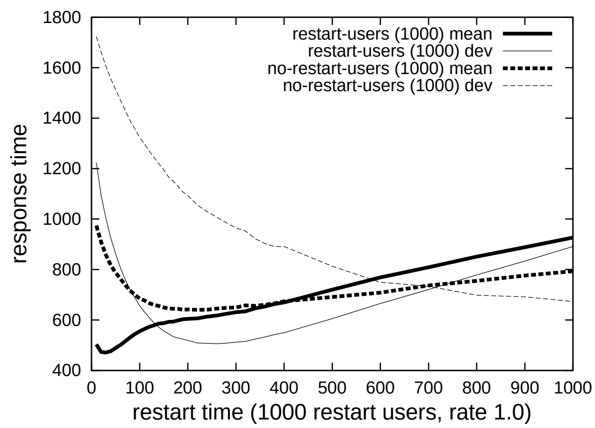
The significant simplification we make is to assume that a restart request always restarts with 0 waiting time, i.e. it keeps restarting until the server it is randomly assigned to happens to be available. We call such requests impatient. An alternative way to view impatient requests is to create an additional queue, the "impatient" queue, to store such requests. And an impatient request is served as long as one of the server is available.

Another simplification is that we no longer require that each user can only generate one request. Instead, we assume that the users generate requests steadily at a constant rate. This simplification removes the feedback loop which adjusts the overall system input rate and makes the analysis much simpler.

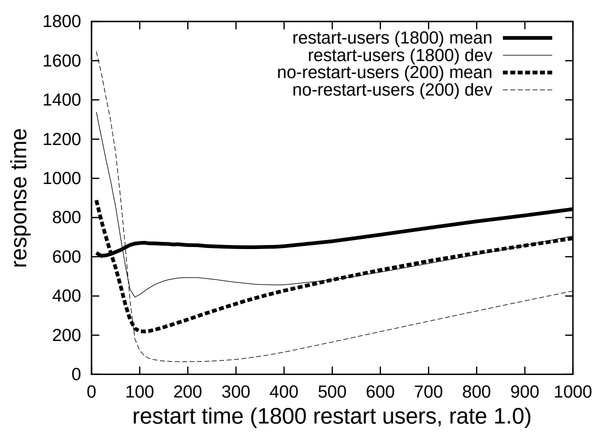
Formally, suppose that there are m identical $M/M/1$ queues, each having a service rate of $\mu = 1$, and there are n clients (players), each



(1) $N_r = 500$



(2) $N_r = 1000$



(3) $N_r = 1800$

Figure 4: Response time with two groups of homogeneous users.

generating requests at the rate of λ/n. There are two kinds of requests, patient or impatient. A patient request is assigned randomly to a server and waits in the queue until being served. There is an additional "impatient" queue for impatient requests. An impatient request would wait in that queue and get served when one of the servers becomes available. This captures the fact that an impatient request has 0 waiting time and tries to catch any available server. Since we are only interested in the expected waiting time, the queuing principle of the impatient queue does not matter. We assume it is an FIFO queue.

The immediate observation is that if all the users are impatient, then there is only a single impatient queue in the system. So the system behaves exactly as the $M/M/m$ queue, which is the most efficient way to use the system, or in the other words, it achieves the social optimum. However, what is interesting to us is the Nash equilibrium of the game where a user may decide each of its requests being patient or not.

In our analysis, we consider mixed strategy, i.e. each user can decide according to a probability whether a request it generates should be patient or impatient. Let p_i be the probability that the requests generated by the client i are patient. Now, the question we would like to answer is whether there is a Nash equilibrium and, if there is, what is the performance at the Nash equilibrium, compared to the social optimum.

Let $\rho = \frac{\lambda}{m\mu} = \frac{\lambda}{m}$, and $p = \sum_{i=1}^{n} p_i/n$. Since all the users generate requests at the same rate, and a request is decided to be a patient request with probability p_i, we can view the system with total incoming rate λ, of which a fraction of p are patient requests, and $1-p$ are impatient requests. Let $T_1(p,\rho)$ and $T_2(p,\rho)$ denote, respectively, the expected waiting time for the patient and impatient requests of such a system. Note that T_1, T_2 are the mean waiting time but not the mean response time, which is the sum of the waiting time and the processing time. It suffices to consider waiting time because the expected processing time is the same for all the requests and all the machines in our model. In the following, we will first attempt to computing T_1 and T_2 and then analyze the property of the game.

