A Dynamic Programming Approach for Pricing Options Embedded in Bonds

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Abstract

The aim of this paper is to price options embedded in bonds in a Dynamic Programming (DP) framework, the focus being on call and put options with advance notice. The pricing of interest rate derivatives was usually done via trees or finite differences. Trees are not really very efficient as they deform crudely the dynamic of the underlying asset(s), here the short term risk-free interest rate. They can be interpreted as elementary DP procedures with fixed grid sizes. For a long time, finite differences presented poor accuracy because of the discontinuities of the bond's value that may arise at decision dates. Recently, remedies were given by d'Halluin et al (2001) via techniques related to flux limiters. DP does not suffer from discontinuities that may arise at decision dates and does not require a time discretization. It may also be implemented in discrete-time models. Results show efficiency and robustness. Suggestions to combine DP and finite differences are also formulated.

1 Introduction

A *bond* is a contract that pays to its holder a known amount, the *principal*, at a known future date, called the *maturity*. A bond may also pay periodically to its holder fixed cash dividends, called the *coupons*. Otherwise, it is known as a *zero-coupon bond*. A bond can be interpreted as a loan with a known principal and interest payments equal to the coupons (if any). The borrower is the *issuer* of the bond and the lender, that is, the holder of the bond, is the *investor*.

Several bonds contain one or several options coming in various flavors. The *call option* gives the issuer of the bond the right to purchase back its debt for a known amount, the *call price*, during a specified period within the bond's life. Several government bonds contain a call feature [see Bliss and Ronn (1995) for the history of callable US Treasury bonds from 1917]. The *put option* gives the investor the right to return the bond to the issuer for a known amount, the *put price*, during a specified period within the bond's life. These options are an integral part of a bond, and cannot be traded alone as is the case for call and put options on stocks (for example). They are said to be *embedded* in the bond. In general, they are of the *American-type* and, thus, allow for early exercising, so that the bond with its embedded options can be interpreted as an American-style interest rate derivative. This paper focuses on call and put options embedded in bonds with *advance notice*, that is, options with exercise decisions prior to exercise benefits.

There are no analytical formulas for valuing American options, even under very simplified assumptions. Numerical methods, essentially trees and finite differences, are usually used for pricing. Recall that trees are numerical representations of discrete-time models and finite differences are numerical solutions of partial differential equations.

As an alternative approach, the pricing of American financial derivatives can be formulated as a Markov Decision process, that is, a stochastic *Dynamic Programming* (DP) problem, as pointed out by Barraquand and Martineau (1995). Here, the DP function, that is, the value of the bond with its embedded options, is a function of the current time and of the current short term risk-free interest rate, namely the *state variable*. This value function verifies a DP recurrence (known also as the *Bellman* equation) via the no-arbitrage principle of asset pricing (Elliott and Kopp, 1999). The key point with DP is to solve efficiently the DP equation, which yields both the bond value and the optimal exercise strategies of its embedded options. For an overview of stochastic DP, see Bertsekas (1987).

As pointed out by Chan et al (1992), most of the alternative dynamics for the short term risk-free interest rate are described by the general stochastic differential equation

$$dr_t = \kappa\left(\overline{r} - r_t\right)dt + \sigma r_t^{\gamma} dB_t, \quad \text{for } 0 \leq t \leq T, \tag{1}$$

where κ, the *reverting rate*, $\overline{r}$, the *reverting level*, σ, the *volatility*, and γ are real parameters, and $\{B\}$ is a standard Brownian motion. No-arbitrage pricing evolves a change of measure and, thus, a change on the dynamic of $\{r\}$ that is discussed later in the text. Table 1 presents various versions of (1) used in the literature.

Table 1: Models for the short term risk-free interest rate

Model	$\overline{r}$	κ	σ	γ
1. Vasicek (1977)				0
2. Brennan-Schwartz (1977)		0		1
3. Brennan-Schwartz (1980)				1
4. Marsh-Rosenfeld (1983)	0			
5. Cox-Ingersoll-Ross (1985)				1/2

