Practical guide to real options in discrete time

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Abstract. Continuous time models in the theory of real options give explicit formulas for optimal exercise strategies when options are simple and the price of an underlying asset follows a geometric Brownian motion. This paper suggests a general, computationally simple approach to real options in discrete time. Explicit formulas are derived even for embedded options. Discrete time processes reflect the scarcity of observations in the data, and may account for fat tails and skewness of probability distributions of commodity prices. The method of the paper is based on the use of the expected present value operators.

Key words: Real options, embedded options, expected present value operators

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1 Introduction

The goal of the paper is to provide a general framework for pricing of real options in discrete time; since option pricing cannot be separated from the problem of the optimal exercise time of an option, we solve for the optimal timing as well. Valuation of real options and optimal exercise strategies explicitly derived in the paper are relevant to many practical situations where individuals make at least partially irreversible decisions. Simple real options cover such situations as timing an investment of a fixed size, the scrapping of a production unit, a capital expansion program, etc. Embedded real options are relevant in such cases as new technology adoption, timing an investment (partially) financed by debt with an

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embedded option to default in the future, human capital acquisition, and many others (see, for example, [14], papers in the volume [8], and the bibliography therein). Usually, continuous time models are used. We believe that the discrete time approach has certain advantages over the continuous time approach to modeling of real options.

The discrete time approach allows one to incorporate in a more tractable way such important aspects of economic reality as "time-to-build". Some options – for example, renewing a labor contract or awarding a grant to a research project – can be exercised only at certain dates (typically, once a year). Further, the data in economics is much more scarce than in finance, where continuous time models were originated and developed. Observations of some variables are available only quarterly or even annually, and it is easier to fit a transition density to the data in a discrete time model than in a continuous time one, when only a list of observations at equally spaced moments in time is available. There are also natural questions, such as the rates of job creation and job destruction (see, e.g., [12] and [10]), which cannot be addressed in the Brownian motion setting under the standard assumption that labor is instantly adjustable, because the rates will be infinite. In the discrete time model, one obtains finite closed form solutions. Finally, discrete time modeling is natural for economics, where agents do not make, say, investment decisions every instant unlike the investors in financial markets.

In fact, the appearance of continuous time models in economics was mainly due to the success of these models in finance, and their tractability. Notice, however, that even in finance, the success of simple models is gone. Relatively simple models use the (geometric) Brownian motion as the underlying stochastic process (see, for example, [16]). However, these models proved to be rather inaccurate, and have been amended in many ways, none of which being as tractable as the Brownian motion model. In particular, nowadays processes with jumps are widely used (see, for instance, [9] and [15]). The normality of commodity price processes is rejected by the data as well (see, for example, [13] or [19]). Assume, for the moment, that we are willing to sacrifice the accuracy of a model for its simplicity, and decide to use the Brownian motion model. The apparent advantage is a well-known manageable scheme: with the help of Ito's lemma write down a second order differential equation for the value of an option, employ economic arguments to add appropriate boundary conditions, such as value matching and smooth pasting, and, using the general solution to the differential equation, reformulate the problem as a system of algebraic equations. In elementary situations, a closed form solution can be derived, and a simple exercise strategy results; in other situations, numerical procedures are available.

Unfortunately, this scheme is uncomplicated only when one considers really simple options. If a part of the option value comes from instantaneous payoffs due at certain future dates, a closed form solution to the optimal exercise problem is no longer available. If we consider an option whose value comes from different streams in different regions of the state space, the resulting system of algebraic equations may involve too many unknowns, and it becomes messy indeed. If we consider embedded options, then there is no general result about the optimal exercise rule, apart from the heuristic smooth pasting condition, and it is not even clear whether the formal solution satisfying the smooth pasting condition exists. In addition, if one incorporates jumps, the intuitive justification for the smooth pasting condition in [14] is lost, and there is no reason to believe that this principle always holds[2]. In fact, it may fail as it was shown in [5] and [6].

From our point of view, the following two drawbacks of the standard approach are conceptually even more important. First, the formulas provided by the standard approach are certain analytic expressions, which have no clear economic meaning by themselves, and the optimal exercise boundaries are just numbers. The classical example is the investment rule formulated in [14] in terms of the correction factor to the Marshallian law: the correction factor is an expression in terms of a root of a certain quadratic polynomial (fundamental quadratic). Second, is it really necessary to write down the equation for the value function, the boundary conditions, and solve the (free) boundary problem? Indeed, the option value by definition is the expected present value (EPV) of an instantaneous payoff or stream of payoffs. Suppose, we have formulas for these expected present values for model payoffs. Then using the argument similar to the backward induction in discrete time models with the finite time horizon, we can calculate the option value step by step. We solve for the value and optimal exercise time (if necessary) of the most distant option in a sequence of embedded options, then we move to the second to last option, etc.