4.2 Expected waiting time of patient requests

We are able to obtain a fairly simple formula for $T_1(p,\rho)$.

Lemma 4.1

$$T_1(p,\rho) = \frac{\rho}{1-p\rho}\,.$$

Proof. Consider how a patient request can be delayed by an impatient request. An impatient request r can only grab a server when the server is free, and the patient requests that are going to be delayed due to the processing of r is exactly those requests that arrive in the busy period starting from the time when r is served. The mean of the total number of requests appearing in that busy period is $\frac{1}{1-p\rho}$ according to the $M/M/1$ queue result. Therefore, the number of patient requests that are delayed by r is $\frac{1}{1-p\rho} - 1 = \frac{p\rho}{1-p\rho}$ on average.

Now consider what happens over a long time interval $[0, T]$. Since we assume $\lambda < 1$, each request experiences a delay with finite mean. Therefore, we can assume that all the requests are processed within time $[0, T + \delta T]$ where $\delta T/T \to 0$ when $T \to \infty$. By symmetry, we know that the impatient requests are evenly divided among the servers. Therefore, each server processes $\frac{(1-p)\lambda T}{m}$ impatient requests, and each causes a delay of $\frac{1}{\mu}$ for $\frac{p\rho}{1-p\rho}$ patient requests. Therefore, the total additional delay caused by impatient requests is:

$$\frac{(1-p)\lambda T}{m} \cdot \frac{p\rho}{1-p\rho} \cdot \frac{1}{\mu} = \frac{(1-p)p\rho^2 T}{1-p\rho}.$$

There are $p\lambda T/m$ patient requests arriving at any server, therefore the mean delay of each patient request is:

$$\begin{aligned} T_1(p,\rho) &= \frac{p\rho}{(1-p\rho)\mu} + \frac{(1-p)p\rho^2 T}{(1-p\rho)} \cdot \frac{m}{p\lambda T} && (1) \\ &= \frac{p\rho}{(1-p\rho)\mu} + \frac{(1-p)\rho}{(1-p\rho)\mu} && (2) \\ &= \frac{\rho}{(1-p\rho)\mu} = \frac{\rho}{1-p\rho}. && (3) \end{aligned}$$

□

¿From the above formula, we can see that $T_1(p, \rho)$ is increasing and convex in terms of both p and ρ.

4.3 Expected waiting time of impatient requests

Computing $T_2(p, \rho)$ is difficult for general p. We consider the extreme cases when $p = 0, 1$. When $p = 0$, i.e. when there is no patient requests, the system is exactly an $M/M/m$ queue. The followinng is a known result about $M/M/m$ queue.

Lemma 4.2

$$T_2(0,\rho) = \frac{(m\rho)^m}{m!m(\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!}(1-\rho)^2 + \frac{(m\rho)^m}{m!}(1-\rho))}.$$

To calculate $T_2(1, \rho) = \lim_{p\to 1} T_2(p, \rho)$, we consider the situation when all but one requests are patient requests. When there is only one server, the impatient request has to wait only when the server is busy upon its arrival, and its waiting time is the time since its arrival to the time when the server becomes free. This is exactly the length of the busy period left of a randomly arriving request. When there are multiple servers, then the waiting time is the minimum of length of the busy period on those servers. When all the requests are patient, the queue state on each server is independent of the other servers. Therefore, we can apply the process for computing the expected value of the minimum of multiple random variables.

