# Optimal Constrained Interest-rate Rules

George W. Evans
University of Oregon

Bruce McGough
Oregon State University

July 2, 2004
Preliminary and Incomplete

**Abstract**

The monetary policy literature has recently devoted considerable attention to Taylor-type rules, in which the interest rate set by the central bank depends on measures of inflation and aggregate output. We show that if policy-makers attempt to choose the optimal rule within a Taylor-type class they may be led to rules that generate indeterminacy and/or instability under learning. This problem is compounded by uncertainty about structural parameters. We advocate a procedure in which policy-makers restrict attention to rules that lie in the determinate stable region for all plausible calibrations, and which minimize the expected loss, computed using structural parameter priors, subject to this constraint.

*JEL classification*: E52, E32, D83, D84.

*Keywords*: Monetary Policy, Taylor Rules, Indeterminacy, E-stability, parameter uncertainty, robust rules.

## 1 Introduction

The development of tractable forward looking models of monetary policy, together with the influential work of [25], has lead to considerable interest in the performance of Taylor-type interest rate rules.[1] These rules take the nominal interest rate as the policy instrument and direct the central bank

[1] For a recent survey and extended analysis, see [24].

to set this rate according to some simple (typically linear) dependence on current, lagged, and/or expected inflation and output gap, and possibly on an inertial term generating interest rate smoothing. Extended Taylor-type rules would allow for a dependence also on observable exogenous shocks.

While these simple policy rules have clear advantages, it has been noted by a number of authors, e.g. [1], [29], [23] and [5], that the corresponding models exhibit indeterminate steady-states for large regions of the reasonable parameter space. This is undesirable because associated with each indeterminate steady-state is a continuum of equilibria depending on extraneous variables known as "sunspots", and the particular equilibrium on which agents ultimately coordinate may not exhibit wanted properties.

The existence of sunspot equilibria raises the question of whether it is plausible that agents will actually coordinate on them if they follow simple adaptive learning rules. Although [28] has shown that stable sunspots can exist in simple overlapping generations models,[2] the sunspot solutions in many calibrated applied models are lacking this necessary stability. For example [7] show that sunspots in the Farmer-Guo model are unstable, and [11] describe a stability puzzle surrounding the lack of stable indeterminacies in a host of non-convex RBC-type models. The New Keynesian Monetary model, however, has led to the discovery of cases involving forward looking Taylor rules in which sunspot solutions are stable under learning for certain representations of these solutions, see [17] and [12].[3]

In [12] we modified the theory of common factor representations to apply to models of monetary policy and found that if the policy rule was forward looking then the associated model exhibited stable common factor sunspots for some parameter values; and further, that this type of stable indeterminacy can exist for reasonable values of the structural and policy parameters. This result raises a natural question: if policy makers are choosing the parameters of their policy optimally (as measured by some standard loss function), is it still possible for the resulting economy to yield stable indeterminacy? This question is the first issue our paper seeks to address. If it is possible for unconstrained optimal policy to lead to this result then policy makers would be well advised to constrain their optimization problem by searching only among those rules that yield stable determinacy. Equally troubling would

[2]For the local stability conditions see [6] and [10].

[3]See [13] for a thorough discussion in the univariate case of the alternative representations of sunspot solutions. This paper shows that "common factor" sunspots can be stable under learning even when other standard representations are not.

be cases where unconstrained optimization leads to unstable determinacy or unstable indeterminacy.

We begin our analysis of optimal policy by simply appending to the models we studied in [12] a government criterion representing a loss in the volatility of output gap and inflation. We consider a several calibrations of the New-Keynesian relations including purely forward-looking and inertial specifications.

For each of a variety of Taylor-type rules (i.e. policy rules that condition only on endogenous variables) we use numerical methods to compute the optimal policy. We find that, in case forward-looking models are considered, the resulting economy may yield stable indeterminacy; it may also yield stable determinacy, unstable indeterminacy, and unstable determinacy, depending on calibrations and policy rules. Thus, when a Taylor-type rule is used, our question is answered and policy makers are strongly cautioned: unconstrained optimal policy may yield undesirable outcomes.[4] On the other hand, we find that inertial specifications may mitigate this danger. When relatively high degrees of inertia are present in the IS and AS relations, optimal policy appears to always render stable determinacy.

Next we turn to the analysis of extended Taylor-type rules, i.e. policy rules that depend on exogenous shocks as well as endogenous variables. [8] showed that these types of rules may be able to implement the unconstrained optimal REE, that is, the REE that, independent of policy, minimizes the government's criterion. In particular, these authors constructed two policy rules consistent with the optimal REE: one depending only on lagged output gap and current shocks, and the other also depending on expectations. When closed with the former rule, the economy was always unstable and possibly indeterminate, but when closed with the latter rule, the economy was always stable and determinate. In our analysis of extended Taylor-type rules, we characterize all possible forward looking rules capable of implementing the optimal REE, and then study their associated stability and determinacy

[4]Some of these issues were previously investigated by [21], who evaluated optimal policy for several classes of Taylor-type rules. Our analysis goes beyond theirs in several ways. They restricted attention to optimal policy given their specific estimated model, whereas we investigate policy across calibrations. Secondly, we consider stability under learning, as well as indeterminacy, and investigate the apparent gains to policy rules inside the unstable and/or indeterminate regions. Third, we examine optimal policy when policy makers have uncertainty about structural parameter values and make explicit allowance for this uncertainty.

properties. We find that "optimal policy" can yield unstable indeterminacy and stable determinacy, as pointed out by Evans and Honkapohja, but that it also may result in stable sunspots. These results are particularly worrying since a numerical search algorithm, if left unconstrained, cannot distinguish between the stable determinate Evans-Honkapohja-rule, and a rule yielding a potentially far worse result.

The dangers described in above demand prescription and we turn to this in Section 5. Most obviously, we advise policy makers to restrict attention to policy rules that result simultaneously in stability and determinacy. However, this necessary restriction may not be sufficient to guarantee good outcomes: a rule that is optimal and yields stable determinacy with respect to one calibration may result in stable sunspots with respect to another calibration. This point holds regardless of whether inertial specifications are considered and whether extended Taylor-type rules are used. To address this problem of parameter uncertainty, we consider the existence of "robust" policy. At issue here is whether there exist policy rules yielding stable determinacy across calibrations. If so, we can employ techniques such as those advocated by [2]. That is, we show how to obtain a policy rule that meets the constraint that it is stable determinate across calibrations and that is the optimal choice within this class, based on prior probabilities for the alternative calibrations. We find that for each form of Taylor-type rules, such a constrained optimal policy exists, and we report its computation.

We next employ the same technique, incorporating structural parameter uncertainty, to extended Taylor-type rules. We first note that, while for a given known calibration, a fully optimal such policy exists that is also stable and determinate, the intersection of these regions across calibrations appears to be empty. This forcefully illustrates the strong and binding nature of the constraint that the chosen policy rule be stable and determinate across calibrations. We then proceed to show how to implement our procedures for choosing the optimal constrained policy when parameter uncertainty is present.

The paper is organized as follows. Section 2 develops the theory necessary construct common factor representations and analyze their stability in the context of multivariate monetary models. Also, the specification of the government's criterion, as well as the method used to compute its value, is described. The work in this section is done with respect to Taylor-type rules, but we note that it is straightforward to apply the same techniques to the extended Taylor-type rules as well. Section 3 presents the results on Taylor-type rules, including figures containing graphs of policy indifference curves. Section 4 presents results on extended Taylor rules. Section 5 presents the analysis of optimal policy under parameter uncertainty and Section 6 concludes.

# 2 Theory

In this section we develop the theory necessary to analyze the stability of sunspot equilibria and the evaluation of the government's objective, thus allowing for the derivation of optimal policy. We begin by specifying the models of interest. Then, for expedience, we choose a particular specification and develop the associated equilibrium representations and learning analysis. It is straightforward to modify this developed theory for application to other model specifications, and thus we omit the details concerning these other models. Finally, we describe the policy maker's problem and show how to compute the value of the criterion for given structural and policy parameters.

## 2.1 Monetary Models and Policy Rules

We study optimal policy using several variants of the New Keynesian Monetary model. All specifications have in common the following forward looking IS-AS curves:

$$IS : x_t = -\phi(i_t - E_t\pi_{t+1}) + \delta E_t x_{t+1} + (1-\delta)x_{t-1} + g_t \quad (1)$$
$$AS : \pi_t = \beta(\gamma E_t\pi_{t+1} + (1-\gamma)\pi_{t-1}) + \lambda x_t + u_t \quad (2)$$

Here $x_t$ is the proportional output gap, $\pi_t$ is the inflation rate, and $g_t$ and $u_t$ are independent, exogenous, stationary, zero mean AR(1) shocks with damping parameters $0 \leq \rho_g < 1$ and $0 \leq \rho_u < 1$ respectively.

The first equation is a formulation of the forward-looking IS curve amended to include inertia. This functional form may be obtained from a linearized model of optimization behavior on the part of consumers. In some cases we also allow for an inertial term $x_{t-1}$ present due to habit formation: see for example [22]. The second equation is the forward-looking Phillips curve. When $\gamma = 1$, equation (2) is the pure forward-looking New Keynesian "AS" relationship based on "Calvo pricing," and employed in [4] and Ch. 3 of [30].[5]

[5] For the version with mark-up shocks see [30] Chapter 6, Section 4.6.