This paper provides a general framework of this sort, with explicit formulas for each model situation, under the assumption that the underlying stochastic process is a random walk (under a risk-neutral measure chosen by the market), X_t, on the real line, that is, $X_t = X_0 + Y_1 + Y_2 + \cdots + Y_t$, where $Y_1, Y_2, \ldots$ are independently identically distributed random variables on a probability space Ω, and X_0 is independent of $Y_1, Y_2, \ldots$ This process specification implies that the dates when observations and/or decisions to exercise options can be made are equally spaced, and time periods are normalized to one. We calculate the EPV's in the following model situations: (1) an instantaneous payoff tomorrow; (2) a perpetual stream of payoffs; (3) an instantaneous payoff which is due when a barrier is crossed from below (or from above); (4) an entitlement for a stream of payoffs which comes into effect when a barrier is crossed from below (or above); (5) a stream of payoffs which will be lost when a barrier is crossed from below (or from above). Using these EPV's, we solve for the optimal exercise time of an option, calculate the expected waiting time till an option will be exercised, and find the optimal gradual capital expansion strategy.

Let $g(X_t)$ be the payoff function, and $q \in (0, 1)$ the discount factor. The first two cases are simple: the calculation of $qE^x[g(X_1)]$, and the summation of an infinite series $\sum_{t=0}^{\infty} q^t E^x[g(X_t)]$, respectively. For $g(X_t)$ a linear combination of exponents, the calculations in the first two cases reduce to the calculation of values of the moment generating function of Y_1, and closed-form solutions result. These observations are fairly standard and commonly used. The novelty of our approach is that the solutions in all the cases above are expressed in terms of the

[2]We are grateful to Avinash Dixit for pointing out to us that this principle may not hold in a discrete time model.

expected present value operators (EPV–operators). For a random walk X, and a given stream of payoffs $g(X_t)$, the EPV–operator (denoted by U^q_X) calculates the EPV of the stream $g(X_t)$:

$$U^q_X g(x) = E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] = E\left[\sum_{t=0}^{\infty} q^t g(X_t) \mid X_0 = x\right].$$

This operator gives the EPV in case (2). Cases (3) and (5) reduce to cases (2) and (4). To express the EPV of a payoff in case (4), in addition to the EPV–operator of X, we need the EPV–operators of the supremum process $\bar{X}_t = \max_{0\le s\le t} X_s$ and the infimum process $\underline{X} = \min_{0\le s\le t} X_s$. These EPV–operators act as follows

$$U^q_{\bar{X}} g(x) = E^x\left[\sum_{t=0}^{\infty} q^t g(\bar{X}_t)\right] \equiv E\left[\sum_{t=0}^{\infty} q^t g(\bar{X}_t) \mid X_0 = x\right],$$

$$U^q_{\underline{X}} g(x) = E^x\left[\sum_{t=0}^{\infty} q^t g(\underline{X}_t)\right] \equiv E\left[\sum_{t=0}^{\infty} q^t g(\underline{X}_t) \mid X_0 = x\right].$$

Let $\mathbf{1}_{[h,+\infty)}$ denote the indicator function of the interval $[h,+\infty)$ and the multiplication operator by the same function. We prove that the EPV of a stream $g(X_t)$ that accrues after the barrier h is crossed from below is the composition of the EPV–operators $U^q_{\bar{X}}$ and $U^q_{\underline{X}}$, and $\mathbf{1}_{[h,+\infty)}$:

$$V(h;x) = (1-q)U^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}U^q_{\underline{X}} g(x). \tag{1.1}$$

Similar formulas are obtained for the other expected present values listed in cases (3)–(5).

Assume that both the supremum and infimum processes are non-trivial, i.e., jumps in both directions happen with non-zero probabilities. Then the form of the solution (1.1) and simple general properties of the EPV–operators allow us to give a very short (approximately a page) proof of the optimal exercise rule: exercise an option which yields a stream of increasing payoffs $g(X_t)$ the first time when the EPV of the stream g under the *infimum process* becomes non-negative [3]. In this case, the *record setting low* payoff process matters. This reflects Bernanke's *bad news principle* spelled out in [3]. Similarly, for a decreasing payoff function g, the *record setting high* payoff process matters: exercise the option the first time when the EPV of the stream g under the *supremum process* becomes non-negative. Notice that a sufficient condition for optimality makes a perfect economic sense: the payoff function $g(X_t)$ is a monotone function of the stochastic factor X_t. *Record setting news principles* were obtained in [4] in the continuous time model, for a special case of payoff streams of the form $Ge^x - C(1-q)$ or $C(1-q) - Ge^x$, by using the well-known results for the perpetual American call and put options, respectively. The present paper extends the record setting news principles to general continuous monotone streams in

[3]For instance, in timing an investment of a fixed size, g is the stream of revenues net of $(1-q)C$, where C is the fixed cost of investment.

discrete time models. If the payoff stream, g, is an increasing function of the stochastic factor X_t, then it is optimal to exercise the right for (respectively, to give up) the stream when the EPV of the stream calculated under the assumption that the process $\{X_t\}$ is replaced by the infimum process becomes non-negative (respectively, non-positive). For a decreasing payoff function $g(X_t)$, one only needs to substitute the supremum process for the infimum process in the above statement.

