Strongly rational expectations equilibria with endogenous acquisition of information

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This paper analyzes conditions for existence of a strongly rational expectations equilibrium (SREE) in models with private information, where the amount of private information is endogenously determined. It is shown that the conditions for existence of a SREE known from models with exogenously given private information do not change as long as it is impossible to use the information transmitted through market prices. In contrast, these conditions are too weak, when there is such learning from prices. It turns out that the properties of the function which describes the costs that are associated with the individual acquisition of information are important in this respect. In case of constant marginal costs, prices must be half as informative than private signals in order for a SREE to exist. An interpretation of this result that falls back on the famous Grossman–Stiglitz–Paradox is also given.

Key words: Eductive Learning, Private Information, Rational Expectations, Strongly Rational Expectations Equilibrium
JEL-Classification: D 82, D 83.

1. Introduction

The concept of a rational expectations equilibrium (REE) is indeed quite ambitious if the underlying severe requirements on agent's information gathering and processing capabilities are considered. It is therefore not surprising that many attempts have been made in order to justify this concept and to state a clear set of assumptions that imply rational expectations on the side of the agents. One such attempt is the concept of a strongly rational expectations equilibrium (SREE) proposed by Guesnerie (1992, 2002). This concept asks, whether an REE can be educed by rational agents, meaning that the REE is the solution of some kind of mental process of reasoning of the agents. A SREE is then a REE that is learned by agents

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using this ”eductive” mental process (equivalently, the REE is said to be eductively stable). As shown by Guesnerie (1992, 2002), eductive learning of rational expectations is possible, if based on a suitably specified game–form of the model, agent’s use an iterative process to eliminate non–best responses from their strategy sets and if this process converges to the REE. It turns out that an REE is not necessarily a SREE, but that additional restrictions have to met for a SREE to exist. Guesnerie (2002) provides an overview over the conditions for existence of SREE that have been derived in various economic contexts.

Among other things, the concept of a SREE has been successfully applied to models with private information, which usually exhibit quite complex rational expectations equilibria. Conditions for existence of a SREE have been derived for models, where agents are unable to use the information transmitted through current market prices by prices (cf. Heinemann (2003)), as well as for models, where this information can be used (cf. Desgranges et al. (2003), Desgranges (1999), Heinemann (2002)). However, a common feature of all these studies is that they assume an exogenously given amount of private information. This means that so far not only the question how this private information comes into the market has been ignored. It also means that by now it has not been analyzed, whether the endogenization of private information acquisition causes additional restrictions an REE must fulfill in order to be eductively stable.

The present paper tries to fill this gap. We will introduce endogenous information acquisition into a simple market model and are able to derive conditions for existence of a SREE given this endogenously acquired information. Regarding the introduction of endogenous information acquisition, we follow the seminal work by Grossman and Stiglitz (1980) and more precisely Verrecchia (1982) who has analyzed rational expectations equilibria with endogenous acquisition of information in a quite similar economic environment. The present analysis considers two different equilibrium concepts that are both reasonable in the framework underlying our analysis. Initially, we will look at equilibria without learning from prices, where agents are not able to use the information transmitted trough current market prices for their own decisions. This model corresponds for example to a situation where every agent makes an irreversible production decision before he/she knows the price (this is the case of the cobweb models). After that, we will consider a more demanding equilibrium concept, where agents are able to use the information revealed by prices. This model is inspired from the well-known literature about REE under asymmetric information à la Grossman (1976).

The central results of the paper can be summarized as follows: As long as equilibria without learning from prices are considered, the opportunity to acquire private information endogenously leads to no conditions for existence of a SREE beyond that known for the case with exogenously given information. This, however, is not true for equilibria with learning from prices. Here, the conditions for existence of a SREE turn out to be stronger than the respective conditions for the case with exogenously given information.

This striking difference between these two results is driven by the following intuition: In the first model where no information is extracted from the price by agents, endogenous acquisition of private information does not create additional difficulties of coordinating expectations. In other words, at the time where he/she makes his/her decision, every agent needs to guess the price to make an optimal choice. To this purpose, it is enough to guess the shape of the supply and demand curves. In particular, no agent is concerned by the precision of the information acquired by others. Hence, the conditions for stability of the REE (i.e. existence of a SREE) are not affected by endogenous acquisition of private information and they depend on the relative slope of the demand and supply curves only.

In the second model where agents use the informational content of the price, the problem is quite different, and endogenous acquisition of private information do create additional difficulties of coordinating expectations. Namely, every agent needs to know the precision of information acquired by others in order to correctly understand the informational content of the price. The condition for existence of a SREE in the model with information transmitted by the price deserves some more comments. This condition states that the price must not be too informative, with respect to the informativeness of the private information acquired by agents. The underlying intuition is that the informational content of the price is determined by the correlation between private information and agents' decisions. As long as the price is not very informative, this correlation is easy to predict. But, when the price is very informative, agents' decisions mainly depend on the beliefs about the informational content of the price, and agents' decision are therefore not easy to predict. This condition for existence of a SREE is quite analogous to the condition in the case with exogenously given precision of private information. Still, it is a more demanding condition. The fact that endogenous information acquisition makes more difficult for a SREE to exist can be explained as follows: As already explained above, when the price is very informative, the REE is not eductively stable because every agent reacts less to his private information than to his beliefs about the information revealed by the price. In this case, given that private information is not very useful to agents, the precision of the private information acquired decreases. This last fact reinforces the stability problem. Namely, agents becomes much less reactive to their private information. Hence, agents' decisions depend more on their beliefs, which corresponds to a greater instability problem.

Lastly, an interesting feature of this stronger condition for existence of a SREE is that it ensures that the problem described by the the Grossman–Stiglitz–Paradox (cf. Grossman and Stiglitz (1980)) cannot occur. This famous paradox claims that existence of informationally efficient markets is impossible, since it is impossible to explain how information comes into the market in the first place. Namely, as long as the price publicly reveals all the relevant information, there is no incentive to acquire costly private information. But, if no one acquires information in order to make an accurate decision, the price cannot aggregate any information. This "solution" of the GS paradox builds on two points. Firstly, our condition for existence of a SREE implies that, as soon as a SREE exists, each firm can educe that there is always a positive amount of private information in the market, because the incentive to free–ride on others' information must be bounded from above. Secondly, when a SREE does not exist (meaning that there is a REE that is not a SREE), the "eductive theory" is not meant to make an accurate prediction of the market outcome. It only states that the REE is not a plausible outcome (or, at least, is not more plausible than many other outcomes). Although we give no formal content to this claim, we conclude that this additional uncertainty should create incentives to acquire private information.

2. A competitive market model

The model that builds the framework of our analysis is a model of a competitive market with a continuum of risk neutral firms in $I = [0,1]$. Market demand X is random, but the inverse demand function is known to the firms:

$$p = \beta - \frac{1}{\phi} X + \varepsilon$$

Here, ε is a normally distributed demand shock with zero mean and precision τ_ε. $\beta > 0$ and $\phi > 0$ are known constants. Every firms faces increasing marginal costs that are affected by the parameter θ. With $x(i)$ denoting the output of firm i, her costs are $c(i) = \theta x(i) + \frac{1}{2}\frac{1}{\psi} x(i)^2$, where $\psi > 0$. The cost parameter θ is unknown to the firms. The firms, however, know that this parameter is drawn from a normal distribution with mean $\overline{\theta}$ and precision τ.

Private information on the side of the firms regarding the unknown parameter is introduced into the model by allowing for endogenous acquisition of information as in Verrecchia (1982) (generalizing the seminal framework of Grossman and Stiglitz (1980)). It is assumed that each firm is able to perform an experiment (independent from experiments of other firms) that reveals additional but costly information regarding the unknown parameter θ. In particular, it is assumed that each firm $i \in I$ can acquire a costly private signal $s(i)$ that reveals additional private information regarding the unknown parameter θ. The private signal is given by $s(i) = \theta + u(i)$, where the signal's noise $u(i)$ is normally distributed with mean zero and precision $\tau(i)_u$. The costs of acquiring a signal with precision $\tau(i)_u$ are given by $K(\tau(i)_u)$ and let $\kappa(\tau(i)_u)$ denote the respective marginal costs. The objective of a firm is to maximize the expected profit, where ex-ante profit $\pi(i)$ of firm i is given by:

$$\pi(i) = [p - \theta]\, x(i) - \frac{1}{2}\frac{1}{\psi}\,[x(i)]^2 - K(\tau(i)_u), \tag{1}$$

Assume that the costs are increasing and convex: $\kappa(\tau(i)_u) \geq 0$ and $\kappa'(\tau(i)_u) \geq 0$ for all $\tau(i)_u \geq 0$.

Throughout the following analysis it will always be assumed that the average of the firm's private signals reveals the unknown value of the unknown parameter by the law of large numbers, such that $\int_0^1 s(i)\, di = \theta$ because $\int_0^1 u(i)\, di = 0$.

In what follows, we will first consider equilibria of this simple market model, where the firms are unable to use the information transmitted trough prices. This simply means, that every firm must decide on her profit maximizing output, before the actual market price becomes known and is unable to condition her supply decision on the market price. An equilibrium concept, where such learning from prices is possible because the information transmitted through prices can be used will be analyzed in section 4.

3. SREE without learning from prices

3.1. *Description of the REE*

We will start here with a brief description of the kind of REE that appears, when decisions are made before the actual market price becomes known. Because of the distributional assumptions made above, this REE takes a quite simple form: In equilibrium, each firm's supply decision $x(i)$ will be a linear function of the estimator for the unknown parameter θ based on public information and — if the firm chooses to acquire private information — the private signal $s(i)$ the firm observes. The decision to acquire information altogether, in turn depends on the marginal costs and benefits associated with private information acquisition. The next result summarizes the properties of the REE:

IT S A SMALL DETAIL, BUT IT MAY BE EASIER FOR THE READER TO HANDLE A POSITIVE PARAMETER α.THEN, DEFINE $\alpha = \psi/\phi$

MH: *I agree with you. But I refrain from a change in the notation at this moment because of the tight deadline (it's quite probable that some α's will be overlooked). In the next version, I will use a positive α.*

Proposition 1. *Let $\alpha = -\psi/\phi < 0$. The model then possesses an unique linear REE with the following properties:*

(i) Each firm $i \in I$ will acquire the same level of precision $\tau(i)^*_u = \tau^*_u = \max\{0, \tilde{\tau}_u\}$ of her private signal $s(i)$. $\tilde{\tau}_u$ is the solution of the equation:

$$\frac{\psi}{2}\frac{1}{(\tau+(1-\alpha)\,\tilde{\tau}_u)^2}=\kappa(\tilde{\tau}_u) \tag{2}$$

Furthermore, information is acquired in equilibrium, i.e., $\tau_u^*>0$*, iff* $\frac{\psi}{2\tau^2}>\kappa(0)$.