Vasicek (1977) used a mean-reverting Ornstein-Uhlenbeck process. This model gives nice distributional results and closed-form solutions for zero-coupon bonds and for several European-style interest rate derivatives. But it has the undesirable property of allowing negative interest rates, though with very low probabilities. Several authors took advantage of its properties to price various interest rate derivatives, often in closed-form. Examples include Jamshidian (1989) and Rabinovitch (1989). Brennan and Schwartz (1977, 1980) are pioneers on the modeling of options embedded in bonds. They let the interest rate move as a geometric Brownian motion without a drift to price the call and the put options (Model 2) and as a mean-reverting proportional process to price the conversion option (Model 3). Model 2 was also used by Dothan (1978) to price bonds in closed-form, and Model 3 by Courtadon (1982) to price several European as well as American options on bonds. Notice that Model 2 is a special case of Model 3 and that the latter includes the geometric Brownian motion of Black and Scholes (1973). Model 4 presents the so-called constant elasticity of variance process (Cox, 1996). It was considered by Marsh and Rosenfeld (1983), among others, as an alternative process for the interest rate. Ingersoll, and Ross (1985) (CIR) used the mean-reverting square-root process to handle the interest rate movements. This model is extendible to several factors, ensures strictly positive interest rates, and gives closed-form solutions for zero-coupon bonds and for some European-style interest rate derivatives. Several authors used the CIR model to price various interest rate derivatives (Richard 1978, Ananthanarayanan and Schwartz 1980, and Schaefer and Schwartz 1984).

For the models described by (1), matching all theoretical bond values with their market counterparts is unfeasible because this gives much more equations than the number of parameters to estimate. A remedy, proposed by Hull and White (1990a), is to augment the model, that is, to add parameters until a calibration becomes possible. This leads to the extended Vasicek and CIR models. Hull and White (1990b, 1993, 1994a, 1994b, 1996) interpreted the finite differences method as a trinomial tree, and priced several interest rate derivatives within their extended models. See also the note by Carverhill (1995) and the response by Hull and White (1995) for a discussion about the performance of these models.

If closed-form solutions could not be derived, the pricing of interest rate derivatives was done via trees or finite differences. Trees are not really very efficient as they deform crudely the dynamic of the interest rate. In fact, they can be interpreted as elementary DP procedures with a fixed grid size. Finite differences, as used since Brennan and Schwartz (1977), presented poor accuracy because of the discontinuities of the bond's value that may arise at decision dates. Recently, remedies were given by d'Halluin et al (2001) via techniques related to flux limiters. Another interesting approach, suggested by Büttler and Waldvogel (1996), prices callable bonds by means of Green's function. But the method by d'Halluin et al (2001), as shown by these authors, looks more general and competitive.

In this paper, DP is presented as a viable alternative for pricing. Unlike finite differences, DP does not suffer from discontinuities that may arise at decision dates, and does not require a time discretization. It also may be implemented in discrete-time models. Comparisons with Büttler and Waldvogel (1996) and with d'Halluin et al (2001) are made. Results show efficiency and robustness. Suggestions to combine DP with finite differences are also discussed.

The rest of the paper is organized as follows. Section 2 presents the DP formulation. Results are given in Section 3. Section 4 concludes.

2 The DP Formulation

2.1 The DP equation

In this section, the DP function and the DP equation for a bond with its embedded call and put options are presented. No-arbitrage pricing is used to assess the DP formulation.

Let $t_0, \ldots, t_n$ be a sequence of dates, where $t_0 = 0$ is the origin, $t_1, \ldots, t_{n-1}$ are the coupon dates, and $t_n = T$ is the maturity of the bond, that is, the date the principal and the last coupon are due. The principal (in dollars) is denoted by P and the coupon (in dollars) by $C = Pc/l$, where c is the coupon rate per year and l the number of coupons paid per year. The periods $\Delta t = t_{m+1} - t_m$, for $m = 1, \ldots, n-1$, are equal except perhaps for the the first one $t_1 - t_0 \leq t_2 - t_1$. Assume also that the exercise decisions of the call and put options are at τ_m and that the exercise benefits are at the coupon dates $t_m = \tau_m + \Delta t$, for $m = n^*, \ldots, n$. The first exercise date τ_{n*} is known. The exercise decision and benefit dates verify $t_{m-1} < \tau_m \leq t_m$, for $m = n^*, \ldots, n$. The lag $\Delta t = t_m - \tau_m$ is called the *notice period* and the increment of time $\tau_{n^*} - t_0$ the *protection period* (against early exercising).

Let C_m and P_m be the *call* and *put prices* at t_m, for $m = n^*, \ldots, n$, respectively. Thus, if the issuer calls back the bond at τ_m, he pays C_m to the investor at t_m, and, similarly, if the investor puts the bond at τ_m, he receives P_m from the issuer at t_m. Assume that the call and put prices verify $0 \leq P_m \leq C_m$, as is usual in practice, and that $C_n = P_n = P$.

Table 2 gives the exercise benefits at time t_m, for $m = n^*, \ldots, n$, by the issuer to the investor under decision pairs at τ_m.