Here $0 < \beta < 1$ is the discount factor. Again, this equation is obtained as the linearization around a steady state. The specification of the AS curve in the case $0 < \gamma < 1$ incorporates an inertial term and is similar in spirit to [14], the Section 4 model of [15], and the Ch. 3, Section 3.2 model of [30], each of which allows for some backward looking elements. Models with $0 < \gamma < 1$ are often called "hybrid" models, and we remark that in some versions, such as [14], $\beta = 1$, so that the sum of forward and backward looking components sum to one, while in other versions $\beta < 1$ is possible.[6]

This structural model may be closed by specifying a policy rule describing how interest rates are set. The region and nature of a model's indeterminacy depends critically on the specification of this policy rule. To better understand the role of this specification, we analyze a number of policy rules, which we parameterize as follows:

$$PR_1 : i_t = \alpha_\pi E_t \pi_t + \alpha_x E_t x_t \quad (3)$$

$$PR_2 : i_t = \alpha_\pi \pi_{t-1} + \alpha_x x_{t-1} \quad (4)$$

$$PR_3 : i_t = \alpha_\pi E_t \pi_{t+1} + \alpha_x E_t x_{t+1} \quad (5)$$

We previously studied the determinacy and stability properties of this set of rules (as well as others) in [12], but here we consider the issue of optimal policy. $\mathrm{PR}_1$, $\mathrm{PR}_2$, and $\mathrm{PR}_3$ are the rules examined by [3]. We have omitted the intercepts for convenience, and in each policy rule $\pi_t$ can be interpreted as the deviation of inflation from its target. These are all Taylor-type rules in the spirit of [25]. We assume throughout that $\alpha_\pi, \alpha_x \geq 0$ and thus the $\alpha_\pi \pi_t$ term in $\mathrm{PR}_1$ indicates the degree to which monetary policy authorities raise nominal interest rates in response to an upward deviation of $\pi_t$ from its target.

Taylor's original formulation specified dependence on current values of endogenous variables; but the assumption that current data on inflation and the output gap are available to policymakers when interest rates are set has been met with criticism: see for example [19]. [3] look at three natural alternatives: a slight modification of Taylor's formulation yields $\mathrm{PR}_1$ in which policy makers condition their instrument on expected values of current inflation and the output gap; in $\mathrm{PR}_2$ policy makers respond to the most recent observed values of these variables; and in $\mathrm{PR}_3$ they respond instead to fore-

[6] To remain consistent with the work of [**?**], when we include inertia in our analysis, we set $\beta = 1$.

casts of future inflation and the output gap.[7]

This list of rules is far from exhaustive. In particular, it is quite natural to include dependence on fundamental shocks, as well as analyze more general rules which nest $PR_1$ - $PR_3$ as special cases; in fact, some forms of these more general rules allow for the implementation of the best possible equilibrium as measured via the government's objective. We consider some of these more general rules below, but to provide better context, we put off their discussion until Section 4.

## 2.2 Determinacy

As usual, the model is said to be determinate if there is a unique nonexplosive REE and indeterminate if there are multiple nonexplosive solutions.[8] The determinacy of a model can be analyzed by writing the reduced form equation as a discrete difference equation with the associated extraneous noise terms capturing the errors in the agents' forecasts of the free variables. If the nonexplosive requirement of a rational expectations equilibrium pins down the forecast errors, that is, if the dimension of the unstable manifold is equal to the number of free variables, then the model is determinate. On the other hand, if the errors are not pinned down, that is, if the dimension of the unstable manifold is less than the number of free variables, these forecast errors can capture extrinsic fluctuations in agents' expectations that are not inconsistent with rationality. In this case, multiple equilibria exist; these types of equilibria are sometimes called sunspots.

The methodology for assessing determinacy is well known, and we refrain from presenting the details. For the monetary models and interest rate rules considered in this paper the specifics are given in our earlier paper [12]. If the model is indeterminate, we can distinguish between the cases of order one and order two indeterminacy, depending on whether they are driven by one or two dimensional extraneous sunspot variables. Furthermore, the set of sunspot solutions has alternative representations, a point that is important if one is interested in whether sunspot solutions are stable under learning.

[7] Because at the moment we are assuming rational expectations and a common information set, we do not need to specify whose forecasts are represented in the interest rate rules (3) and (5). We will return to this matter when we discuss the economy under learning.

[8] By "nonexplosive" we mean that the conditional expectation of the absolute value of future variables is uniformly bounded over the horizon. For a detailed discussion of this and related concepts see [13].

Again, these issues are discussed at length in [12].

## 2.3 Learning

If the model is determinate, so that there is a unique non-explosive REE (rational expectations equilibrium), it is desirable that the solution be stable under learning. By this we mean that there is convergence to the solution if private agents in the economy estimate and update the coefficients of their forecast functions using least squares regressions. Because the models are self-referential, i.e. the evolution of the economy depends on how agents form expectations, the stability of an REE under least squares learning cannot be taken for granted.

More specifically, the structural model combined with the interest rate rule can be written in reduced form as follows:

$$y_t = AE_t^* y_{t+1} + BE_t^* y_t + Cy_{t-1} + D\hat{g}_t, \tag{6}$$

where $y_t' = (x_t, \pi_t)$. We now write $E_t^*$ to indicate that we no longer impose rational expectations, and at issue is how agents form their time $t$ expectations $E_t^*$. In the determinate case the unique nonexplosive solution takes the form

$$y_t = a + by_{t-1} + c\hat{g}_t, \tag{7}$$

for particular values of $\bar{a}, \bar{b}$ and $\bar{c}$.

Under least squares learning (7) is treated as the econometric specification of a forecasting rule, the parameters of which are estimated updated by the private agents. The specification is often referred to as a Perceived Law of Motion (PLM). Combining these regressors into the vector $X_t' = (1, y_{t-1}', \hat{g}_t')$ and writing the parameters as $\Theta = (a, b, c)$, the PLM can be written as $y_t = \Theta' X_t$. Under learning agents obtain least squares estimates $\Theta_t = (a_t, b_t, c_t)$ using data through time $t$ and then use the estimated PLM to form their forecasts $E_t^* y_{t+1}$, which in turn influence the path of $y_t$. The question is then whether or not $(a_t, b_t, c_t) \rightarrow (\bar{a}, \bar{b}, \bar{c})$ as $t \rightarrow \infty$. If so, we say that the solution is stable under learning.

We use expectational stability as our criterion for judging whether agents may be able to coordinate on specific solutions, including in particular sunspot equilibria. This is because, for a wide range of models and solutions, E-stability has been shown to govern the local stability of REE under least squares learning. In many cases this correspondence can be proved, and in cases where this cannot be formally demonstrated the "E-stability principle" has been validated through simulations. For a thorough discussion of E-stability see [7].

The E-stability technique is based on a mapping from the PLM to the corresponding Actual Law of Motion (ALM) parameters. For the case at hand, if agents believed in the PLM $(a, b, c)$ then their corresponding forecasts would be given by $E_t^* y_{t+1} = a + bE_t^* y_t + cE_t^* \hat{g}_{t+1}$. Using $E_t^* y_t = a + by_{t-1} + c\hat{g}_t$, and assuming for convenience that $\rho$ is known so that $E_t^* \hat{g}_{t+1} = \rho\hat{g}_t$, yields

$$E_t^* y_{t+1} = (I_2 + b)a + b^2 y_{t-1} + (bc + c\rho)\hat{g}_t.$$

Inserting $E_t^* y_t$ and $E_t^* y_{t+1}$ into (6) and solving for $y_t$ as a linear function of an intercept, $y_{t-1}$ and $\hat{g}_t$ yields the corresponding ALM parameters induced by the PLM.

$$a \rightarrow A(I_2 + b)a + Ba \quad (8)$$

$$b \rightarrow Ab^2 + Bb + C \quad (9)$$

$$c \rightarrow A(bc + c\rho) + Bc + D. \quad (10)$$

Equations (8)-(10) defines a mapping from PLM parameters $\Theta$ to the ALM parameter $T(\Theta)$. The REE $\bar{\Theta} = (\bar{a}, \bar{b}, \bar{c})$ is a fixed point of this map and it is said to be E-stable if it is locally asymptotically stable under the differential equation

$$\frac{d\Theta}{d\tau} = T(\Theta) - \Theta. \quad (11)$$

The E-stability principle tells us that E-stable representations are locally learnable for Least Squares and closely related algorithms. That is, if $\Theta_t$ is the time $t$ estimate of the coefficient vector $\Theta$, and if $\Theta_t$ is updated over time using recursive least squares, then $\bar{\Theta}$ is a possible convergence point, i.e. locally $\Theta_t \rightarrow \bar{\Theta}$, if and only if $\bar{\Theta}$ is E-stable. Computing E-stability conditions is often straightforward, involving computation of eigenvalues of the Jacobian matrices of (11).

Determinacy and stability under learning are clearly desirable properties for a policy rule. If a policy rule yields indeterminacy then in addition to the intended REE there exist other solutions depending on sunspot variables that may be substantially inferior, in terms of the policy makers objective function. If the policy rule yields determinacy but is unstable under learning, then the economy will fail to converge to the intended solution. The earlier literature has shown that these are independent properties and so both must be checked.

A further issue of considerable interest is whether, in the case of indeterminacy, the sunspot solutions are stable under learning. Recent research has found that sunspot solutions can in some cases be stable under learning in monetary models of the type considered here, and that stability can depend on the particular representation of the solution that forms the basis of the agents' PLM.[9] For example, "General Form Representations," which here would take the form

$$y_t = a + by_{t-1} + hy_{t-2} + c\hat{g}_t + f\hat{g}_{t-1} + e\xi_t,$$

with the sunspot $\xi_t$ an arbitrary one or two-dimensional martingale difference sequence, appear never to be stable under learning. However "Common Factor Representations,"

$$y_t = a + by_{t-1} + c\hat{g}_t + d\zeta_t,$$

where $\zeta_t = \lambda\zeta_{t-1} + \breve{\varepsilon}_t$ is an exogenous one or two-dimensional sunspot with "resonant frequency" parameter $\lambda$, have in some cases been found to be stable under learning.