(ii) Each firm $i\in I$ *will use the same supply function* $x(i)=\psi\,[\gamma_0^*+\gamma_1^*\,s(i)]$*, where the weights* γ_0^* *and* γ_1^* *are functions of the model parameters:*

$$\gamma_0^*=\frac{\beta}{1-\alpha}-\frac{1}{1-\alpha}\frac{\tau}{\tau+(1-\alpha)\,\tau_u^*}, \qquad \gamma_1^*=-\frac{\tau_u^*}{\tau+(1-\alpha)\,\tau_u^*}$$

Proof. See Appendix. □

Existence and uniqueness of a linear equilibrium in various cases of CARA/Gaussian settings is a very common result, and this result deserves few comments only. A market equilibrium with private information acquisition (i.e., $\tau_u^*>0$) will therefore exist only if the marginal benefit of information acquisition at zero (i.e., $\frac{\psi}{\tau^2}$), is greater than the marginal cost of information acquisition at zero (i.e., $\kappa(0)$). In what follows, we assume that this condition is satisfied. Thus, there always exists a nontrivial REE, where individual acquisition of information takes place. For simplicity, we will also sometimes make the assumption that the marginal costs of information acquisition are constant, such that $\kappa(\tau(i)_u))=\bar{\kappa}>0$ for all $\tau(i)_u>0$. Under this assumption, a REE with information acquisition (i.e., $\tau(i)_u^*=\tau_u^*>0$ for all $i\in I$) exists if and only if $Q\equiv\sqrt{\frac{\psi}{2\kappa}}>\tau$. From the equilibrium condition (2) we obtain that in this case the equilibrium amount of information acquisition is $\tau_u^*=\frac{Q-\tau}{1-\alpha}$.

3.2. *Existence of a SREE*

Since detailed descriptions of of the concept of a SREE are already available in the literature (cf. Guesnerie (2002)), it is adequate to limit the present analysis to an informal and pragmatic treatment of this concept and the game–theoretical issues that are involved here. The fundamental question associated with the concept of a SREE is whether the assumptions of individual rationality and common knowledge are sufficient to predict a particular REE as an outcome of a model. Therefore it is necessary to look at a suitable game–form of the model and to analyze the best responses of the individual firms to actions taken by other firms in order to derive conditions for eductive stability. If we confine our analysis to linear supply functions, such that an individual firm's supply is given by $x(i)=\psi\,[\gamma(i)_0+\gamma(i)_1\,s(i)]$, the respective best response mapping can be summarized by the equations listed in the following Lemma:[3]

[3] Optimal output of a firm is given by $x(i)=\psi\,\mathrm{E}[p-\theta\,|\,s(i)]$. Hence, this linear supply rule assumes that $\mathrm{E}[p-\theta\,|\,s(i)]=\gamma(i)_0+\gamma(i)_1\,s(i)$.

Lemma 1. *If aggregate behavior is summarized by the coefficients* $\gamma_0 = \int_0^1 \gamma(j)_0\, dj$ *and* $\gamma_1 = \int_0^1 \gamma(j)_1\, dj$*, the best response* $\gamma(i)_0$, $\gamma(i)_1$ *as well as* $\tau(i)_u$ *of a firm* $i \in I$ *is uniquely defined by the following equations:*

$$\gamma(i)_0 = \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\,\frac{\tau}{\tau + \tau(i)_u}\,\overline{\theta} \tag{3a}$$

$$\gamma(i)_1 = (\alpha\gamma_1 - 1)\,\frac{\tau(i)_u}{\tau + \tau(i)_u} \tag{3b}$$

$$0 = \psi\,\frac{1}{2}\left[\frac{\gamma(i)_1}{\tau(i)_u}\right]^2 - \kappa(\tau(i)_u) \tag{3c}$$

This Lemma (that is central to the study of stability, as will soon be clear) calls for several comments:

(i) The best response mapping defined by the above Equations (3a)–(3c) map the three real parameters $(\gamma_0, \gamma_1, \tau_u)$ into the three real parameters $(\gamma(i)_0, \gamma(i)_1, \tau(i)_u)$ charactering the best response of firm i (where the aggregate value $\tau_u = \int_0^1 \tau(j)_u\, dj$ is defined analogously to γ_0 and γ_1).

(ii) Notice that $(\gamma(i)_0, \gamma(i)_1, \tau(i)_u)$ is not affected by τ_u, i.e., the precision of the information acquired by others. Intuitively, firm i makes his decision supply considering (1) his information on θ (that is s_i only as there is no learning from the price), and (2) his information on the price, that consists in the market clearing equation $p = \beta + \alpha[\gamma_0 + \gamma_1 \int_0^1 s(j)\, dj] + \varepsilon$, where ε and $\int_0^1 s(j)\, dj = \theta$ are unknown. Thus, given that the precision of the aggregate information $\int_0^1 s(j)\, dj$ on θ does not depend on the individual precisions $\tau(j)_u$ (it is infinite), the decision made by firm i does not depend on the $\tau(j)_u$ either.

MH: *I have changed your notation, because the symbol T is already used later on in the paper. I decided to use the symbol F.*

(iii) Furthermore, the decision of firm i can be separated into two successive problems. To see this, notice first that equation (3c) defines $\tau(i)_u$ as a function of $\gamma(i)_1$ and rewrite $\tau(i)_u = F(\gamma(i)_1)$. Then, given F, equations (3a)–(3b) define $(\gamma(i)_0, \gamma(i)_1)$ as a function of (γ_0, γ_1), namely:

$$\gamma(i)_0 = \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\,\frac{\tau}{\tau + T(\gamma(i)_1)}\,\overline{\theta} \tag{4a}$$

$$\gamma(i)_1 = (\alpha\gamma_1 - 1)\,\frac{F(\gamma(i)_1)}{\tau + F(\gamma(i)_1)} \tag{4b}$$

It follows that, on the one hand, the supply function of firm i is determined by the expected aggregate supply curve (characterized by (γ_0, γ_1)), and, on the other hand, the precision of the private information is determined on the basis of $\gamma(i)_1$ through the map T. This last remark will have essential consequences to the stability problem.

We can now turn attention to the question of the strong rationality (or stability) of the REE. By definition, the REE γ_0^*, γ_1^* and τ_u^* is a fixed point of the best response mapping (3a)–(3c). In particular, equation 3b) implies that $\gamma_1^* \leq 0$. Again, a detailed account of the analytical characterization of SREE is given in Guesnerie (2002). We just recall here that this REE is a SREE (or, equivalently is ”eductively stable”) if and only if it is a locally stable stationary point of the dynamical system made up from this best response mapping. Now, with respect to this dynamical system, the eigenvalues λ_1, λ_2 and λ_3 of the Jacobian matrix at the equilibrium point can be computed as follows: [4]

MH: *Okay, it seems to be a good idea to present the relevant computations in the appendix.*

$$\lambda_1 = 0\,, \quad \lambda_2 = \alpha\,, \quad \lambda_3 = \frac{\alpha\left(\gamma_1^2\psi + \kappa'(\tau_u)\,\tau_u^3\right)}{\gamma_1^2\psi + \kappa'(\tau_u)\,\tau_u^2(\tau+\tau_u)}$$

Since we have assumed that marginal costs of information acquisition are increasing or at least constant, we have $\kappa'(\tau_u) \geq 0$ for all τ_u. Therefore, we always have $\lambda_3 < \alpha$ and the condition $|\alpha| < 1$ is necessary and sufficient for local stability ands thus for existence of a SREE:

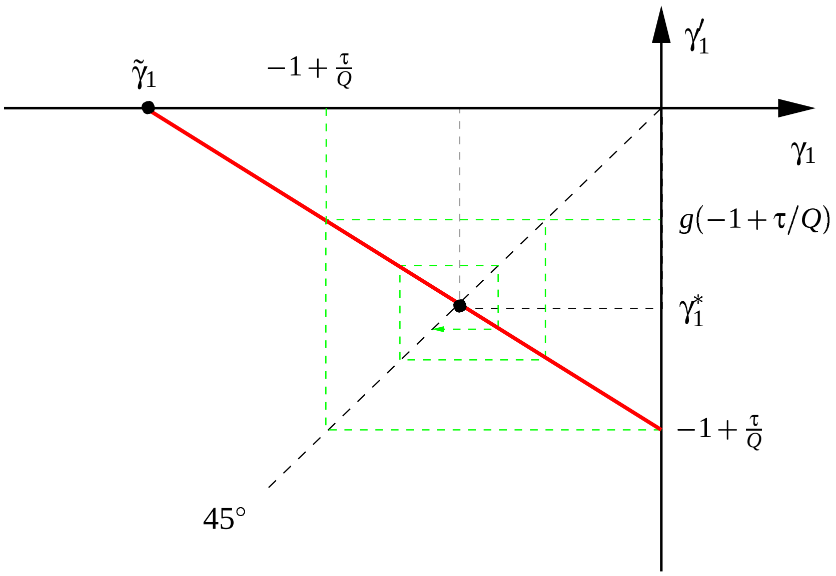
Proposition 2. *The linear REE with private information acquisition is locally eductively stable if and only if* $|\alpha| < 1$

Basically, this conditions requires $-1 < \alpha < 0$, since $\alpha = -\psi/\phi$, where ψ and ϕ are positive constants. Interestingly, this condition is exactly identical to the respective stability condition for the case with exogenously given private information (cf. Heinemann (2003)). As long as equilibria without learning from prices are considered, we therefore get no change with regard to the conditions for existence of a SREE, even if we allow for endogenous acquisition of information.

To understand the intuition for this surprising result, we go back to the above comments to Lemma 1. In these comments, we have explained that the first ”supply” decision is made independently of the precisions of private information, and that the ”precision” decision does not add further ”expectational difficulties”, i.e. the choice of $\gamma(i)_1$ requires to guess other's behavior, but, once $\gamma(i)_1$ has been chosen, $\tau(i)_u$ is simply $F[\gamma(i)_1]$, it does not depend on others' behavior $(\gamma(i)_0, \gamma(i)_1)$. Summing up, (1) the decision of firm i is not affected by expectation about others' precision of information $\int_0^1 \tau(j)_u\, dj$, and (2) the choice of $\tau(i)_u$ is unambiguously made, once $\gamma(i)_1$ has been chosen. Thus, as far as the question of expectations coordination is concerned, it is enough to look at the system (γ_0, γ_1), and it is useless to look

[4] Details on the computation of the eigenvalues of this dynamical system are given in the appendix at the end of the proof of Lemma 1.

Fig. 1. *Graphical representation of the SREE condition*



at τ_u. This intuition is clearly emphasized in the specific case with constant marginal costs described below.

If it is assumed that marginal costs are constant, it is quite easy to give a graphical representation of the stability condition and the iterative process that leads to the REE. Under the assumption that $\kappa(\tau_u) = \bar{\kappa}$ such that $\kappa' = 0$, Equation (3c) gives $\tau(i)_u = -Q\gamma(i)_1$, where Q was already defined above. Then, the best response mapping (3a)-(3b) rewrites as the following linear system:

$$\gamma(i)_0 = \beta + \alpha\gamma_0 - \frac{\tau}{Q}\bar{\theta} \tag{5a}$$

$$\gamma(i)_1 = \alpha\gamma_1 - 1 + \frac{\tau}{Q} \equiv g(\gamma_1) \tag{5b}$$

where the variables τ_u and $\tau(i)_u$ do not appear, as explained above. The first equation characterizes the dynamics of γ_0, while the second describes the dynamics of γ_1.