Let $b(t)$ denote the p.d.f. for the busy period distribution of the $M/M/1$ queue with arrival rate ρ and service rate 1. It is known that

$$b(t) = \frac{1}{t\sqrt{\rho}} e^{-(1+\rho)t} I_1(2t\sqrt{\rho})\,,$$

where $I_1(x)$ is the Bessel function of the first kind,

$$I_1(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{k!(k+1)!}\,.$$

Now, we calculate the probability that the waiting time of impatient request does not exceed t. Let X, X_1 denote the random variable of the waiting time of the impatient request for m and 1 servers, respectively. Then,

$$\text{Prob}[X \le T] = 1 - \text{Prob}[X_1 > T]^m\,.$$

Let B be the mean of the busy period of an $M/M/1$ queue. Since $B = \frac{1}{1-\rho}$, we have that:

$$\text{Prob}[X_1 > T] = \frac{\rho}{B} \int_T^{\infty} (t-T)b(t)dt = \rho(1-\rho) \int_T^{\infty} (t-T)b(t)dt\,.$$

If we let $f_1(\cdot), f(\cdot)$ denote the p.d.f. of X_1 and X respectively, then

$$f_1(T) = \rho(1-\rho) \int_T^{\infty} b(t)dt\,,$$

and

$$\begin{aligned} f(T) &= m\text{Prob}[X_1 > t]^{m-1} \cdot f_1(t) \\ &= m\rho^m(1-\rho)^m \int_T^{\infty} b(t)dt \left(\int_T^{\infty} (t-T)b(t)dt \right)^{m-1} \end{aligned}$$

Therefore, , $T_2(1,\rho)$ is given by the following formula.

$$T_2(1,\rho) = \int_0^{\infty} t \cdot f(t)dt\,.$$

Since $b(t)$, the p.d.f. of the busy period distribution of an $M/M/1$ queue, has a quite complicated form so the numerical solution is susceptible to errors. We show that for two servers, we can use the tool of Laplace transform to obtain a simpler form, which is an interesting result in its own right.

We first calculate the distribution of X_1. Suppose the impatient request sees N requests waiting in queue 1 when it enters the system, where N is a random variable. From [3] we know that N is geometrically distributed

$$\text{Prob}[N = n] = (1-\rho)\rho^n, \quad n = 0, 1, \ldots, \tag{4}$$

and has the generating function

$$f_N(x) = \frac{1-\rho}{1-\rho x}. \tag{5}$$

The time the request has to wait for queue 1 to become empty can thus be represented as a sum of busy period:

$$X_1 = \sum_{i=1}^{N} X_{Bi}. \tag{6}$$

X_{Bi} is the time it takes the server to decrease the queue size by one for the i'th time, which has the same distribution as the busy period. The Laplace transform of X_1 is

$$\hat{f}_1(s) = E[e^{sX_1}] = E[E[e^{sX_1}|N]] = E[E[e^{sX_B}]^N] = E[\hat{f}_B(s)^N], \tag{7}$$

where $\hat{f}_B(s)$ is the Laplace transform of $f_B(t)$, the density of the busy period X_B. It can be deduced that [3]

$$\hat{f}_B(s) = \frac{1}{2\rho}(\rho + 1 + s - \sqrt{(\rho+1+s)^2 - 4\rho}). \tag{8}$$

Notice that the last step of (7) is nothing but the moment generating function of N calculated at $x = \hat{f}_B(s)$, so

$$\begin{aligned}
\hat{f}_1(s) &= \frac{1-\rho}{1-\rho\hat{f}_B(s)} \\
&= \frac{1-\rho}{1-(\rho+1+s-\sqrt{(\rho+1+s)^2-4\rho})/2} \\
&= \frac{2(1-\rho)}{1-\rho-s+\sqrt{(\rho+1+s)^2-4\rho}} \\
&= \frac{1-\rho}{2s}[-1+\rho+s+\sqrt{(\rho+1+s)^2-4\rho}].
\end{aligned} \tag{9}$$

When there are two queues, the impatient request can be served when either queue becomes empty, so its waiting time is

$$X = \min(X_1, X_2), \tag{10}$$

where the subscripts 1 and 2 stand for queue 1 and queue 2. Because the two queues are independent and identical, X's CDF

$$F(t) = \text{Prob}[X \le t] = 1 - (1 - F_1(t))^2. \tag{11}$$

Then $T_2(1,\rho) = EX$, which is

$$\begin{aligned} EX &= \int_0^\infty t\,dF(t) = -\int_0^\infty t\,d(1 - F(t)) \\ &= \int_0^\infty (1 - F(t))dt = \int_0^\infty (1 - F_1(t))^2 dt. \end{aligned} \tag{12}$$

If we let

$$g(t) \equiv 1 - F_1(t), \tag{13}$$

then (12) can be written as

$$EX = \int_0^\infty g^2(t)dt = \int_0^\infty e^{-st} g^2(t)dt\Big|_{s=0} = \hat{g^2}(0), \tag{14}$$

where $\hat{g^2}(s)$ is the Laplace transform of $g^2(t)$.