The reader may think that the calculation of the action of the expected present value operators is difficult. For a general random walk, this is really the case. However, one of the advantages of the discrete time setting is that the transition density of a random walk can be approximated by exponential polynomials on each half-axis with desired accuracy and simplicity, although there is certainly a trade-off between the two. The family of transition densities given by exponential polynomials is fairy flexible, and such densities can account for fat tails and skewness observed in empirical distributions of commodity prices. For a transition density of this family, and $g(x)$ given by exponential polynomials (different exponential polynomials on different intervals of the real line are admissible), all the functions $U^q_X g(x)$, $U^q_{\bar{X}} g(x)$ and $U^q_{\underline{X}} g(x)$ are of the same class as $g(x)$. Therefore, we can use general formulas derived for model situations several times (embedded options), and each time we will calculate the integrals of a simple structure. All the tools needed for the calculation of these integrals are the fundamental theorem of calculus and integration by parts. It is unnecessary to write down differential equations and solve them. In fact, after the roots of a certain polynomial are found (we will call it the *fundamental polynomial* by analogy with the *fundamental quadratic* in the Brownian motion case: see [14]), all the calculations reduce to straightforward algebraic manipulations. Even in the case of embedded options of complicated structure, there is no need to consider systems of algebraic equations, and timing an option reduces to the calculation of a (unique) zero of a monotone function. At the same time, the method of the paper allows one to solve the optimal exercise problems for much wider classes of payoffs than the standard approach does.

The rest of the paper is organized as follows. In Section 2, we present general formulas for EPV's in model situations (1)–(5) and determine the optimal exercise time and the waiting time till an option is exercised. In Section 3, explicit formulas for the results of Section 2 are presented. In Section 4, timing a capital expansion program is solved by reduction to a sequence of investment opportunities of fixed size, and formulas for the optimal capital stock and the value of the firm are derived. The last formula shows clearly that the value of the firm does not evolve as a Markovian process. A similar result can be obtained in the continuous time model. Therefore the value of a firm cannot be modeled as a Gaussian process as it is often done in finance. Technical details and proofs are given in the appendices.

2 Model situations

2.1 Instantaneous payoff and payoff stream

The standing assumption of the paper is that the underlying stochastic process X_t is a random walk on $\mathbf{R}$, which has the transition density, p (the method of the paper can be applied to random walks on lattices as well). The transition operator, T, is defined by

$$(Tf)(x) = E^x[f(X_1)] \equiv E[f(X_1)|\, X_0 = x].$$

Given p, one calculates the EPV of a stochastic payoff tomorrow:

$$E^x[qg(X_1)] = q(Tg)(x) = q\int_{-\infty}^{+\infty} p(y)f(x+y)dy.$$

To compute the EPV of a stochastic payoff t periods from now, we use the Markov property of a random walk:

$$E^x[q^t g(X_t)] = q^t(T^t g)(x).$$

The next step is to calculate the EPV of a stream of payoffs:

$$U_X^q g(x) = E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] = \sum_{t=0}^{\infty} q^t (T^t g)(x).$$

Under the condition $q \in (0,1)$, the operator $I - qT$ is invertible in $L^\infty(\mathbf{R})$, and the inverse is $(I-qT)^{-1} = \sum_{t=0}^{\infty} q^t T^t$, hence

$$U_X^q g(x) = (I-qT)^{-1} g(x), \tag{2.1}$$

for a bounded measurable g. Under additional conditions on the transition density, (2.1) can be extended to unbounded g. From (2.1), we conclude that

$$U_X^q (I-qT) = (I-qT)U_X^q = I. \tag{2.2}$$

This equation allows one to express an instantaneous payoff $G(X_t)$ in terms of the EPV of a stream and vice versa:

$$g(X_t) = (I-qT)G(X_t), \tag{2.3}$$
$$G(X_t) = U_X^q g(X_t). \tag{2.4}$$

Clearly, equation (2.4) is the value, at date t, of the stream of payoffs that starts at date t as well. To evaluate such a stream at the initial date, we compute