Table 2: Exercise benefits under decision pairs

	Investor	
Issuer	Put	Hold
Call	Sub-optimal	C_m
Not to Call	P_m	No exercise

At the maturity date $t_n = T$, the value of the bond is equal to

$$v_{t_n}(r) = P + C, \quad \text{for all } r, \tag{2}$$

where $r = r_{t_n}$ is the current interest rate at time t_n.

By the no-arbitrage principle of asset pricing, the value of the bond with its embedded options at time τ_m, for $m = n, \ldots, n^*$, is

$$v_{\tau_m}(r) \tag{3}$$
$$= \begin{cases} E_{m,r}\left[C_m e^{-\int_{\tau_m}^{t_m} r_t dt} + C e^{-\int_{\tau_m}^{t_m} r_t dt}\right] & \text{if the issuer calls} \\ E_{m,r}\left[v_{\tau_{m+1}}\left(r_{\tau_{m+1}}\right) e^{-\int_{\tau_m}^{\tau_{m+1}} r_t dt} + C e^{-\int_{\tau_m}^{t_m} r_t dt}\right] & \text{if the investor holds} \\ E_{m,r}\left[P_m e^{-\int_{\tau_m}^{t_m} r_t dt} + C e^{-\int_{\tau_m}^{t_m} r_t dt}\right] & \text{if the investor puts} \end{cases},$$

where $E_{m,r} = E\left[\cdot \mid r_{\tau_m} = r\right]$ points out the conditional expectation operator under the so-called *risk-neutral probability measure* with the convention that $\tau_{n+1} = t_n = T$. The optimal exercising strategies are as follows. The issuer will call the bond at τ_m if its (net) holding value exceeds the exercise benefit, that is,

$$E_{m,r}\left[v_{\tau_{m+1}}\left(r_{\tau_{m+1}}\right) e^{-\int_{\tau_m}^{\tau_{m+1}} r_t dt}\right] > E_{m,r}\left[C_m e^{-\int_{\tau_m}^{t_m} r_t dt}\right] = C_m \rho\left(r, \tau_m, t_m\right),$$

where $\rho\left(r, \tau_m, t_m\right)$ is the discount factor over the period $\left[\tau_m, t_m\right]$ at $r_{\tau_m} = r$. This holds for $r < r_c^m$, where r_c^m is the *break-even interest rate* associated to the call option at step m. On the other side, the investor will put the bond at τ_m if the exercise benefit exceeds the (net) holding value, that is,

$$E_{m,r}\left[P_m e^{-\int_{\tau_m}^{t_m} r_t dt}\right] = P_m \rho\left(r, \tau_m, t_m\right) > E_{m,r}\left[v_{\tau_{m+1}}\left(r_{\tau_{m+1}}\right) e^{-\int_{\tau_m}^{\tau_{m+1}} r_t dt}\right].$$

This holds for $r > r_p^m$, where r_p^m is the *break-even interest rate* associated to the put option at time τ_m. Otherwise, the bond will be held at least for another period and the value of the bond will coincide with its holding value, denoted by $v_{\tau_m}^h(\cdot)$. Notice that it cannot be optimal for the issuer to call and for the investor to put simultaneously since $0 \leq P_m \leq C_m$.

At the origin, the value of the bond is

$$v_{\tau_0}(r) = E_{0,r}\left[v_{\tau_{n^*}}\left(r_{\tau_{n^*}}\right) e^{-\int_{\tau_0}^{\tau_{n^*}} r_t dt}\right] + C\left[\rho\left(r, \tau_0, t_{n^*}\right) + \cdots + \rho\left(r, \tau_0, t_1\right)\right], \tag{4}$$

with the convention that $\tau_0 = t_0$.

Equations (2)-(4) define the stochastic DP formulation. This is the DP function at maturity, at a given step m, for $m = n, \cdots, n^*$, and at origin. Solving the DP equation(s) backwards from the maturity to the origin yields both the initial value of the bond and the optimal exercise strategies of its embedded options.

2.2 The DP procedure

Solving the DP equation requires to approximate the DP function in some way. Piecewise linear interpolations are used.