The problem now boils down to calculating $\hat{g^2}(0)$. We first study some of the properties of $\hat{g}(s)$. Because $F_1(t)$ is the integral of $f_1(t)$, we have from (13) that

$$\hat{g}(s) = \frac{1 - \hat{f}_1(s)}{s}. \tag{15}$$

We claim that $s = 0$ is not a pole of $\hat{g}(s)$. This is clear from the expansion of $\hat{f}_1(s)$ near $s = 0$:

$$\hat{f}_1(s) = 1 - \frac{\rho}{(1-\rho)^2} s + O(s^2), \tag{16}$$

which implies

$$\lim_{s \to 0} \hat{g}(s) = \frac{\rho}{(1-\rho)^2}. \tag{17}$$

Hence $\hat{g}(s)$ is analytical at $s = 0$. However, because there is a square root in $\hat{f}_1(s)$ (see (9)), $\hat{g}(s)$ is not analytical on the entire complex plane. Define

$$\rho_1 = -(1 + \sqrt{\rho})^2, \quad \rho_2 = -(1 - \sqrt{\rho})^2. \tag{18}$$

Then the square root in (9) can be written as

$$\sqrt{(\rho + 1 + s)^2 - 4\rho} = \sqrt{(s - \rho_1)(s - \rho_2)}. \tag{19}$$

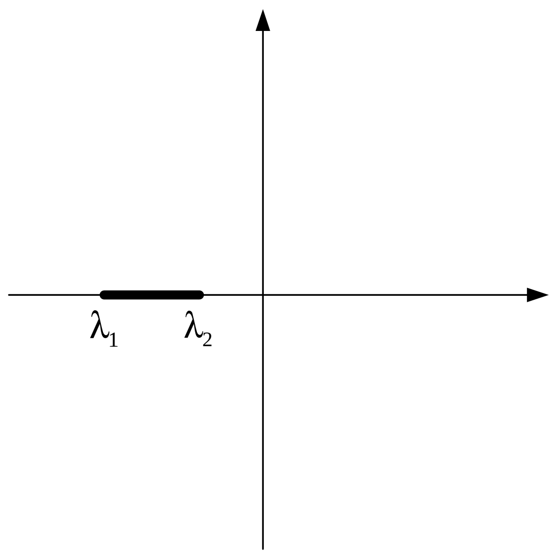


Figure 5: Branch cut of $\hat{f}_1(s)$ and $\hat{g}(s)$.

If we make a branch cut along the line segment $[\rho_1, \rho_2]$, then both $\hat{f}_1(s)$ and $\hat{g}(s)$ are analytical on the plane with the branch cut removed (Fig. 5).

The Laplace transform of the product $g^2(t)$ can be represented as a convolution product of $\hat{g}(s)$ and itself:

$$\begin{aligned}
\hat{g^2}(s) &= \int_0^\infty g^2(t)e^{-st}dt \\
&= \int_0^\infty \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{g}(s')e^{s't}ds'\right) g(t)e^{-st}dt \qquad (20) \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{g}(s')ds' \int_0^\infty g(t)e^{-(s-s')t}dt \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{g}(s')\hat{g}(s-s')ds', \qquad (21)
\end{aligned}$$

where (20) is the formula of inverse Laplace transform. c can be any real number larger than the convergence abscissa of $\hat{g}(s)$, which is ρ_2 in our case. In particular, we can take $c = 0$ because $\rho_2 < 0$. Thus

$$\hat{g^2}(0) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \hat{g}(s)\hat{g}(-s)ds. \qquad (22)$$