$$E^x[q^t G(X_t)] = q^t T^t U_X^q g(x). \tag{2.5}$$

2.2 Payoff stream that starts to accrue at a random time

The list of the EPV's presented in the previous subsection conveys no new information to any person familiar with the basics of economics or finance. Suppose that we want to price the stream of payoffs that starts to accrue after the underlying stochastic variable, X_t, crosses a certain barrier, h. In this case, we need to replace the deterministic time, t, on the LHS in (2.5) with a random time, τ, and (2.5) is no longer valid. Fortunately, it is still possible to compute the value $E^x[q^\tau f(X_\tau)]$ in terms of the EPV's of some payoff streams. To do this, we need to distinguish between two cases: X_t crosses h from below, and X_t crosses h from above. Denote $\tau^+ = \tau^+_h = \min\{t \mid X_t \ge h\}$, and consider

$$V^+(x;h) = E^x\left[\sum_{t=\tau^+}^{\infty} q^t g(X_t)\right].$$

Under natural conditions on g,

$$V^+(x;h) = E^x\left[q^{\tau^+} U^q_X g(X_{\tau^+})\right] = (1-q)U^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}U^q_{\underline{X}}g(x). \qquad (2.6)$$

Equation (2.6) holds due to the quite deep result, known as the Wiener-Hopf factorization formula, that establishes the relationship among the EPV–operators U^q_X, $U^q_{\bar{X}}$, and $U^q_{\underline{X}}$:

$$U^q_X = (1-q)U^q_{\bar{X}}U^q_{\underline{X}} \qquad (2.7)$$

(for details, see Subsection 3.2). The Wiener-Hopf factorization formula (2.7) per se has no economic intuition behind it (and neither does Ito's lemma, both being general deep mathematical results), but using this formula, one obtains values of real options and optimal exercise strategies in economically meaningful terms.

Similarly, for the case of crossing from above, set $\tau^- = \tau^-_h = \min\{t \mid X_t \le h\}$ and

$$V^-(x;h) = E^x\left[\sum_{t=\tau^-}^{\infty} q^t g(X_t)\right].$$

Then the value of such a stream is given by

$$V^-(x;h) = E^x\left[q^{\tau^-} U^q_X g(X_{\tau^-})\right] = (1-q)U^q_{\underline{X}}\mathbf{1}_{(-\infty,h]}U^q_{\bar{X}}g(x). \qquad (2.8)$$

It is easy to see that (2.8) is just a version of (2.6): change the direction on the real line, then the supremum process becomes the infimum process, and vice versa.

Notice that (2.6) and (2.8) price the streams of payoffs that are described in case (4). The model situations in case (3) reduce to these formulas if we express an instantaneous payoff as the EPV of a stream using (2.3):

$$\begin{aligned} E^x\left[q^{\tau^+} G(X_{\tau^+})\right] &= (1-q)U^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}U^q_{\underline{X}}(I-qT)G(x); \\ E^x\left[q^{\tau^-} G(X_{\tau^-})\right] &= (1-q)U^q_{\underline{X}}\mathbf{1}_{(-\infty,h]}U^q_{\bar{X}}(I-qT)G(x). \end{aligned}$$

2.3 Payoff stream that is lost at a random time

The case when the right to the stream $g(X_t)$ is lost when X_t crosses h from below is reduced to the cases considered above:

$$\begin{aligned} V_+(x;h) &= E^x\left[\sum_{t=0}^{\tau^+-1} q^t g(X_t)\right] \\ &= E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] - E^x\left[\sum_{t=\tau^+}^{\infty} q^t g(X_t)\right] \\ &= U^q_X g(x) - (1-q)U^q_{\bar{X}}\mathbf{1}_{[h,+\infty)}U^q_{\underline{X}}g(x). \end{aligned}$$

The Wiener-Hopf factorization formula (2.7) allows us to proceed:

$$\begin{aligned} V_+(x;h) &= (1-q)U^q_{\bar{X}}\left(U^q_{\underline{X}}g(x) - \mathbf{1}_{[h,+\infty)}U^q_{\underline{X}}g(x)\right) \\ &= (1-q)U^q_{\bar{X}}\mathbf{1}_{(-\infty,h)}U^q_{\underline{X}}g(x). \end{aligned} \tag{2.9}$$

In a similar way, if the right for the stream is lost on crossing h from above, we derive

$$V_-(x;h) = E^x\left[\sum_{t=0}^{\tau^--1} q^t g(X_t)\right] = (1-q)U^q_{\underline{X}}\mathbf{1}_{(h,+\infty)}U^q_{\bar{X}}g(x). \tag{2.10}$$