For example, consider the dynamics of γ_1 as depicted in figure 1. To draw the figure, denote γ_1^* the equilibrium weight of private information (such that $\gamma_1^* = g(\gamma_1^*)$) and $\tilde{\gamma}_1 = -\frac{1}{\alpha}\left(-1+\frac{\tau}{Q}\right)$ the root of g, and notice that existence of an REE with $\tau_u^* > 0$ implies $Q > \tau$ such that $g(0) < 0$. Thus, whenever the stability condition stated in Proposition 2 is satisfied, such that $-1 < \alpha < 0$, we have $\tilde{\gamma}_1 < -1+\frac{\tau}{Q}$. Figure 1 can then be used to describe the iterative process of elimination of non-best responses that converges to this REE. This process is illustrated in the figure, starting from the assumption that it is common knowledge that no firm uses a weight $\gamma(i)_1$ greater than zero. [5] This necessarily implies that $\gamma_1 \leq 0$ and from the figure it can be seen that in this case no firm will ever choose a weight $\gamma(i)_1$ which smaller than $-1+\frac{\tau}{Q}$. [6] From this, however, it in turn follows that γ_1 must be greater than $-1+\frac{\tau}{Q}$, which implies that no firm i will use a weight $\gamma(i)_1 > g(-1+\tau/Q)$. It is easily verified that this process converges to the equilibrium γ_1^*, whenever $-1 < \alpha < 0$.

4. Eductive stability with learning from prices

4.1. *The case of exogenously given information*

Let us now turn to the second equilibrium concept, where learning from current prices is possible. It is reasonable to start this analysis with a brief discussion of a version of the model, where the amount of private information is given. This enables us to build on some known results and to illustrate, where these known results have to be modified if endogenous acquisition of information is allowed for. The analysis is based on the initially considered model with risk neutral firms and it is assumed that each firm's signal has precision $\tau_u > 0$. When there is learning from prices, the firms are able to use the information transmitted through the actual market price for their own decisions. Hence, profit maximizing output for a firm $i \in I$ is now given by $x(i) = \psi\,[p - \mathrm{E}[\theta\,|\,s(i), p]]$. In analogy to the financial market models considered by Desgranges (1999) and Heinemann (2002), it can then be established that there exists an unique linear REE in this model with learning from prices.

Proposition 3. *Let again $\alpha = -\psi/\phi < 0$. The model with learning from prices then possesses an unique linear REE, where every firm uses a linear supply function $x(i) = \psi\,[(1-\gamma_2^*)\,p - \gamma_0^* - \gamma_1^*\,s(i)]$. The coefficients γ_0^* and γ_1^* and γ_2^* are solutions to the equations:*

$$\gamma_0^* = \frac{\beta\,\alpha\,\gamma_1^*\,\tau_\varepsilon + \tau\,\bar{\theta} - \alpha^2\,\gamma_1^*\,\gamma_0^*\,\tau_\varepsilon}{\tau + \tau_u + \alpha^2\,{\gamma_1^*}^2\,\tau_\varepsilon}\,, \quad \gamma_2^* = -\frac{\gamma_1^*\,\alpha\,(1-\alpha(1-\gamma_2^*))\tau_\varepsilon}{\tau + \tau_u + \alpha^2\,{\gamma_1^*}^2\,\tau_\varepsilon}$$

and γ_1^* is the unique solution of the polynomial $H(\gamma_1^*) \equiv \gamma_1^*\left[(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau_\theta + \tau_u\right] = \tau_u$.

Conditions for existence of a SREE in this model with exogenously given private information are derived by Heinemann (2003). For convenience the respective conditions are reproduced in the following proposition:

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[5] From equation (3c) it follows that this is equivalent to the assumption that it is common knowledge that $\tau(i)_u \geq 0$ for all i.

[6] Using equation (3c) it can be shown that with respect to the amount of information that is acquired this means that no firm will acquire information with precision greater than $\tau_u = (Q - \tau)$.

Proposition 4.

(i) *The rational expectations equilibrium* $\gamma_0^*, \gamma_1^*, \gamma_2^*$ *is a SREE if and only if* $\alpha^2\,(\gamma_1^*)^2\,\tau_\varepsilon < \tau_u$.

(ii) *The condition (i) for existence of a SREE is equivalent to the condition that in the rational expectations equilibrium the market price* p *is less informative regarding* θ *than the private signals.*

4.2. *Conditions for existence of a SREE with endogenous acquisition of information*

Starting from the above described rational expectations equilibrium with exogenously given information, it is quite easy to derive the respective equilibrium conditions for the model with endogenous information acquisition. The reason is, that all the conditions stated in Proposition 3 remain essentially valid. The only modification consists in an additional condition which describes the optimal equilibrium amount of private information acquisition:

AGAIN THIS IS A DETAIL, BUT IT MAY BE EASIER FOR THE READER TO KEEP $+\gamma_1 s(i)$ AS THE COEFFICIENT FOR THE SIGNAL IN THE SUPPLY AND NOT TO CHANGE THIS INTO $-\gamma_1 s(i)$.

MH: *You are right. But at this stage, I'm afraid was change this, because this will probably bring about many errors.*

Proposition 5. *In the model with learning from prices and endogenous information acquisition exists an unique linear REE, where every firm uses a linear supply function* $x(i) = \psi\,[(1-\gamma_2^*)\,p - \gamma_0^* - \gamma_1^*\,s(i)]$.

(i) *Each firm* $i \in I$ *will acquire the same level of precision* $\tau(i)_u^* = \tau_u^* = \max\{0, \tilde{\tau}_u\}$ *of her private signal* $s(i)$. $\tilde{\tau}_u$ *is the solution of the equation:*

$$\frac{\psi}{2}\left(\frac{\gamma_1^*}{\tilde{\tau}_u}\right)^2 = \kappa(\tilde{\tau}_u)$$

(ii) *The coefficients* γ_0^* *and* γ_1^* *and* γ_2^* *are given as in Proposition 3, that is* γ_1^* *is the unique solution of the polynomial* $H(\gamma_1^*) \equiv \gamma_1^*\left[(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau_\theta + \tau_u\right] = \tau_u$ *and:*

$$\gamma_0^* = \frac{\beta\,\alpha\,\gamma_1^*\,\tau_\varepsilon + \tau\,\overline{\theta}}{\tau + \tau_u + \alpha^2\,\gamma_1^{*2}\,\tau_\varepsilon + \alpha^2\,\gamma_1^*\,\tau_\varepsilon}$$
$$\gamma_2^* = -\frac{\gamma_1^*\,\alpha\,(1-\alpha)\tau_\varepsilon}{\tau + \tau_u + \alpha^2\,\gamma_1^{*2}\,\tau_\varepsilon + \alpha^2\,\gamma_1^*\,\tau_\varepsilon}$$

Proof. See Appendix. □

We now again ask, whether the assumptions of individual rationality and common knowledge are sufficient for a justification of this REE. In order to derive the respective conditions for existence of eductive stability, we have again to look a the best responses of the individual firms to actions taken by other firms. As in the preceeding section, we confine our analysis to linear supply functions, such that an individual firm's supply is given by $x(i) = \psi\,[(1-\gamma(i)_2)\,p - \gamma(i)_0 - \gamma(i)_1\,s(i)]$. The respective best response mapping is then as summarized in the following Lemma:

Lemma 2. *If aggregate behavior is summarized by the coefficients $\gamma_0 = \int_0^1 \gamma(j)_0\, dj$, $\gamma_1 = \int_0^1 \gamma(j)_1\, dj$ and $\gamma_2 = \int_0^1 \gamma(j)_2\, dj$, the best response of a firm $i \in I$ is:*

$$\gamma'(i)_0 = \frac{\beta\alpha\gamma_1\tau_\varepsilon + \tau\bar{\theta} - \alpha^2\gamma_1\gamma_0\tau_\varepsilon}{\tau + \tau'_u(i) + \alpha^2\gamma_1^2\tau_\varepsilon} \tag{6a}$$

$$\gamma'(i)_1 = \frac{\tau'_u(i)}{\tau + \tau'_u(i) + \alpha^2\gamma_1^2\tau_\varepsilon} \tag{6b}$$

$$\gamma'(i)_2 = -\frac{\gamma_1\alpha(1-\alpha(1-\gamma_2))\tau_\varepsilon}{\tau + \tau'_u(i) + \alpha^2\gamma_1^2\tau_\varepsilon} \tag{6c}$$

$$\psi\frac{1}{2}\left(\frac{\gamma'(i)_1}{\tau'_u(i)}\right)^2 = \kappa(\tau'_u(i)) \tag{6d}$$

Proof. See Appendix. □

The REE is strongly rational, if the map defined by equations (6a)–(6d) is contracting. The required stability analysis can be simplified, however, since a closer look at this system reveals, that equations (6b) and (6d) can be analyzed independently from equations (6a) and (6c). This means that given stability of the subsystem (6b) and (6d) around γ_1^*, τ_u^*, it would sufficient to find stability conditions for the remaining two linear equations (6a) and (6c). Indeed, with respect to these latter two equations this is a simple task, because the respective equations are the same as for the case with exogenously give information such that the stability condition coincides with the condition stated in Proposition 4 above. [7] Indeed, the system formed by the two equations (6a) and (6c) is stable if and only if,

$$\left|\frac{-\alpha^2\gamma_1^*\tau_\varepsilon}{\tau + \tau_u^* + \alpha^2\gamma_1^{*2}\tau_\varepsilon}\right| < 1$$

I AGREE THAT THIS MAY BE DELETED.
MH: *It might indeed be helpful to the reader. So why should it be deleted (we can still delete it if a referee asks us to do so)?*

As $\frac{\tau_u^*}{\tau+\tau_u^*+\alpha^2\gamma_1^{*2}\tau_\varepsilon} = \gamma_1^*$ at the REE, this rewrites $\alpha^2\gamma_1^{*2}\tau_\varepsilon < \tau_u^*$.

It therefore remains to analyze the subsystem (6b) and (6d) in order to check whether there are any consequences of endogenous acquisition of information on the conditions for existence of a SREE. Now, the eigenvalues λ_1, λ_2 of the Jacobian matrix of this system evaluated at the REE are:

$$\lambda_1 = 0, \qquad \lambda_2 = -\frac{2\alpha^2\gamma_1^{*2}\tau_\varepsilon}{\tau + \tau_u^* + \alpha^2\gamma_1^{*2}\tau_\varepsilon - (1-\gamma_1^*)W}$$

[7] Heinemann (2003) shows that the stability conditions stated in Proposition 4 is in fact necessary and sufficient for stability of the subsystem (6a) and (6c) when γ_1 is fixed. For convenience, the complete stability analysis is given in the appendix as a proof of Proposition 6.

where $W \equiv \frac{\psi \frac{\gamma_1^*}{\tau_u^{*2}}}{\kappa' + \psi \frac{\gamma_1^{*2}}{\tau_u^{*3}}} \in [0, \frac{\tau_u^*}{\gamma_1^*}]$, depending on the value of κ'. Some computations show that $\lambda_2 < 0$. To see this point, notice first that the equilibrium condition (6b), $\tau + \tau_u^* + \alpha^2 \gamma_1^{*2} \tau_\varepsilon = \frac{\tau_u^*}{\gamma_1^*}$ implies that the intersection point of the two curves $\tau + \tau_u^* + \alpha^2 \gamma_1^{*2} \tau_\varepsilon$ and $\frac{\tau_u^*}{\gamma_1^*}$ satisfies $\gamma_1^* > 0$ and $\tau + \tau_u^* < \frac{\tau_u^*}{\gamma_1^*}$, that is $\gamma_1^* < \frac{\tau_u^*}{\tau + \tau_u^*} < 1$. Then, using again the equilibrium condition $\tau + \tau_u^* + \alpha^2 \gamma_1^{*2} \tau_\varepsilon = \frac{\tau_u^*}{\gamma_1^*}$,

$$\lambda_2 = -\frac{2\,\alpha^2 \gamma_1^{*2} \tau_\varepsilon}{\frac{\tau_u^*}{\gamma_1^*} - (1 - \gamma_1^*)\, W}$$

As $W \in [0, \frac{\tau_u^*}{\gamma_1^*}]$, the denominator of the above expression is greater than

$$\frac{\tau_u^*}{\gamma_1^*} - (1 - \gamma_1^*)\, W \geq \frac{\tau_u^*}{\gamma_1^*} - (1 - \gamma_1^*)\, \frac{\tau_u^*}{\gamma_1^*} = \tau_u^* > 0$$

MH: *Is it possible to relegate some of the computational steps to the appendix and to present and discuss just the sufficient condition? We should think about that.*

Hence, $\lambda_2 < 0$.