## 2.4 Government's Behavior

The model is closed via inclusion of the policy rule; however, the parameters of the policy rule are still free. These parameters may be pinned down by imposing optimizing behavior on the part of the government. As is standard in the literature, we assume, for the government's criterion, a loss function that is quadratic in $\pi$ and $x$.[10] The government chooses its policy parameters to minimize this criterion subject to the structural model of the economy. For example, if we are analyzing $\text{PR}_1$, then we assume the government faces the following problem:

$$\min_{\alpha_x,\alpha_\pi} \psi Var(x|\alpha) + Var(\pi|\alpha) \tag{12}$$

such that the interest rate is determined by $\text{PR}_1$ and equations (1) and (2) hold. Here $\psi$ is the relative weight assigned to the variance of the output gap,

[9] See [17] and [12].

[10] This is consistent with a second order approximation to expected average utility: see [30].

and $Var(\cdot|\alpha)$ is the unconditional variance of "$\cdot$" given the policy parameters. Note the number of choice variables, i.e. the number of policy parameters, available to the government depends on the policy rule being considered. Here the government has only two choice variables, but for the more general rules considered below this number may increase to nine.

The value of the government's objective may be computed by determining the rational expectations equilibrium associated to the relevant policy parameters $\alpha$; however, in the indeterminate case, this value is not well defined due to the presence of multiple equilibria. Thus we are required to choose an equilibrium from the many available; for the analysis in this paper we choose the "minimal state variable" solution, i.e. a solution of the form $y_t = a + by_{t-1} + c\hat{g}_t$, and in case there are multiple such solutions, we take the loss to equal the minimum of across solutions of this form.[11]

We think of the government's problem described above as being unconstrained. It is unconstrained in the sense that the government, when choosing its optimal policy, does not restrict attention to the region corresponding to stable determinacy. It is our contention that when not constrained the solution may advocate a policy yielding instability, indeterminacy, or both.

# 3 Results on Taylor-type Rules

Our central concern in this section is to investigate the possibility that unconstrained optimization may result in undesirable outcomes. More specifically, we seek to determine whether simply choosing the policy that imparts an MSV solution minimizing the government's objective can imply indeterminacy, instability, or even the presence of stable sunspots. We will find that for forward-looking specifications of the model, all these outcomes are possible, as well as is the prefered outcome of stable determinacy, and thus we are led to the recommendation that optimal policy should be formed subject to the constraints implied by stability and determinacy.

Analytic results are not tractable and so we proceed numerically. We restrict attention to Taylor rules of the form (3)-(5) and analyze each policy with respect to three different calibrations of the parameters in the IS-AS curves, as due to [29], [5], and [20], as well as a fourth *Variant* calibration consistent with estimates reported in the literature; the relevant parameter

[11] If no lagged variables are present in the structural model or in the policy rule then the (unique) minimal state variable solution takes the form $y_t = a + c\hat{g}_t$.

values are given in Table 1 below. All calibrations have in common $\rho_g = \rho_u = .9$. For each calibration we consider two inertial specifications: the usual purely forward looking specification in which $\beta = .99$ and $\delta = \gamma = 0$; and a lagged specification in which $\beta = 1$ and $\delta = \gamma = .5$, which is largely consistent with the estimations of [22].[12] Finally, the conditional variance of $g$ and $u$ must be specified in order to compute the value of the government's objective. For all analysis presented, we take this variance to be .1.[13]

**Table 1: Calibrations**[14]

| Name | $\phi$ | $\lambda$ |
|---|---|---|
| W | 1/.157 | .024 |
| V | 1/.157 | .3 |
| CGG | 4 | .075 |
| MN | .164 | .3 |

For each policy rule, calibration and inertial specification, and for objective weights $\psi \in \{.1, 1, 10\}$, a lattice was analyzed in the region of policy space given by $0 \leq \alpha_x, \alpha_\pi \leq 5$, where $\alpha_x$ and $\alpha_\pi$ are meant to represent the weight on output gap and inflation respectively regardless of the specification of the policy rule. The stability and determinacy properties of the model corresponding to each lattice point were computed, and the value of the government's objective was determined. These values were then used to numerically compute contours, hence a graphical representation of the government's indifference curves was obtained. Finally, a numerical optimization algorithm was used to compute the optimal policy parameters constrained to lie with the specified $5 \times 5$ benchmark policy space.

## 3.1 General Results

Table 2 below presents a complete summary of the results obtained in our numerical analysis for the non-inertial specification. In this table is recorded

[12]Setting $\beta = 1$ in case of inertia in the Phillips curve imposes that the sum of the weights on inflation is unity as is consistent with many, but not all, of the associated theoretical models.

[13]Not surprisingly, altering the value of the conditional variance appears only to change the value of the government's objective at the optimum, and not the parameter values corresponding to optimal policy or the stability and determinacy properties of the associated economy.

[14]The calibrations are for quarterly data, and so the CGG estimates have been adjusted accordingly.

the value of the government's objective, and the stability and determinacy properties of the equilibrium associated to optimal policy, for all permutations of policy rules, calibrations, and objective weights. To identify the stability and determinacy properties, we use the notation SD (stable determinacy), UD (unstable determinacy), SI (stable indeterminacy), and UI (unstable indeterminacy). For example, under the W calibration, using $PR_1$, and assuming $\psi = 1$, the optimal policy yields a stable determinate equilibrium and results in an objective value of 42.25. An objective value marked with an asterisk indicates that across rules it is the smallest value associated to that calibration and objective weight: see Section 3.4 below.

Table 2: Forward-Looking Model

| Calibration | PR | $\psi = .1$ | | $\psi = 1$ | | $\psi = 10$ | |
|---|---|---|---|---|---|---|---|
| W | 1 | 29.84* | SD | 42.25* | SD | 44.09* | SD |
| | 2 | 31.39 | UI | 134.22 | UI | 1094.03 | UI |
| | 3 | 29.84* | SI | 42.25* | SI | 44.09* | SI |
| V | 1 | .58 | SD | 5.17 | SD | 25.21* | SD |
| | 2 | .53* | UI† | 4.59* | UI | 38.02 | UD |
| | 3 | .58 | SI | 5.17 | SI | 25.21 | SI* |
| CGG | 1 | 7.73 | SD | 30.07* | SD | 42.31* | SD |
| | 2 | 7.38* | SD | 40.73 | SD | 295.25 | UI |
| | 3 | 7.73 | SI | 30.07* | SI | 42.31* | SI |
| MN | 1 | 1.70 | SD | 6.17 | SD | 28.40* | SD |
| | 2 | 1.52* | SD | 5.99* | SD | 28.54 | SD |
| | 3 | 2.03 | SD | 6.46 | SD | 28.99 | SD |

Note: Those values of the objective marked with an asterisk represent the minimum across policy rules for fixed calibration and lag structure. Also the symbol † indicates a solution that is very near the origin. In these cases there is typically a solution not near the origin and within the $5 \times 5$ space that yields a value for the objective function close to the optimum. This alternate solution may be SD or UI.

This table indicates the main result of this paper: unconstrained optimal policy may produce SD, UD, SI, or UI. In particular, the constraint of stable determinacy may well be binding and must therefore be imposed when computing optimal monetary policy – not only do regions of UI, UD, and SI exist, but indeed optimization algorithms may seek them out; against this possibility policy makers must stand guard.

The table also provide a caveat to this result, though perhaps not a particularly important one. If policy makers are confident that the model's structural parameters are consistent with the findings of McCallum and Nelson, then they should have no fear of choosing policy associated to undesirable outcomes. However, given the varied opinion in the literature of the appropriate values for the structural parameters, it seems likely that policy makers would not, or at least should not, feel that confident about their estimates.

We now turn to some case specific results. Originally, Taylor specified an interest rate rule conditioned on current levels of inflation and output gap. However, as mentioned above, some regard this rule as infeasible as the Fed is unlikely to have access to the necessary data. The rules specified by (3) - (5) are feasible variants of Taylor's formulation, and perhaps the most closely related to Taylor's rule is $\mathrm{PR}_1$, which, we remind the reader, is given by

$$PR_1 : i_t = \alpha_x E_t x_t + \alpha_\pi E_t \pi_t.$$

The stability and determinacy properties of the purely forward-looking New Keynesian model (1), (2) closed with $\mathrm{PR}_1$ have been characterized analytically by [3]. They found that the regions in policy space corresponding to determinacy and stability coincide – in particular, there are no stable sunspots – and this desirable feature inclined the authors to recommend this rule.[15] The argument in favor of $\mathrm{PR}_1$ is considerably strengthened by the results in the table. Under this rule, for all calibrations and governmental objectives concerned, the resulting optimal policy is stable and determinate; and, this is the only rule for which SD always obtains.

Figure 1 presents our numerical analysis under the W calibration with $\psi = .1$. In this and all figures the contours represent the indifference curves for the government. Regions corresponding to SD, SI, etc. are separated by bold curves. In Figure 1 there are two regions: UI, corresponding to the southwest corner of the figure; and SD in the complement. The large black dot represents the location of the optimal policy parameters as determined by the search algorithm.

Figure 1, W PR1 $\psi = .1$ No Lag Here

We note that the optimal policy chosen by the search algorithm (and consistent with the contours) lies on the eastern boundary of our artificially

[15] [12] extended the result of Bullard and Mitra to include inertia in the Phillips Curve, the further strengthening the argument for rules of $\mathrm{PR}_1$ form.

constrained $5 \times 5$ policy space. And indeed, this constraint is binding; if the search algorithm considers a $100 \times 100$ grid, it selects a point again near the boundary (and again, SD). On the other hand, the flatness of the government's objective, as is evidenced by the contours, imparts little benefit to expanding the parameter space: the optimal value of the objective for the $100 \times 100$ grid is 29.8348 and for the $5 \times 5$ grid is 29.8351, a difference of .001%.