2.4 Optimal exercise time

2.4.1 Time to enter

Suppose now that g is the stream of payoffs specified for a real option, for example, an option to invest capital C into a technology. If an entrepreneur chooses to invest at time τ, then the technology will produce a commodity at rate G, starting from time $\tau+1$, ever afterward. Investment is irreversible. One may view C as the present value of a deterministic stream of expenditures, to which the investor commits at time τ. By (2.3), this stream is $c = (1-q)C$. The output is sold on the spot at the market price $P_t = e^{X_t}$. Obviously, we are facing the situation to which (2.6) applies with $g(X_t) = qGE_t[e^{X_{t+1}}] - (1-q)C$. In Appendix A, we show that if g is non-decreasing and $U^q_{\underline{X}}g(x)$ changes sign only once, the optimal exercise (log) price (the optimal investment threshold in our example), h^* is a unique solution to the equation

$$w(x) = (U^q_{\underline{X}}g)(x) = 0. \tag{2.11}$$

We stress that contrary to the Brownian motion model and other models with continuous trajectories, the exercise boundary in the present model is not necessarily the price at which the option is exercised. It is optimal to exercise the option the first time τ such that $X_\tau \geq h^*$. For the case of irreversible investment, (2.11) is equivalent to

$$E^x\left[\sum_{t=0}^{\infty} q^t\left(qGe^{\underline{X}_{t+1}}-(1-q)C\right)\right]=0. \tag{2.12}$$

The last equation says that investment becomes optimal when the EPV of the project calculated under the assumption that the original price process is replaced by the infimum process becomes non-negative. If the investment threshold is chosen optimally, then the value of the option to invest, as given by (2.6), is

$$V^+(x;h^*)=(1-q)U^q_{\bar{X}}\mathbf{1}_{[h^*,+\infty)}U^q_{\underline{X}}g(x).$$

We use (2.12) to write

$$C=e^{h^*}E\left[\sum_{t=1}^{\infty} q^t Ge^{\underline{X}_t}\mid X_0=0\right]. \tag{2.13}$$

Therefore,

$$\begin{aligned}
E^x\left[\sum_{t=0}^{\infty} q^t\left(qGe^{\underline{X}_{t+1}}-(1-q)C\right)\right] &= e^xE\left[\sum_{t=1}^{\infty} q^t Ge^{\underline{X}_t}\mid X_0=0\right]-C\\
&= (e^x-e^{h^*})Ce^{-h^*},
\end{aligned}$$

and

$$\begin{aligned}
V^+(x;h^*) &= Ce^{-h^*}(1-q)(U^q_{\bar{X}}\mathbf{1}_{[h^*,+\infty)}(\cdot)(e^{\cdot}-e^{h^*}))(x)\\
&= Ce^{-h^*}(1-q)E^x\left[\sum_{t=0}^{\infty} q^t(e^{\bar{X}_t}-e^{h^*})_+\right],
\end{aligned}$$

where $(e^{\bar{X}_t}-e^{h^*})_+=\max\{(e^{\bar{X}_t}-e^{h^*}),0\}$. Now we can express Tobin's Q as

$$Q(x)=\frac{V^+(x;h^*)}{C}=\frac{E^x\left[\sum_{t=0}^{\infty} q^t(e^{\bar{X}_t}-e^{h^*})_+\right]}{e^{h^*}/(1-q)}.$$

Hence Tobin's Q is the ratio of the EPV of the stream of payoffs $(e^{\bar{X}_t}-e^{h^*})_+$ and the EPV of the perpetual stream e^{h^*}. Notice that the investment threshold is determined by the infimum process, and Tobin's Q is determined by the threshold and supremum process.

Alternatively, using (2.13), we can write the value $V^+(x;h^*)$ as

$$V^+(x;h^*)=(1-q)E\left[\sum_{t=1}^{\infty} q^t Ge^{\underline{X}_t}\mid X_0=0\right]E^x\left[\sum_{t=0}^{\infty} q^t(e^{\bar{X}_t}-e^{h^*})_+\right].$$

The last representation factors out contributions of the infimum and supremum price processes to the value of the option to invest. The first expectation on the RHS decreases as the probability of downward jumps in prices increases, and the second expectation increases with the probability of upward jumps. If both probabilities increase, the overall effect on the value $V^+(x;h^*)$ is ambiguous.

2.4.2 Time to exit

The option considered above is similar to the perpetual American call option. In this subsection, we consider the optimal exercise strategy for a generalization of the perpetual American put option, which is relevant to irreversible decisions such as scrapping a production unit or exit; it also applies to the optimal timing of default on a debt.