Let $a_0 < a_1 < \ldots < a_p < a_{p+1} = +\infty$ be a set of points that form a partition of $\mathbb{R}$. Given an approximation $\widetilde{v}_{\tau_{m+1}}$ of the bond's value $v_{\tau_{m+1}}(\cdot)$ at the points a_k and step $m+1$, a piecewise linear interpolation gives

$$\widehat{v}_{\tau_{m+1}}(r) = \sum_{i=0}^{p} \left(\alpha_i^{m+1} + \beta_i^{m+1} r\right) I\left(a_i < r \leq a_{i+1}\right),$$

where $I(\cdot)$ is the indicator function. Its local coefficients α_i^{m+1} and β_i^{m+1} are obtained by solving the linear equations

$$\widetilde{v}_{\tau_{m+1}}(a_i) = \widehat{v}_{\tau_{m+1}}(a_i), \quad \text{for } i = 1, \ldots, p-1, \tag{5}$$

whereas at $i = 0$ and $i = p$, they are identical to those of the adjacent interval. Others approximations, such as high order polynomials or splines may be used. See de Boor (1978) for a general discussion.

Assume now that $\widehat{v}_{\tau_{m+1}}(\cdot)$ is known, and so are its local coefficients. The DP function at step m becomes

$$\begin{aligned}
\widetilde{v}_{\tau_m}^h(a_k) &= E_{m,a_k}\left[\widehat{v}_{\tau_{m+1}}\left(r_{\tau_{m+1}}\right) e^{-\int_{\tau_m}^{\tau_{m+1}} r_t dt}\right] + C\rho\left(a_k, \tau_m, t_m\right) \qquad (6) \\
&= \sum_{i=0}^{p} \left(\alpha_i^{m+1} E_{m,a_k}\left[I\left(a_i \leq r_{t_{m+1}} < a_{i+1}\right) e^{-\int_{\tau_m}^{\tau_{m+1}} r_t dt}\right] + \right. \\
&\quad \left. \beta_i^{m+1} E_{m,a_k}\left[r_{\tau_{m+1}} I\left(a_i \leq r_{\tau_{m+1}} < a_{i+1}\right) e^{-\int_{\tau_m}^{\tau_{m+1}} r_t dt}\right]\right) + \\
&\quad C\rho\left(a_k, \tau_m, t_m\right) \\
&= \sum_{i=0}^{p} \left(\alpha_i^{m+1} A_{k,i}^m + \beta_i^{m+1} B_{k,i}^m\right) + C\rho\left(a_k, \tau_m, t_m\right),
\end{aligned}$$

where $\widetilde{v}_{\tau_m}^h(a_k)$ is the approximate holding value of the bond at $a_k = r_{\tau_m}$, and the $A_{k,i}^m$'s and $B_{k,i}^m$'s are transition parameters that are model specific. Now, given $\widetilde{v}_{\tau_m}^h(a_k)$, one may compute by (3) $\widetilde{v}_{\tau_m}(a_k)$, for $k = 1, \ldots, p$, interpolate and get by (5) $\widehat{v}_{\tau_m}(r)$, for all r. Solving in that way the DP equations backwards from the maturity to the origin yields both the initial value of the bond and the optimal exercise strategies of its embedded options.

From (6), the efficiency of the DP procedure depends directly on the efficiency of computing the discount factor and the transition parameters. For models where these ingredients can be computed in a closed-form, the bond with its embedded options may be priced practically in a quasi-closed form. Examples, as shown in the next subsection, include the Vasicek and CIR models. In addition, for these models, the $A_{k,i}^m$'s and $B_{k,i}^m$'s depend on the step m only through $\tau_{m+1} - \tau_m$, and the discount factor $\rho(a_k, \tau_m, t_m)$ on τ_m and t_m only through $t_m - \tau_m$. This is of course a desirable property that will accelerate further the DP procedure.

2.3 Transition Parameters

Clearly, the $A_{k,i}^m$'s and $B_{k,i}^m$'s are related to the conditional distribution

$$\left(r_{t'}, \int_{t'}^{t''} r_t dt\right) \mid r_{t'} = r, \text{ for } 0 \leq t' \leq t \leq t'' \leq T. \tag{7}$$

Theorem 2 and Proposition 3 characterize the conditional distribution in (7) and derive the exact associated transition coefficients for the Vasicek model. Proposition 4 give similar results for the CIR model.

Lemma 1 *For f and g two real functions continuously differentiable in $[t', t'']$, for $0 \leq t' \leq t'' \leq T$, and $\{W\}$ a standard Brownian motion, one has*

$$\begin{aligned}
& \int_{t'}^{t''} \left(\int_{t'}^{u} f(t)\, g(u)\, dW(t)\right) du \\
= & \int_{t'}^{t''} \left(\int_{t'}^{t''} f(t)\, g(u)\, I\left(t \in [t', u]\right) dW(t)\right) du \\
= & \int_{t'}^{t''} \left(\int_{t'}^{t''} f(t)\, g(u)\, I\left(u \in [t, t'']\right) du\right) dW(t) \\
= & \int_{t'}^{t''} \left(\int_{t}^{t''} f(t)\, g(u)\, du\right) dW(t)\,.
\end{aligned}$$

