A functional auto-regression as a model of interest rate dynamics

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Abstract

We introduce a novel dynamical model of the interest rate term structure based on the concept of functional auto-regression. We show how parameters of this auto-regressions can be estimated, and how it can be used for prediction and simulation purposes. We also provide conditions for the existence of the equivalent martingale measure. The model is applied to the example of term structure in Eurodollar futures. It is shown that consistently with the literature the dynamics of the forward rates can be described using 3 prototypical curve shapes, which we estimate from data. We discuss the properties of the coefficient operator and the covariance matrix of the autoregression. It is found that there is an inseparable dynamic interaction between different factors.

1 Introduction

The forward rate is the equilibrium interest rate, at which borrowing at a future date can be arranged today. It can be either inferred from the current prices of different maturity bonds, or observed directly in prices of forward interest rate contracts. For one thing the study of forward rates is important since the market of interest rate futures contracts is large and growing. The number of futures contracts on interest rates traded on U.S. Exchanges grew more than 20 times during the last two decades: from 12.5 millions in 1980 to 248.7 millions in 2000. For another, the study of forward rate curve dynamics is important for pricing other interest rate securities and for managing risk of fixed-income portfolios. Finally, models of term structure shed light on how investors form their expectations about future interest rates. In this paper we construct a model of forward rates dynamics based on functional autoregression:

$$f_t(T) - \overline{f}(T) = \rho \left[f_{t-\delta}(T) - \overline{f}(T) \right] + \varepsilon_t(T), \tag{1}$$

where t is the calendar time, measured in discrete increments δ, forward rate $f_t(T)$ is a Hilbert-space valued random variable, and ρ is a linear operator on

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this Hilbert space. We show how to estimate this model and why it is useful for both pricing interest rate products and forecasting the forward rate curve.

The models of forward curve dynamics come in 3 varieties. Equilibrium models start with assumption that there are several factors, typically short and long interest rates, that determine the prices of all other bonds. The process of the factors is specified in a parametric form and estimated. The bonds in these models may be mispriced. The prominent contributions in this tradition are [11], [2] and [4]. Arbitrage-free models start from the assumption that all bonds are priced correctly and focus on how to price interest rate derivatives. The prominent contributions include [7], [8] and [6]. The most well-known of those models, Heath-Jarrow-Morton (HJM) model, add the insight that modeling forward rate curve evolution instead of bond yields makes pricing derivatives easier. The focus in these papers is, however, on novel methods of pricing, not on estimation. The random string models relax the assumption that evolution of term structure should be driven by a finite number of factors. The prominent contributions are by [9], [5], and [10]. In their current form, however, they are ill-suitable for estimation purposes.

We model interest rate dynamics by representing it as a particular case of a functional auto-regression. The theory of functional auto-regressions has made a significant progress in last years (see monograph [1]) and we capitalize on it. We develop a strategy of estimation of the forward rate auto-regression and prove its consistency.

Next we show how the model can be applied in prediction of future forward rates, evaluation of portfolio risk, and pricing of interest rate securities. With respect to the question whether our model can be made arbitrage-free, we provide a result on the existence of the martingale measure in our model.

In an empirical section our method is illustrated by an application to the data on Eurodollar future rates. We estimate the functional auto-regression and extract the factors. We find that the factor dynamic cannot be separated in 3 components independent from each other.

2 Definitions and Model

Let $P_t(T)$ is the time t price of a coupon-free bond with maturity at time T. The function is assumed differentiable in T, and the forward rate is defined:

$$f_t(T) = -\frac{\partial \log P_t(T)}{\partial T}. \tag{2}$$

At each moment of time we have a curve of forward rates, indexed by the time remaining to maturity. We are going to model the time evolution of this curve as a stochastic process with values in continuous functions.

By definition, the functional auto-regression is described by equation (1), where $f_t(T)$ denotes a Hilbert space valued random variable, $\varepsilon_t(T)$ is a strong white noise, and ρ is a bounded linear operator. See [1] for definitions of these concepts.

The functional auto-regression (1) was thoroughly studied in the recent literature on estimation of linear processes in function spaces. It describes forward rates in a more flexible way than HJM model because, first, the covariance matrix of the shocks is allowed to be non-singular. And second, the functional auto-regression allows mean reversion in forward rates.