Thus, strong rationality is equivalent to $\lambda_2 > -1$, that rewrites (using again the equilibrium condition (6b)):

$$2\,\alpha^2 \gamma_1^{*2} \tau_\varepsilon < \frac{\tau_u^*}{\gamma_1^*} - (1 - \gamma_1^*)\, W$$

As $\frac{\tau_u^*}{\gamma_1^*} - (1 - \gamma_1^*)\, W \geq \tau_u^*$, it follows that $\alpha^2 \gamma_1^{*2} \tau_\varepsilon < \tau_u^*/2$ is a sufficient condition for stability. It can be shown that informativeness of the market price p in a REE (i.e., $1/Var(\theta|p) - 1/Var(\theta)$) is given by $\tau_p^* \equiv \alpha^2\, (\gamma_1^*)^2\, \tau_\varepsilon$ (cf. Heinemann (2002)). Thus, we have exactly shown that a sufficient condition for stability is $\tau_p^* < \tau_u^*/2$: the precision of the information revealed by the equilibrium price is less than half the precision of the private signal ($\tau_u^* = 1/Var(\theta|s(i)) - 1/Var(\theta)$). A necessary and sufficient condition for stability is

$$2\,\tau_p^* < \tau_u^* \frac{\frac{\kappa'}{\gamma_1^*} + \psi \frac{\gamma_1^{*2}}{\tau_u^{*3}}}{\kappa' + \psi \frac{\gamma_1^{*2}}{\tau_u^{*3}}}$$

One sees that, given that $0 < \gamma_1^* < 1$, the condition $2\,\tau_p^* < \tau_u^*$ is necessary only when $\kappa' = 0$, that is: marginal costs are constant.

THIS IS NOT ELEGANT AT ALL. IS THERE ANYTHING BETTER? IS IT OF ANY INTEREST?... STILL, I HOPE YES: THERE IS SOME INTUITION TO BE GAINED FROM THIS.

MH: *Its not easy to interprete this expression since τ_u^* as well as τ_p^* that appear on the rhs depend on ψ and κ'. Perhaps it is possible to find an alternative representation which says the same (that is to derive some comparative static results). But this must wait until the next revision.*

Using again $\frac{\tau_u^*}{\gamma_1^*} = \tau + \tau_u^* + \tau_p^*$, the stability condition becomes:

$$2\,\tau_p^* < \frac{\kappa' \left(\tau + \tau_u^* + \tau_p^*\right) + \psi \frac{1}{\left(\tau + \tau_u^* + \tau_p^*\right)^2}}{\kappa' + \psi \frac{1}{\tau_u^* \left(\tau + \tau_u^* + \tau_p^*\right)^2}}$$

$$\kappa' < \psi \frac{1 - 2\frac{\tau_p^*}{\tau_u^*}}{(\tau + \tau_u^* + \tau_p^*)^2 (\tau_p^* - \tau - \tau_u^*)}$$

This mainly says that κ' must not be too large, nor ψ too small. The fact that κ' must not be too large can be easily understood: a small κ' implies that this is not very costly to adjust $\tau_u(i)$ for firm i so that this quantity cannot be easily predicted by others.

The implications of these results for existence of a SREE are summarized in the next Proposition:

MH: *I'm not sure whether this is what you have in mind regarding this proposition.*

Proposition 6.

(i) If private information is endogenously acquired, a sufficient condition for the rational expectations equilibrium $d^* = \left(\gamma_0^*, \gamma_1^*, \gamma_2^*\right)$ to be a SREE is $\alpha^2\,(\gamma_1^*)^2\,\tau_\varepsilon < \frac{1}{2}\tau_u^*$.

(ii) If marginal costs are constant such that $\kappa(\tau(i)_u) = \bar{\kappa}$, the rational expectations equilibrium $d^* = \left(\gamma_0^*, \gamma_1^*, \gamma_2^*\right)$ is strongly rational if and only if $\alpha^2\,(\gamma_1^*)^2\,\tau_\varepsilon < \frac{1}{2}\tau_u^*$.

(iii) The condition for existence of a SREE is equivalent to the condition that in the rational expectations equilibrium the market price p is at most half as informative regarding θ than the private signals.

Proof. See Appendix. □

The condition stated in Proposition 6 is obviously stronger than the respective condition for existence of a SREE with exogenously given information which is stated in Proposition 4. Thus, contrary to the above considered case without learning form prices, the presence of endogenous information acquisition in the model with learning from prices implies that conditions for existence of a SREE have to be qualified. Contrary to the condition for existence of a SREE with exogenously given private information, the condition stated in Proposition 6 contains two endogenous variables, namely γ_1^* and τ_u^*. Since according to (6d) we must have $\gamma_1^* = \frac{\tau_u^*}{Q^2}$, we can, however, state an alternative and equivalent condition which contains only one endogenous variable: A SREE exists if and only if $\frac{Q^2}{2\alpha^2\tau_\varepsilon} > \tau_u^*$. Taken seriously, the condition for eductive stability stated in Proposition 6 is a condition for local stability. As we will, however, demonstrate soon, this condition is also necessary and sufficient for global stability. In the end this means that there is no need to impose ‘credible restrictions’ on the individual strategy sets as in Guesnerie (1992).

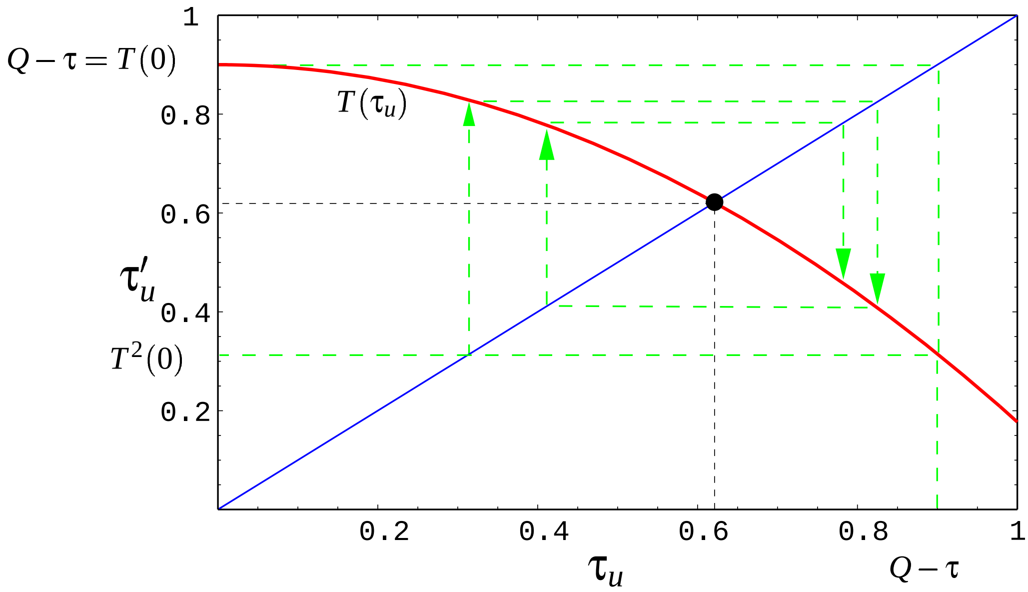
I DID NOT ADD ANYTHING IN THE SEQUEL. STILL, I READ IT AND I HAVE THREE REMARKS: FIRST, I AGREE WITH THE RESULTS, 2. WE SHOULD SPEAK OF ”MAXIMAL” TWO CYCLES OR STH LIKE THIS (or assume or show existence of a unique cycle, otherwise there may be several ”embedded” cycles and the dynamics converge to the larger one). 3. IN the appendix, proof of prop 1, write γ_1 **instead of** γ_2.

MH: *ad 3) a silly mistake. ad 2) I do not really understand what you mean, because I'm sure that existence of a cycle with period two is proven. Probably the the term ”two cycle” was confusing??*

In the remainder of the paper, we will not address the issue of increasing marginal costs. Instead we will look at some numerical examples which serve to illustrate the so far derived results.

Example 1: Consider a numerically specified version of the model where $\alpha = -0.85$, $\psi = 1$, $\tau = 0.1$ and $\tau_\varepsilon = 1$. Marginal costs of information acquisition are constant and given

Fig. 2. *Best response mapping $T(\tau_u)$ for example 1*



by $\bar{\kappa} = 0.5$. From equations (6b) and (6d), the values for γ_1 and τ_u in a rational expectations equilibrium can be computed as: $\gamma_1^* = 0.621197$ as well as $\tau_u^* = 0.621197$.[8] Now let us look first at the case where the amount of private information is exogenously given and equal to τ_u^*. According to Proposition 4, the condition for existence of a SREE is $\alpha^2 \tau_\varepsilon \gamma_1^{*2} < \tau_u^*$, which is satisfied since $\alpha^2 \tau_\varepsilon \gamma_1^{*2} = 0.278803$.

If we now consider the case where information is endogenously acquired, the stronger stability condition according to Proposition 6 is $\alpha^2 \tau_\varepsilon \gamma_1^{*2} < \frac{1}{2}\tau_u^*$, which is also satisfied, since $\tau_u^*/2 = 0.310599$. Thus, in this case the fact that information acquisition is endogenously determined is not relevant for existence of a SREE.

The best way to illustrate the best response dynamics is to substitute equation (6b) into (6d). Equation (6d) can be used to eliminate γ_1, since it implies that $\gamma_1^2 = \frac{2\bar{\kappa}}{\psi}\tau_u^2$. With $Q \equiv \sqrt{\frac{\psi}{2\bar{\kappa}}}$, the resulting equation gives the individual best response $\tau'_u(i) = T(\tau_u)$ to the '*amount of information in the market*' τ_u:[9]

$$\tau'_u(i) = T(\tau_u) = Q - \tau - \frac{\alpha^2 \tau_\varepsilon}{Q^2}\tau_u^2 \tag{7}$$

Figure 2 shows how this function $T(\tau_u)$ looks like in case of the underlying numerical specification. Since τ_u is necessarily nonnegative, it is common knowledge that $\tau_u \geq 0$. As

[8] The respective values for γ_0^* and γ_2^* are omitted here. They are not of interest, since they do not appear in the stability conditions.

[9] Equation (6d) implies $\gamma(i)_1 = \tau_u(i)/Q$. Integrating over all $i \in I$ then gives $\gamma_1 = \tau_u/Q$, where $\tau_u = \in_0^1 \tau_u(i)\,di$.

the figure reveals, it is always individually optimal to acquire information, if no other firm does it. If $\tau_u = 0$, the corresponding maximum amount of private information a firm will ever acquire is given by $T(0) > 0$. Since it is therefore also common knowledge that $\tau_u < T(0)$, it follows that no firm will choose $\tau(i)_u < T(T(0)) \equiv T^2(\tau_u)$. This means that no firm expects the amount of information in the market to be such large that it becomes optimal to stop the individual acquisition of information. As indicated in the figure, the dynamics that result if this kind of reasoning is iterated, are similar to the well known cobweb–dynamics. The condition stated in Proposition 6 then ensures that these dynamics converge to the REE precision τ_u^*. In this case each firm can educe that only the precision REE $\tau_u^* = 0.621197$ constitutes possible solution under individual rationality and common knowledge.

