Security Transaction Differential Equation

——A Transaction Volume/Price Probability Wave Model

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This draft: June 10, 2004

Abstract

Financial market is a typical complex system because it is an open trading system and behaved by a variety of interacting agents. The consequence of the interaction appears quite complex and nonlinear. Therefore, how to observe this system and find a simplified methodology to describe it is, probably, a key to understand and solve the problem. In this paper, the author observes a stationary transaction volume distribution over a trading price range, studied the relationship between the volume and price of transaction through the amount of it in stock market. The probability of accumulated trading volume (i.e. actual supply/demand quantity or transaction volume) that distributes over a trading price range gradually emerges kurtosis near a transaction price mean value in a transaction body system when it takes a longer trading time, regardless of actual trading price fluctuation path, time series, or total transaction volume in the time interval. The volume and price behaves a probability wave toward an equilibrium price, driven by an actual supply/demand quantity restoring or regressive force that can be represented by a linear potential (an autoregressive item in mathematics). In terms of physics, the author derives a time-independent security transaction probability wave differential equation and obtains an explicit transaction volume distribution function over the price, the distribution of absolute zero-order Bessel eigenfunctions, when the supply and demand quantity is dynamic in a stable transaction body system. By fitting and testing the function with intraday real transaction volume distributions over the price on a considerable number of individual stocks in Shanghai 180 Index, the author demonstrates its validation at this early stage, and attempts to offer a micro and dynamic probability wave theory on the price volatility with actual supply/demand quantity (transaction volume) in financial economics.

JEL classification numbers: G12; D50; C51; C52

Keywords: Complex system; Restoring or regressive force; Linear potential (an autoregressive item); Probability wave differential equation; The distribution of absolute zero-order Bessel eigenfunctions

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The Object, Reference, and Benchmark of Study

Financial market is a typical complex system because it is an open trading system and behaved by a variety of interacting agents, for example, retail traders, enterprises, bankers, security companies, insurance corporations, funds, other investment institutions, and even government agencies etc. The consequence of the interaction among all of the agents appears quite complex and nonlinear. Therefore, how to observe this system and find a simplified methodology to describe it is, probably, a key to understand and solve the problem.

In his Nobel Memorial Lecture, Arrow (1972) wrote: "From the time of Adam Smith's *Wealth of Nations* in 1776, one recurrent theme of economic analysis has been the remarkable degree of coherence among the vast numbers of individual and seemingly separate decisions about the buying and selling of commodities". According to a static and qualitative description in general equilibrium theory, oversupply quantity and excess demand quantity produces a driving force toward an equilibrium price in an economic system (see figure 1)[①].

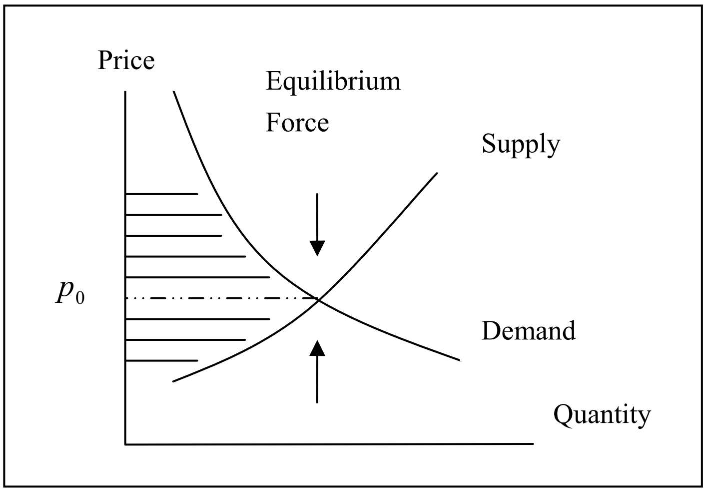

Figure 1: Equilibrium force toward an equilibrium price p_0

Soros (1987) guessed: In natural sciences, the phenomenon most similar to that in financial economics probably exists in quantum physics, in which scientific observation generated Heisenberg's uncertainty principle…… Unfortunately, it is impossible for economics to become a science…… "To date, no approximately correct and falsifiable deterministic model of market dynamics exists" (McCauley and Küffner, 2004). Inspired by his guess, the author, however, is engaged in academic study on the volume and price of transaction on individual stocks in terms of physics (Shi, 2001) and has produced successful outcomes (Shi, 2002, 2004).

In this paper, the object of study is that the probability of accumulated trading volume (i.e. transaction volume or actual supply/demand quantity) distributes over a price range in intraday transactions on individual stocks. The reference or the origin of coordinate is the price zero point.

[①] According to the static model, there is no trading at all unless the price is rational or equilibrium (McCauley and Küffner, 2004).

And, the benchmark is amount of the transaction, a holonomic constraint (a generalized velocity independent constraint, i.e. $\Phi(p,t)=0$) between the volume and price of transaction at a price coordinate.

According to classic dynamics, the number of degrees of freedom in generalized coordinates (volume and price) is equal to that the number of the coordinates 2 minus the number of independent constraint 1. So, the author chooses the price as an independent generalized coordinate to describe the volume/price behavior as a whole.

Ⅰ. Transaction (or Actual Supply/Demand) Energy Hypothesis

A transaction body system is defined as a set of interacting transactions in a given time interval, for example, a set of intraday transactions on a stock. Transaction volume v means accumulated trading volume or actual supply/demand quantity at a price p in a transaction body system, if not specified. Obviously, it varies over a price range or there is a kind of distribution.

What the price volatiles upward and downward in reference to an equilibrium price in a transaction body system is like to what a body of mass m slides back and forth to an equilibrium position in a restoring force system (see figure 2 (a)). In addition, the probability of accumulated trading volume (actual supply/demand quantity) that distributes over its price range gradually emerges the maximum around the price mean value in intraday transactions on an individual stock regardless of its price fluctuation path, time series, and total transaction volume. This distribution is similar to that of a large volume of particles in a ground state in a harmonic oscillator system in quantum mechanics (see figure 2 (b)). The gradually emerging phenomenon observed in the volume and price behavior is the same as that observed in double-slit electron interference in physics. Therefore, the author assumes that the volume/price behaves a probability wave toward an equilibrium price, driven by a restoring force[①]. In this way, the author finds a simplified methodology to describe the volume/price behavior in this complex system.

