A Comparison of Optimal Control Solutions in a Labor Market Model

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Abstract

In this paper a variety of computational optimal control techniques are compared using a nonlinear labor market model.

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1 Introduction

In this paper we compare a variety of computational optimal control techniques to undertake the same problem. The problem we use for the comparison is that of determining the optimal number of long-term unemployed to enter Active Labour Market Programs (ALMPs) as a policy tool to reduce unemployment. The model we use is nonlinear with the nonlinearities inherent in the model due to a Nash bargaining rule between employers and workers, and Cobb-Douglas technology in a job-matching function. The model is complicated by distinguishing between various duration classes of short-term unemployment. These features make the model difficult for the application of optimal control techniques.

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The optimal control methods we use are, firstly, standard linear-quadratic techniques based on a liberalization of the model and the optimal control applied to the full nonlinear model. This approach has been frequently used in the literature of applying optimal control to nonlinear models. The second approach is that of applying nonlinear optimization to the stacked over time nonlinear model. This approach has also been used in the literature. The final approach we use is that of model based predictive control. This approach has recently been used with large complicated engineering models, but has not been applied as frequently to economic models.

2 The Model

The labour market model we use is based on a standard matching model framework. The model is developed and described in Herbert and Leeves (2003). A summary description of variables in the model is given in Tables 1, 2 and 3. The model consists of equations for wages, vacancies, job-matching, unemployment and the dynamics of stock movements. The model includes endogenous job creation and job destruction (UEV, NEV, ENV, ENV, EUV) and exogenous flows of workers between states. The time-step for the model is one month.

2.1 Wages

The wages equation is derived from the bargaining of employers and workers following a Nash bargaining rule. The resulting equation is:

$$w_t = \frac{\beta(y_{t-1} + c - \tau)\xi_{0,t} + (1-\beta)b\xi_{1,t}}{r + (1-\beta)(\xi_{2,t} + \xi_{3,t}) + \beta\xi_{4,t}} \tag{1}$$

where

$$\begin{aligned}
\xi_{0,t} &= r + \frac{F_{EU,t-1}}{E_{t-1}} + \frac{F_{UE,t-1}}{U_{t-1}} \\
\xi_{1,t} &= r + \xi_{2,t} + \xi_{3,t} \\
\xi_{2,t} &= \frac{\pi_{qn}F_{EN,t-1} + \pi_{qu}F_{EU,t-1}}{E_{t-1}} \\
\xi_{3,t} &= \frac{F_{UEV,t-1} + F_{NEV,t-1}}{V_{t-1}} \\
\xi_{4,t} &= \frac{F_{UE,t-1}}{U_{t-1}} + \frac{F_{EU,t-1}}{E_{t-1}}
\end{aligned}$$

2.2 Vacancies

Vacancies are also derived from the employer-worker bargaining and result in:

$$V_t = \frac{(F_{UEV,t-1} + F_{NEV,t-1})(y_{t-1} - w_{t-1} - \tau)}{c(r + \xi_{2,t-1})} \tag{2}$$

2.3 Job Matching Function

The job matching function uses Cobb-Douglas technologies to match the unemployed to vacancies. It differentiates between the short-term unemployed (U_S), the long-term unemployed who have participated in ALMPs (LP) and the long-term unemployed who have not participated in the ALMPs (LNP). The matching function is:

$$F_{UEV,t} = c_m V_{t-1}^{1-\alpha}(U_{S,t-1} + \theta(U_{LNP,t-1} + \sigma U_{LP,t-1})^{\alpha} \tag{3}$$

The policy choice (or control) is λ, where $\lambda = U_{LP}/(U_{LNP} + U_{LP})$.

2.4 Unemployment

Short-term unemployment is divided into 12 one-month duration classes, with

$$U_{1,t} = UI_t \tag{4}$$

and

$$U_{k,t} = (1 - \pi_{S,t})U_{k-1,t-1} \tag{5}$$

Total short-term unemployment is the summation of all duration classes:

$$U_{S,t} = \sum_{k=1}^{12} U_{k,t} \tag{6}$$

Long-term unemployment is given by:

$$U_{L,t} = U_t - U_{S,t} - \lambda_{t-1}U_{L,t-1} + \lambda_{t-6}U_{L,t-6} \tag{7}$$

There is an escape probability from short-term unemployment (π_S) and long-term unemployment (π_L). They are represented respectively by:

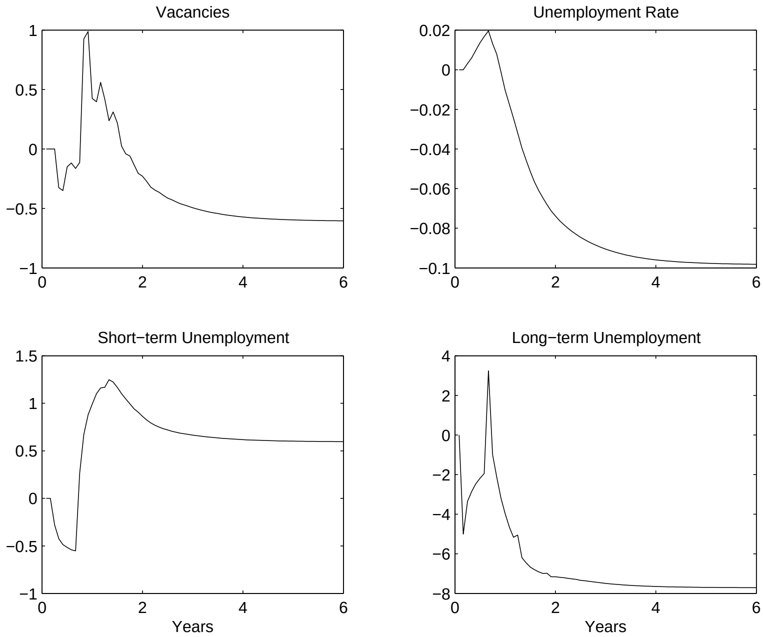

Fig. 1. Expansion in Labour Market Programs. (% Deviations from baseline.)

$$\pi_{S,t} = \frac{UO_t}{U_{S,t-1} + \theta U_{L,t-1}} \tag{8}$$

$$\pi_{L,t} = \theta \pi_{S,t} \tag{9}$$

$$\text{where } UO_t = F_{UEV,t} + F_{UEX,t} + UN_t \tag{10}$$

2.5 Job Creation and Destruction

The equations of motion that close the model are:

$$VI_t = V_t - V_{t-1} + VO_{U,t} + VO_{EX,t} \tag{11}$$

$$U_t = U_{t-1} + F_{EUV,t} + UI_{EX,t} - F_{UEV,t} - UO_{EX,t} \tag{12}$$

$$E_t = E_{t-1} + F_{UEV,t} + EI_{EX,t} - F_{EUV,t} - EO_{EX,t} \tag{13}$$

2.6 Calibration

The model is calibrated using Australian data for 1998 to produce a baseline steady-state. The numerical values of the calibration are given in Table 4. Figure 1 presents a step response for the model with an expansion in ALMPs from the baseline 5% to 10%. The results are presented as deviations from the baseline.

Table 1
Endogenous Variables in the Model.

Description	Variable
Wages	w
Vacancies	V
Inflow of vacancies	VI
Total unemployment	U
Total employment	E
Short-term unemployment	U_S
Monthly short-term unemployment	$U_k, \quad k = 1, ..., 12$
Long-term unemployment	U_L
Probability of escape from unemployment	π
Total flow from employment to unemployment	F_{EU}
Flow from unemployment to newly created jobs; job creation	F_{UEV}
Flow from employment to not-in-labour-force; job destruction	F_{ENV}
Flow from not-in-labour-force to newly created jobs; job creation	F_{NEV}
Flow of new vacancies to unemployed workers	VO_U

Table 2
Exogenous Variables in the Model.

Description	Variable
Flow of vacancies to non-unemployed workers	VO_{EX}
Outflow from unemployment to employment other than job creation	F_{UEX}
Outflow from employment to unemployment other than job destruction	F_{EUX}
Added value per worker	y
Interest rate	r
Lump-sum taxes	τ
Unemployment benefits	b
Proportion of long-term unemployed in ALMPs	λ

3 The Control Methods

3.1 Purpose of the Control

We consider the unemployment rate to be the output of the model. The control objective is to generate a policy for ALMPs that reduces the unemployment

Table 3
Flows that Contain Endogenous and Exogenous Components.

Description	Variable
Flow from employment to unemployment; job destruction	F_{EUV}
Total flow from unemployment to employment	F_{UE}
Total outflow from unemployment	UO
Total outflow from employment	EO
Total inflow to unemployment	UI
Total inflow to employment	EI

Table 4
Numerical values for baseline model (*=Number of persons/jobs x 1000, annual averages).

Variable	Value	Variable	Value
U*	750	F_{EU}*	600
F_{ENV}*	300	F_{UEV}*	200
F_{NEV}*	400	F_{EUV}*	300
E*	8,600	UO*	4,600
V*	75	r	0.03
θ	0.9	c	0.9
α	0.5	EO*	4,200
VI*	600	τ	0.08
VO_U*	200	y	1
VO_{EX}*	400	b	$0.3w_{ss}$

rate to 5% and has no participants in ALMPs. This unemployment target is a substantial reduction from the baseline rate of nearly 8%, and is a target that will be difficult for any method to achieve.