Consider a firm in a deteriorating environment. The firm's profit, $f(X_t)$, is falling, on average, so that it may become optimal to discontinue operations at some point in time, and sell the firm's inventory for the scrap value C. If the firm's manager makes a decision to scrap the inventory at time τ, the operations will be discontinued at time $\tau+1$, and the scrap value will be available at time $\tau+2$. When the firm discontinues its operations, it looses the stream of profits whose EPV is

$$E^x\left[\sum_{t=\tau+1}^{\infty} q^t f(X_t)\right] = E^x[q^{\tau+1}U^q_X f(X_{\tau+1})].$$

The scrap value, C, can be viewed as a deterministic stream of payoffs, $c = (1-q)C$, that starts to accrue at $\tau+2$. Hence, at time τ, the option value of exit is $U^q_X g(X_\tau)$, where $g(x) = q^2(1-q)C - qTf(x)$. If the decision to exit is made when the underlying stochastic factor, X_t, crosses a barrier h from above, then the option value to exit is given by (2.8). Suppose that $g(x)$ is non-increasing and $U^q_{\bar{X}} g(x)$ changes sign only once. Then the optimal exercise price of the option is a unique solution, call it h_*, of the equation

$$w(x) = (U^q_{\bar{X}} g)(x) = 0 \tag{2.14}$$

(see Appendix A). For g in the case under consideration, (2.14) is equivalent to

$$E^x\left[\sum_{t=0}^{\infty} q^t(q^2(1-q)C - qf(\bar{X}_{t+1}))\right] = 0, \tag{2.15}$$

or

$$f(x) + q^2 C = U^q_{\bar{X}} f(x). \tag{2.16}$$

The LHS in (2.16) is the EPV of the instantaneous payoff received if the operations are terminated, the RHS is the EPV of the stream of profits evaluated under the assumption that the original stochastic process is replaced by the supremum process. Rule (2.16) says that it is optimal to exit when the difference between the former and the latter EPV's becomes non-negative.

Suppose that $f(x) = Ge^x$, then from (2.15), one obtains

$$q^2 C = e^{h_*} E\left[\sum_{t=1}^{\infty} q^t G e^{\bar{X}_t} \mid X_0 = 0\right],$$

whence

$$E^x\left[\sum_{t=0}^{\infty} q^t(q^2(1-q)C - qf(\bar{X}_{t+1}))\right] = q^2 C e^{-h_*}(e^{h_*} - e^x),$$

and from (2.8), we derive

$$\begin{aligned}
V^-(x;h_*) &= q^2Ce^{-h_*}(1-q)\left(U^q_{\underline{X}}\mathbf{1}_{(-\infty,h_*]}(\cdot)(e^{h_*}-e^{\cdot})\right)(x) \\
&= q^2Ce^{-h_*}(1-q)E^x\left[\sum_{t=0}^{\infty} q^t(e^{h_*}-e^{\underline{X}_t})_+\right] \\
&= (1-q)E\left[\sum_{t=1}^{\infty} q^tGe^{\bar{X}_t} \mid X_0=0\right]E^x\left[\sum_{t=0}^{\infty} q^t(e^{h_*}-e^{\underline{X}_t})_+\right].
\end{aligned}$$

2.5 Expected waiting time

We consider timing the investment of a fixed size once again. Assume that the spot log-price x is less than h^*, and consider the waiting time R_x till the investment is made. This is the random variable defined by

$$R_x = \min\{t > 0 \mid X_t \geq h^*\}.$$

The expected waiting time can be calculated as follows:

$$\begin{aligned}
E[R_x] &= E^x\left[\sum_{t=0}^{\infty}\mathbf{1}_{(-\infty,h^*)}(\bar{X}_t)\right] = \lim_{q\to 1-0} E^x\left[\sum_{t=0}^{\infty} q^t\mathbf{1}_{(-\infty,h^*)}(\bar{X}_t)\right] \\
&= \lim_{q\to 1-0}\left\{E^x\left[\sum_{t=0}^{\infty} q^t\right] - \lim_{q\to 1-0} E^x\left[\sum_{t=0}^{\infty} q^t\mathbf{1}_{[h^*,+\infty)}(\bar{X}_t)\right]\right\} \\
&= \lim_{q\to 1-0}\left\{(1-q)^{-1} - U^q_{\bar{X}}\mathbf{1}_{[h^*,+\infty)}(x)\right\}. \qquad (2.17)
\end{aligned}$$

Notice that here one must consider the process under the historical measure and not a risk-neutral one. Explicit formulas for the expected waiting time are presented in Appendix C.

3 Explicit formulas for model situations

In this section, we show how to calculate the EPV when the transition density is given by exponential polynomials.

3.1 Instantaneous payoff and payoff stream

3.1.1 Calculation of the expected present value of an instantaneous payoff at a deterministic moment in the future

Let p be the transition density of the random walk $X_t = X_0 + Y_1 + \cdots + Y_t$, and let

$$M(z) = E\left[e^{zY_1}\right] = \int_{-\infty}^{+\infty} e^{zx}p(x)dx$$

be the moment generating function of Y_1. If G is an exponential function: $G(x) = e^{zx}$, then the transition operator, T, acts as a multiplication operator by the number $M(z)$, and we obtain $qE^x[e^{zX_1}] = qTG(x) = qM(z)e^{zx}$. Similarly, one easily calculates the expected present value of a payoff $G(X_t)$ at a deterministic moment t in the future: $q^tE^x[e^{zX_t}] = ((qT)^tG)(x) = (qM(z))^te^{zx}$. Equations (2.3) and (2.4) can be used to represent a payoff G as the expected present value of the stream $g(X_t) = ((I - qT)G)(X_t)$; $G(X_t) = U_X^q g(X_t)$. In a special case of an exponential function $G(x) = e^{zx}$, with z satisfying the condition $1 - qM(z) > 0$, we have $g(x) = (1 - qM(z))G(x)$.