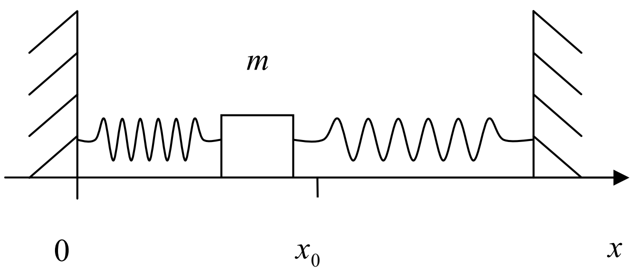

Figure 2 (a): A mass m slides back and forth in a restoring force system
(It is similar to what the price volatiles upward and downward)

[①] In this dynamic model, there is transaction volume (or actual supply/demand quantity) distribution over a price range (reference to the shaded area in figure 1). Here, an equilibrium price does not mean a rational price.

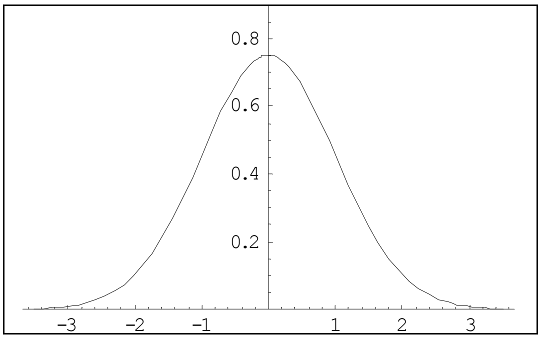

Figure 2 (b): The distribution of a large volume of particles in the stable (or ground) state of a harmonic oscillator

Let

$$E \equiv PE + (1-P)E \equiv T + W, \tag{1}$$

where E is transaction energy or liquidity energy at a price in a transaction body system, which is equal to $\frac{pv}{t^2} = \frac{pv_t}{t} = pv_{tt}$; P is an accumulated trading volume probability that is equal to the ratio of accumulated trading volume v at a correspondent price p to total accumulated trading volume V over a trading price range; T is transaction dynamic energy or distribution energy and W is interacting transaction energy or potential energy at the price p (the interaction between the volume v and $V-v$), respectively, of which the potential energy produces a driving force toward an equilibrium price in the system. They are

$$T \equiv PE = P(pv_{tt}) = \left(\frac{v}{V} v_{tt}\right) p = \frac{v_t^2}{V} p \tag{2}$$

and $$W \equiv (1-P)E = \frac{V-v}{V} E = \frac{(V-v)\cdot v}{Vt^2} p. \tag{3}$$

Thus,

$$-E + \frac{v_t^2}{V} p + W(p) = 0, \tag{4}$$

where the equation (4) is a transaction energy hypothesis in its differential expression. It means that transaction energy is always equal to the sum of its dynamic energy and potential energy over a price range, which could be analogous to a conserved energy system in physics.

If $v/V = 1$, i.e. there is no price volatility in a given trading time interval, then, $W(p) = 0$; If $W(p) > 0$, then, $v/V < 1$, i.e. there is price volatility in the time interval. The potential energy can represent for the price volatility or imbalance between the supply quantity and demand quantity (see Section Ⅱ in details). In addition, if $(v/V)_{\max} << (V-v)/V$ is satisfied over a trading price range in a transaction body system, then, its transaction energy mean value or the sum of transaction dynamic energy is subject to $\overline{E} = \sum_i T_i << \sum_i W_i$ over the price range, and becomes an error item in comparison with $\sum_i E_i$ or $\sum_i W_i$.

Ⅱ. Linear Potential (Energy) and Actual Supply/Demand Quantity Restoring Force

In Section Ⅰ, the author assumes that the volume/price behaves a probability wave toward an equilibrium price, driven by a restoring force that is produced by the interaction between the volume v and $V - v$. Let us study how the force is produced in a financial economics perspective, and find what can represents for it in terms of physics.

According to economic theory, buying or selling behavior is constrained. The demand quantity is constrained by budget or the amount of cash hold by buyers, while the supply quantity is constrained by the volume of product (or stock) hold by sellers. The actual supply/demand quantity or transaction volume is constrained by the amount of transaction and its price.

How does the volume/price behave in a set of intraday transaction on an individual stock? Assume that the stock initial price is its equilibrium price, and the total amount of cash used in buying stock (demand quantity) and the total volume of stock to be sold (supply quantity) are given in this transaction body system.

Reference to initial price, if the balance is constantly maintained between the supply quantity and demand quantity on intraday transaction, then, the trading price is always equal to its initial price or equilibrium price, and has no volatility at all (Here, the transaction potential energy is equal to zero). Any a variable is exclusively determined by the other two among the volume, price, and amount of transaction.

Reference to an equilibrium price, the supply quantity is frequently not equal to the demand quantity as a consequence of separate individual buying and selling decisions in an actual stock trading scenario. If the demand quantity is greater than the supply quantity reference to its equilibrium price in a short term interval, then, the price volatiles positively and deviates from its equilibrium price. The more the price deviates from it, the higher the price is. The same amount of remained cash could buy fewer volume of stock. The demand quantity is reduced. When the demand quantity is less than supply quantity, the price will volatile negatively and show regressive. It adjusts toward its equilibrium price. When reached it, the price volatiles continuously in the same direction and deviate from it again because of price motion inertia. The more the price deviates from it, the lower the price is. The same amount of remained cash could buy more volume of stock. The demand quantity is increased. When the demand quantity is greater than supply quantity once again, the price will volatile in a positive direction simultaneously and adjust toward its equilibrium price again until it returns to its equilibrium price once a more time. Hereafter, the price will repeatedly deviate from its equilibrium price and adjust toward it.