3 Estimation

The natural estimator for the mean forward curve, $\overline{f}(T)$, is

$$\widehat{f}(T) = \frac{1}{n} \sum_{i=1}^{n} f_{i\delta}(T). \tag{3}$$

To define an estimator of ρ, we first introduce the empirical covariance function:

$$C_n(T_1, T_2) = \frac{1}{n} \sum_{i=1}^{n} f_{i\delta}(T_1) f_{i\delta}(T_2) - \widehat{f}(T_1)\widehat{f}(T_2), \tag{4}$$

and the empirical cross-covariance function:

$$D_n(T_1, T_2) = \frac{1}{n-1} \sum_{i=1}^{n-1} f_{i\delta}(T_1) f_{(i+1)\delta}(T_2) - \widehat{f}(T_1)\widehat{f}(T_2). \tag{5}$$

From the model we know that true cross-covariance function

$$D = E_t \left[f_t f_{t+\delta} \right] = E_t \left[f_t(\rho f_t + \varepsilon_{t+1}) \right] = \rho C. \tag{6}$$

Consequently, it is natural to define the estimator of ρ as follows:

$$\rho_n = D_n C_n^{-1}. \tag{7}$$

Unfortunately, this estimator behaves badly for a singular or near-singular C. The regularization method, advocated by Bosq, is to use π_{k_n}, the projector on a set of eigenvectors of C_n, associated with the k_n largest eigenvalues. Let $\widetilde{C}_n$ and $\widetilde{D}_n$ be the empirical covariance and cross-covariance operators restricted to the span of the largest eigenvalues. Then define

$$\rho_n = \widetilde{D}_n \widetilde{C}_n^{-1} \tag{8}$$

on the span of the eigenvalues and zero on its orthogonal complement. Under certain conditions this estimator is consistent.

Assume that ρ is a Hilbert-Schmidt operator. Let $a_1 = (\lambda_1 - \lambda_2)^{-1}$, and

$$a_i = \max \left\{ \frac{1}{\lambda_{i-1} - \lambda_i}, \frac{1}{\lambda_i - \lambda_{i+1}} \right\} \text{ for } i > 1, \tag{9}$$

where λ_i are eigenvalues of the covariance operator C ordered in the decreasing order.

Theorem 1. *If for some $\beta > 1$,*

$$\lambda_{k_n}^{-1} \sum_{i=1}^{k_n} a_i = O(n^{1/4} (\log n)^{-\beta}), \tag{10}$$

then ρ_n is strongly consistent in operator norm induced by L^2 norm:

$$\|\rho_n - \rho\|_{L^2} \to 0 \text{ a.s.} \tag{11}$$

Proof: This is a restatement of Theorem 8.7 in [1].

The condition of the theorem require that the eigenvalues of the covariance matrix do not drop to zero too fast, and that the eigenvalues are not too close to each other.

4 The martingale measure drift

For pricing purposes we need to adjust the drift of the forward rates process using the condition that the prices of bonds follow martingales. For a fixed time to maturity , the process of discounted bond prices is a martingale if:

$$E_t\left[e^{-r\delta}P_{t+\delta}(T-\delta)\right] = P_t(T), \tag{12}$$

where

$$e^{-r\delta} \equiv P_t(t+\delta) = \exp\left(-\int_0^\delta f_t(\tau)d\tau\right). \tag{13}$$

Therefore, the following condition must hold:

$$E_t\left[\frac{P_{t+\delta}(T-\delta)}{P_t(T)}\right] = \exp\left(\int_0^\delta f_t(\tau)d\tau\right). \tag{14}$$

We are especially interested in the case when δ is small. We will assume that the discrete-time process of forward rates under the martingale measure,

$$f_{t+\delta}(T) - f_t(T) = \widetilde{\mu}_t^{(\delta)}(T) + \varepsilon_{t+\delta}(T), \tag{15}$$

can be represented as a sampling of a continuous-time process. Assume also that the expectation function $\widetilde{\mu}_t^{(\delta)}$ can be written as $\widetilde{\mu}_t\delta$, and the covariance function of $\varepsilon_{t+\delta}(T)$ can be written as $c_t\delta$ to the first order in δ.