While the benefits of using $PR_1$ are evident, many economists advocate specifying a rule depending on expectations of future inflation, arguing that anticipatory responses may diminish the usual policy lag and also may help anchor agents' expectations. Our version of a forward looking policy rule is (5), as given by

$$i_t = \alpha_x E_t x_{t+1} + \alpha_\pi E_t \pi_{t+1}.$$

Before giving the results we discuss the interpretation of this rule under learning. Under least squares learning private agents are assumed to recursively estimate the parameters of their PLM and use the estimated forecasting model to form the expectations $E_t^* \pi_{t+1}$ and $E_t^* x_{t+1}$ that enter into their decisions as captured by the IS and AS curves. Under $PR_3$ forecasts also enter into the policy rule. Because we are now relaxing the rational expectations assumption, one can in principle distinguish between the forecasts of the private sector, which enter the IS and AS curves, and the forecasts of the Central Bank, which enter policy rule $PR_3$. We will instead adopt the simplest assumption for studying stability under learning, which is that the forecasts for the private sector and the Central Bank are identical. This can either be because private agents and the Central Bank use the same least squares learning scheme, or it could be because one group relies on the others' forecasts. In the latter case, for example, the Central Bank might be setting interest rates as a reaction to private sector forecasts, as in [1] or [9].

The homogeneous expectations assumption was adopted in [3].[16] They found, in case of a purely forward looking AS curve, that determinate equilibria were stable under learning, and also that for some parameter values, there may be stable MSV solutions associated to indeterminate steady-states. In [12] we extended this result to include Phillips curves with explicit inertia,

[16] The implications of heterogeneous expectations in the context of the New Keynesian monetary model is examined in [18]. This issue is further discussed in [9]

and further showed that stable common factor sunspot equilibria may exist.[17]

Now, back to results. A quick glance at Table 2 reveals a picture quite different from $PR_1$. Here, we see that optimizing policy makers may choose rules that result in SI, which implies the existence of stable sunspots. Indeed, only the MN calibration is free of this possibility. Figure 2 presents the results for $PR_3$ under precisely the same calibration, etc. as was used in Figure 1. And notice that almost all the features of the graph, including the shape and values of the level curves, and the location of and value at the optimum, are essentially identical. However, much of the region which, in Figure 1 corresponded to SD, here corresponds to SI.

Figure 2, W PR3 $\psi = .1$ No Lag Here

The existence of stable indeterminacy associated to rules of the form $PR_3$ is troubling because policy makers consider forecasts of future variables when considering policy moves; though, the mere presence of bad outcomes for some policy choices is not necessarily damning. However, the results presented here cast aside any doubt that the existence of stable sunspots is a minor concern; these results imply not only the presence of bad outcomes, but moreover that unconstrained optimizing behavior may in fact result in these outcomes obtaining.

The lagged version of the Taylor rule, $PR_2$, as given by

$$i_t = \alpha_x x_{t-1} + \alpha_\pi \pi_{t-1},$$

yields some interesting behavior not witnessed with $PR_1$ or $PR_3$. As noted by Bullard and Mitra, there exist determinate cases for which the REE is not stable under learning. We find that in fact these cases may be selected by optimizing policy makers. As an example, consider Figure 3. Here we see an optimum within the region of unstable determinacy.

Figure 3, V PR2 $\psi = 10$ Here

## 3.2 Flatness of the Objective

As mentioned above in our discussion of Figure 1, there is a tendency for the government's objective function to be very flat near the optimum. This

[17] [17] found stable finite state Markov sunspots associated to $PR_3$; see [12] for a discussion of the relationship between their result and ours.

has the potential benefit of rendering precision irrelevant when attempting to determine the optimal policy, but also this flatness may be detrimental due to the difficulty of pinning down an optimal rule in the presence of multiple nearly optimal rules.

As an example, consider again the W calibration with either $PR_1$ or $PR_3$ and with $\psi = .1$. Figure 2 suggests that the objective is nearly flat for a non-empty sub-region of the benchmark space, and perhaps even constant along a positively sloped line. To analyze this possibility more closely, we had the optimization algorithm solve the policy problem twice, thus yielding two different optimal policies (both yielding essentially the same value for the objective). These two points were used to construct a line with the following specification: $\alpha_x = .433\alpha_\pi - .365$. We then allowed $\alpha_\pi$ to vary from 1 to 5, used our constructed line to choose $\alpha_x$, and computed the value of the government's objective. The result is plotted in Figure 4. Here a dashed line indicates the corresponding model is SD and a solid line indicates SI. As suspected, the government's objective is almost constant across these parameters.[18]

Figure 4 here

## 3.3 The Inertial Specifications

Analysis of the inertial specifications reveals that including lags in the IS and AS relations may mitigate the negative results obtained in the forward-looking model. Indeed for the relatively high levels of inertia considered in our calibrations, optimal policy always resulted in a determinate model with a stable equilibrium. On the other hand, low levels of inertia present results similar to those obtained in the purely forward-looking case: optimal policy may be consistent with stable sunspots. This raises the interesting issue of how much inertia is required to preclude "bad" optimal policy; we intend to investigate this question more carefully in a future version of the paper.

## 3.4 Optimum across Rules

We have thus far considered the implications of policy making via unconstrained optimization, under the restriction that policy makers are compelled

[18]Extending the line to $\alpha_\pi = 100$ does not alter this finding.

to use a rule of a specified functional form. It is reasonable to assume, however, that policy makers may choose among rules of different functional forms when making decisions; and further, that allowing for this possibility may overturn the generally negative results obtained above. In particular, perhaps for each calibration and objective weight, the optimum across rules yields stable determinacy. The information provided by the table allows us address this question. In the table, for fixed calibration and objective weight, the rule(s) yielding the lowest loss value are marked with an asterisk. We find that even optimizing across rules does not provide a foolproof solution. For example, in case of the variant calibration, the optimum across rules yields UI for $\psi = .1, 1$. Also, for many specifications, $PR_1$ and $PR_3$ produce the same or nearly the same minimum value of the government's objective, but where as $PR_1$ yields SD, in many of these cases $PR_3$ results in stable sunspots.

# 4 Results on Extended Taylor-type Rules

The previous section analyzed policy rules which depended on the model's endogenous variables; however, such a restriction is not necessary. Indeed it may be possible for policy makers to view fundamental shocks and if these shocks contain information orthogonal to that provided by the endogenous variables then policy makers would do well to condition their policy accordingly.

In this section we model optimal extended Taylor-type rules in precisely the same way we modeled Taylor-type rules above. The government takes the form of the rule, as well as the structural model of the economy, as given, and chooses the parameters of the rule to minimize its loss function.

Rules depending on fundamental shocks have been studied by [8].[19] These authors showed that the optimal unconstrained equilibrium – that is, the REE yielding the minimum value of the loss function, independent of the policy rule: see equation (15) below – may be implemented by two different policy rules, each dependent on fundamental shocks; and furthermore, the stability properties of the optimal equilibrium and the determinacy properties of the associated model depend on the form of the policy rule chosen. Because of the close connection between their work and ours, we review their work

[19] Giannoni and Woodford have also studied optimal policy rules dependant upon exogenous shocks for a host of IS and AS specifications. See [16] for details.

briefly here. Then, noting that a modification of the government's first order conditions nicely connects their work to ours, we proceed to characterize all possible rules that implement the optimal equilibrium, as well as to consider how the stability and determinacy properties change across policy rules.

## 4.1 The Results of Evans and Honkapohja

Using the same non-inertial New-Keynesian structural model employed here, and assuming the timeless perspective, Evans and Honkapohja (EH) impose that optimal monetary policy is derived from the following objective:

$$E_t \sum_{s=0}^{\infty} \beta^s (\pi_{t+s}^2 + \psi x_{t+s}^2). \tag{13}$$

As shown by Woodford and others, under commitment the optimal REE must satisfy

$$\lambda \pi_t = -\psi (x_t - x_{t-1}). \tag{14}$$

This dynamic equation may be combined with the AS curve to obtain a representation of the unique optimal REE given by

$$y_t = A y_{t-1} + B \hat{g}_t, \tag{15}$$

where $y = (x, \pi)'$, $\hat{g} = (g, u)'$, and both the second column of $A$ and the first column of $B$ are zero: see equations (9), (10), in [8].

To obtain an interest rate rule consistent with this optimal REE, the representation (15) may be used to form expectations, which may then be imposed in the IS curve; the associated optimal interest rate rule is thus obtained, having the form

$$i_t = \delta_x x_{t-1} + \frac{1}{\phi} g_t + \delta_u u_t. \tag{16}$$

However, EH proceed to show that the economy described by (16) together with the structural IS-AS curves (1) and (2), may be indeterminate, and the equilibrium represented by (15) is *never* stable.

An alternate construction of an interest rate rule consistent with the optimal REE proceeds as follows. Solve the optimality condition for $\pi_t$ and impose this equation into the AS curve (2), isolating $x_t$. Combining the resulting equation with the IS curve (1) and solving for $i_t$ yields an interest rate rule of the form

$$i_t = \hat{\alpha}_x^f E_t x_{t+1} + \hat{\alpha}_\pi^f E_t \pi_{t+1} + \hat{\alpha}_x^L x_{t-1} + \hat{\alpha}_g^{\hat{g}} g_t + \hat{\alpha}_u^{\hat{g}} u_t, \tag{17}$$

where the hats on the policy parameters are meant to distinguish this rule – the EH-Rule – from the general rule (19) below.[20] EH show that, when combined with the IS-AS curves (1) and (2), the above rule yields a stable determinate equilibrium, where, of course, the unique REE is necessarily the optimal REE (15).

## 4.2 The Modified EH-Rule

Following Woodford and others, Evans and Honkapohja choose a discounted sum of expected future losses as their objective, and impose commitment by assuming the timeless perspective. This is in contrast to our model, which takes as the objective a loss in unconditional variances. Though different, our objective is closely related to the discounted sum (13). Indeed, our objective is simply a scalar multiple of the average value of (13), where the average is taken across initial conditions.