In a transaction body system, the price deviates from its equilibrium price because of imbalance between the supply quantity and demand quantity in a short term. The deviation changes the imbalance and causes the price regression. When the price returns to its equilibrium price, it will continue its volatility in the same direction because of price motion inertia until it appears regressive again and then returns to its equilibrium price. It demonstrates that there exists an actual supply/demand quantity (i.e. transaction volume) regressive force that drives the price upward and downward toward its equilibrium price as a result of interaction among total transaction behavior in a transaction body system whenever the imbalance between the supply quantity and demand quantity determines the price volatility direction reference to present trading price as a consequence of seemingly separate individual supply and demand decisions[①].

From the equation (1) or (4), we know that the potential energy is in proportion to the price and is a linear potential. In addition, we assume that the volume/price behaves a probability wave toward an equilibrium price in a set of intraday transactions on a stock, driven by an actual supply/demand quantity regressive force. Therefore, the linear potential (an autoregressive item in mathematics), which can represent for an actual supply/demand quantity regressive force, is expressed by

$$W(p) = A(p - p_0) \approx A\left(p - \overline{p}\right), \qquad (5)$$

where p_0 is an equilibrium price, $\overline{p}$ is its price mean value, and A is a transaction coefficient that could either be a constant or not over a price range.

Putting the equation (5) into the equation (4) and differentiating it, we then define actual supply/demand quantity (or transaction volume) regressive force quantitatively in this holonomic constraint system as

$$F = -\frac{\partial W}{\partial p} = -A = -\left(v_{tt} - \frac{v_t^2}{V}\right) = -\left(1 - \frac{v}{V}\right)v_{tt} \qquad (6)$$

where F is the regressive force, which is also called as a restoring force. Thus, we see that the coefficient A is the magnitude of actual supply/demand quantity (transaction volume) restoring force in a transaction body system. The minus sign means that the force is always toward to its equilibrium price (reference to figure 1).

From definition (6), we can conclude that the restoring force is equal to zero in an absolute equilibrium transaction body system, in which the supply and demand quantity is constantly balanced or $v/V = 1$ is satisfied.

Ⅲ. Probability Wave Differential Equation and Its Solution

Suppose that the transaction volume/price probability wave function $\psi(p)$ (Derbes, 1996) is

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① Reference to present price, we define that the price volatility is positive if demand quantity is greater than supply quantity, whereas it is negative if demand quantity is less than supply quantity.

$$\psi(p) = R \cdot e^{iS/B} \tag{7}$$

where R is the wave amplitude, S its action or Hamiltonian principal function, and B a constant to make its phase dimensionless.

In an interacting transaction body system, the volume/price behavior satisfies a Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(p, \frac{\partial S}{\partial p}) = 0\,, \tag{8}$$

where $H(p, \frac{\partial S}{\partial p})$ is Hamiltonian.

The action derived from the Hamilton-Jacobi equation (8) is

$$S = \alpha\left(pv_t\right) - Et + \beta\,, \tag{9}$$

where α and β are any constants. In convenience, let $\alpha = 1$ and $\beta = 0$, then, we define

$$S \equiv pv_t - Et\,. \tag{10}$$

Thus, generalized momentum or transaction momentum in the price coordinate is

$$Q \equiv \frac{\partial S}{\partial p} = v_t\ . \tag{11}$$

According to security trading regulation, transaction priority is given to or constrained by the price first and the time first regardless of participant's rationality. If current trading price is p_c, then, the next trading priority is offered to minimize the price volatility with respect to p_c. Therefore, we have its mathematic expression as

$$\delta \int G(\psi, p)dp = 0\,, \tag{12}$$

where $G(p,\psi)$ is its energy functional in a transaction body system. The actual transaction price volatility path is chosen by its energy functional $G(p,\psi)$ to minimize its wave function ψ with respect to price variations.

Substituting the (11) into the (4) and using the (12), we, therefore, derive a transaction probability wave differential equation as,

$$\frac{B^2}{V}\left(p\frac{d^2\psi}{dp^2} + \frac{d\psi}{dp}\right) + \left[E - W(p)\right]\psi = 0 \tag{13}$$

or

$$\frac{B^2}{V}\left(p\frac{d^2\psi}{dp^2}+\frac{d\psi}{dp}\right)+\left[E-A(p-p_0)\right]\psi=0 \ . \tag{14}$$

When $E = pv_{tt}$, we choose a natural unit ($\frac{V}{B^2}=1$) and obtain a transaction volume distribution function over a price range from the equation (14) as

$$\left|\psi_m(p)\right| = C_m\left|J_0[\omega_m(p-p_0)]\right| , \qquad (m=0,1,2\cdots) \tag{15}$$

where $J_0[\omega_m(p-p_0)]$ are zero-order Bessel eigenfunctions, p is the price, $\left|\psi_m(p)\right|$ are the probability of accumulated trading volume at the price p (reference to figure 3), C_m are normalized constants, and ω_m are eigenvalues ($\omega_m > 0$) and satisfy

$$\omega_m^2 = v_{tt} - A = v_{tt} + F = v_{tt}P = \frac{v}{V}v_{tt} = const. \tag{16}$$