Example 2: Let us now look at a slightly modified version of the above considered example, where the precision of the noise is given by $\tau_\varepsilon = 1.3$ instead of $\tau_\varepsilon = 1.0$. From equations (6b) and (6d), the values for γ_1 and τ_u in a rational expectations equilibrium can be computed as $\gamma_1^* = 0.58193$ and $\tau_u^* = 0.58193$. If the case, where the amount τ_u^* of private information is assumed to be exogenously given, is again considered first, it turns out that a SREE exists, since $\alpha^2 \tau_\varepsilon \gamma_1^{*2} = 0.31807$, which is smaller than τ_u^*. However, if information is endogenously acquired, the stronger stability condition is $\alpha^2 \tau_\varepsilon \gamma_1^{*2} < \frac{1}{2}\tau_u^*$, which is not satisfied, since $\tau_u^*/2 = 0.290965$. Thus, we have here an example where a SREE exists if the amount of private information is exogenously given, but does not exist if there is endogenous acquisition of information.

This numerical example is especially interesting because it gives rise to a special kind of best response dynamics. To see this look at figure 3 where again the function $T(\tau_u)$ is plotted. In addition, however, we have now also plotted the function $T^2(\tau_u) \equiv T(T(\tau_u))$. As can be seen, this function possesses two additional fixed points, $\tau_1^* = 0.203623$ and $\tau_2^* = 0.861057$. Notice too that the associated 2–cycle is stable. If we repeat the argumentation used in the discussion of the first example, we therefore get a process which converges to this 2–cycle: Since $T(0) > 0$, an individual firm will always acquire information, if there is no information in the market. As from this it follows that it is common knowledge that $\tau_u < T(0)$, no firm will choose a precision smaller than $T^2(0)$. Hence, no firm expects so much information in the market that it becomes optimal to stop the individual acquisition of information. Iterating this argument allows only to eliminate precisions outside the interval $[\tau_1^*, \tau_2^*]$ as being incompatible with rationality and common knowledge. Thus, the REE τ_u^* is not a SREE, since all precisions in the interval $[\tau_1^*, \tau_2^*]$ still constitute possible solutions under rationality.

Fig. 3. *Best response mapping $T(\tau_u)$ for example 2*

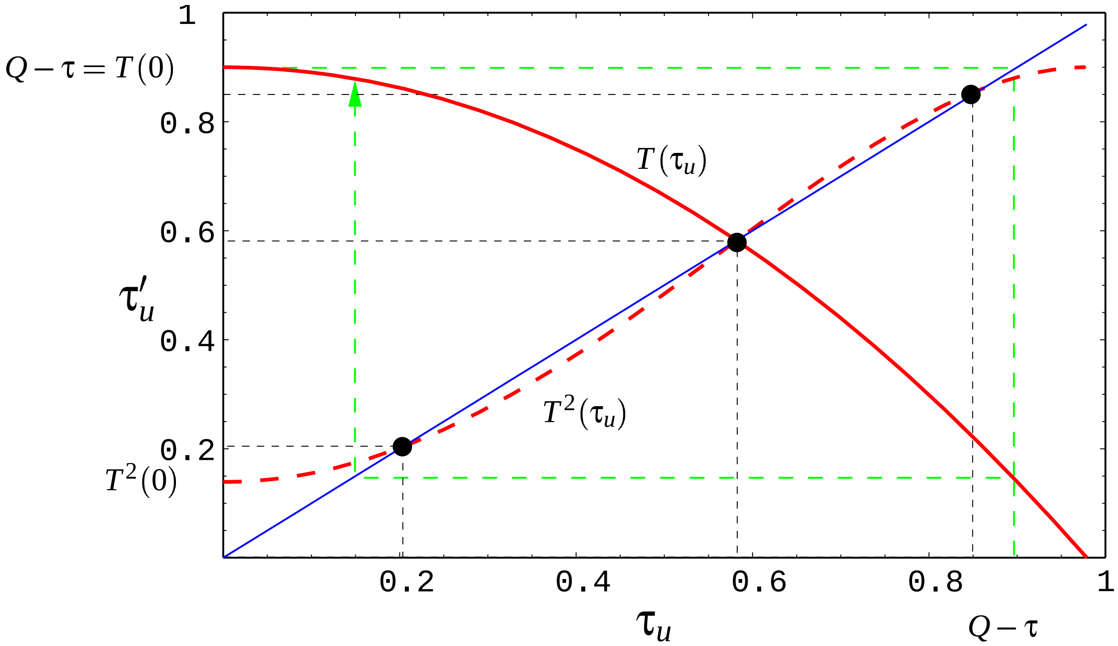
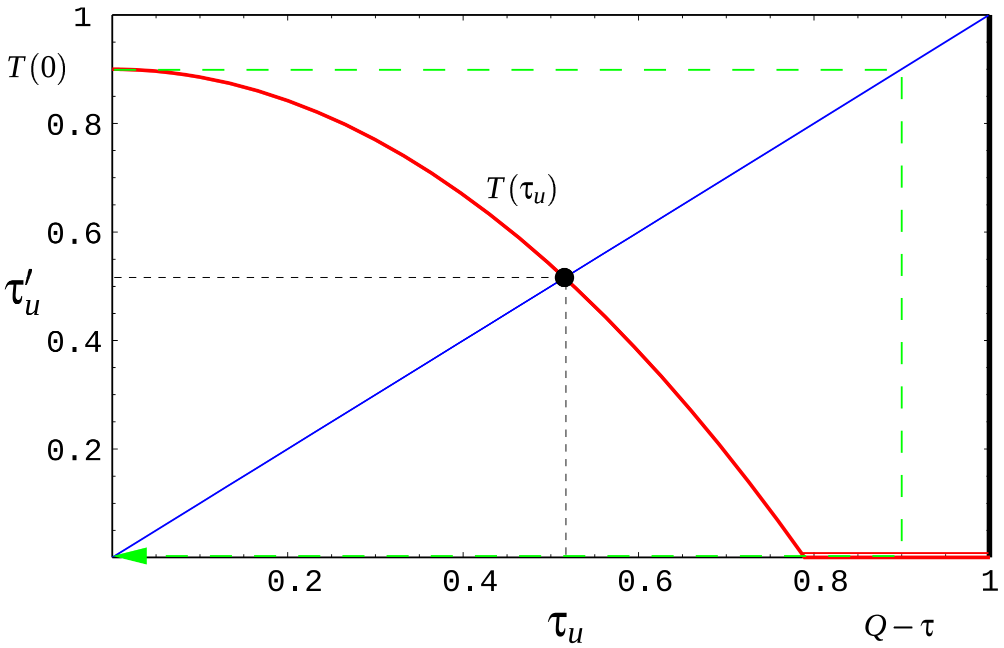


Fig. 4. *Best response mapping $T(\tau_u)$ for example 3*



It should be noted at this point that this kind of dynamics does not necessarily emerge if the stability condition of Proposition 6 is violated. If this condition does not hold it is also possible that $T^2(0) = T(T(0)) < 0$. This is the case in the third and last example, we will now discuss.

Example 3: In this example, the precision of noise is even larger than in example 2, namely $\tau_\varepsilon = 2.0$. In the REE we have $\gamma_1^* = 0.515703$ and $\tau_u^* = 0.515703$. Since $\alpha^2\,\tau_\varepsilon\,\gamma_1^{*2} = 0.384297$, this REE is still eductively stable, if information is assumed to be exogenously given, but not (since $\tau_u^*/2 = 0.257851$), when information acquisition is endogenous. The best response function $T(\tau_u)$ depicted in figure 4 reveals that in this example we have $T^2(0) < 0$. This example then gives rise to an interpretation which is quite similar to the famous Grossman–Stiglitz–Paradox (cf. Grossman and Stiglitz (1980)). As the figure shows, it is individually optimal to acquire private information, whenever there is no information in the market, i.e. $T(0) > 0$. If, however, each firm acquires this amount of private information such that $\tau(i)_u = T(0) = Q - \tau$ for all $i \in I$, there is so much information in the market, that it is individually optimal to stop the acquisition of information. This is indeed quite similar to the Grossman–Stiglitz–Paradox where the impossibility of informationally efficient markets is claimed. The underlying problem there is that the information revealed by prices makes it unattractive to spend resources for the acquisition of private information. The same problem appears here, where it is not possible to rule out the possibility that no firm acquires information privately, since prices are too informative. Notice that this can occur only because the stability condition stated in Proposition 6 is violated, which as we have argued actually requires that prices are not too informative.

As the discussion of the three examples revealed, the properties of the mapping $T^2(\tau_u) = T(T(\tau_u))$ are of crucial importance for dynamics of the eductive process. Moreover, this discussion has shown that possible to restrict the analysis of the dynamics of the best response mapping $T(\tau_u)$ to $\tau_u \in [0, Q - \tau]$, since τ_u must be nonnegative and no firm will ever acquire information with precision larger than $T(0) = Q - \tau$. Given this restriction, the following proposition establishes how the properties of $T(\tau_u)$ and $T^2(\tau_u)$ are linked to the above derived stability condition:

Proposition 7.

Consider the best response mapping $T(\tau_u)$, with τ_u restricted to the set $S = [0, Q - \tau]$.

(i) The REE precision τ_u^* is the unique stable fixed point of the mapping $\tau_u' = T(\tau_u)$ in S if and only if $\alpha^2\,\gamma_1^{*2}\,\tau_\varepsilon < \frac{1}{2}\,\tau_u^*$.

(ii) If this condition is violated, τ_u^* is an unstable fixed point and there exists a stable two–cycle $\underline{\tau}_u^* = T(\overline{\tau}_u^*) = T(T(\underline{\tau}_u^*))$ in S with $0 < \underline{\tau}_u^* < \tau_u^* < \overline{\tau}_u^* < Q - \tau$, whenever $1 > \frac{\alpha^2\,\tau_\varepsilon}{Q^2}\,\tau_u^* + \left[\frac{\alpha^2\,\tau_\varepsilon}{Q^2}\,\tau_u^*\right]^2$. Otherwise, τ_u^* is still unstable and there exists no two–cycle in S.

Proof. See Appendix. □

This Proposition clarifies that whenever a SREE exists, the corresponding best response mapping will always look like the one depicted in figure 2. Moreover, since the precision acquired by an individual firm is necessarily restricted to the set $[0, T(0)]$, the existence of a SREE and hence stability of τ_u^* implies global stability of this fixed point. Given this global stability of τ_u^* and hence γ_1^*, it is then easy to establish that the dynamics of the remaining two best responses (6a) and (6c) are globally stable too. Since these responses are linear functions of γ_0 and γ_2, respectively, stability will always be global. All in all this means that our condition for existence of a SREE is in fact necessary and sufficient as it implies that the underlying dynamics are globally stable.

4.3. *What if no SREE exists?*

Thus far we have been able to derive an explicit condition for existence of a SREE in our model with endogenous information acquisition. Whenever the condition stated in Proposition 6 is satisfied, rationality and common knowledge turn out to be necessary and sufficient to consider the REE γ_0^*, γ_1^*, γ_2^*, τ_u^* as the unique reasonable solution of the model. It has, however, also been shown that not every REE will be a SREE too. It thus remains to discuss, what we can say in cases, where the condition of existence is not satisfied.