The integral of (22) can be calculated with the aid of the contour shown in Fig. 6. Because $\hat{g}(s)\hat{g}(-s)$ is analytical on the entire complex

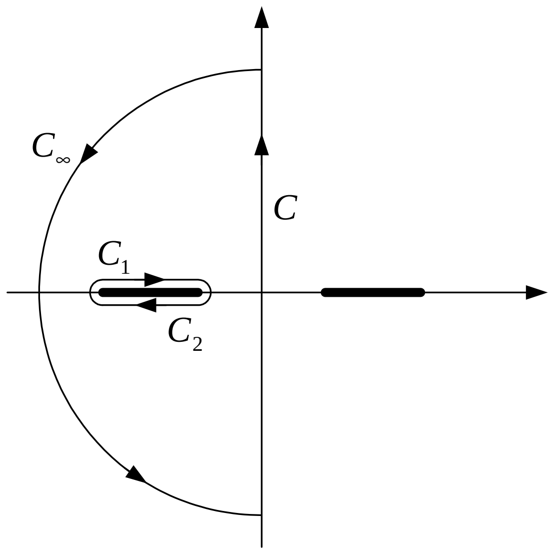


Figure 6: Contour of integral.

plane with the two branch cuts removed, we have

$$\left(\int_C + \int_{C_\infty} + \int_{C_1} + \int_{C_2}\right) \hat{g}(s)\hat{g}(-s)ds = 0. \tag{23}$$

As $s \to \infty$, $\hat{f}_1(s) \to 1-\rho$ uniformly. As a result $\hat{g}(s)$ is of the order s^{-1} when s is large, and $\hat{g}(s)\hat{g}(-s)$ is of the order s^{-2}. The length of the half circle is of the order s, so the integral on the path C_∞ is of the order s^{-1} which goes to zero when $s \to \infty$. Therefore

$$\int_{-i\infty}^{i\infty} \hat{g}(s)\hat{g}(-s)ds = -\left(\int_{C_1} + \int_{C_2}\right) \hat{g}(s)\hat{g}(-s)ds. \tag{24}$$

Because $\hat{g}(s)$ has the same real part on the upper and lower edges of the branch cut, the real parts of the two integrals along C_1 and C_2 with opposite directions cancel out. Only the image part of $\hat{g}(s)$ counts for the integral. The image part of $\hat{g}(s)$ on the upper edge and the lower edge differ only in the sign, so

$$\begin{aligned} & \int_{-i\infty}^{i\infty} \hat{g}(s)\hat{g}(-s)ds \\ = & \; -2i\int_{C_1} \mathrm{Im}[\hat{g}(s)]\hat{g}(-s)ds \\ = & \; 2i\int_{C_1} \frac{1}{s}\mathrm{Im}[\hat{f}_1(s)]\hat{g}(-s)ds \end{aligned}$$

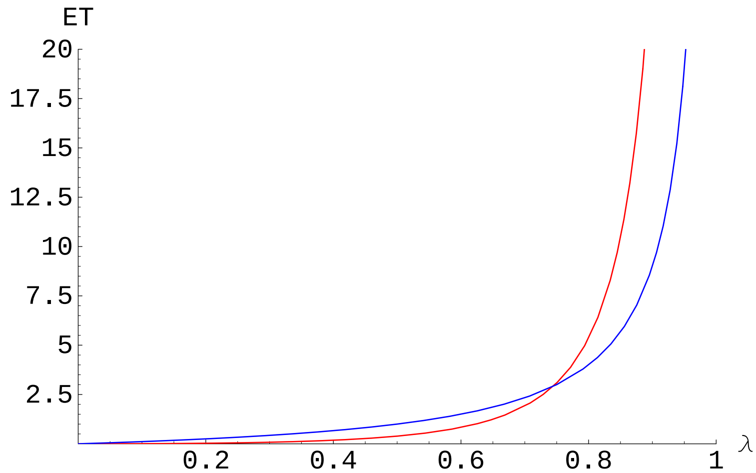


Figure 7: Expected waiting time versus incoming rate. The red line is for the impatient request and the blue line is for patient requests. The two lines cross at $\rho = 0.744$.