Proof. One can use the integration by parts theorem in stochastic calculus (Øksendal, 1995). Define the function $h(t) = f(t) \int_t^{t''} g(u)\, du$ and transform the right hand integral as

$$\begin{aligned}
\int_{t'}^{t''} \left(\int_t^{t''} f(t)\, g(u)\, du \right) dW(t) &= \int_{t'}^{t''} h(t)\, dW(t) \\
&= h(t'')\, W(t'') - h(t')\, W(t') - \\
&\quad \int_{t'}^{t''} \frac{\partial h}{\partial t}(t)\, W(t)\, dt \\
&= -\int_{t'}^{t''} f(t')\, g(u)\, W(t')\, du \\
&\quad + \int_{t'}^{t''} f(t)\, g(t)\, W(t)\, dt \\
&\quad - \int_{t'}^{t''} \left(\int_t^{t''} \frac{\partial f}{\partial t}(t)\, g(u)\, W(t)\, du \right) dt.
\end{aligned}$$

Now, one can use the same theorem to transform $\int_{t'}^{u} f(t)\, g(u)\, dW(t)$ and thereafter the left hand integral as

$$\begin{aligned}
\int_{t'}^{t''} \left(\int_{t'}^{u} f(t)\, g(u)\, dW(t) \right) du &= \int_{t'}^{t''} f(u)\, g(u)\, W(u)\, du - \\
&\quad \int_{t'}^{t''} f(t')\, g(u)\, W(t')\, du - \\
&\quad \int_{t'}^{t''} \left(\int_{t'}^{u} \frac{\partial f}{\partial t}(t)\, g(u)\, W(t)\, dt \right) du.
\end{aligned}$$

The final result comes from the basic properties of multi-dimensional real integrals. ■

Theorem 2 *For the Vasicek model,*

$$\left(r_{t''}, \int_{t'}^{t''} r_u du\right) \mid r_{t'} = r, \text{ for } 0 \leq t' \leq t'' \leq T,$$

is normal with mean

$$\mu(r) = (\mu_1(r), \mu_2(r)) = \left(\overline{r} + e^{-\kappa \Delta t}(r - \overline{r}), \overline{r}\Delta t + \frac{1 - e^{-\kappa \Delta t}}{\kappa}(r - \overline{r})\right)$$

and variance

$$\Sigma = \begin{bmatrix} \sigma_1^2 = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\Delta t}\right) & \sigma_{12} = \frac{\sigma^2}{2\kappa^2}\left(1 - 2e^{-\kappa\Delta t} + e^{-2\kappa\Delta t}\right) \\ \sigma_{21} = \sigma_{12} & \sigma_2^2 = \frac{\sigma^2}{2\kappa^3}\left(-3 + 2\kappa\Delta t + 4e^{-\kappa\Delta t} - e^{-2\kappa\Delta t}\right) \end{bmatrix},$$

where $\Delta t = t'' - t'$.

Proof. From the Vasicek model, one can apply Ito's lemma to the process $\phi(t, r_t) = e^{\kappa t} r_t$, for $0 \leq t \leq T$, and show that

$$r_u = \overline{r} + e^{-\kappa(u-t')}(r - \overline{r}) + \sigma \int_{t'}^{u} e^{-\kappa(u-t)} dB(t),$$

and consequently that

$$\begin{aligned} \int_{t'}^{t''} r_u du &= \overline{r}\Delta t + \frac{1 - e^{-\kappa\Delta t}}{\kappa}(r - \overline{r}) + \sigma \int_{t'}^{t''} \left(\int_{t'}^{u} e^{-\kappa(u-t)} dB(t)\right) du \\ &= \overline{r}\Delta t + \frac{1 - e^{-\kappa\Delta t}}{\kappa}(r - \overline{r}) + \sigma \int_{t'}^{t''} \left(\int_{t}^{t''} e^{-\kappa(u-t)} du\right) dB(t). \end{aligned}$$

The last equality comes from Lemma 1. Conditioning on the information available at time t', one can decompose $r_{t''}$ and $\int_{t'}^{t''} r_u du$ into a deterministic part and a random part. The latter part turns out to be a limit of linear combinations of the same standard Brownian motion taken at different points in time. The random variables $r_{t''}$ and $\int_{t'}^{t''} r_u du$, conditioned on $r_{t'} = r$, are thus jointly normal.