Let us begin with the case where the mapping $T(\tau_u)$ possesses a stable two–cycle. This means that the assumptions of common knowledge and rationality restrict the set of possible individual precisions to the set $[\underline{\tau}_u^*, \overline{\tau}_u^*]$. Since $\gamma(i)_1 = \sqrt{Q}^{-1}\tau(i)_u$, set of individual weights for this private information is accordingly restricted to the set $[\underline{\gamma}_1^*, \overline{\gamma}_1^*] = [\sqrt{Q}^{-1}\underline{\tau}_u^*, \sqrt{Q}^{-1}\overline{\tau}_u^*]$.

It remains to ask whether it is also possible to restrict the remaining two weights γ_0 and γ_2 to particular sets. This requires to analyze the dynamical properties of the two equations (6a) and (6c) for all values for γ_1 that are reasonable, i.e., for all $\gamma_1 \in [\underline{\gamma}_1^*, \overline{\gamma}_1^*]$. The next proposition establishes the respective result:

Proposition 8.

(i) Assume that there exists a stable two–cycle $0 < \underline{\tau}_u^* < \tau_u^* < \overline{\tau}_u^* < Q - \tau$ *such that* γ_1 *can be restricted to the set* $[\underline{\gamma}_1^*, \overline{\gamma}_1^*]$. *Let* $[\underline{\gamma}_0^*, \overline{\gamma}_0^*]$ *as well as* $[\underline{\gamma}_2^*, \overline{\gamma}_2^*]$ *denote the set of fixed points of equations (6a) and (6c) given* $\gamma_1 \in [\underline{\gamma}_1^*, \overline{\gamma}_1^*]$. *These sets then represent all values for the weights* γ_0 *as well as* γ_2 *that are compatible with common knowledge and rationality.*

(ii) If there exists neither a SREE nor a stable two–cycle, the assumptions of rationality and common knowledge impose no further restrictions on the weights γ_0 *and* γ_2

Proof. See Appendix. □

Fig. 5. *Best response dynamics for the weight γ_0.*

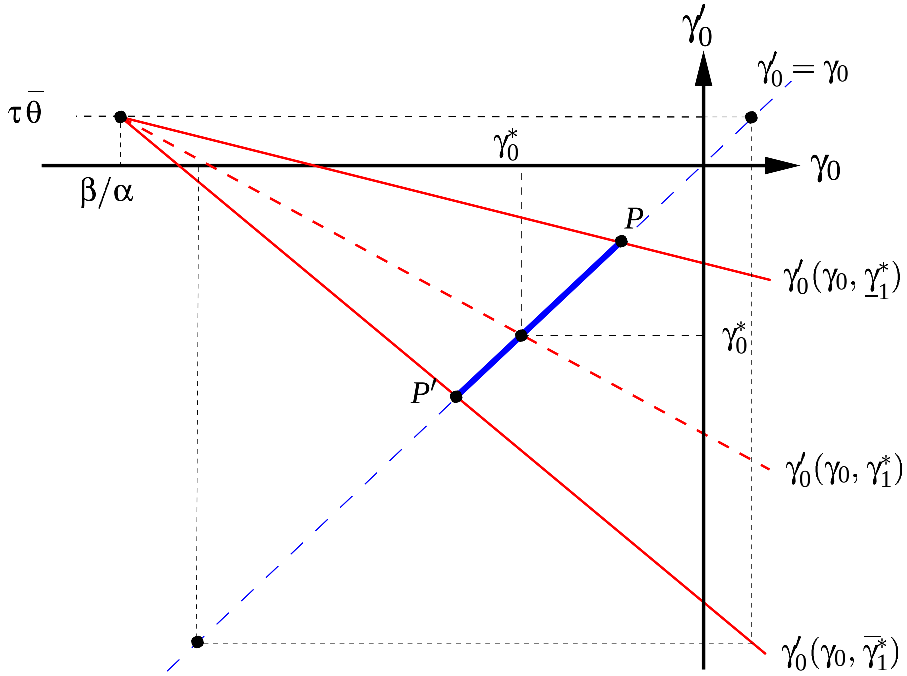
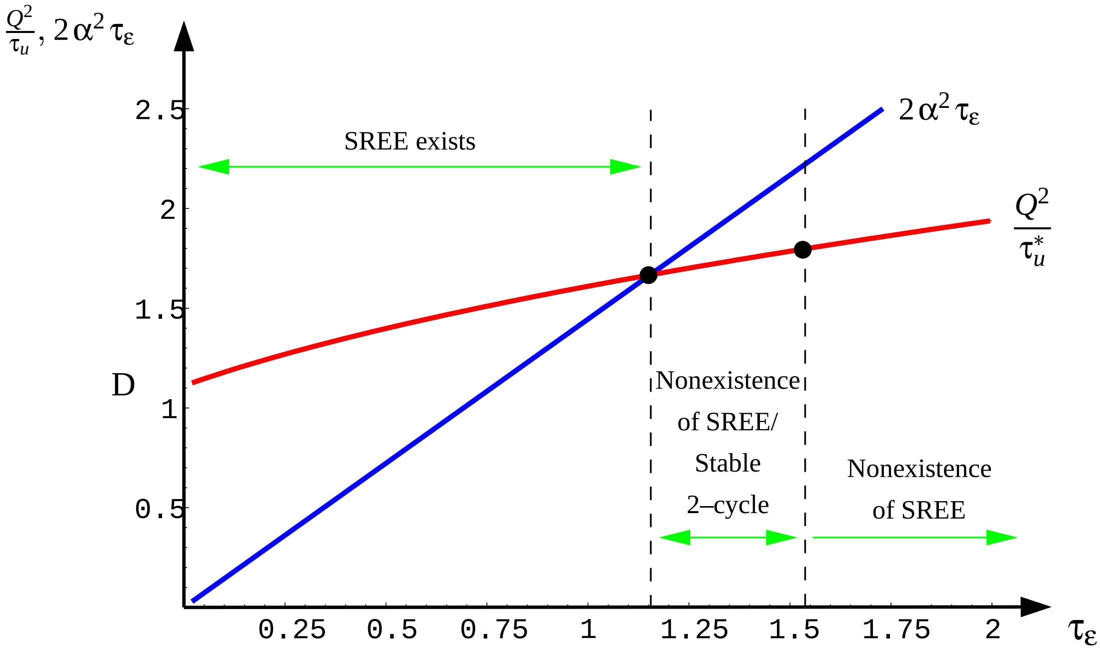


Figure 5 serves to illustrate this result. The figure shows the best response function for the weight γ_0 according to equation (6a) in case of a two–cycle for all values of γ_1 within the set $[\underline{\gamma}_1^*, \bar{\gamma}_1^*]$. The result stated in Proposition 8 builds on the fact that the maximum of slopes of these best responses (i.e. the slope of the straight line $\gamma_0'(\gamma_0, \bar{\gamma}_1^*)$) is less then one in absolute value, whenever a stable two–cycle exists. Given this it is possible to restrict the set of weights γ_0 that are compatible with rationality and common knowledge to values corresponding to the line segment between the points P and P' in the figure.[10] Linearity of the best response function (6a) then implies that this is possible starting from any set of possible weights γ_0. In a similar fashion it can be shown that regarding the weight γ_2 there exists a set $[\underline{\gamma}_2^*, \bar{\gamma}_2^*]$ of weights such that all γ_2 within this set are compatible with rationality and common knowledge. Thus, even if there may exist no SREE, the assumptions of rationality and common knowledge allow to restrict the set of possible supply functions that will be used by rational firms, when a stable two–cycle exists. If even this is not the case, that is, if not even a stable two–cycle exists, it is still possible to restrict the weights γ_1 and the precisions τ_u of the privately acquired information, but the best response mappings (6a) and (6c) are unstable for some of the reasonable values for γ_1 and τ_u. This means any values for γ_0 and γ_2 are compatible with rationality and common knowledge in this case.

[10] The underlying argument is quite similar to the one presented in section 3 in the discussion of figure 1.

Fig. 6. *Properties of the REE dependent on the noise precision* τ_ε



Going back to the above presented numerical examples, figure 6 allows to summarize these results. The figure shows the two components of the condition for existence of a SREE in dependence on the precision τ_ε of the noise in the model. The equilibrium amount of private information which is acquired in a REE decreases as τ_ε rises. Hence, starting from $Q^2/\tau_u^* = Q^2/(Q-\tau)$ for $\tau_\varepsilon = 0$, we have $Q^2/\tau_u^* \to \infty$ as $\tau_\varepsilon \to \infty$. This means that there exists an upper bound for τ_ε such that a SREE exists only if the precision of the noise smaller than this respective upper bound. If τ_ε is larger than this upper bound, no SREE exists, but at least a stable two–cycle exists, whenever the condition stated in Proposition 7 is satisfied. This condition requires $\frac{Q^2}{\tau_u^*} > \alpha^2\tau_\varepsilon\left(1+\alpha^2\tau_\varepsilon\frac{\tau_u^*}{Q^2}\right)$. In this case, the assumptions of rationality and common knowledge at least allow to restrict the behavior of the firms. Even this becomes impossible, whenever τ_ε is still larger, even though an REE always exists as long as τ_ε remains bounded. For $\tau_\varepsilon \to \infty$, however, the Grossman–Stiglitz Paradox in its original shape appears: As prices become fully informative, no firm has an incentive to acquire information the information prices will reveal. In this case no REE exists. Viewed from this perspective, the main result of the paper can be summarized as follows: Whenever prices are fully informative, no REE exists, since there is no incentive to acquire information. As the informativeness of prices becomes smaller, e.g. because τ_ε becomes smaller, a REE exists, but it may be impossible to justify this REE using the assumptions of rationality and common knowledge, since no SREE exist. Only if the informational content of prices falls short of a certain upper bound, it is at least possible to predict that rational individual behavior will be restricted to a particular set of actions and only for an even lower informativeness of prices a SREE exists.

5. Conclusions

In the present paper, we have shown how known results for existence of SREE must be modified, if models with endogenously acquired private information are considered. While this assumptions does not lead to modifications of the respective conditions for existence in when there is no learning from prices, it turns out we arrive at stronger conditions if there is such learning. In particular, it was shown that prices in a REE need to be half as informative than private signals for a SREE to exist in case of learning from prices, whereas it is sufficient for prices to be less informative than private signals without such learning. It was also possible to give an interpretation of the result that falls back on the well known Grossman–Stiglitz Paradox of the impossibility of informationally efficient markets. Viewed from this perspective, our result says that for existence of a SREE markets have to show a minimum level of informational inefficiency.

Future work on this subject will allow for increasing marginal costs of information acquisition in order to check the robustness of the result. Moreover, it should be analyzed whether the results carry over to financial market models with learning from current prices, where risk aversion of traders is allowed for.