$$= \quad i\int_{C_1} \frac{1-\rho}{s^2}\sqrt{s-\rho_1}\sqrt{\rho_2 - s}\,\hat{g}(-s)ds \tag{25}$$

Finally we have

$$T_2(1,\rho) = EX(\rho) = \hat{g^2}(0) = \int_{\rho_1}^{\rho_2} \frac{1-\rho}{2\pi s^2}\sqrt{s-\rho_1}\sqrt{\rho_2 - s}\,\hat{g}(-s)ds. \tag{26}$$

A numerical computation of the integral is shown in Fig. 7.

To summarize, we have that

Lemma 4.3

$$T_2(1,\rho) = \int_0^\infty t\cdot f(t)dt\,.$$

In particular, when $m = 2$,

$$T_2(1,\rho) = \hat{g^2}(0) = \int_{\rho_1}^{\rho_2} \frac{1-\rho}{2\pi s^2}\sqrt{s-\rho_1}\sqrt{\rho_2 - s}\,\hat{g}(-s)ds\,.$$

4.4 Nash equilibrium of the restart game

By the above analysis, we can compute the critical values ρ_1 and ρ_2 where $T_1(0,\rho_1) = T_2(0,\rho_1)$ and $T_1(1,\rho_2) = T_2(1,\rho_2)$. Those values are of importance because of the following statement.

m	2	3	4	5	6	7	8	9	10	100
ρ_1	0.618	0.748	0.810	0.847	0.871	0.889	0.902	0.912	0.921	0.991
ρ_2	0.744	0.875	0.925	0.950	0.964	0.973	0.979	0.983	0.987	

Table 1: The critical value ρ_1 and ρ_2

Lemma 4.4 *1. When $\rho \leq \rho_1$, for any $0 \leq p \leq 1$, $T_1(p, \rho) \geq T_2(p, \rho)$. Or when $\lambda \leq m\mu\rho_1$, it is always better off to be impatient, i.e. $p_i = 0$, for $i = 1, \ldots, n$, is the Nash equilibrium in this case.*

2. When $\rho \geq \rho_2$, for any $0 \leq p \leq 1$, $T_1(p, \rho) \leq T_2(p, \rho)$. Or when $\lambda \geq m\mu\rho_2$, it is always better off to be patient, i.e. $p_i = 1$, for $i = 1, \ldots, n$, is the Nash equilibrium.

By Lemma 4.1 and 4.2, we can solve, numerically, for ρ_1. It is the root of the following equation:

$$\frac{(m\rho)^m}{m!m(\sum_{k=0}^{m-1}\frac{(m\rho)^k}{k!}(1-\rho)^2 + \frac{(m\rho)^m}{m!}(1-\rho))} = \frac{\rho}{1-p\rho}.$$

Table 1 shows the numerical solution to the above equation for various m's. The value of ρ_1 approaches 1 when $m \to \infty$, which is consistent with the intuition that when the number of servers increases, it is more likely that there exists a free server.

So when $\rho \leq \rho_1$, the Nash equilibrium is achieved when all the p_i's are 0. In this case, the Nash equilibrium is also the social optimum.

Similarly, we calculate ρ_2 numerically, and the solution is again given in Table 1. In Figure 8, we plot the critical values ρ_1, ρ_2 as a function of the number of servers. When $m \to \infty$, both ρ_1 and ρ_2 approach to 1.

In the regime of $\rho \geq \rho_2$, the Nash equilibrium is achieved when $p_i = 1$ for all the i's. The mean waiting time at the Nash equilibrium is $\frac{m\rho}{1-m\rho}$, the mean waiting time given by an $M/M/1$ queue, while the social optimum happens at when all the requests are impatient. There is a loss of efficiency at the Nash equilibrium. When $m = 2$, $T_2(0, \rho) = \frac{\rho^2}{1-\rho^2}$. Therefore, for two servers, the efficiency of the Nash equilibrium is $T_2(0, \rho)/T_1(1, \rho) = \frac{\rho}{1+\rho}$. When the number of servers increases, the efficiency decreases.