Now, from basic properties of stochastic integrals (Øksendal, 1995), one can derive the conditional mean and the conditional variance of the vector

$\left(r_{t''}, \int_{t'}^{t''} r_u du\right)$. Its conditional mean is

$$\begin{aligned} & E\left[\left(r_{t''}, \int_{t'}^{t''} r_u du\right) \mid r_{t'} = r\right] \\ = & \left(\overline{r} + e^{-\kappa\Delta t}\left(r - \overline{r}\right), \overline{r}\Delta t + \frac{1 - e^{-\kappa\Delta t}}{\kappa}\left(r - \overline{r}\right)\right), \end{aligned}$$

since the centered random vector

$$\left(\int_{t'}^{t''} e^{-\kappa(u-t)} dB(t), \int_{t'}^{t''}\left(\int_{t}^{t''} e^{-\kappa(u-t)} du\right) dB(t)\right),$$

is independent of $\{r_t, t \leq t'\}$. The conditional variance of $r_{t''}$ is

$$\begin{aligned} \mathrm{Var}\left[r_{t''} \mid r_{t'} = r\right] &= E\left[\left(\int_{t'}^{t''} \sigma e^{-\kappa(t''-t)} dB(t)\right)^2 \mid r_{t'} = r\right] \\ &= \sigma^2 \int_{t'}^{t''} e^{-2\kappa(t''-t)} dt = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\Delta t}\right). \end{aligned}$$

The conditional variance of $\int_{t'}^{t''} r_u du$ is

$$\begin{aligned} & \mathrm{Var}\left[\int_{t'}^{t''} r_u du \mid r_{t'} = r\right] \\ = & E\left[\left(\int_{t'}^{t''}\left(\int_{t}^{t''} \sigma e^{-\kappa(u-t)} du\right) dB(t)\right)^2 \mid \mathcal{F}(t')\right] \\ = & \sigma^2 \int_{t'}^{t''}\left(\int_{t}^{t''} e^{-\kappa(u-t)} du\right)^2 dt = \frac{\sigma^2}{2\kappa^3}\left(-3 + 2\kappa\Delta t + 4e^{-\kappa\Delta t} - e^{-2\kappa\Delta t}\right). \end{aligned}$$

The conditional covariance between $r_{t''}$ and $\int_{t'}^{t''} r_u du$ is

$$\begin{aligned} & \mathrm{Cov}\left[r_{t''}, \int_{t'}^{t''} r_u du \mid r_{t'} = r\right] \\ = & E\left[\int_{t'}^{t''} \sigma e^{-\kappa(t''-t)} dB(t) \int_{t'}^{t''}\left(\int_{t}^{t''} \sigma e^{-\kappa(u-t)} du\right) dB(t) \mid r_{t'} = r\right] \\ = & \sigma^2 \int_{t'}^{t''} e^{-\kappa(t''-t)}\left(\int_{t}^{t''} e^{-\kappa(u-t)} du\right) dt = \frac{\sigma^2}{2\kappa^2}\left(1 - 2e^{-\kappa\Delta t} + e^{-2\kappa\Delta t}\right). \end{aligned}$$

■

Proposition 3 *For the Vasicek model, from Theorem 2, one has*

$$A_{k,i}^{\Delta t} = e^{-\mu_2(a_k)+\sigma_2^2/2} \left[\Phi\left(x_{k,i}\right) - \Phi\left(x_{k,i-1}\right)\right]$$

and

$$\begin{aligned} B_{k,i}^{\Delta t} &= e^{-\mu_2(a_k)+\sigma_2^2/2} \left[\left(\mu_1\left(a_k\right) - \sigma_{12}\right)\left(\Phi\left(x_{k,i}\right) - \Phi\left(x_{k,i-1}\right)\right) - \right. \\ &\quad \left. \sigma_1 \left(e^{-x_{k,i}^2} - e^{-x_{k,i-1}^2}\right) / \sqrt{2\pi}\right], \end{aligned}$$

where $\Delta t = t'' - t'$, Φ *is the cumulative density function of the standard normal distribution, and*

$$x_{k,j} = \left(a_j - \mu_1\left(a_k\right) + \sigma_{12}\right)/\sigma_1, \textit{ for } j \in \{i-1, i\}.$$