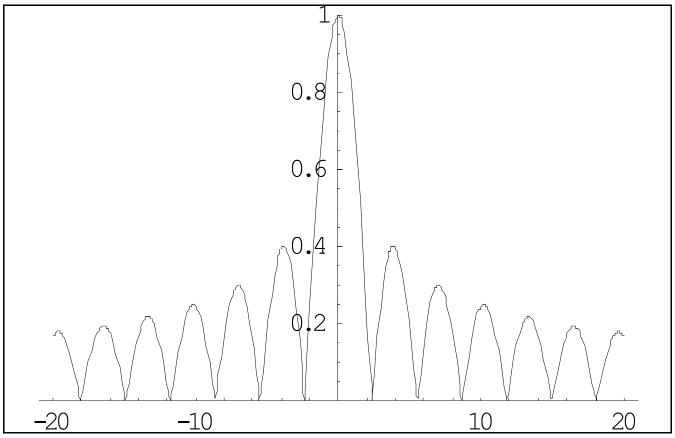


Figure 3: The absolute zero-order Bessel eigenfunctions

When transaction energy or the magnitude of restoring force is a constant over a price range, we choose a natural unit ($\frac{V}{B^2}=1$) and have the volume distribution functions over a price range from the equation (14) as

$$\left|\psi_m(p)\right| = C_m e^{-\sqrt{A_m}\left|p-p_0\right|}\cdot\left|F\left(-m,1,2\sqrt{A_m}\left|p-p_0\right|\right)\right| \tag{17}$$

with $$\sqrt{A_m} = \frac{E_m}{1+2m} = const. > 0 \qquad (m=0,1,2\cdots) \tag{18}$$

where m is the order of the eigenfunctions, $F(-m,1,\sqrt{A_m}\left|p-p_0\right|)$ is a confluent hypergeometric function or the first Kummer's function, and A_m is the magnitude of actual supply/demand quantity restoring force or eigenvalues.

When $m = 0$, $\quad \left| F\left(0,1,2\sqrt{A_m}\left|p - p_0\right|\right)\right| \equiv 1$, (19)

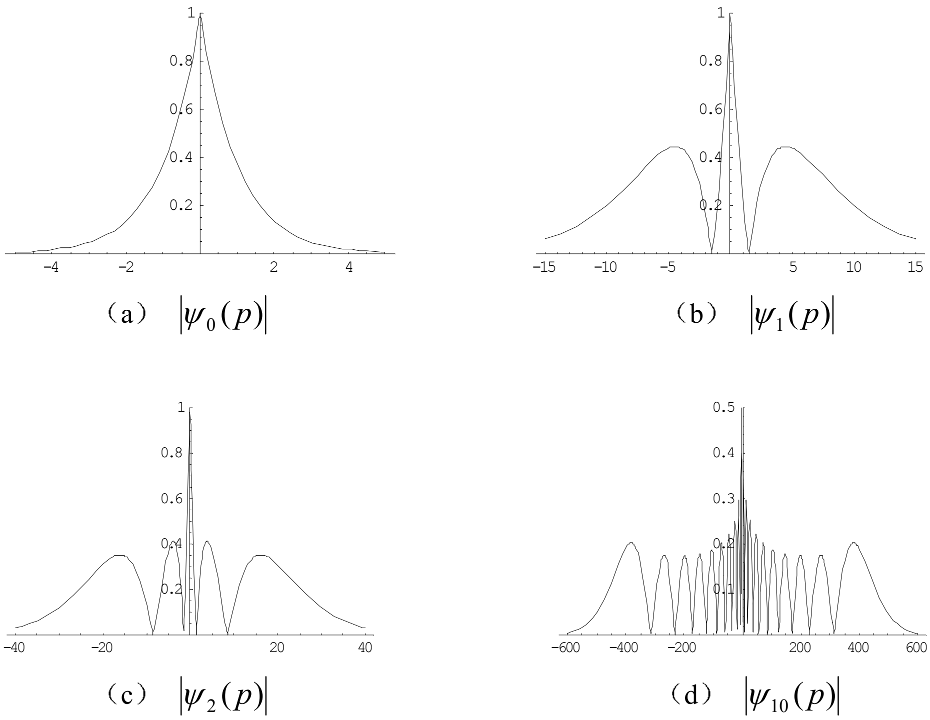
it is an exponent distribution, which is sufficiently covered by the function (15) (Compare figure 3 with figure 4 (a) and see the sample fitness in figure 5 (c));

When $m = 1$, $\quad \left|\psi_1(p)\right| = C \cdot e^{-\sqrt{A_1}|p-p_0|}\left|\left(1 - 2\sqrt{A_1}\left|p - p_0\right|\right)\right|$; (20)

When $m = 10$, $\quad \left|\psi_{10}(p)\right| = C \cdot e^{-\sqrt{A_{10}}|p-p_0|} \cdot \left|F\left(-10,1,\sqrt{A_{10}}\left|p - p_0\right|\right)\right|$, (21)

it is almost a uniform distribution over a price range (see figure 4 (d)). The volume/price behaves a random characteristic.

The eigenfunctions (17) are plotted ($m = 0,1,2 and 10$) in figure 4.



(a) $\left|\psi_0(p)\right|$ (b) $\left|\psi_1(p)\right|$

(c) $\left|\psi_2(p)\right|$ (d) $\left|\psi_{10}(p)\right|$

Figure 4: The eigenfuntions when transaction restoring force is a constant over a price range

Ⅳ. Empirical Results

The author uses the function (15) to fit intraday real transaction volume distribution samples over the price on the first 30 individual stocks in Shanghai 180 Index in June, 2003. There are total 630 (30×21) samples. However, 11 samples are halt samples and one sample data is lost. Used samples are 618. Figure 5 illustrates the fitting results of three typical samples, figure 6 shows the fitting goodness distribution of used samples, and figure 7 is their significant test result at 95% level.