3.1.2 Expected present value of an ongoing project: capital and labor are fixed

Consider a firm that produces a commodity which is sold on the spot at the price $P_t = e^{X_t}$. We impose the following restriction on the price process:

$$qM(1) < 1. \tag{3.1}$$

Restriction (3.1) is appropriate when capital, K, and labor, L, employed by the firm are fixed or capped. Indeed, for K and L fixed, the EPV of the revenue stream grows each period by factor $M(1) = E\left[e^{Y_1}\right]$, and it is discounted back at rate q. Hence, the EPV of the stream of revenues is given by

$$P_0G(K,L)\sum_{j=0}^{\infty}(qM(1))^t = \frac{P_0G(K,L)}{1 - qM(1)}, \tag{3.2}$$

where $G(K,L)$ is the production function of the firm. For the series in (3.2) to converge, it is necessary and sufficient that (3.1) holds.

3.1.3 The case of costlessly adjustable labor

In other situations, more stringent conditions than (3.1) may be necessary. For instance, consider a firm with the Cobb-Douglas production function $G(K,L) = dK^\theta L^{1-\theta}$, which faces the fixed labor cost w, and can instantly and costlessly adjust labor. A similar situation was considered in [2] for a continuous time model of irreversible investment. At a given price level of the output, P, the firm chooses L to maximize $PdK^\theta L^{1-\theta} - wL$. From the F.O.C.

$$(1-\theta)PdK^\theta L^{-\theta} - w = 0,$$

we find $L = K\big((1-\theta)Pd/w\big)^{1/\theta}$, and hence the revenue is $R = AP^{1/\theta}$, where

$$A = Kw\left(\frac{(1-\theta)d}{w}\right)^{1/\theta}\cdot\frac{\theta}{1-\theta}.$$

We conclude that the EPV of the revenue stream

$$AP_0\sum_{t=0}^{\infty}(qM(1/\theta))^t = \frac{AP_0}{1 - qM(1/\theta)} \tag{3.3}$$

is finite iff $qM(1/\theta) < 1$. Notice that if increases in prices are anticipated, that is, $M(1) > 1$, then $M(1/\theta) > M(1)$ (recall that the moment generating function is convex), and therefore, the condition $qM(1/\theta) < 1$ is more stringent than (3.1).

3.1.4 The case of stochastic prices and operational costs

One can allow for a stochastic operational cost as well, and the model remains quite tractable provided both prices and variable costs depend on the same stochastic factor, call it X_t. A simple example (with fixed capital and labor) would be $R(X_t) = e^{X_t}G(K, L)$, and $C(X_t) = a + be^{\gamma X_t}$, where $a, b, \gamma > 0$, so that the profit is

$$g(X_t) = R(X_t) - C(X_t) = e^{X_t}G(K, L) - a - be^{\gamma X_t}.$$

These examples show that it may be necessary to consider payoffs of various kinds; hence, some general regularity conditions on g are needed.

3.1.5 Calculation of $U^q_X g(X)$ for general payoffs

The EPV of a stream $g(X_t)$ can be easily calculated when g is an exponential polynomial, that is, a sum of products of exponents and polynomials. For $g(X_t) = e^{zX_t}$, the result is

$$U^q_X e^{zx} = E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] = \frac{e^{zx}}{1 - qM(z)}, \tag{3.4}$$

provided $1 - qM(z) > 0$. By linearity, (3.4) extends to g a linear combination of exponential functions, and more generally, for integrals of exponential functions. Hence, if an explicit formula for the moment generating function is available, $U^q_X g(X)$ can be calculated for all payoff streams g of interest.

Sufficient conditions for $U^q_X g(x)$ of a measurable stream $g(X_t)$ to be finite are

$$|g(X_t)| \leq C\exp(\sigma^+ X_t), \quad X_t \geq 0, \tag{3.5}$$
$$|g(X_t)| \leq C\exp(\sigma^- X_t), \quad X_t \leq 0, \tag{3.6}$$

where constant C is independent of X_t, and $\sigma^\pm$ satisfy

$$1 - qM(\sigma^\pm) > 0. \tag{3.7}$$

These conditions are necessary if g is monotone on each half-axis.