Theorem 2. *Suppose condition (14) is satisfied for a continuous time process of bond prices with time to maturity T. Then the following condition must hold:*

$$\int_0^T \widetilde{\mu}_t(\tau)d\tau = f_t(T) - f_t(0) + \frac{1}{2}\int_0^T\int_0^T c_t(\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{16}$$

Corollary. *Suppose condition (14) is satisfied for all times to maturity. Then the forward rate curve must be differentiable with respect to the time to maturity, and the drift and covariance function must satisfy the following condition:*

$$\widetilde{\mu}_t(T) = \frac{\partial f_t(T)}{\partial T} + \int_0^T c(T,\tau)d\tau. \tag{17}$$

From the practical point of view, the presence of the derivative with respect to the time to maturity in the equation for the drift require interpolation of the forward rate curve by a smooth curve. After the interpolation, the pricing algorithm is straightforward. First, compute the drift of the process under the risk-adjusted measure. Then, simulate the process of forward rate using this drift. Finally, calculate the price of the derivative security in question by averaging the discounted payoff under these simulations.

5 Existence of an equivalent martingale measure

It is important to figure out what restrictions on and guarantee the existence of a martingale measure. Indeed, it is a result of Harrison and Pliska that the martingale measure exists if and only if the market is free from arbitrage opportunities. By an arbitrage opportunity we mean a self-financing strategy that brings non-negative return with probability one and positive return with positive probability. The market with arbitrage opportunities cannot be in equilibrium because any rational investor would be willing to take a position in the arbitrage opportunity.

A sufficient condition for existence of martingale measure can be obtained using an infinite-dimensional version of the Girsanov theorem (see [3]).

Let forward rates follow an infinite-dimensional diffusion:

$$df = Af\,dt + BdW, \tag{18}$$

where A is a linear (perhaps unbounded) operator on the space U of forward rates, W is a cylindrical Wiener process taking values in a Hilbert space H, and B is an operator from H to U. And let us assume that under martingale measure the evolution of forward rates has to be described by a similar diffusion equation:

$$df = \widetilde{A}f\,dt + BdW, \tag{19}$$

Theorem 3. *The continuous- time process of forward rates in (18) admits an equivalent martingale measure if there is such a process (19) that*

i) (19) defines a martingale measure;

ii) $(A - \widetilde{A})f \in \operatorname{Im} B$*;*

iii) for some $\gamma > 0$, $\sup_{t\in[0,\bar{t}]} E \exp\left\{\gamma \left\|B^{-1}(A - A)f\right\|\right\} < \infty$.

6 Application

6.1 Data

We use daily settlement data on the Eurodollar futures traded on Chicago Mercantile Exchange. The Eurodollar futures contract is an obligation to deliver a 3-month deposit of $1,000,000 in a bank account outside of the United States. The available contracts has delivery dates that starts from several first months after the current date and the go each quarter up to 10 years into the future.

We obtained the data from Commodity Research Bureau. The available data start in 1982, however we use only the data starting in 1994 when the trading on 10-year contract appeared. Available datapoints were interpolated by cubic splines and the resulting curve were sampled in 30 day intervals speed up the estimation. Datapoints with less than 90 or more than 3480 days to expirations were removed. Consequently 114 points per curve and 2507 valid dates were available for estimation.

6.2 Results of Estimation

6.2.1 Factors

The results of estimation suggest that the rank of operator is around 3. This corresponds well to the finding in the previous empirical literature that dynamics of the term structure can be decomposed into evolution of 3 main factors: level, slope and curvature. Figure 1 shows the eigenvectors of the covariance matrix of forward rates that corresponds to the highest eigenvalues.

Figure 1. Eigenvectors
Figure 1 is here.
Note: The eigenvectors are estimated using the daily data on the Eurodollar forward rates.

Clearly the eigenvectors can be interpreted as the level, slope, and curvature factors.

6.2.2 Dynamics of factor loadings

Figure 2 illustrates how loadings on the factors evolve over time.

Figure 2. Time Evolution of Factor Loadings
Figure 2 is here.
Note: On the horizontal axis the calendar time in working days since January 1994. On the vertical line is the value of the factor loading. The blue line is for the loading on the "level", the green is for the "slope", and the red is for the "curvature". The factor curves were estimated using the entire data period from January 1994 till December 2003.

It is clear from this picture that the "level" is the most important contributor to the interest rates, followed by the "slope" and the "curvature". The mean-reverting nature of the dynamics is also apparent. Figure 3 shows the scatterplot of factor loadings:

Figure 3. Scatterplot of factor loadings
Figure 3 is here.
Note: The level loading is on the horizontal axis, the slope loading is on the vertical axis.

6.3 Estimates of coefficient operator

Operator maps the term structure curve to a linear combination of the factor curves. The action of operator on the factor curves themselves is given by the matrix in Table 1:

Table 1. The estimate of $\rho - I$ in the basis of eigenvectors.