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Appendix

Proof of Proposition 1. We will first show, that there exists an unique linear equilibrium with given precisions $\tau_u(i) \geq 0$ for all $i \in I$. After that, we derive the optimal individual amount of information acquisition in such an equilibrium. Assume that each $i \in I$ firm uses the linear strategy $x^*(i) = \psi\,[\gamma(i)_0 + \gamma(i)_1\, s(i)]$. With $\int_0^1 \gamma(i)_0\, di = \gamma_0$, and $\int_0^1 \gamma(i)_1\, di = \gamma_1$ as well as $x^s = \int_0^1 x^*(i)\, di = \psi[\gamma_0 + \gamma_1\,\theta]$ such strategies result in the market price:

$$p = \beta + \alpha\,[\gamma_0 + \gamma_1\,\theta] + \varepsilon$$

where $\alpha = -\psi/\varphi$. From this it follows $p - \theta = \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\,\theta + \varepsilon$ and the respective conditional expectation of a firm $i \in I$ results as:

$$\begin{aligned}
\mathrm{E}[p-\theta\,|\,s(i),w] &= \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\ \mathrm{E}[\theta\,|\,s(i),w] \\
&= \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\ \left[\frac{\tau(i)_u}{\tau + \tau(i)_u}\, s(i) + \frac{\tau}{\tau + \tau(i)_u}\,\hat{\theta}\right] \\
&= \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\,\frac{\tau}{\tau + \tau(i)_u}\,\hat{\theta} + (\alpha\gamma_1 - 1)\,\frac{\tau(i)_u}{\tau + \tau(i)_u}\, s(i) \qquad \text{(A.1)}
\end{aligned}$$

Therefore, in an equilibrium the individual coefficients must satisfy the following two equations:

$$\gamma(i)_0 = \beta + \alpha\gamma_0 + (\alpha\gamma_2 - 1)\,\frac{\tau}{\tau + \tau(i)_u}\,\hat{\theta} \qquad \text{(A.2a)}$$

$$\gamma(i)_1 = (\alpha\gamma_1 - 1)\,\frac{\tau(i)_u}{\tau + \tau(i)_u} \qquad \text{(A.2b)}$$

Now assume $\tau(i)_u = \tau_u^*$ for all $i \in I$. Eqs. (A.2a) and (A.2b) can then be solved for the equilibrium coefficients:

$$\gamma_0^* = \frac{\beta}{1-\alpha} - \frac{1}{1-\alpha}\,\frac{\tau}{\tau + (1-\alpha)\,\tau_u^*} \qquad \text{(A.3a)}$$

$$\gamma_1^* = -\,\frac{\tau_u^*}{\tau + (1-\alpha)\,\tau_u^*} \qquad \text{(A.3b)}$$

It remains to derive the optimal individual amount of information acquisition. Assume that the costs associated with information acquisition are given by $K(\tau(i)_u)$ and let $\kappa(\tau(i)_u)$ denote the respective marginal costs. Profit $\pi(i)$ of firm i in an equilibrium is then given by:

$$\begin{aligned}
\pi(i) &= [p-\theta]\,x(i) - \frac{1}{2}\,\frac{1}{\psi}\,[x(i)]^2 - K(\tau(i)_u) \\
&= \psi\left[\beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\,\theta + \varepsilon\right]\left[\gamma(i)_0 + \gamma(i)_1\, s(i)\right] - \frac{1}{2}\,\psi\left[\gamma(i)_0 + \gamma(i)_1\, s(i)\right]^2 - K(\tau(i)_u)
\end{aligned}$$

(A.4)

We can write Eq. (A.4) as follows:

$$\pi(i) = \psi\left[\beta + (\alpha\gamma_1 - 1)[\theta - \bar{\theta}] + (\alpha\gamma_1 - 1)\bar{\theta} + \alpha\gamma_0 + \varepsilon\right]\left[\beta + \gamma(i)_0 + \gamma(i)_1[s(i) - \bar{\theta}] + \gamma(i)_1\bar{\theta}\right]$$
$$-\frac{1}{2}\psi\left[\beta + \gamma(i)_0 + \gamma(i)_1[s(i) - \bar{\theta}] + \gamma(i)_1\bar{\theta}\right]^2 - K(\tau(i)_u)$$

Taking expectations then yields:

$$\mathrm{E}[\pi(i)] = \psi\left(\beta + (\alpha\gamma_1 - 1)\bar{\theta} + \alpha\gamma_0\right)\left(\beta + \gamma(i)_0 + \gamma(i)_1\bar{\theta}\right) + \psi\gamma(i)_1(\alpha\gamma_1 - 1)\frac{1}{\tau}$$
$$-\frac{\psi}{2}\left(\gamma(i)_0 + \gamma(i)_1\bar{\theta}\right)^2 - \frac{\psi}{2}\gamma(i)_1^2\left(\frac{\tau + \tau(i)_u}{\tau(i)_u\tau}\right) - K(\tau(i)_u) \tag{A.5}$$

The first order conditions with respect to $\gamma(i)_0$, $\gamma(i)_1$ and $\tau_u(i)$ are:

$$\frac{\partial\mathrm{E}[\pi(i)]}{\partial\gamma(i)_0} = \psi\left(\beta + (\alpha\gamma_1 - 1)\bar{\theta} + \alpha\gamma_0\right) - \psi\left(\gamma(i)_0 + \gamma(i)_1\bar{\theta}\right)$$
$$\frac{\partial\mathrm{E}[\pi(i)]}{\partial\gamma(i)_1} = \psi\bar{\theta}\left(\beta + (\alpha\gamma_1 - 1)\bar{\theta} + \alpha\gamma_0\right) + \psi(\alpha\gamma_1 - 1)\frac{1}{\tau} - \psi\bar{\theta}\left(\gamma(i)_0 + \gamma(i)_1\bar{\theta}\right) - \psi\gamma(i)_1\frac{\tau + \tau(i)_u}{\tau(i)_u\tau}$$
$$\frac{\partial\mathrm{E}[\pi(i)]}{\partial\tau(i)_u} = \psi\frac{1}{2}\gamma(i)_1^2\frac{1}{\tau(i)_u^2} - \kappa(\tau(i)_u)$$

We obtain the following solutions:

$$\gamma(i)_0 = \beta + \alpha\gamma_0 + (\alpha\gamma_1 - 1)\frac{\tau}{\tau + \tau(i)_u}\hat{\theta} \tag{A.6a}$$
$$\gamma(i)_1 = (\alpha\gamma_1 - 1)\frac{\tau(i)_u}{\tau + \tau(i)_u} \tag{A.6b}$$
$$0 = \psi\frac{1}{2}\left[\frac{\gamma(i)_1}{\tau(i)_u}\right]^2 - \kappa(\tau(i)_u) \tag{A.6c}$$

Under the assumption of an equilibrium with $\gamma(i)_1 = \gamma_1^*$ and $\tau(i)_u = \tau_u^* > 0$, substitution of Eq. (A.6b) into (A.6c) gives:

$$\frac{\psi}{2}\frac{1}{(\tau + (1 - \alpha)\tilde{\tau}_u)^2} = \kappa(\tilde{\tau}_u) \tag{A.7}$$

Eq. (A.7) will not necessarily possess a solution with $\tau_u^* > 0$. In such a case, the respective solution is $\tau_u^* = 0$. Together with the above derived Eqs. (A.3a) and (A.3b) the REE is then completely described. □

Proof of Lemma 1. The best response mapping has already been derived while proving Proposition 1. It is given by Eqs.(A.6a)–(A.6c).

The total differentials of these equations evaluated at the REE are given by:

$$d\gamma'(i)_0 + (\alpha\gamma_1^* - 1)\frac{\tau}{[\tau + \tau_u^*]^2}\hat{\theta}\,d\tau'(i)_u = \alpha\,d\gamma_0 + \alpha\frac{\tau}{\tau + \tau_u^*}\hat{\theta}\,d\gamma_1 \tag{A.8a}$$
$$d\gamma'(i)_1 - (\alpha\gamma_1^* - 1)\frac{\tau_u^*}{[\tau + \tau_u^*]^2}d\tau'(i)_u = \alpha\frac{\tau_u^*}{\tau + \tau_u^*}d\gamma_1 \tag{A.8b}$$
$$W\,d\gamma'(i)_1 - d\tau_u'(i) = 0\,, \tag{A.8c}$$

where $W \equiv \frac{\psi \frac{\gamma_1}{\tau_u^2}}{\kappa' + \psi \frac{\gamma_1{}^2}{\tau_u^3}}$. Using matrices, this system can be formulated as follows:

$$\begin{bmatrix} 1 & 0 & \frac{(\alpha\gamma_1^* - 1)\,\tau}{[\tau + \tau_u^*]^2}\,\hat{\theta} \\ 0 & 1 & -\frac{(\alpha\gamma_1^* - 1)\,\tau_u^*}{[\tau + \tau_u^*]^2} \\ 0 & W & -1 \end{bmatrix} \begin{pmatrix} d\gamma'(i)_0 \\ d\gamma'(i)_1 \\ d\tau'(i)_u \end{pmatrix} = \begin{bmatrix} \alpha & \frac{\alpha\tau}{\tau + \tau_u^*}\,\hat{\theta} & 0 \\ 0 & \frac{\alpha\tau_u^*}{\tau + \tau_u^*} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} d\gamma_0 \\ d\gamma_1 \\ d\tau_u \end{pmatrix}$$

We write this system as $x' = Px$, where $P = A^{-1}B$. Since it turns out that P is a triangular matrix, its eigenvalues are equal the elements on its main diagonal. The respective eigenvalues λ_1, λ_2 and λ_3 are:

$$\lambda_1 = 0, \quad \lambda_2 = \alpha, \quad \lambda_3 = -\frac{\alpha\left(\gamma_1^2\,\psi + \kappa'(\tau_u)\,\tau_u^3\right)}{\gamma_1^2\,\psi + \kappa'(\tau_u)\,\tau_u^2(\tau + \tau_u)}$$

□

Proof of Proposition 5. The proof parallels the proof of Proposition 1. Starting point again is that profit $\pi(i)$ of firm i in an equilibrium is then given by:

$$\pi(i) = [p - \theta]\,x(i) - \frac{1}{2}\frac{1}{\psi}\,[x(i)]^2 - K(\tau(i)_u) \tag{A.9}$$

Since we assume a linear supply rule and are only interested in the optimal choice of $\tau(i)_u$, only the last two terms of the profit equation are relevant. With $C(i)$ denoting these two terms, we get:

$$\begin{aligned} C(i) &= -\frac{1}{2}\frac{1}{\psi}\,[x(i)]^2 - K(\tau(i)_u) \\ &= -\frac{1}{2}\,\psi\left[(1 - \gamma(i)_2)\,p - \gamma(i)_0 - \gamma(i)_2\,s(i)\right]^2 - K(\tau(i)_u) \end{aligned}$$

Since expectations are conditioned on the price p, taking expectations and differentiating with respect to $\tau_u(i)$ yields

$$\frac{\partial\,\mathrm{E}[C(i)\,|\,p]}{\partial\tau(i)_u} = \psi\,\frac{1}{2}\,\gamma(i)_2^2\,\frac{1}{\tau(i)_u^2} - \kappa(\tau(i)_u) \tag{A.10}$$

Thus, the necessary condition for the optimal amount of information acquisition is the same as in the case, where there is no learning from prices. □

Proof of Lemma 2. The best response mapping for the case with a given amount of private information (i.e., Eqs. (6a)– (6c) is derived in Heinemann (2003). The additional Eq. (6d) has just been just derived in the above given proof of Proposition 5. □