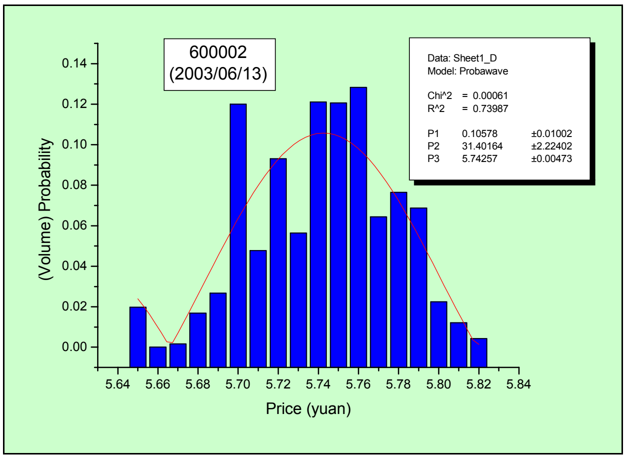


Figure 5 (a): Fitting goodness in sample 1
(Its price mean value is RMB5.74 yuan)

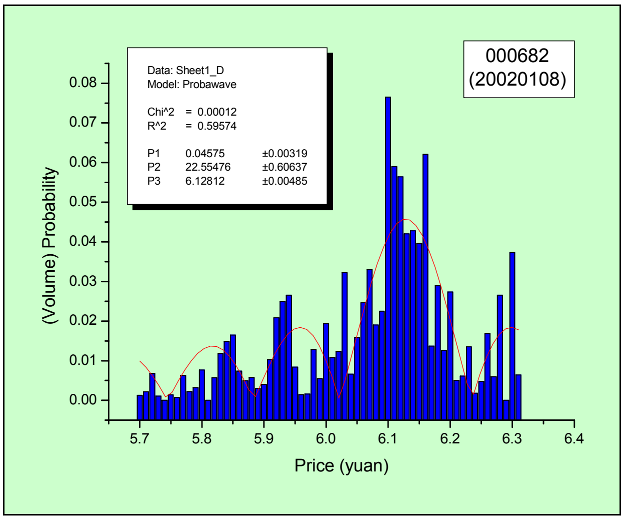


Figure 5 (b): Fitting goodness in sample 2
(Its price mean value is RMB6.08 yuan)

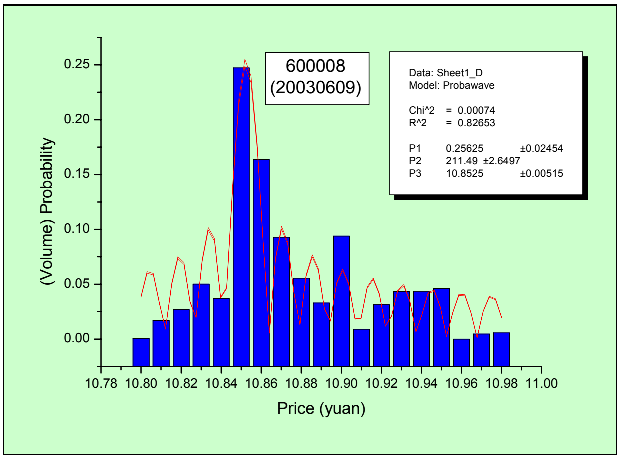


Figure 5 (c): Fitting goodness in sample 3
(Its price mean value is RMB10.87 yuan)

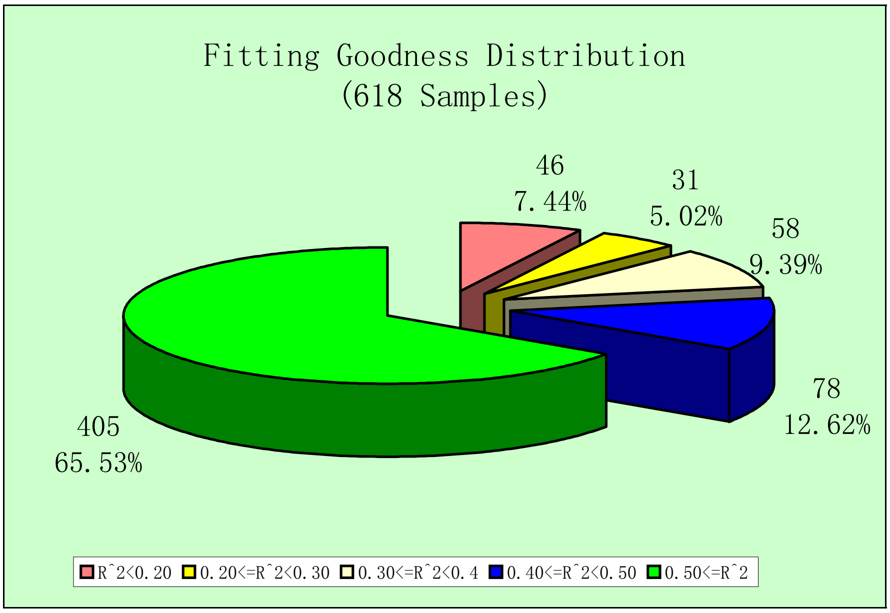


Figure 6: Fitting goodness distribution

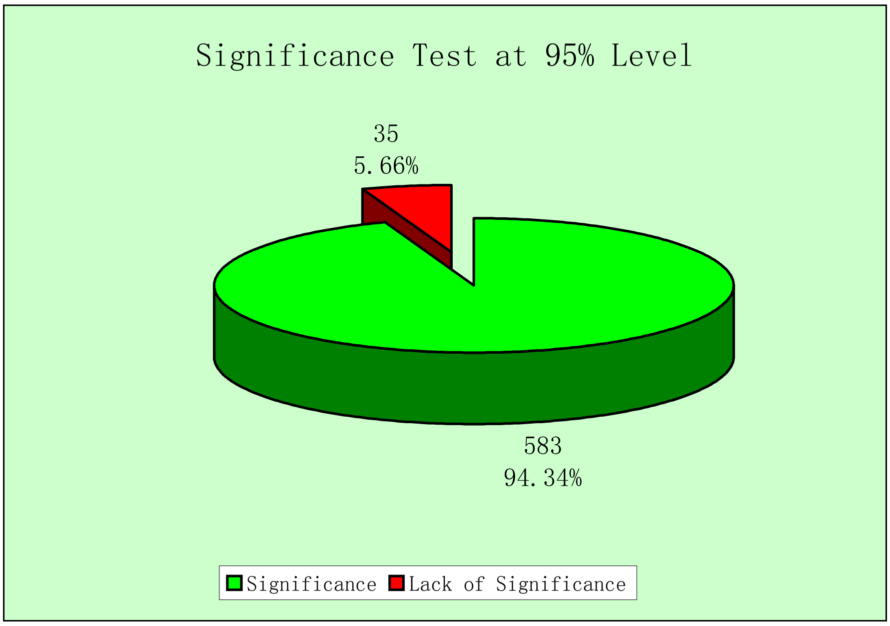


Figure 7: Test in significance (F test)