Because our objective differs from the discounted sum analyzed by EH, and because the timeless perspective is not designed to be fully optimal (more on this point in a moment) it is possible that rules exist that provide performance as measured by our objective superior to the EH-Rule. In fact, Jensen and McCallum (JM) [Cite Here] show that there are even relations of the form (14) that are superior to the timeless perspective. In particular, they recommend a relation of the form

$$\lambda \pi_t = -\psi(x_t - \beta x_{t-1}), \tag{18}$$

and document numerically its superior performance.

The existence of relations like (18) that yield lower average objective values than the timeless perspective is not surprising when one recalls the timeless perspective is not designed fully optimal. Indeed the value of the objective (13) depends upon the initial state of the economy, and the solution to the associated optimization problem advises the government to condition its policy on this initial state differently than it conditions its policy on

[20] The policy parameters $\hat{\alpha}$ may be written in terms of the structural parameters.

subsequent realizations. It is precisely for this reason that the fully optimal policy is not time consistent.

In the sequel, we will be interested in characterizing optimal policy rules, as well as comparing the numerically computed objective value implemented by constrained rules with the objective evaluated at the optimal REE. Because our government's objective is equivalent to the average discounted objective (13), we take the JM relation (18) as defining the optimal REE. It is then straightforward to compute the associated rule of the form (17), which we call the *modified EH-Rule*, and use it to implement the optimal REE.[21] It is also easy to compute an alternative modified rule of the form (16), which leads us to the next section.

### 4.3 The Optimal Policy Manifold

That there are two possible policy rules consistent with the optimal REE begs the question, "what does the collection of all policy rules consistent with the optimal REE look like, and what are the associated stability and determinacy properties?" We address this question by characterizing the collection of all policy rules (restricted within a certain class) that are capable of implementing the optimal REE, where here and for the remainder of the section, the optimal REE refers to the REE obtained by combining (18) with the AS curve under the assumption of no inertia. We postulate a general policy rule of the form

$$i_t = \alpha^f E_t y_{t+1} + \alpha^L y_{t-1} + \alpha^{\hat{g}} \hat{g}_t, \tag{19}$$

where $\alpha^f = (\alpha^f_x, \alpha^f_\pi)$, $\alpha^L = (\alpha^L_x, \alpha^L_\pi)$, and $\alpha^{\hat{g}} = (\alpha^{\hat{g}}_g, \alpha^{\hat{g}}_u)$. Imposing rationality implies that (19) must reduce to (16) when the optimal REE (15) is used to form expectations.[22] Thus our goal is to find policy parameters $\alpha$ which allow for this reduction.

[21] We compared the value of the government's objective when evaluated at the REE implemented by the modified EH-rule with the value of the government's objective when evaluated at the REE implemented by the policy recommended by the search algorithm under the condition that the modified EH-Rule was available as a choice for the search algorithm. We found that the modified EH-Rule and the search algorithm yielded essentially identical results, and further that these results were better than the result obtained when the original EH-Rule was employed.

[22] The coefficients in (15) and (16) are now assumed modified to account for (18).

Using the optimal REE (15), we may form expectations and impose these expectations into (19). The resulting policy rule depends only on $y_{t-1}$ and $\hat{g}_t$ and thus defines a map $T : \mathbb{R}^6 \to \mathbb{R}^{1\times2} \oplus \mathbb{R}^{1\times2}$ as determined by

$$i_t = T_1(\alpha)y_{t-1} + T_2(\alpha)\hat{g}_t. \tag{20}$$

It is straightforward to compute

$$\begin{aligned} T_1(\alpha) &= \alpha^f A^2 + \alpha^L \\ T_2(\alpha) &= \alpha^f (AB + B\rho) + \alpha^g \end{aligned}$$

The policy parameters $\alpha$ are consistent with the optimal REE provided (20) is the same as (16), or, more precisely

$$\begin{aligned} T_1(\alpha) &= (\delta_x, 0) \\ T_2(\alpha) &= (\frac{1}{\phi}, \delta_u). \end{aligned}$$

It is not difficult to show that optimal policy requires $\alpha_g^{\hat{g}} = 1/\phi$ and $\alpha_\pi^L = 0$. The remaining four policy parameters face only two constraints, suggesting that, depending on regularity conditions, the collection of optimal policy rules is characterized by a 2-manifold in 4-space. In fact, one can use a program such as Mathematica to solve for any two of the policy parameters in terms of the remaining policy parameters and structural parameters, thus fully parameterizing the manifold of optimal policy.[23]

To study the impact on stability and determinacy of using alternate optimal policy rules we parameterize the optimal policy manifold by solving for $\alpha_x^L$ and $\alpha_u^{\hat{g}}$ in terms of $\alpha_x^f$, $\alpha_\pi^f$ and structural parameters. Then, for each point on a lattice over the $5 \times 5$ $(\alpha_\pi^f, \alpha_x^f)$ benchmark policy space we computed the stability and determinacy properties of the model closed with the corresponding optimal policy: see Figure 5. For this figure, the V parameterization was used, with the modification that $\lambda = 1$; admittedly this is a value of $\lambda$ that is larger than estimates found in the literature, but it is consistent with certain theoretical models: see for example (CITE HERE). We use this calibration for emphasis and note that while less dramatic, the

[23]The parameterization obtained using Mathematica is too complex to be worth recording here.

same conclusions apply to other calibrations. The large dot is the location of the modified EH-rule.[24]

Figure 5 Here

Figure 5 demonstrates that while rules capable of implementing the optimal REE abound, at least for the modified V calibration a large proportion of these rules have associated to them either stable or unstable indeterminacy. This result is punctuated by the location of the EH-rule: while it does lie in the region of stable determinacy, this region is a small oasis surrounded by a sea of trouble, and because all policies represented in this figure implement the optimal REE, an unconstrained optimizer can not distinguish between the oasis and the sea.

## 4.4 Discussion

In Section 3 on Taylor-type rules, we used numerical analysis to compute the REE yielding the minimum value of the government's loss function subject to the constraint that the policy rule take on a certain functional form. In this section, we found that by relaxing that functional form to include fundamental shocks, the unconstrained optimal REE may be obtained; and furthermore, multiple rules are consistent with its implementation.[25] However, the properties of the economy depend upon the rule chosen to implement the optimal REE: some rules yield stable determinacy while others yield stable or unstable indeterminacy. An unconstrained optimization program will not distinguish between such rules as the objective is flat across the associated regions in parameter space. Therefore, this result strongly advocates imposing stability and determinacy constraints when searching for optimal policy.

[24]Note that in contrast to the exercises producing the figures in the previous section, when producing Figure 5 the policy parameters not referenced in the figure (such as $\alpha_x^L$, etc) are allowed to vary; in fact, they are required to vary in order to maintain a rule consistent with the optimal REE.

[25]Woodford has pointed out that the optimal rule must be history dependant, and therefore can not be strictly forward-looking. In particular, it must depend on lagged variables: as we see here, our optimal rule always depends on $x_{t-1}$.

# 5 Robust Optimal Policy

The results of the previous sections warn of the need to constrain policy choices, and we now turn to the implementation of this warning. For a given calibration and functional form for the policy rule, computing the optimal constrained policy is straightforward: solutions can visualized using the indifference curves and graphic representations of the model's characteristics that were presented earlier; and numerically, constrained optimization algorithms may be employed. Furthermore, in case a non-inertial model is taken as well representing the economy, the optimal REE may be implemented using the modified EH-rule. However, the wisdom of restricting attention to a particular calibration is questionable given the varied estimates of structural parameters available in the literature. Importantly, a rule which performs well with respect to one calibration, may perform quite poorly with respect to another. For example, suppose the true parameters are consistent with the non-inertial Woodford calibration, and $\psi = 1$. If the government uses $PR_3$, believes the parameters are in accordance with the MN calibration, and restricts attention to rules which result in stable determinacy, then the associated optimal policy, when combined with the IS and AS relations under Woodford calibration, will result in an economy exhibiting stable sunspots. This observation holds regardless of the presence of inertia. Similarly, if the government attempts to implement fully optimal policy using the modified EH-rule, stable sunspots will also obtain. We conclude that robustness with respect to alternate calibrations is critical.

The method of robust analysis we prescribe when faced with model uncertainty is based on the work of [2]. Put simply, these authors suggest assigning a distribution over possible models consistent with the priors of the policy maker. The value of the policy maker's objective may then be computed as the expected value of the objective conditioned on this prior distribution.

We implement the method suggested by [2] in our constrained setting by assignment probabilities to the various calibrations. We consider two sets of priors: one which includes inertial models (called "lag models" below), and one which gives positive weight only to purely forward looking models. We assign the weights according to the following table:

**Table 3: Weights**

| | W | V | CGG | MN | W-Lag | V-Lag | CGG-Lag | MN-Lag |
|---|---|---|---|---|---|---|---|---|
| No lag | .3 | .1 | .3 | .3 | 0 | 0 | 0 | 0 |
| Lag | .15 | .05 | .15 | .15 | .05 | .15 | .15 | .15 |

We use a procedure in the spirit of Brock et al method to compute optimal robust policy subject to the constraint that the policy rule have a specified functional form.