-0.21	0.1	1.48
-0.12	-0.43	-0.77
0.00	-0.05	-0.74

Note: The coefficients are estimated from the daily data and all coefficients are multiplied by 100 for convenience.

What is surprising about this matrix is that it is non-symmetric and that the off-diagonal elements are quite large compared with diagonal elements. This observation suggests two possible explanations. The first one is that the dynamic of the model is more complex than the simple mean reversion. The second one is that the estimates of the off-diagonal elements are not sufficiently precise. First, we address the issue of dynamics.

Operator ρ have 1 real and 2 complex eigenvalues. All of them are less than 1 in absolute value so the operator is stable in the sense that it corresponds to a stable dynamic system: the deviation from the mean tends to the zero eventually The complexity of the dynamics can be seen from Figure 4.

Figure 4. Dynamics of factor loadings.
Figure 4 is here.
Note: The horizontal axis shows the loading on the "level" factor. The vertical axis shows the loading on the "slope" factor. The loading on the curvature factor is not shown. Its initial value was set equal to 1.

This picture shows evolution of the factor loadings with different initial values in the absence of external noise, so it illustrates impulse response function for a particular impulse. While the loadings are seen to converge to zero eventually, the convergence is not monotonic. The "level" initially increases before converging to zero, and the slope at some point becomes negative.

We can address the concern about the precision of the estimates by plotting the results of the estimation as the number of data increases:

Figure 5. The evolution of matrix entries of the estimate of operator .
Figure 5 is here.
Note: The operator is estimated using the daily data on the Eurodollar forward rates. The estimation is on the rolling basis so it uses all the information available at the time of estimation.

This chart suggest that some entries in the coefficient operator are indeed unstable over time. It turns out that these are all entries that are over the main diagonal. The entries on the main diagonal and below are stable.

7 Further Research

It appears that the main direction of further research is identifying what causes the regime changes in interest rates dynamics.

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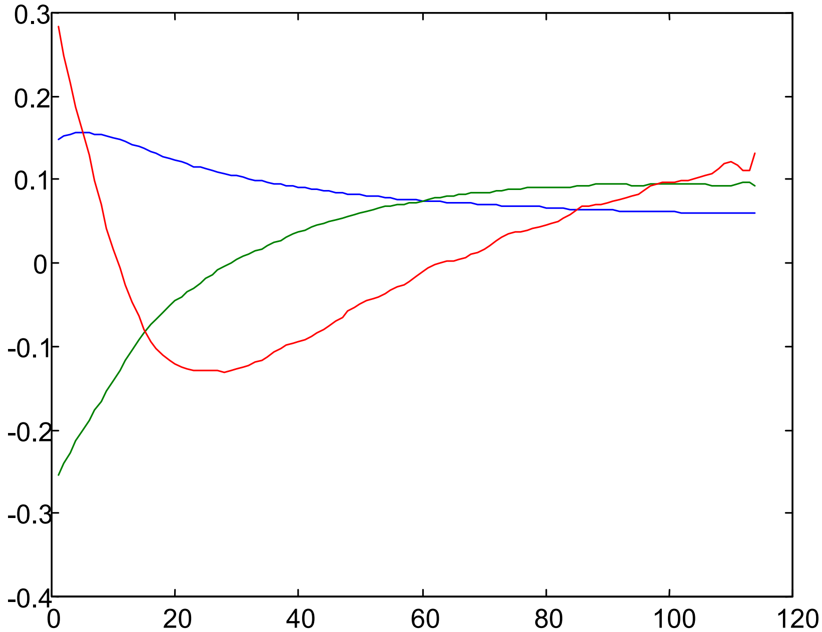
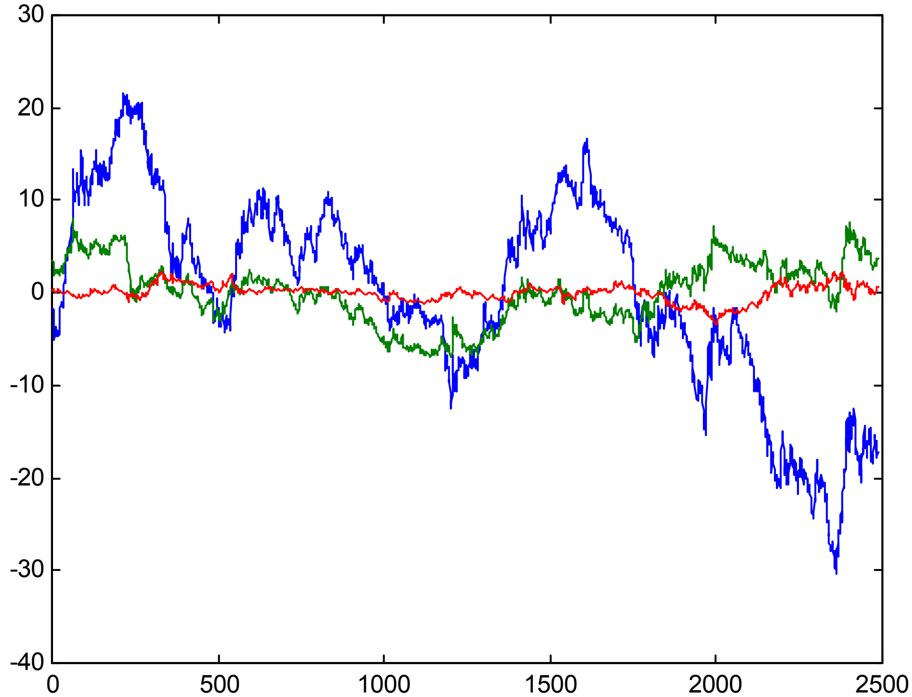
Figure 1. Eigenvectors.