Ⅴ. Analysis on the Samples Lacking Significance

35 samples among 618 lack significance at 95% level. They are characterized by at least two maximum volume probability values over its price range (see figure 8). The imbalance between supply quantity and demand quantity has been enlarged too much, referencing to an equilibrium price, so as to cause their equilibrium price change in those transaction body systems. The volume/price waves around from one equilibrium price to another one. They are not stable. The volume distribution over the price is the superposition of two functions as follow:

$$|\psi_m(p)| = C\left(|J_0[\omega_m(p - p_{01})]| + |J_0[\omega_m(p - p_{02})]|\right). \qquad (22)$$

Figure 8 illustrates the fitting results of two samples by the function (22), figure 9 shows the fitting goodness distribution among the 35 samples, and figure 10 is the test result in significance at 95% level.

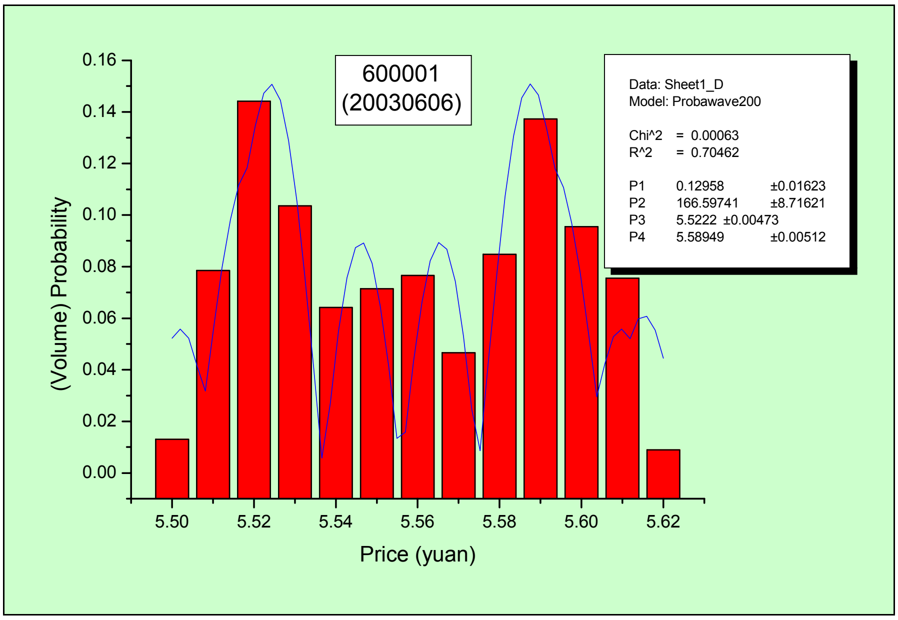


Figure 8 (a): Fitting goodness by the superposition of two functions with an eigenvalue

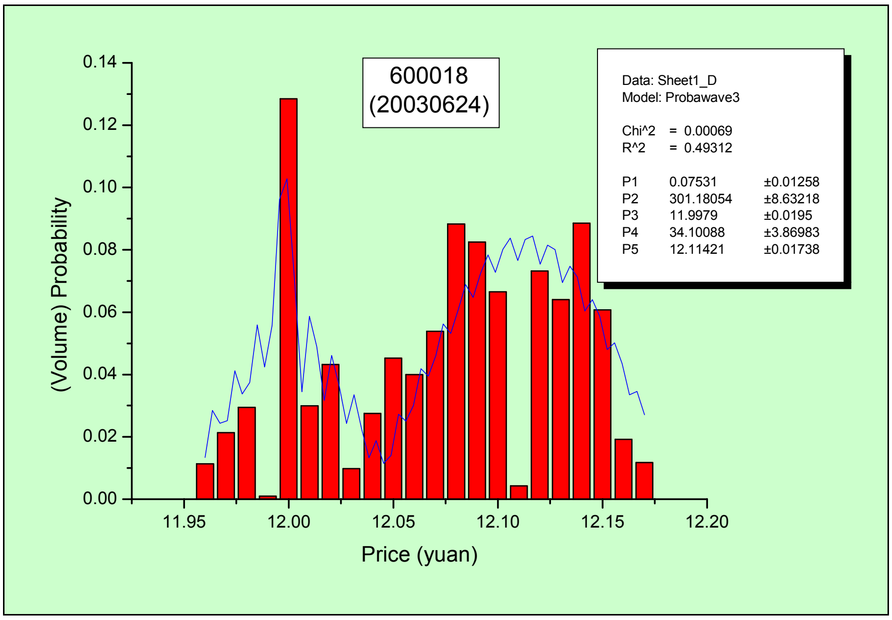


Figure 8 (b): Fitting goodness by the superposition of two functions with two eigenvalues