In the above, we have characterized the Nash equilibrium when the system is lightly or heavily loaded. In both cases, there exist a homogeneous pure strategy Nash equilibrium. What about the case when $\rho_1 < \rho < \rho_2$?

When $\rho_1 < \rho < \rho_2$, there exists $0 < p_\rho < 1$ such that $T_1(p, \rho) = T_2(p, \rho)$. It is tempting to conclude that if all the users decide with

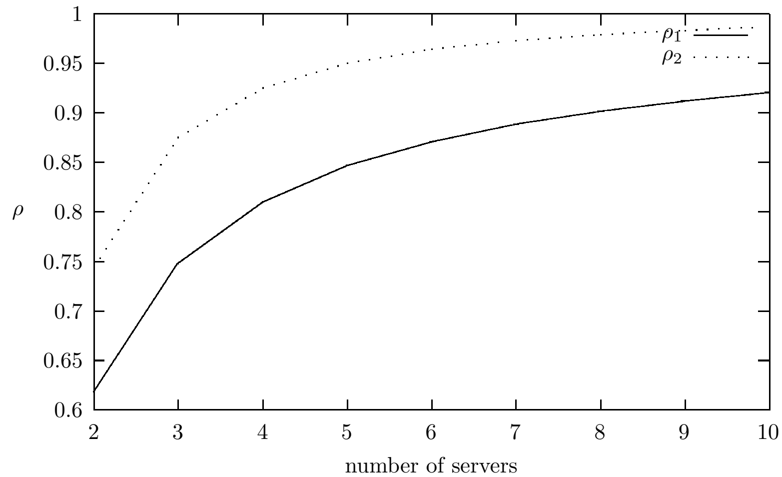


Figure 8: The plot of critical values ρ_1, ρ_2.

probability p that a request is patient, then it is a Nash equilbrium because it does not seem to make difference when a request switches from patient to impatient and vice versa. This is indeed the case when there are infinitely many users and each user generates one request. The reason is that an individual user's decision will have no effect on the other requests. However, when there are finite number of users, it is not the case by the following analysis.

We will have to need an assumption which we have observed consistently from simulation results but are, unfortunately, unable to prove:

Hypothesis. For any fixed $\rho \in (\rho_1, \rho_2)$, T_1, T_2 are continuous, monotonically increasing, and convex with respect to p. That is, $T_1'(p), T_2'(p), T_1''(p), T_2''(p) \geq 0$. Further, $T_1(p) - T_2(p)$ is monotonically increasing.

The above hypothesis implies that there exists a unique $0 < p_\rho < 1$ such that $T_1(p_\rho) = T_2(p_\rho)$. This follows from that $T_1(0) < T_2(0)$, $T_1(1) > T_2(1)$, and $T_1 - T_2$ is continuous and monotonically increasing.

Let $p = \frac{\sum_{i=1}^n p_i}{n}$. The expected waiting time $T^{(i)}$ of the i-th player is:

$$T^{(i)} = p_i T_1(p) + (1 - p_i) T_2(p)\,.$$

For a combination of strategies $p_1, \ldots, p_n$, it is a Nash equilibrium if for any $1 \leq i \leq n$ and $0 \leq x \leq 1$:

$$T^{(i)} \leq x T_1\left(p + \frac{x - p_i}{n}\right) + (1 - x) T_2\left(p + \frac{x - p_i}{n}\right)\,.$$

In particular, when the users are homogeneous, i.e. $p_i = p$, it is:

$$pT_1(p)+(1-p)T_2(p) \leq xT_1\left(\frac{n-1}{n}p+\frac{1}{n}x\right)+(1-x)T_2\left(\frac{n-1}{n}p+\frac{1}{n}x\right).$$

Let $q = (1-\frac{1}{n})p$. Then a homogeneous strategy where $p_i \equiv p$ is a Nash equilibrium if the function $F_p(x)$

$$F_p(x) = xT_1\left(q+\frac{1}{n}x\right)+(1-x)T_2\left(q+\frac{1}{n}x\right),$$

achieves the minimum at $x = p$.