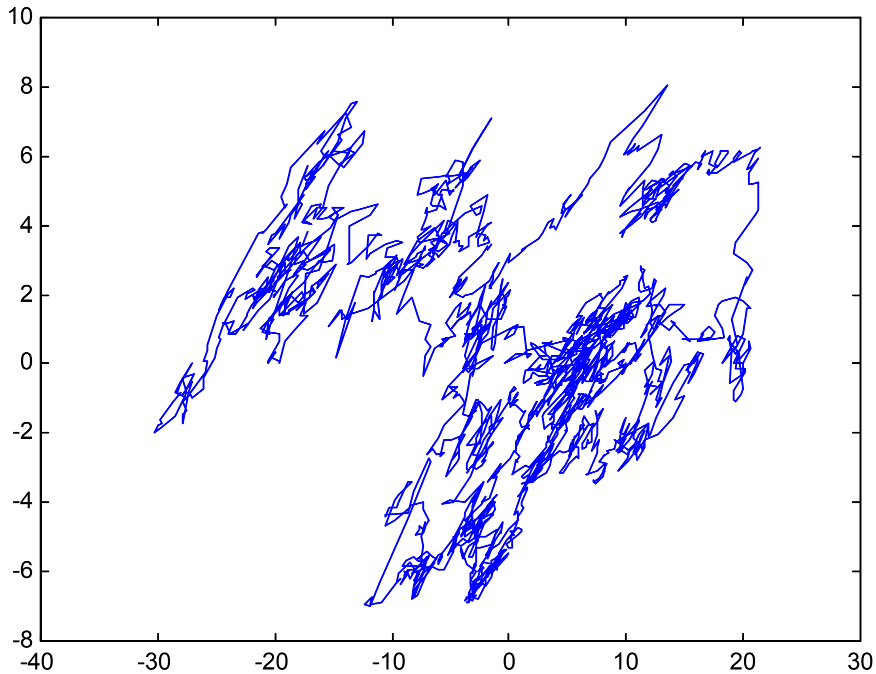
Note: The eigenvectors are estimated using the daily data on the Eurodollar forward rates.