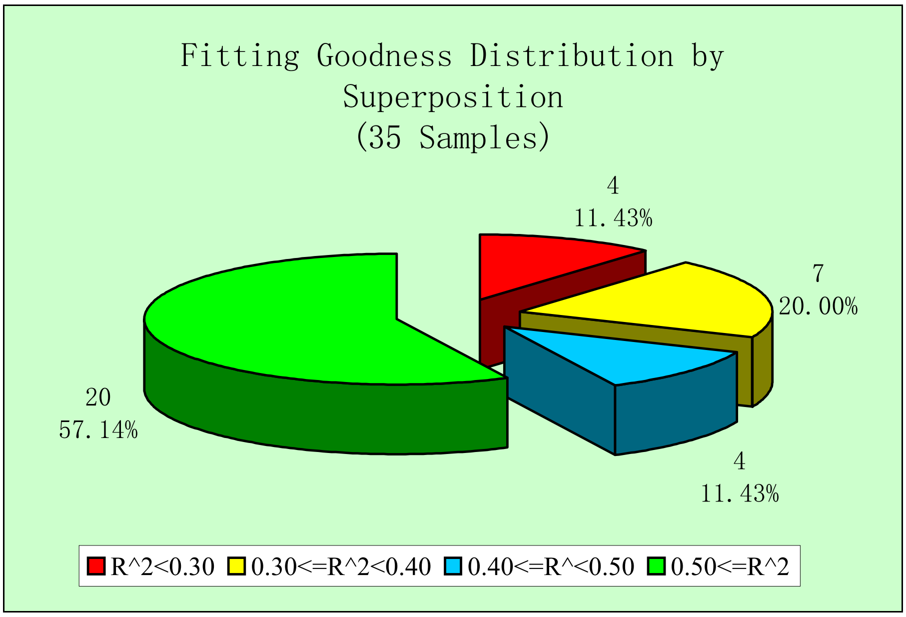


Figure 9: Fitting goodness distribution in terms of superposition functions

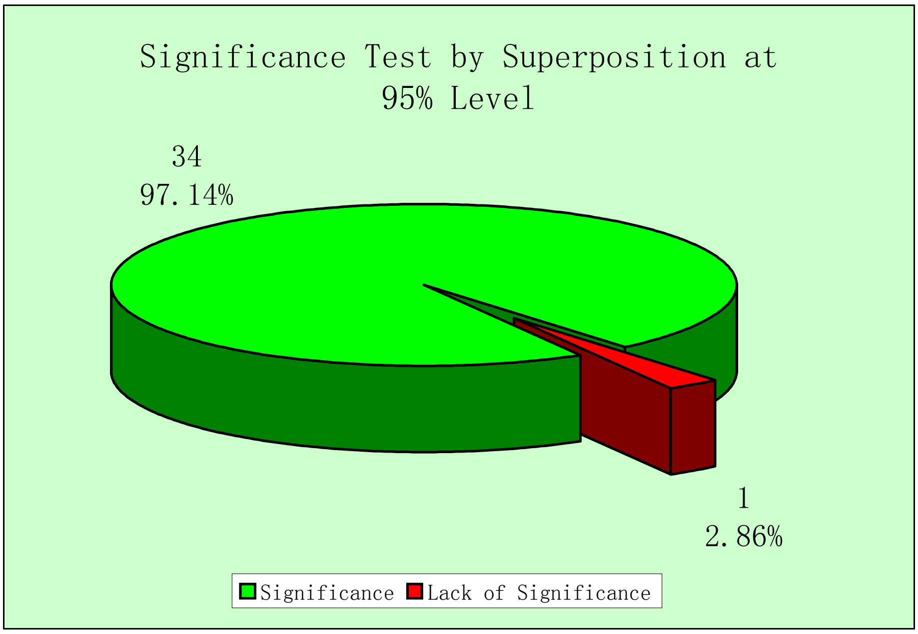


Figure 10: Test in significance at 95% level for superposition functions

The rest one sample lacking significance is an approximate random distribution. It is the case that its transaction energy or the magnitude of restoring force is a constant over a trading price range, though it is rare (1/618=0.16% in this paper). When the sample is fitted by the first order function (20), the test shows significance at 95% level (compare figure 11 with figure 12).

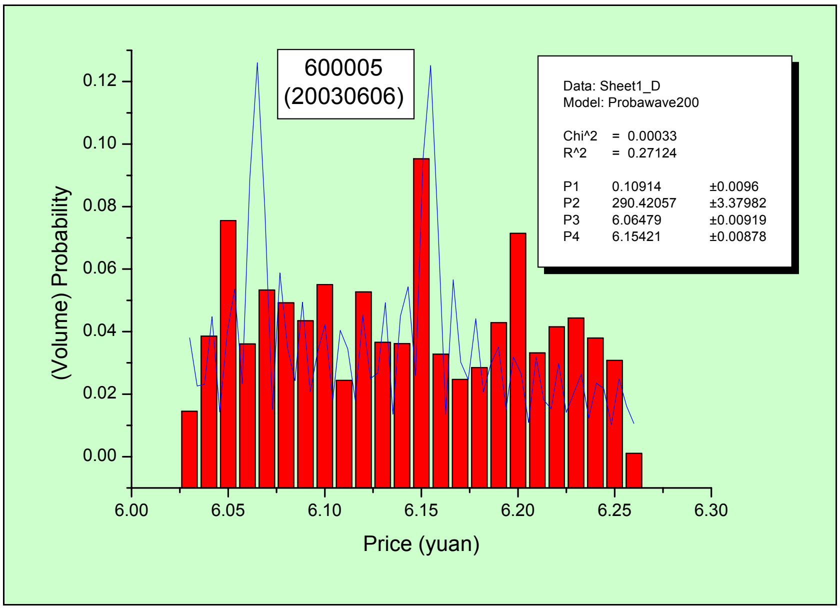


Figure 11: The sample fitted by the function (22) lacks significance at 95% level

($R^2 = 0.27 < R_{crit}^2 = 0.29$)