Proposition 4 *For the CIR model, the conditional distribution in (7) is known by its Laplace transform. For* $\Delta t = t'' - t'$, *one has*

$$A_{k,i}^{\Delta t} = \rho\left(a_k, \Delta t\right)\left(\sum_{t=0}^{+\infty} e^{-\lambda_k/2} \frac{\left(\lambda_k/2\right)^t}{t!}\left(\Phi_{d+2t}\left(\frac{a_{i+1}}{\mu}\right) - \Phi_{d+2t}\left(\frac{a_i}{\mu}\right)\right)\right),$$

and

$$\begin{aligned} B_{k,i}^{\Delta t} &= \rho\left(a_k, \Delta t\right)\mu \sum_{t=0}^{\infty} e^{-\lambda_k/2}\frac{\left(\lambda_k/2\right)^t}{t!}\left[-2\left(a_{i+1}\phi_{d+2t}(\frac{a_{i+1}}{\mu}) - a_i\phi_{d+2t}(\frac{a_i}{\mu})\right)\right. \\ &\quad \left. +(d+2t) \times \left(\Phi_{d+2t}(\frac{a_{i+1}}{\mu}) - \Phi_{d+2t}(\frac{a_i}{\mu})\right)\right], \end{aligned}$$

where $\phi_{d+2t}(\cdot)$ *and* $\Phi_{d+2t}(\cdot)$ *are respectively the density function and the cumulative density function of the* χ^2 *distribution with* $d+2t$ *degrees of freedom,* $\rho\left(a_k, \Delta t\right) = \rho\left(a_k, t', t''\right)$ *is the discount factor over* $[t', t'']$ *at* $a_k = r_{t'}$,

$$\begin{aligned} \gamma &= \sqrt{\kappa^2 + 2\sigma^2},\ \mu = \frac{\sigma^2(e^{\gamma\Delta t} - 1)}{2\left[(\gamma+\kappa)(e^{\gamma\Delta t} - 1) + 2\gamma\right]},\ d = \frac{4\kappa\overline{r}}{\sigma^2}, \textit{ and} \\ \lambda_k &= \frac{8\gamma^2 e^{\gamma\Delta t} a_k}{\sigma^2\left[(\gamma+\kappa)(e^{\gamma\Delta t} - 1) + 2\gamma\right](e^{\gamma\Delta t} - 1)}. \end{aligned}$$

3 Results

The numerical investigation compares our results to those of Büttler and Waldvogel (1996) and d'Halluin et al (2001). The example is taken from Büttler and Waldvogel (1996). Thereafter, the work by d'Halluin et al is denoted by DFVL, and by Büttler and Waldvogel by BW.

The security to be priced is a 4.25% callable bond issued by the Swiss Confederation with a life period 1987-2012. At the pricing date $t_0 = 0$, December 23, 1991, the bond had a maturity $T = t_n = 20.172$ years with $n = 21$, a principal $P = 1$ \$, a coupon $C = 0.0425$ \$ coming once per year with a first coupon coming at $t_1 = 0.172$, a notice period of 2 months, that is, $\Delta t = t_m - \tau_m = 0.1666$, and a protection period $t_{n^*} = 10.172$ years with $n^* = 11$. The call prices are $C_{11} = 1.025$ \$, $C_{12} = 1.020$ \$, $C_{13} = 1.015$ \$, $C_{14} = 1.010$ \$, $C_{15} = 1.005$ \$, and $C_{16} = \cdots = C_{21} = 1$ \$.

The parameters $\overline{r}$, κ, σ were estimated by fitting the theoretical term structure at the pricing date. They are then adjusted via the *price of risk*, denoted here by q, to catch the dynamic of the interest rate under the risk-neutral probability measure, required by the no-arbitrage pricing. The models in (1) become

$$dr_t = \kappa^* \left(\overline{r}^* - r_t\right) dt + \sigma^* r_t^\gamma dB_t, \quad \text{for } 0 \leq t \leq T.$$

For the Vasicek model, one has $\overline{r}^* = \overline{r} + q\sigma/\kappa$, $\kappa^* = \kappa$, $\sigma^* = \sigma$, and $\gamma = 0$ and, for the CIR model, one has $\overline{r}^* = \overline{r}\kappa/(\kappa + q)$, $\kappa^* = \kappa + q$, $\sigma^* = \sigma$, and $\gamma = 1/2$. Table 2 gives the characteristics of the model at the pricing date.

Table 3: Input data for the Vasicek and CIR models

	Vasicek	CIR
$\overline{r}$	0.0348468515	0.0348468515
κ	0.44178462	0.54958046
σ	0.13264223	0.38757496
q	0.21166329	-0.40663675

.

The points a_k, for $k = 2, \ldots, p-1$, are selected to be equally spaced with $a_1 = \overline{r}^* - 6\sigma\left[r_T\right]$ and $a_p = a_1 = \overline{r}^* + 6\sigma\left[r_T\right]$ for the Vasicek model, and, $a_1 = 10^{-6}$ and $a_p = 3$, for the CIR model.