## 5.1 Robust Taylor-type Rules

We begin by considering robust rules that depend only on endogenous variables. Specifically, we consider $PR_1 - PR_3$, with the additional possibility of including an interest rate smoothing term parameterized by $\theta$. It is straightforward to verify that for each policy rule among $PR_1 - PR_3$, and for each weight $\psi$, there is a policy pair $\alpha$ so that the associated model is stable determinate for all calibrations, thus the constraint set of our optimization problem is non-empty for all policy rules. Using Matlab to perform the optimization, subject to stability, determinacy, and $\alpha_x, \alpha_\pi \in [0, 5]$, $\theta \in [0, 1]$, we obtained the results presented in the following tables corresponding to $\psi = 1$:

Table 4.1: Robust Taylor-Type Constrained Optimal Policy: No Inertia

| Policy Rule | Restriction | $\alpha_x^f$ | $\alpha_\pi^f$ | $\theta$ | Value |
|---|---|---|---|---|---|
| $PR_1$ | $\theta = 0$ | 5 | 3.78 | – | 27.05 |
| | $\theta \in [0, 1]$ | 1.38 | .81 | 1 | 25.44 |
| $PR_2$ | $\theta = 0$ | .16 | 1.19 | – | 78.6 |
| | $\theta \in [0, 1]$ | .31 | .39 | 1 | 54.45 |
| $PR_3$ | $\theta = 0$ | .26 | 1.36 | – | 52.38 |
| | $\theta \in [0, 1]$ | .48 | 1.05 | .51 | 31.96 |

Table 4.2: Robust Taylor-Type Constrained Optimal Policy: Inertia

| Policy Rule | Restriction | $\alpha_x^f$ | $\alpha_\pi^f$ | $\theta$ | Value |
|---|---|---|---|---|---|
| $PR_1$ | $\theta = 0$ | 2.44 | 5 | – | 112.85 |
| | $\theta \in [0, 1]$ | .69 | 1.16 | 1 | 109.96 |
| $PR_2$ | $\theta = 0$ | .18 | 1.60 | – | 162.37 |
| | $\theta \in [0, 1]$ | .35 | .94 | 1 | 145.40 |
| $PR_3$ | $\theta = 0$ | .24 | 1.49 | – | 129.58 |
| | $\theta \in [0, 1]$ | .53 | 1.34 | .67 | 109.44 |

Notice that, when the smoothing term $\theta$ is set equal to zero then regardless of whether the inertial models are considered, $PR_1$ performs significantly better than $PR_3$. This reflects the prevalence of indeterminacy when policy rules depend on forward expectations. In particular, while for a given set of policy parameters $\alpha$ the government objective evaluates essentially the same for either $PR_1$ or $PR_3$ (compare Figures 1 and 2), the set of parameters $\alpha$ corresponding to stable determinacy under $PR_3$ is significantly smaller than the set under $PR_1$. For example, according to Table 4.1, the optimal value of $\alpha$ under $PR_1$ is $(5, 3.78)$. If $PR_3$ is parameterized using this value of $\alpha$ then resulting objective value is approximately 27, just like with $PR_1$; however, the associated model is indeterminate.

Interestingly, the inclusion of a smoothing term appears to significantly mitigate this effect. This is especially true in case the lag calibrations are considered. In this case, the optimal $PR_3$ with smoothing produces essentially the same objective value as the optimal $PR_1$ with smoothing.

Further observations about the optimal value of $\theta$ are warranted. While it is natural that the inclusion of an additional degree of freedom in the policy rule should only improve performance, we find that in many cases, the improvement appears substantial. For example, in the purely forward looking case, including the smoothing term in $PR_3$ reduced the government's loss by 39%. Also, notice that in all cases, $\theta$ was chosen larger than zero, and in four of six cases, the upper constraint placed on $\theta$ was reached. This suggests that super-inertial rules may actually be optimal.

## 5.2 Robust Extended Taylor-type Rules

Precisely the same method can be employed to obtain robust extended Taylor-type rules. We begin by restricting attention to rules dependant only upon lagged endogenous variables and current fundamentals, and then proceed to relax the restriction until the full rule, given by (21), is analyzed:

$$i_t = \theta i_{t-1} + \alpha^f E_t y_{t+1} + \alpha^c E_t y_t + \alpha^L y_{t-1} + \alpha^{\hat{g}} \hat{g}_t; \tag{21}$$

For this exercise, we set $\psi = 1$. Note that because the extended rules nest the non-extended rules, we again have that the relevant constraint set is non-empty. We impose the following restrictions on the policy parameters:

$$0 \le \theta \le 1, 0 \le \alpha^f, \alpha^c, \alpha^L \le 5, -5 \le \alpha^{\hat{g}} \le 5.$$

The following table is obtained:

Table 5: Robust Extended Taylor-Type Rules

| Model | $\alpha_x^f$ | $\alpha_\pi^f$ | $\theta$ | $\alpha_x^c$ | $\alpha_\pi^c$ | $\alpha_x^L$ | $\alpha_\pi^L$ | $\alpha_g^{\hat{g}}$ | $\alpha_u^{\hat{g}}$ | Value |
|---|---|---|---|---|---|---|---|---|---|---|
| No Lag | – | – | – | – | – | .18 | 1.14 | .44 | .52 | 76.04 |
| Lag | – | – | – | – | – | .20 | 1.50 | .35 | .91 | 157.83 |
| No Lag | – | – | – | 4.88 | .445 | 0 | 2.51 | 2.47 | 5 | 25.25 |
| Lag | – | – | – | 2.28 | 2.50 | 0 | 2.73 | 1.34 | -5 | 108.96 |
| No Lag | – | – | 1 | 3.22 | .47 | 0 | .25 | -.51 | 5 | 24.54 |
| Lag | – | – | .87 | 1.6 | 2.15 | 0 | 1.11 | .32 | -.5 | 108.70 |
| No Lag | .1 | .19 | .96 | 1.67 | 0 | 0 | 0 | .04 | 5 | 23.58 |
| Lag | .68 | 2.10 | .74 | .18 | .035 | 0 | 0 | .48 | -3.24 | 108.1 |

A cell with a dash (–) indicates the associated policy parameter was set equal to zero.

Observe that relaxing the constraint on the functional form of the policy rule to include current expectations significantly improved the rule's performance regardless of whether inertial models were considered; however, subsequent relaxing of the this constraint had relatively minor impact. On the other hand, when $\theta$ was allowed to vary, it value was chosen to be large, often near one. This suggests that while fully optimal policy may be implemented without an interest rate smoothing term when parameter values are known with certainty (as we saw in Section 4), a non-zero smoothing term may be advisable in case of parameter uncertainty. We further observe that often the constraint on $\alpha_u^{\hat{g}}$ was binding, which indicates that perhaps a larger region should be considered. Finally, notice that including the forward looking terms in the policy rule did not much lower the value of the government's objective, thus suggesting that including a dependence on expectations of future inflation is not critical to achieve near optimal policy.

In case only forward looking models are considered, it is conceivable that a fully optimal policy may be implemented. As we saw in Section 4, the collection of policies implementing the optimal REE is a 2-manifold in 6-space. The shape and location of the manifold are calibration dependant, as is the subregion of the manifold corresponding to stable determinacy. It is a-priori possible that the intersection of these stable determinate subregions across calibrations is non-empty; if so, a single rule would implement the optimal REE, and the lowest possible value of the weighted objective would be obtained.

Whether the intersection of numerically computed 2-manifolds in 6-space is empty is, in general, difficult to determine, and for the four calibrations considered here, our current results do not directly imply a void intersection.[26] On the other hand, it is straightforward to compute optimal value of the weighted objective: simply implement the modified EH-rule for each calibration and evaluate. If the intersection of the stable determinate subregions is non-empty, is should be possible for our search algorithms to find a policy yielding an objective value that closely approximates the optimal one. Computation reveals the optimal value of the weighted objective to be approximately 17. As this value is considerably lower than the minimum value obtained in Table 5, we conclude that the intersection is likely empty, at least over the region considered by our search algorithm, and thus implementing a fully optimal rule is not possible when alternate calibrations are simultaneously considered.

## 5.3 Discussion

Researchers are at odds over the correct calibration of New Keynesian models, and because of this, a variety of calibrations should be considered when searching for a good policy rule. To simultaneously consider multiple calibrations, we employed the method recommended by [2], and thus weight each calibration according to the specified prior probability that it is accurate. We then computed, for each policy rule $\mathrm{PR}_i$, the optimal policy parameters subject to the constraint that the economy be stable and determinate for all calibrations. An important point of this exercise was to emphasize that the constraint set was not empty, so that indeed Brock et al's method could be

[26]If the V calibration is modified so that $\lambda = 1$ then it is straightforward to show that the intersection is indeed empty.

employed, given this constraint.

Although these results are preliminary, they do suggest that this is a feasible and fruitful way to think about the optimal choice of interest-rate rules, taking into account both determinacy and stability constraints and structural parameter uncertainty.

# 6 Conclusion

We have demonstrated the potential for policy makers, attempting to choose the optimal rule within a class, to be directed towards rules that lie in the indeterminacy and/or instability regions. Taylor-type rules have an appealing simplicity, with the key inflation and output coefficients traditionally chosen based on plausible rules of thumb. One might expect that improved performance would be obtained by choosing these policy parameters optimally for a given calibrated model. Paradoxically this may not be the case, because searching for the optimal policy rule with a given class may fail to deliver a rule that produces a determinate equilibrium that is stable under learning. It is therefore imperative that the search for optimal policy rules be constrained to the determinate stable region. One might think that this problem would be avoided by considering a class of extended Taylor rules that is sufficiently general that it includes fully optimal solutions. However, we have also seen that this class will contain some "optimal" rules that are subject to indeterminacy and/or instability problems.

These problems are compounded by the issue of structural parameter uncertainty. Policy rules which lead to determinacy and stability under learning and which are fully optimal for one set of structural parameters can lead to indeterminacy or instability for another set of parameters. We therefore advocate a "robust" optimization procedure, in which policy makers select the optimal constrained rule. Such a rule is computed as the one which minimizes the policy-makers expected loss, based on prior probabilities for the structural parameters, but which is constrained to satisfy the condition that it lies within the stable, determinacy region for every calibration that has positive probability.

## References

[1] B.S. Bernanke and M. Woodford. Inflation forecasts and monetary policy. *Journal of Money, Credit, and Banking*, 24:653–684, 1997.

[2] William A. Brock, Steven N. Durlauf, and Kenneth D. West. Policy evaluation in uncertain environments. Working Paper SSRI 2003-15, University of Wisconsin - Madison, 2003.

[3] James Bullard and Kaushik Mitra. Learning about monetary policy rules. *Journal of Monetary Economics*, 49:1105–1129, 2002.

[4] Richard Clarida, Jordi Gali, and Mark Gertler. The science of monetary policy: A new keynesian perspective. *Journal of Economic Literature*, 37:1661–1707, 1999.