Proof of Proposition 6. The relevant dynamical system is given by Eqs. (6a) – (6d). The total differentials of these equations evaluated at the REE are given by:

$$d\gamma'(i)_0 + \frac{\gamma_0^*}{Z}\,d\tau'(i)_u = -\frac{\alpha^2\gamma_1^{*2}\,\tau_\varepsilon}{\tau_u^*}\,d\gamma_0 + \frac{[\beta\tau_\varepsilon - \alpha\gamma_0^* - 2\alpha\gamma_1^*\gamma_0^*]\,\alpha\tau_\varepsilon}{Z}\,d\gamma_1 \tag{A.11}$$

$$d\gamma'(i)_1 - \frac{1 - \gamma_1^*}{Z}\,d\tau'(i)_u = -\frac{2\,\alpha^2\gamma_1^{*2}\,\tau_\varepsilon}{Z}\,d\gamma_1 \tag{A.12}$$

$$d\gamma'(i)_2 + \frac{\gamma_2^*}{Z}\,d\tau'(i)_u = \frac{\alpha\tau_\varepsilon(1 - \alpha(1 - \gamma_2^*)) + 2\,\alpha^2\gamma_1^*\gamma_2^*\,\tau_\varepsilon}{Z}\,d\gamma_1 - \frac{\alpha^2\gamma_1^{*2}\,\tau_\varepsilon}{\tau_u^*}\,d\gamma_2 \tag{A.13}$$

$$W\,d\gamma'(i)_1 - d\tau_u'(i) = 0, \tag{A.14}$$

where $Z = \tau + \tau_u^* + \alpha^2\,\gamma_1^{*2}\,\tau_\varepsilon$ and $W \equiv \frac{\psi \frac{\gamma_1}{\tau_u^2}}{\kappa' + \psi \frac{\gamma_1{}^2}{\tau_u^3}}$. Using matrices, this system can be formulated as follows:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{\gamma_0^*}{Z} \\ 0 & 1 & 0 & -\frac{1-\gamma_1^*}{Z} \\ 0 & 0 & 1 & \frac{\gamma_2^*}{Z} \\ 0 & W & 0 & -1 \end{bmatrix} \begin{pmatrix} d\gamma'(i)_0 \\ d\gamma'(i)_1 \\ d\gamma'(i)_2 \\ d\tau'(i)_u \end{pmatrix} =$$

$$\begin{bmatrix} -\frac{\alpha^2\gamma_1^{*2}\tau_\varepsilon}{\tau_u^*} & \frac{[\beta\tau_\varepsilon - \alpha\gamma_0^* - 2\alpha\gamma_1^*\gamma_0^*]\,\alpha\tau_\varepsilon}{Z} & 0 & 0 \\ 0 & -\frac{2\alpha^2\gamma_1^{*2}\tau_\varepsilon}{Z} & 0 & 0 \\ 0 & \frac{\alpha\tau_\varepsilon[(1-\alpha(1-\gamma_2^*))+2\alpha\gamma_1^*\gamma_2^*]}{Z} & -\frac{\alpha^2\gamma_1^{*2}\tau_\varepsilon}{\tau_u^*} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} d\gamma_0 \\ d\gamma_1 \\ d\gamma_2 \\ d\tau_u \end{pmatrix}$$

We write this system as $x' = Px$, where $P = A^{-1}B$. Since it turns out that P is a triangular matrix, its eigenvalues are equal the elements on its main diagonal. The respective eigenvalues $\lambda_1 \dots \lambda_4$ are:

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = -\frac{\alpha^2\gamma_1^{*2}\tau_\varepsilon}{\tau_u^*}, \quad \lambda_4 = -\frac{2\alpha^2\gamma_1^{*2}\tau_\varepsilon}{Z-(1-\gamma_1^*)W}$$

The condition for stability of this dynamical system and, thus, the condition for existence of a SREE sis that all eigenvalues are less than one in absolute value.

If we now assume that marginal costs of information acquisition are constant and equal to $\bar{\kappa}$, such that $\kappa' = 0$, we have $W = \frac{\tau_u}{\gamma_1}$. From equation (6b) it follows that in equilibrium $W = \tau + \tau_u\alpha^2\gamma_1^2\tau_\varepsilon$ and from this we get $(1-\gamma_1)W = \tau + \alpha^2\gamma_1^2\tau_\varepsilon$. Stability in this case requires:

$$\frac{\alpha^2\gamma_1^{*2}\tau_\varepsilon}{\tau_u^*} < 1 \quad \text{and} \quad \frac{2\alpha^2\gamma_1^{*2}\tau_\varepsilon}{\tau_u^*} < 1,$$

where the second inequality is obviously stronger, such that our stability condition in fact is:

$$2\frac{\alpha^2\gamma_1^2\tau_\varepsilon}{\tau_u} < 1 \quad \Rightarrow \quad \alpha^2\gamma_1^2\tau_\varepsilon < \frac{1}{2}\tau_u$$

□

Proof of Proposition 7. The proof proceeds in two steps. The first step is to find a condition that ensures $T^2(0) > 0$ and the second step is to derive the relevant properties of the mapping $T^2(\tau)$.

i) According to Eq. (7) $T^2(\tau_u)$ is given by:

$$T^2(\tau_u) = Q - \tau - \frac{\alpha^2\tau_\varepsilon}{Q^2}\left(Q - \tau - \frac{\alpha^2\tau_\varepsilon}{Q^2}\tau_u^2\right)^2 \tag{A.15}$$

From this we get $T^2(0) = (Q-\tau)\left[1 - \frac{\alpha^2\tau_\varepsilon}{Q^2}(Q-\tau)\right]$. As an REE with $\tau_u^* > 0$ requires $Q - \tau > 0$, $T^2(0) > 0$ if and only if:

$$1 > \frac{\alpha^2\tau_\varepsilon}{Q^2}(Q-\tau) \tag{A.16}$$

Now, from Eq. (7) it also follows that in an REE we have $\tau_u^* = T(\tau_u^*)$ which is equivalent to:

$$(Q-\tau) = \tau_u^* + \frac{\alpha^2\tau_\varepsilon}{Q^2}\tau_u^*$$

With this, the inequality stated in (A.16) becomes:

$$1 > \frac{\alpha^2\,\tau_\varepsilon}{Q^2}\left[\tau_u^* + \frac{\alpha^2\,\tau_\varepsilon}{Q^2}\,\tau_u^*\right] = \frac{\alpha^2\,\tau_\varepsilon}{Q^2}\,\tau_u^* + \left[\frac{\alpha^2\,\tau_\varepsilon}{Q^2}\,\tau_u^*\right]^2 \tag{A.17}$$

Since $\gamma_1^{*2} = \left(\frac{\tau_u^*}{Q}\right)^2$, the stability condition stated in Proposition 6 is equivalent to $\frac{\alpha^2\,\tau_\varepsilon}{Q^2}\,\tau_u^* < \frac{1}{2}$. Therefore, inequality (A.17) necessarily holds if a SREE exists. This inequality may also hold in cases were no SREE exists. To see this rewrite (A.17) as:

$$\frac{Q^2}{\tau_u^*} > \alpha^2\,\tau_\varepsilon\left(1 + \alpha^2\,\tau_\varepsilon\,\frac{Q^2}{\tau_u^*}\right) \tag{A.18}$$

In case of SREE, the right hand side of this equation is smaller than to $\frac{3}{2}\alpha^2\,\tau_\varepsilon$, while the left hand side is greater than $2\,\alpha^2\,\tau_\varepsilon$ such that (A.18) is satisfied. In contrast, in case of an REE with $\alpha^2\,\tau_\varepsilon = \frac{Q^2}{\tau_u^*}$, no SREE exists, i.e., the left hand side is smaller than $2\,\alpha^2\,\tau_\varepsilon$, while the right hand side equals $2\,\alpha^2\,\tau_\varepsilon$. This implies that there exist REE with $\alpha^2\,\tau_\varepsilon < \frac{Q^2}{\tau_u^*}$ which satisfy the inequality (A.18).

ii) Now, look again at $T^2(\tau_u)$ as stated in Eq. (A.15). It is straightforward to show that (i) $T^2(\tau_u)$ is monotone and increasing and that (ii) $T^2(\tau_u)$ has at most two inflection points:

a) With respect to the derivative with respect to τ_u, $T^{2'}(\tau_u)$, we get:

$$T^{2'}(\tau_u) = T'(\tau_u)\,T'(T(\tau_u)) \geq 0 \quad \text{for all:} \quad 0 \leq \tau_u < Q - \tau$$

because $T'(\tau_u) < 0$.

b) The second derivative with respect to τ_u, $T^{2''}(\tau_u)$, is given by:

$$T^{2''}(\tau_u) = 4\left(\frac{\alpha^2\,\tau - \varepsilon}{Q^2}\right)^2\left[\tau_u\,T'(\tau_u) + T(\tau_u)\right]$$

From this it follows that $T^{2''}(\tau_u) = 0$, if $T(\tau_u) = \tau_u\,T'(\tau_u)$ which is equivalent to:

$$(Q - \tau) = \tau_u^2\,\frac{\alpha^2\,\tau_\varepsilon}{Q^2}$$

With $Q - \tau > 0$, this equation possesses two real roots, such that there are at most two points of inflection where $T^{2''}(\tau_u) = 0$.

Now, existence of a SREE implies $|T'(\tau_u)| < 1$ and hence $|T^{2'}(\tau_u)| < 1$ such that no two cycle exists. Whenever we have $|T'(\tau_u)| > 1$, but inequality (A.17) still holds, we have $T^2(0) > 0$ too such that (since $|T^{2'}(\tau_u)| > 1$) a stable two–cycle exists in the set $S = [0, Q - \tau]$. Finally, if inequality (A.17) does not hold, no such two–cycle exists in the set $S = [0, Q - \tau]$.

□

Proof of Proposition 8. The slopes of the best responses (6a) and (6c) for a given value of γ_1 are given by:

$$\frac{\partial\gamma_0'}{\gamma_0} = \frac{\partial\gamma_2'}{\gamma_2} = -\frac{\alpha^2\,\gamma_1^*\,\tau_\varepsilon}{\tau + \tau_u' + \alpha^2\,\gamma_1^2\,\tau_\varepsilon} \equiv \Gamma$$

It must be shown that this slope is smaller than one in absolute value for the maximum value, the weight γ_1 can attain, if and only if $T^2(0) > 0$.

Let $\bar{\tau}_u$ denote the precision for which $T(\bar{\tau}_u) = 0$:

$$(Q - \tau)\,\frac{Q^2}{\alpha^2\,\tau_\varepsilon} = \bar{\tau}_u^2$$

This precision implies that $\gamma_1 = \bar{\tau}_u/Q$, which is the maximum value, γ_1 can attain as well as $\tau_u' = 0$. In this case, the slope is given by:

$$\begin{aligned}
\Gamma(\bar{\tau}_u) &= -Q\,\frac{\alpha^2\,\tau_\varepsilon\,\bar{\tau}_u}{Q^2\,\tau\alpha^2\,\tau_\varepsilon\,\bar{\tau}_u^2} = -Q\,\frac{\alpha^2\,\tau_\varepsilon\,\bar{\tau}_u}{Q^2\,\tau\alpha^2\,\tau_\varepsilon\left[(Q-\tau)\,\frac{Q^2}{\alpha^2\,\tau_\varepsilon}\right]} \\
&= -\frac{1}{Q^2}\,\alpha^2\,\bar{\tau}_u\,\tau_\varepsilon = -\frac{1}{Q^2\,\bar{\tau}_u}\,\alpha^2\,\bar{\tau}_u^2\,\tau_\varepsilon = -\frac{1}{Q^2\,\bar{\tau}_u}\,\alpha^2\,\tau_\varepsilon\,(Q-\tau)\,\frac{Q^2}{\alpha^2\,\tau_\varepsilon} \\
&= -\frac{Q-\tau}{\bar{\tau}_u}
\end{aligned} \tag{A.19}$$

From (A.19) it follows that $|\Gamma| < 1$ if and only if $Q - \tau < \bar{\tau}_u$, and because $T(\tau_u)$ is monotone decreasing, this requires $T^2(0) = T(T(0)) = T(Q - \tau) > 0$. □