Figure 2. Time Evolution of Factor Loadings



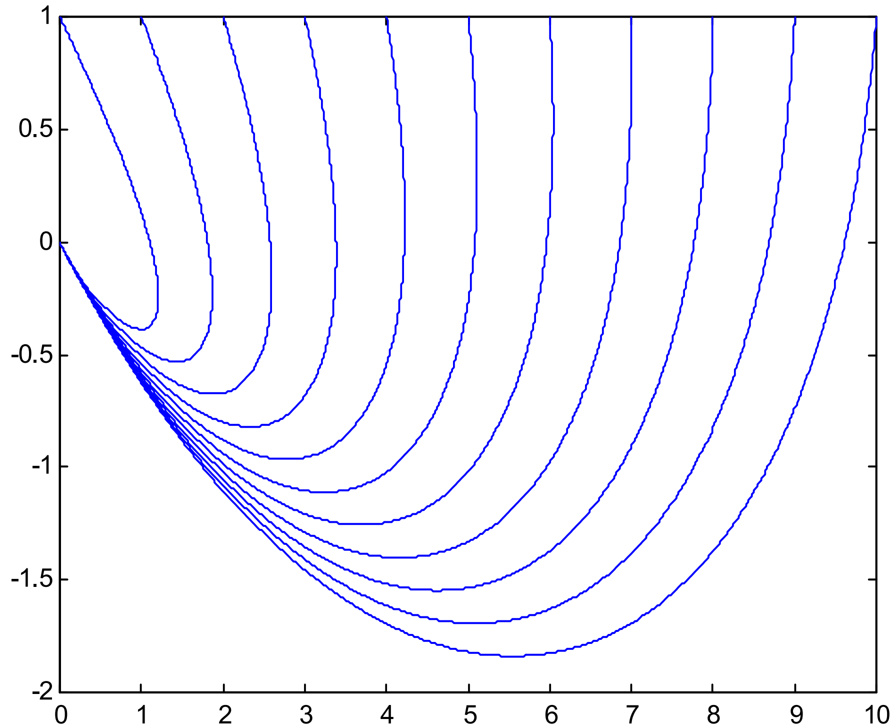
Note: On the horizontal axis the calendar time in working days since January 1994. On the vertical line is the value of the factor loading. The blue line is for the loading on the "level", the green is for the "slope", and the red is for the "curvature". The factor curves were estimated using the entire data period from January 1994 till December 2003.

Figure 3. Empirical dynamics of factor loadings.



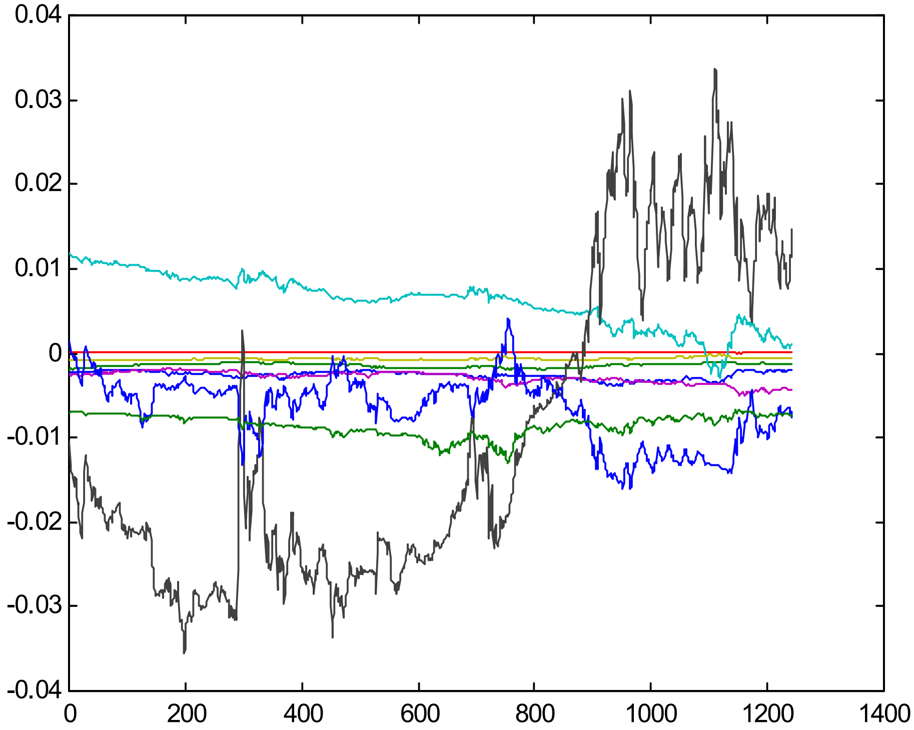
Note: The horizontal axis shows the loading on the “level” factor. The vertical axis shows the loading on the “slope” factor.

Figure 4. Dynamics of factor loadings.



Note: The horizontal axis shows the loading on the “level” factor. The vertical axis shows the loading on the “slope” factor. The loading on the curvature factor is not shown. Its initial value was set equal to 1.

Figure 5. The evolution of matrix entries of the estimate of operator ρ.



Note: The operator ρ is estimated using the daily data on the Eurodollar forward rates. The estimation is on the rolling basis so it uses all the information available at the time of estimation.