[5] Richard Clarida, Jordi Gali, and Mark Gertler. Monetary policy rules and macroeconomic stability: Evidence and some theory. *Quarterly Journal of Economics*, 115:147–180, 2000.

[6] George W. Evans and Seppo Honkapohja. On the local stability of sunspot equilibria under adaptive learning rules. *Journal of Economic Theory*, 64:142–161, 1994.

[7] George W. Evans and Seppo Honkapohja. *Learning and Expectations in Macroeconomics*. Princeton University Press, Princeton, New Jersey, 2001.

[8] George W. Evans and Seppo Honkapohja. Monetary policy, expectations and commitment. mimeo, www.valt.helsinki.fi/RAKA/seppo.hmtl, and http://darkwing.uoregon.edu/∼gevans/, May 2002.

[9] George W. Evans and Seppo Honkapohja. Adaptive learning and monetary policy design. *Journal of Money, Credit and Banking,* forthcoming, 2003.

[10] George W. Evans and Seppo Honkapohja. Existence of adaptively stable sunspot equilibria near an indeterminate steady state. *Journal of Economic Theory*, 111:125–134, 2003.

[11] George W. Evans and Bruce McGough. Indeterminacy and the stability puzzle in non-convex economies. mimeo, 2002.

[12] George W. Evans and Bruce McGough. Monetary policy, indeterminacy and learning. mimeo, http://darkwing.uoregon.edu/∼gevans/, 2003.

[13] George W. Evans and Bruce McGough. Stable sunspot solutions in models with predetermined variables. *Journal of Economic Dynamics and Control,* forthcoming, 2003.

[14] Jeff Fuhrer and George Moore. Inflation persistence. *Quarterly Journal of Economics*, 110:127–159, 1995.

[15] Jordi Gal and Mark Gertler. Inflation dynamics: A structural econometric approach. *Journal of Monetary Economics*, 44:195–222, 1999.

[16] M. Giannoni and M. Woodford. Optimal interest-rate rules: Ii. applications. *NBER Working Paper Series, Number 9420*, 2002.

[17] Seppo Honkapohja and Kaushik Mitra. Are non-fundamental equilibria learnable in models of monetary policy? Working paper, www.valt.helsinki.fi/RAKA/seppo.hmtl, 2001.

[18] Seppo Honkapohja and Kaushik Mitra. Performance of monetary policy with internal central bank forecasting. manuscript, www.valt.helsinki.fi/RAKA/seppo.hmtl, 2002.

[19] Bennett T. McCallum. Issues in the design of monetary policy rules. In Taylor and Woodford [27], chapter 23, pages 1483–1530.

[20] Bennett T. McCallum and Edward Nelson. Performance of operational policy rules in an estimated semi-classical model. In Taylor [26], pages 15–45.

[21] Julio J. Rotemberg and Michael Woodford. Interest-rate rules in an estimated sticky price model. In Taylor [26], pages 55–119.

[22] Frank Smets. Maintaining price stability: How long is the medium term? *Journal of Monetary Economics*, 50:1293–1309, 2003.

[23] Lars E. O. Svensson and Michael Woodford. Implementing optimal policy through inflation-forecast targeting. Working paper, 1999.

[24] Lars E.O Svensson. What is wrong with taylor rules? using judgment in monetary policy through targeting rules. mimeo, 2001.

[25] J.B. Taylor. Discretion versus policy rules in practice. *Carnegie-Rochester Conference Series in Public Policy*, 39:195–214, 1993.

[26] J.B. Taylor, editor. *Monetary Policy Rules*. University of Chicago Press, Chicago, 1999.

[27] J.B. Taylor and M. Woodford, editors. *Handbook of Macroeconomics, Volume 1*. Elsevier, Amsterdam, 1999.

[28] Michael Woodford. Learning to believe in sunspots. *Econometrica*, 58:277–307, 1990.

[29] Michael Woodford. Optimal monetary policy inertia. *The Manchester School, Supplement*, 67:1–35, 1999.

[30] Michael Woodford. *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press, Princeton, NJ, 2003.