Our code is written in C, compiled with GCC, and executed with a Laptop Pentium 4 running under Windows XP. CPU times reported here are in seconds.

Table 4 and Table 5 report the price of the straight bond associated to the callable bond described above obtained with DP with increasing grid sizes. The column entitled "Formula" reports its exact value using the closed-form solution under the Vasicek and CIR models. The last 2 columns report the prices obtained by BW and DFVL.

Table 4: Convergence of the DP procedure for the Vasicek model

	Prices of the Straight Bonds					
(r, p)	600	1200	2400	Formula	BW	DFVL
0.01	0.92746	0.92743	0.92742	0.92742	0.9274	0.92739
0.02	0.90899	0.90896	0.90896	0.90895	0.9089	0.90892
0.03	0.89091	0.89089	0.89088	0.89088	0.8908	0.89084
0.04	0.87322	0.87319	0.87319	0.87318	0.8731	0.87315
0.05	0.85590	0.85588	0.85587	0.85587	0.8558	0.85583
0.06	0.83895	0.83893	0.83892	0.83892	0.8389	0.83887
0.07	0.82236	0.82233	0.82233	0.82233	0.8223	0.82228
0.08	0.80612	0.80610	0.80609	0.80609	0.8060	0.80604
0.09	0.79022	0.79020	0.79020	0.79019	0.7901	0.79014
0.10	0.77466	0.77464	0.77464	0.77464	0.7746	0.77458

Table 5: Convergence of the DP procedure for the CIR model

	Prices of the Straight Bonds					
(r, p)	600	1200	2400	Formula	BW	DFVL
0.01	0.95537	0.95528	0.95526	0.95525	0.9552	0.95527
0.02	0.93166	0.93157	0.93154	0.93154	0.9315	0.93155
0.03	0.90858	0.90848	0.90846	0.90845	0.9084	0.90846
0.04	0.88610	0.88601	0.88599	0.88598	0.8859	0.88599
0.05	0.86422	0.86414	0.86411	0.86411	0.8641	0.86411
0.06	0.84292	0.84284	0.84282	0.84281	0.8428	0.84281
0.07	0.82219	0.82211	0.82208	0.82208	0.8220	0.82207
0.08	0.80200	0.80192	0.80190	0.80189	0.8018	0.80188
0.09	0.78235	0.78227	0.78225	0.78224	0.7822	0.78223
0.10	0.76322	0.76314	0.76312	0.76311	0.7631	0.76309

Table 4 and Table 5 report a clear convergence of the prices obtained by DP to the exact prices. The precision of DP compares avantageously with DFVL and BW. BW reported only 4 digits for bond's prices.

Table 6 reports the price of the callable bond. The price of the corresponding embedded call option can be obtained by

$$v_{t_0}^{\text{call option}}(r) = v_{t_0}^{\text{straight bond}}(r) - v_{t_0}^{\text{callable bond}}(r),$$

where $r = r_{t_0}$ is the current interest rate at t_0. CPU times are reported for a grid size of $p = 1200$, and are given only as indicators since the code were not ran on the same machine.

Table 6: Compared prices for the Vasicek and CIR models

Prices of Callable Bonds						
	Vasicek model			CIR model		
r	BW	DFVL	DP	BW	DFVL	DP
0.01	0.8556	0.84282	0.84285	0.9392	0.93926	0.93921
0.02	0.8338	0.82627	0.82630	0.9159	0.91598	0.91595
0.03	0.8223	0.81010	0.81009	0.8933	0.89333	0.89330
0.04	0.8062	0.79420	0.79423	0.8712	0.87127	0.87125
0.05	0.7904	0.77868	0.77871	0.8498	0.84980	0.84978
0.06	0.7749	0.76348	0.76351	0.8289	0.82890	0.82888
0.07	0.7598	0.74860	0.74862	0.8085	0.80855	0.80854
0.08	0.7450	0.73403	0.73406	0.7887	0.78874	0.78873
0.09	0.7305	0.71977	0.71980	0.7694	0.76945	0.76945
0.10	0.7163	0.70578	0.70583	0.7507	0.75067	0.75067
CPU	> 200	$12 - 14$	$2 - 3$	> 200	$12 - 14$	$2 - 3$

The DP procedure runs extremely fast with CPU times less than 3 seconds. CPU times for a similar grid size are within 191 and 629 seconds for BW and within 10 to 15 seconds for DFVL. Prices obtained by DP with a grid size $p = 2400$ are practically the same as those reported in Table 6. CPU times are within 10 to 12 seconds for DP, 1735 to 2520 seconds for BW, and 20 to 30 seconds for DFVL.

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