Assuming the hypothesis, we can show that for any $0 \leq p \leq 1$, $F_p(x)$ is strictly convex.

$$\begin{aligned} F'_p(x) &= T_1\left(q+\frac{1}{n}x\right)+\frac{x}{n}T'_1\left(q+\frac{1}{n}x\right) && (27)\\ &\quad -T_2\left(q+\frac{1}{n}x\right)+\frac{1-x}{n}T_2\left(q+\frac{1}{n}x\right). && (28)\end{aligned}$$

$$\begin{aligned} F''_p(x) &= \frac{2}{n}\left[T'_1\left(q+\frac{x}{n}\right)-T'_2\left(q+\frac{x}{n}\right)\right] && (29)\\ &\quad +\frac{x}{n^2}T''_1\left(q+\frac{x}{n}\right)+\frac{1-x}{n^2}T''_2\left(q+\frac{x}{n}\right). && (30)\end{aligned}$$

Since T_1-T_2 is increasing and $T''_1, T''_2 > 0$, we have that $F''_p(x) > 0$, i.e. F_p is a strictly convex function for any $p > 0$.

Now, consider $F'_0(0) = T_1(0) - T_2(0) + \frac{1}{n}T'_2(0)$. Since F_0 is strictly convex, $F_0(0)$ is the minimum of $F_0(x)$ if and only if $F'_0(0) \geq 0$. Therefore, when $F'_0(0) \geq 0$, the strategy $p_i \equiv 0$ is a Nash equilibrium.

When $F'_0(0) < 0$, we consider $F'_{p_\rho}(p_\rho)$.

$$\begin{aligned} F'_{p_\rho}(p_\rho) &= T_1(p_\rho)+\frac{p_\rho}{n}T'_1(p_\rho)-T_2(p_\rho)+\frac{1-p_\rho}{n}T'_2(p_\rho) && (31)\\ &= \frac{p_\rho}{n}T'_1(p_\rho)+\frac{1-p_\rho}{n}T'_2(p_\rho) > 0\,. && (32)\end{aligned}$$

Therefore, there exists $p \in (0, p_\rho)$ such that $F'_p(p) = 0$. Since $F_p(\cdot)$ is strictly convex, $F_p(x)$ achieves the minimum at $x = p$. Thus $p_i \equiv p$ is a Nash equilibrium. According to Equation (32), $F'_{p_\rho}(p_\rho) \to 0$ when $n \to \infty$. Therefore, $p_i \approx p_\rho$ is a Nash equilibrium when there are a large number of users.

To summarize, when $\rho_1 < \rho < \rho_2$, there exists a unique homogeneous Nash equilibrium at $p < p_\rho$. At the Nash equilibrium, the impatient requests have longer waiting time than the patient requests.

5 Conclusion

In this paper, we have shown the characteristics of the Nash equilibrium of using restart strategy in a queueing system. We did it through experimental results and the analysis of a simplified model. Apparently, there are many open questions, such as finding the Nash equilibrium of the game described in our experimental setup and proving the hypothesis for the simplified model.

References

[1] G. I. Falin and J. G. C. Templeton. *Retrial Queues*. Chapman & Hall, 1997.

[2] B. A. Huberman and T. Hogg. Distributed computation as an economic system. *Journal of Economic Perspectives*, pages 141–152, 1995.

[3] L. Kleinrock. *Queueing Systems*, volume I and II. John Wiley & Sons, 1976.

[4] E. Koutsoupias and C. Papadimitriou. Worst-case equilibria. *Lecture Notes in Computer Science*, 1563:404–413, 1999.

[5] S. M. Maurer and B. A. Huberman. Restart strategies and Internet congestion. *Journal of Economic Dynamics and Control*, 25:641–654, 2001.

[6] N. Nisan and A. Ronen. Algorithmic mechanism design. *Games and Economic Behavior*, 35:166–196, 2001.

[7] C. H. Papadimitriou. Algorithms, games, and the Internet. In *ACM Symposium on Theory of Computing*, pages 749–753, 2001.

[8] T. Roughgarden and E. Tardos. How bad is selfish routing? In *IEEE Symposium on Foundations of Computer Science*, pages 93–102, 2000.