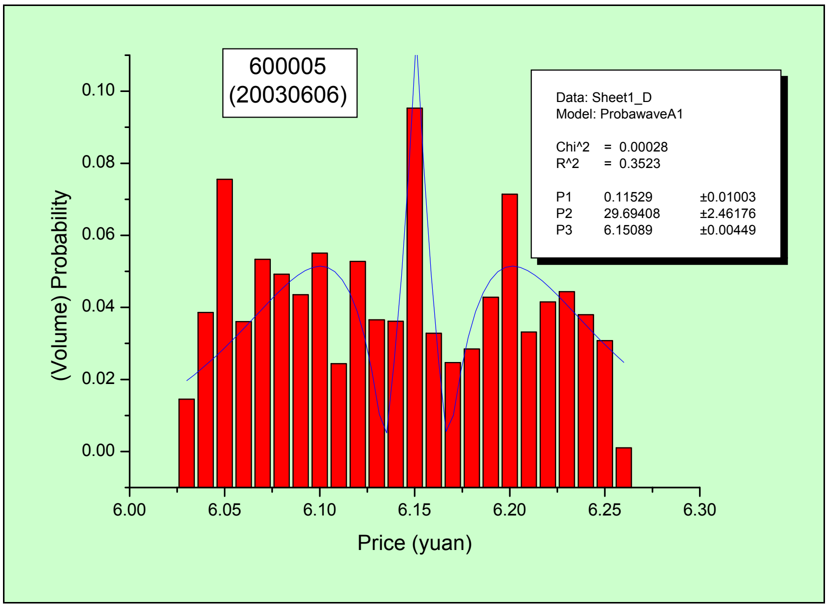


Figure 12: The sample fitted by the function (20) shows significance at 95% level

($R^2 = 0.35 > R_{crit}^2 = 0.16$)

Ⅵ. Conclusions

The author draws some main conclusions from the volume/price probability wave model as follows:

1. The author observes and demonstrates that there exists a stationary volume distribution over a price range on an individual stock in great majority intraday transaction situations. The probability of accumulated trading volume (i.e. actual supply/demand quantity or transaction volume) that distributes over its price range gradually emerges the maximum or kurtosis around the price mean value in intraday transactions on an individual stock regardless of its price fluctuation path, time series, and total transaction volume. It is that the volume/price deviates from an equilibrium price frequently and shows price path uncertainty in a short term, but adjusts toward it and exhibits a relatively stable volume distribution over a price range in a longer term. It behaves a kind of probability wave toward an equilibrium price, driven by an actual supply/demand quantity restoring force or a supply/demand quantity imbalance restoring force.
2. A linear potential (energy) is an autoregressive item in mathematics and can represent actual supply/demand quantity (i.e. transaction volume) restoring force that is quantitatively and measurably defined in a transaction body system. If its equilibrium price is equal to the price mean value and the magnitude of restoring force is a constant over a trading price range in a long term, then, the potential mean value is equal to zero. This could explains why "most long-term return anomalies tend to disappear", which was defended by Fama (1998). Moreover, if $(v/V)_{\max} << (V-v)/V$ is satisfied over a trading price range in a transaction body system, then, its transaction energy mean value or the sum of transaction dynamic energy becomes an error item in comparison with the sum of the linear potential energy. The transaction dynamic energy is an explicit expression for ARCH.
3. Had established and tested the transaction volume/price probability wave model or equation (13) or (14), the author concludes that although the volume/price behaves apparently quite complex and nonlinear as a consequence of interaction among a variety of agents, it is holonomically constrained by the amount of transaction and governed by a second order linear differential equation (a Sturm-Liouville equation). The normalized $|\psi(p)|$ is the transaction volume probability at the price p in the equation. From this model, the author obtains a volume distribution function over a price range, the distribution of absolute zero-order Bessel eigenfunctions (see figure 3). To the author's knowledge, it is a new explicit distribution function.
4. The function gives a tentative explanation for shaded area in figure 1. It is the distribution of actual supply/demand quantity (accumulated trading volume or transaction volume) over a price range in an economic system when its supply and demand is in a dynamic state. The probability of total actual supply/demand quantity over the price range above its equilibrium price in an economic system is total excess demand quantity probability reference to its equilibrium price, whereas the probability over the price range below its equilibrium price is total excess supply quantity probability. And, the trading is most active, rather than the least,

at equilibrium price.

5. In this model, the eigenvalue indicates quantitatively that a set of intraday transactions on an individual stock, driven by a variable restoring force toward an equilibrium price over a price range, is a relatively stable system ($v_{tt} + F = const.$) in an intraday interval. Meanwhile, a set of discrete eigenvalues and equilibrium prices in eigenfunctions show intrinsic uncertain and changing characteristics in a transaction body system, and could be used in analyzing and describing its possible behavior or state. The volume/price behaves both relatively stable and frequently changing in stock market, measured by a set of eigenvalues and equilibrium prices.
6. There exist unstable transaction body systems when the imbalance between supply quantity and demand quantity has been enlarged too much, referencing to an equilibrium price, so as to cause equilibrium price change in the system. The volume/price waves around from one equilibrium price to another one. Therefore, the volume distribution over the price is the superposition of functions (15). In addition, the volume/price random walk is characterized by the probability wave model, though it is rare.
7. Reference to a transaction energy equilibrium price, the transaction energy in drop is the same as that in rise, but its volume probability in the drop is greater than that in the rise. It means that the supply quantity is greater than the demand quantity and its volume distribution over the price range is skewed in a transaction energy equilibrium system. This conclusion will help us to recognize and eventually accept indispensable market makers or reasonable price manipulators in stock market.
8. The asymptotic behavior in the probability wave function (15) describes Mandelbrot's early findings (Mandelbrot, 1963) and gives some reasonable explanations for heavy tail, cluster, and memory in the price volatility. The price volatility has its memory like a probability wave. The greater an eigenvalue ($\omega^2 = \frac{v}{V} v_{tt} = v_{tt} + F$) is in the eigenfunction, the more the zero points exists over a given price range in the eigenfunction. Therefore, it shows clear cluster and heavy tail.
9. Transaction momentum is critically defined in a price coordinate. It is transaction volume per given time interval. It determines the price mean value in a transaction body system through its weight and therefore determines its price rise or drop in general. From this, the author suggests using the momentum and the ratio of supply quantity to demand quantity to measure and make early warning on risk in an asset market.
10. This model is validated at current stage. The author attempts to offer a micro and dynamic probability wave theory on the price volatility with actual supply/demand quantity in financial economics. It is a reasonable solution to Soros's guess.

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