The input to the model (or control) is the proportion of the long-term unemployed on ALMPs (λ). The control target is zero. There is no constraint within the model on this proportion so the control method should incorporate a realistic constraint. We use the constraint of $0 \leq \lambda \leq 0.5$ so that a maximum of 50% of the long-term unemployed can participate in ALMPs in any one month.

We follow the convention of using a quadratic objective function where the optimization aims to reduce the sum of squared deviations from the output (unemployment) and control (λ) targets. The deviations are weighted and we

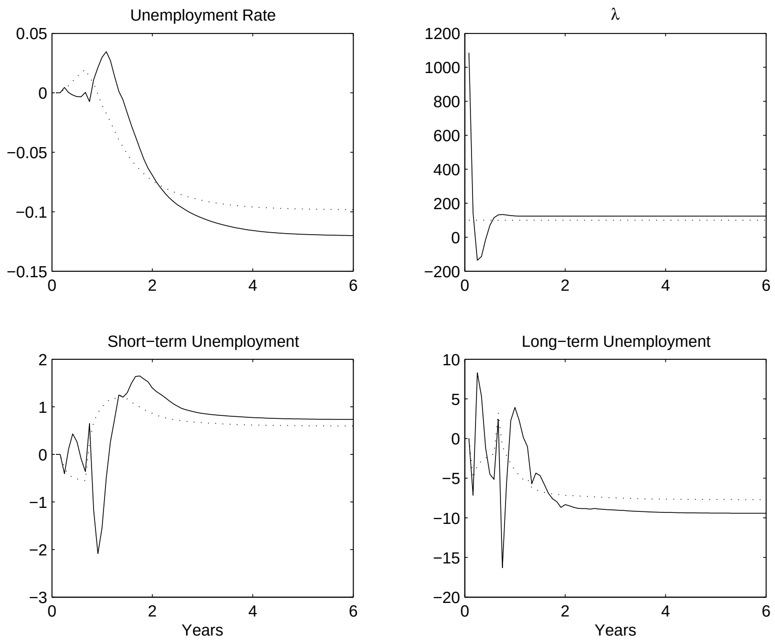

Fig. 2. Linear Model Based Control. (% deviation from baseline, except λ. Dashed Line is the Response from Figure 1.)

use weights of unity. This is a conventional quadratic output tracking objective (social loss) function.

3.2 Linear Model

For the linear model based control method we use the standard linear-quadratic output tracking approach (Herbert, 1998) with a linear model derived from the full nonlinear model. The linear model is numerically estimated by perturbing the input to the full nonlinear model.

The advantage of the linear-quadratic approach is that the optimization can be analytically solved, and the solution programmed. The control policy is developed from the linear model dynamically at each time step. We apply this control to the full nonlinear model.

The disadvantage of this method is that it is dependant upon the accuracy of linear model and that constraints on the control cannot be included explicitly as part of the control generation procedure. In the literature the usual approach of implementing constraints is to varying the relative weights between the output and control tracking errors in the objective function.

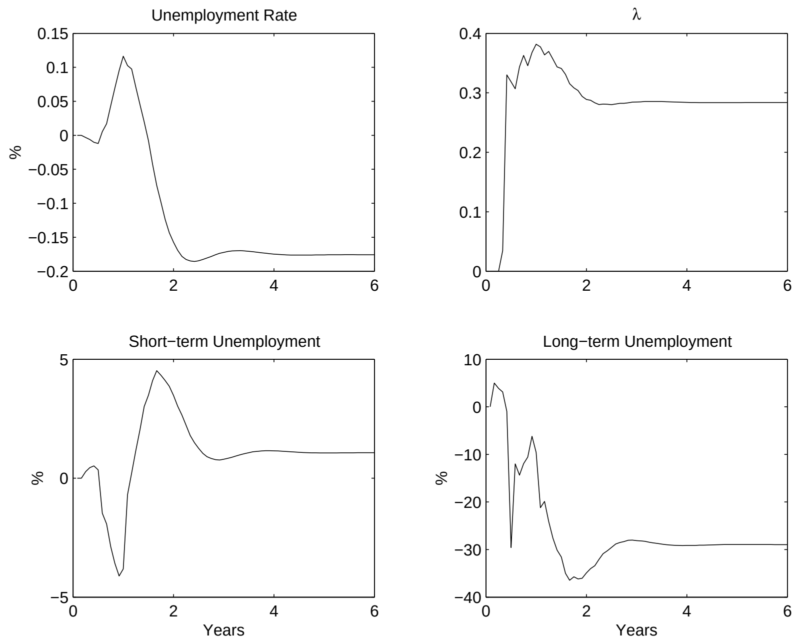

Fig. 3. Model Based Predictive Control. (% deviation from baseline, except λ).