Figure 1: Woodford Calibration, $PR_1$, No lag, $\psi = .1$

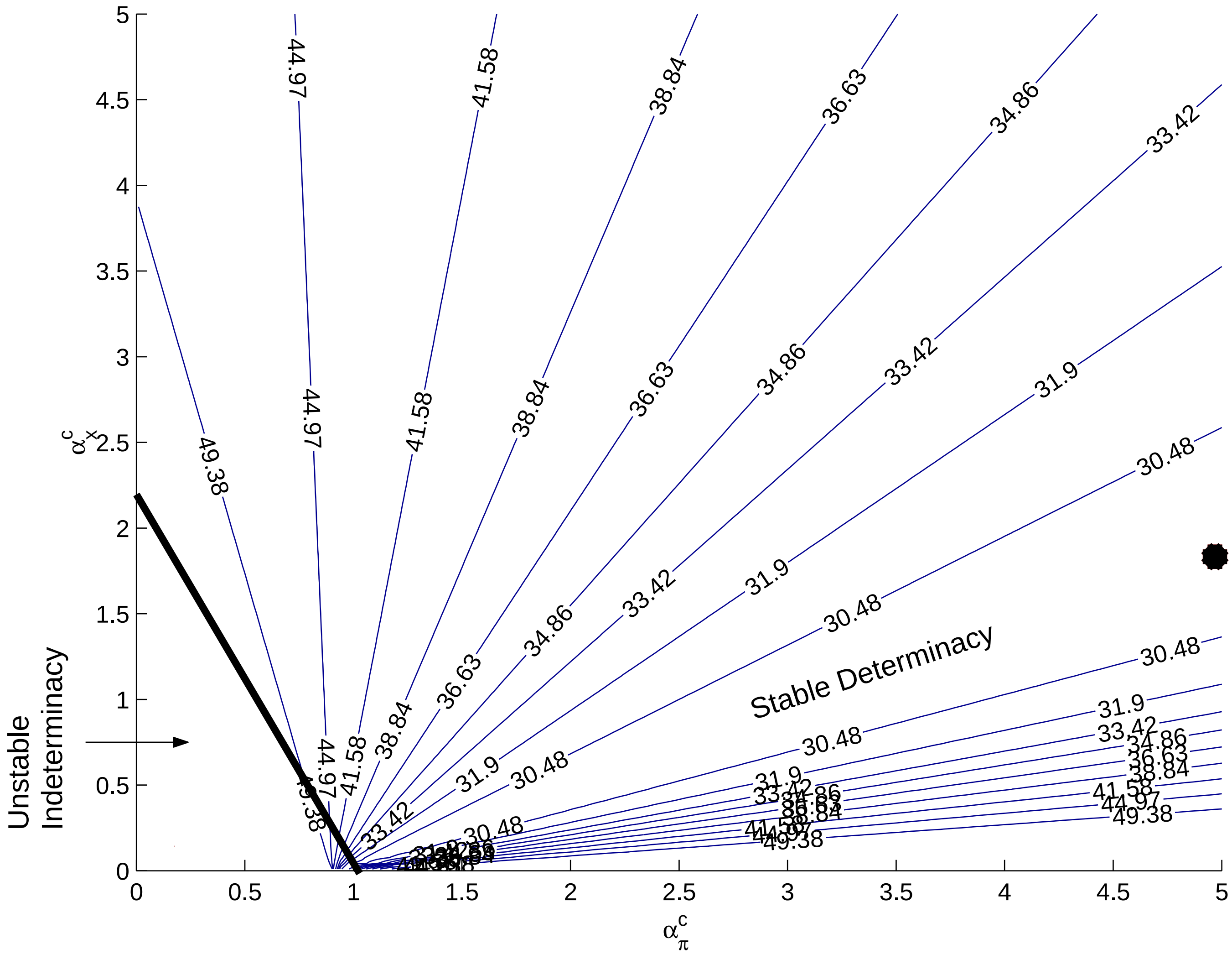

Figure 2: Woodford Calibration, $PR_3$, No Lag, $\psi$=.1

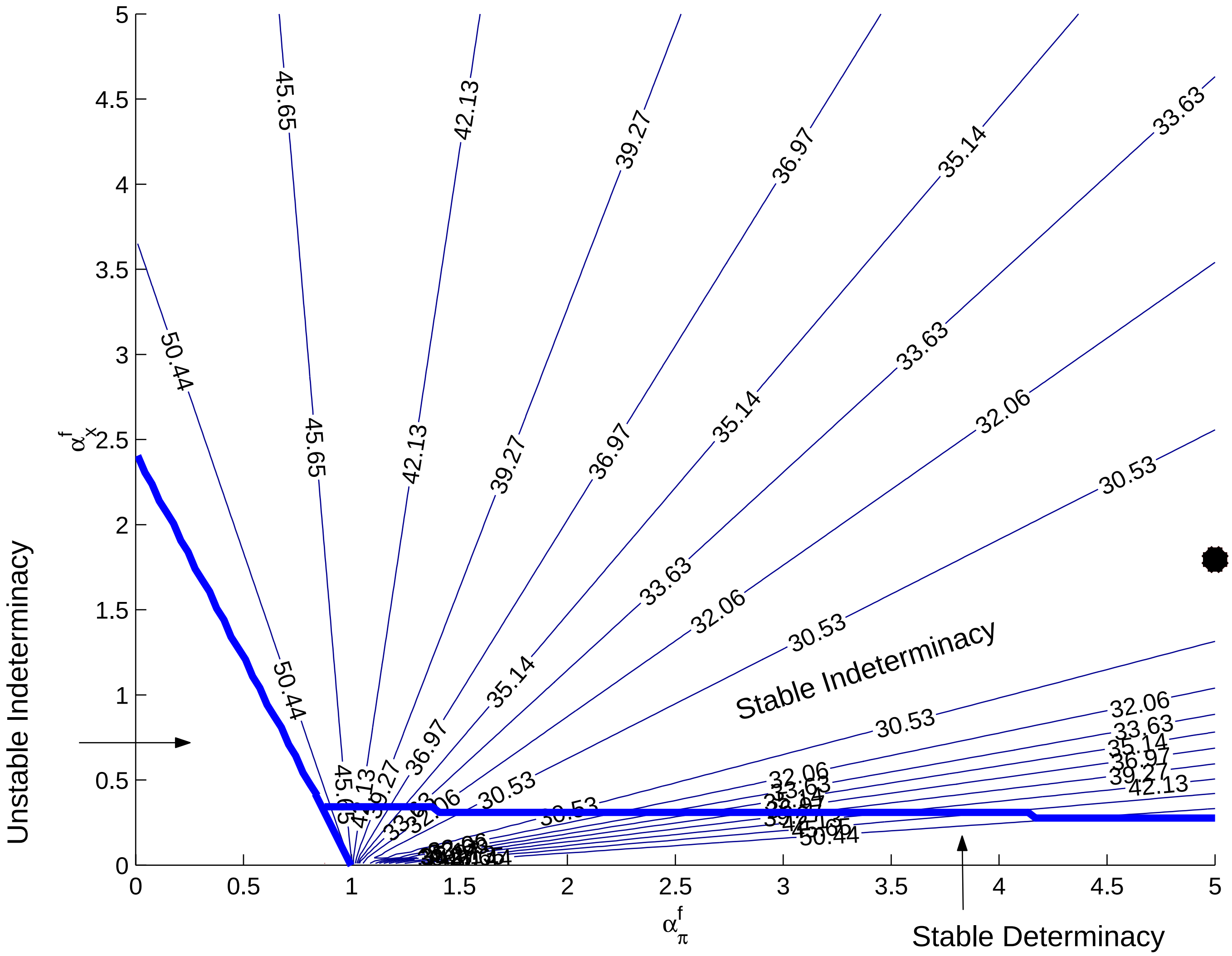

Figure 3: V Calibration, $PR_2$, $\psi$=10

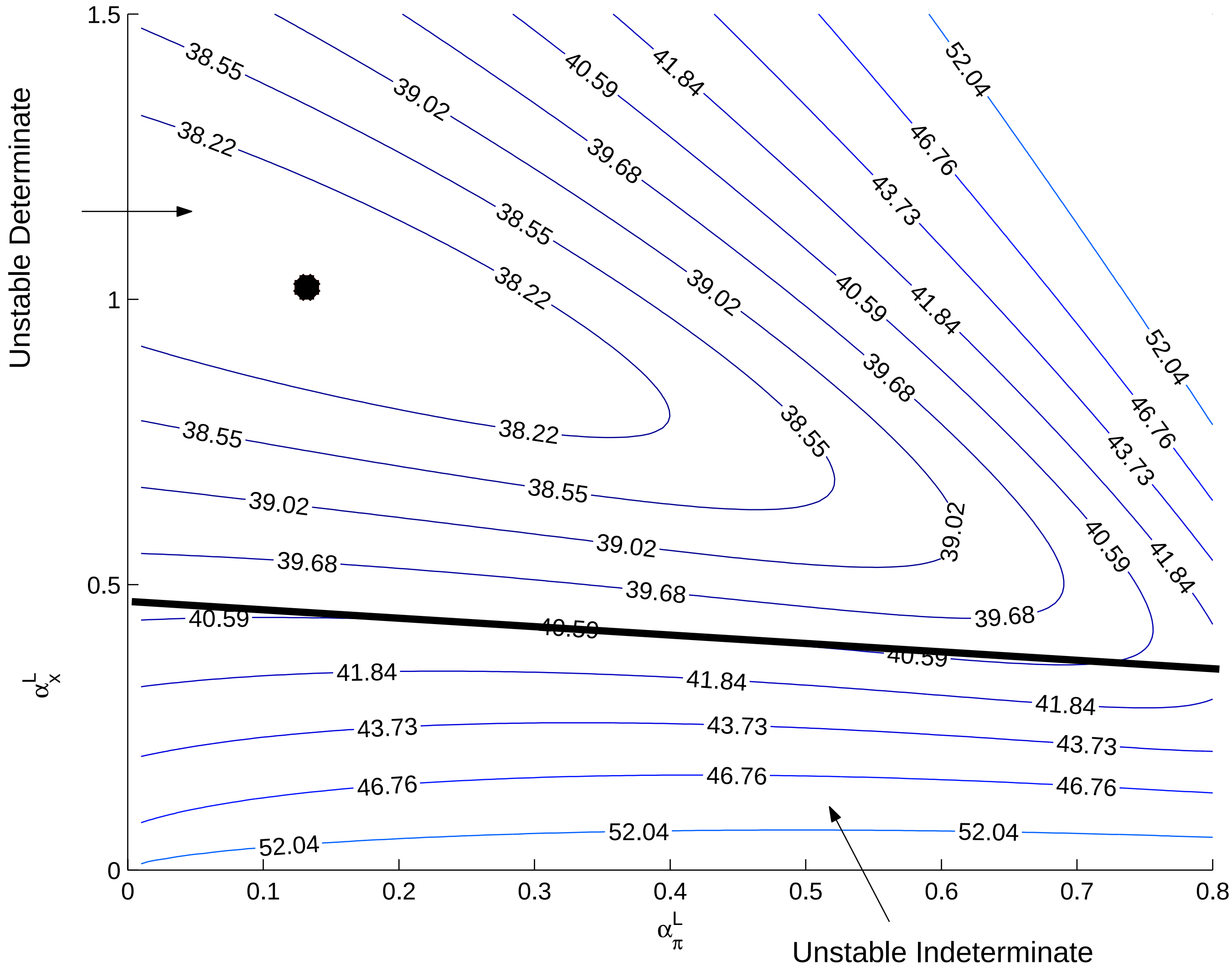

Figure 4: W Cal., No Inertia, $\alpha_x = .433 \times \alpha_\pi - .365$

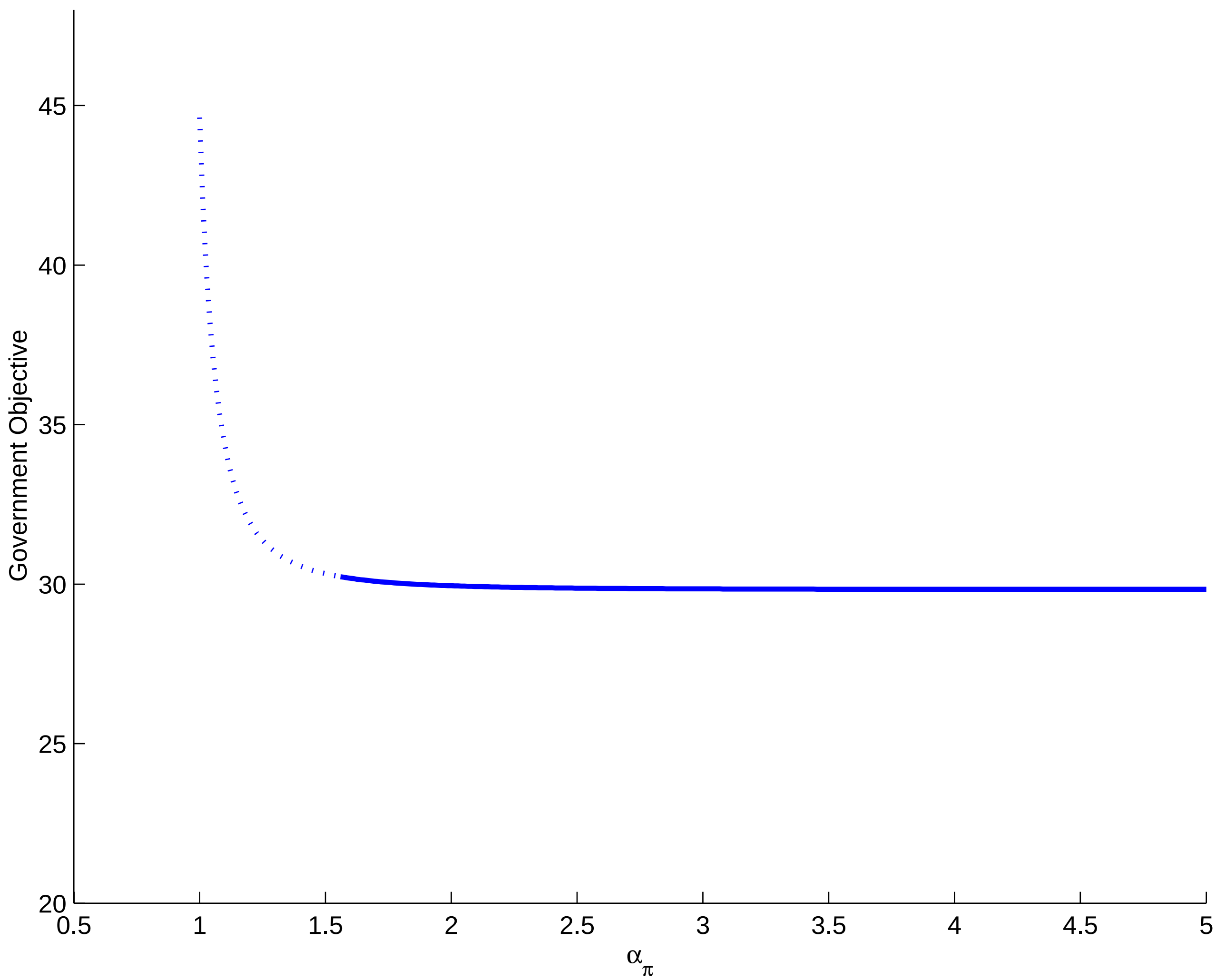

Figure 5: Optimal Manifold, No Lag, $\lambda = 1$, $\phi = 6.37$, $\beta = .99$, $\psi = 1$