3.3 Model Based Predictive Control

Model based predictive control (MBPC) is a suite of control methods where a linear model is used to predict the future output of the full nonlinear model; and the control rule is developed from the prediction is applied to the full nonlinear model. As with the linear-quadratic approach, the control is developed dynamically (at each time-step) and is applied to the full nonlinear model at the same time-step; but unlike the linear-quadratic approach the linear model is regenerated dynamically as part of the control rule generation. Details of the methods can be found in Rossiter (2003) and we use the approach developed in Herbert and Bell (2004).

The MBPC approach attempts to overcome the accuracy of the linear model in the linear-quadratic method by using a time-varying linear model. In the results here, we also add a learning component so the linear model adjusts according to its difference in predicted output from the nonlinear model (Herbert, 1998).

The MBPC approach also allows for explicit constraints to be added in the control rule generation. The constraints can be placed on the level and rate of change of the output as well as the control. Such constraints are much more realistic with labor market models where there are plausible policy strategies and un-modelled components.

MBPC is more complex mathematically than the linear-quadratic approach.

Fig. 4. Nonlinear Optimization. (% deviation from baseline, except λ).

The MBPC algorithm results in a quadratic programming problem.

3.4 Nonlinear Optimization

The nonlinear optimization approach uses the full nonlinear model and stacks it up over time. The model effectively becomes a static model for the entire time horizon rather than a dynamic model which is solved one time-period after another. The same variable at a different time-period is treated as a different variable so that the number of equations in the original model is multiplied by the number of time-steps in the time horizon resulting in a much larger model. The resulting stacked model is solved once. To determine the optimal control, the objective function is stacked over the time horizon. A nonlinear optimization algorithm is then used to minimize the stacked objective function against the stacked model.

The advantage in this approach is that it is more general with constraints or other requirements simply included in the code that implements the objective function. The objective function need not be quadratic.

Another advantage with the nonlinear optimization approach is that the is a single optimization, and the entire trajectory for the output and control is found in the single optimization.

The disadvantage is that it is a large optimization problem that generally has to be solved by numeric methods. We use purely numeric optimization approaches including quasi-Newton and direct simplex searches.

4 Results

4.1 Policy Strategy

The results from the linear-quadratic control method are given in Figure 2. The lack of the constraint on the control can be clearly seen as the suggested changes in ALMPs commencements fall outside the range of a plausible policy strategy, with the maximum value of λ of over 1000 being nonsensical.

The results from the MBPC control method are given in Figure 3. When compared to Figure 2, we see that (a) the values of λ are realistic (b) better results are obtained for the unemployment rate and long term unemployment without any detriment to short term unemployment and (c) there is less oscillatory transient dynamics.

The results for the nonlinear optimization method are given in Figure 4. The first point to note is that these results are for a shorter time horizon than the previous methods. This is due to the fact that for the longer time horizon none of the optimization algorithms was successful in reducing the objective function. When compared to the linear-quadratic and MBPC methods we see that (a) no steady-state value for λ is achieved and (b) the results for the unemployment rate and long-term unemployment are not as good as with the previous methods.

4.2 Computational Effort

The programming effort and the computational effort of the program for solving the linear-quadratic method is not extensive. Basically it is using linear algebra and the solving of matrix Ricatti equations for a time step. The optimal control is then applied to the full nonlinear model. This procedure continues for each time-step in the time horizon.

The MBPC method requires more effort. At each time step the full nonlinear model is predicted for a predetermined time window and the from the output a predictive model is developed. From the predictive model the optimal control is generated and it is applied to the current time-step of the full nonlinear model. The optimal control generation requires the solution of a quadratic programming problem. The MBPC method thence requires more programming and computational effort than the linear-quadratic approach.

The nonlinear optimization requires the most effort. For the results presented in Figure 4 the Nelder-Mead direct simplex search algorithm was used. The algorithm required over 20,000 evaluations of the stacked nonlinear model to produce these results. Using a quasi-Newton optimization approach the algorithm failed to reduce the objective function after 100,000 evaluations of the stacked model.

5 Conclusion

This paper compared three methods of deriving optimal control for the same nonlinear labor market model. The methods were the standard linear-quadratic control with a derived linear model and control applied to the nonlinear model; model based predictive control; and a fully nonlinear optimization with the model stacked over time. The paper found that for this model the MBPC produced the best results.

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