If we confine ourselves to a wide and fairly flexible class of probability densities given by exponential polynomials on each half-axis, then for a general g, the calculation of $U^q_X g(X)$ reduces to simple integration procedures. Consider first the case when the transition density is of the form

$$p(x) = c^+\lambda^+ e^{-\lambda^+ x}\mathbf{1}_{[0,+\infty)}(x) + c^-(-\lambda^-)e^{-\lambda^- x}\mathbf{1}_{(-\infty,0]}(x), \tag{3.8}$$

where $c^+, c^- > 0$, and $\lambda^- < 0 < \lambda^+$. If we want to have a continuous p, we must require that $c^+\lambda^+ + c^-\lambda^- = 0$, and then the normalization requirement $M(0) = 1$ leads to $c^+ = \lambda^-/(\lambda^- - \lambda^+), c^- = \lambda^+/(\lambda^+ - \lambda^-)$. We have constructed a two-parameter family of probability densities.The moment generating function is

$$M(z) = \frac{c^+\lambda^+}{\lambda^+ - z} + \frac{c^-\lambda^-}{\lambda^- - z} = \frac{-\lambda^-\lambda^+}{\lambda^+ - \lambda^-}\left[\frac{1}{\lambda^+ - z} - \frac{1}{\lambda^- - z}\right].$$

It is easily seen that $1 - qM(z)$ has two real roots: $\beta^- \in (\lambda^-, 0)$, and $\beta^+ \in (0, \lambda^+)$, which are the roots of the quadratic equation

$$z^2 - (\lambda^+ + \lambda^-)z + (1-q)\lambda^+\lambda^- = 0. \tag{3.9}$$

We find

$$\beta^\pm = 0.5 \cdot \left(\lambda^+ + \lambda^- \pm \sqrt{(\lambda^+ + \lambda^-)^2 - 4(1-q)\lambda^-\lambda^+}\right), \tag{3.10}$$

and represent $1/(1 - qM(z))$ in the form

$$\frac{1}{1 - qM(z)} = \frac{a^+}{\beta^+ - z} + \frac{a^-}{\beta^- - z}, \tag{3.11}$$

where $a^\pm = (\beta^\pm - z)/(1 - qM(z))|_{z=\beta^\pm} = 1/qM'(\beta^\pm)$. Set $g(x) = e^{zx}$. Using (3.8) and (3.11), it is straightforward to check that the RHS in (3.4) is equal to the RHS in the equation below

$$U_X^q g(x) = a^+ \int_0^\infty e^{-\beta^+ y} g(x+y)dy - a^- \int_{-\infty}^0 e^{-\beta^- y} g(x+y)dy. \tag{3.12}$$

By linearity, (3.12) holds for linear combinations of exponents, and more generally, for wide classes of functions which can be represented as integrals of exponents e^{zx} w.r.t. z. One can also use more than one exponential on each axis, and obtain more elaborate probability densities (see Appendix B for details).

3.2 Payoff stream that starts to accrue at a random time

As an example, consider the problem of timing an investment of a fixed size (see Subsection 2.4.1). We need to compute the value given by (2.6), where

$$g(X_t) = qGE_t[e^{X_{t+1}}] - (1-q)C = qM(1)Ge^{X_t} - (1-q)C.$$

The operator $(1-q)U_X^q$ acts on an exponential function e^{zx} as the multiplication operator by the number $(1-q)/(1-qM(z))$:

$$(1-q)U_X^q e^{zx} = \frac{1-q}{1-qM(z)} e^{zx}.$$

Similarly, the $(1-q)U_{\bar{X}}^q$ and $(1-q)U_{\underline{X}}^q$ act on an exponential function e^{zx} as multiplication operators by certain numbers, call them $\kappa_q^+(z)$ and $\kappa_q^-(z)$:

$$(1-q)U_{\bar{X}}^q e^{zx} = \kappa_q^+(z)e^{zx}, \quad (1-q)U_{\underline{X}}^q e^{zx} = \kappa_q^-(z)e^{zx}. \tag{3.13}$$

These numbers are

$$\kappa_q^+(z) = (1-q)\ (U^q_{\bar{X}} e^{zx})\big|_{x=0} = (1-q)E\left[\sum_{t=0}^{\infty} q^t g(\bar{X}_t) \mid X_0 = 0\right],$$

and

$$\kappa_q^-(z) = (1-q)\ (U^q_{\underline{X}} e^{zx})\Big|_{x=0} = (1-q)E\left[\sum_{t=0}^{\infty} q^t g(\underline{X}_t) \mid X_0 = 0\right],$$

respectively. The Wiener-Hopf factorization formula (see, e.g., [18]) states that

$$\frac{1-q}{1-qM(z)} = \kappa_q^+(z)\kappa_q^-(z). \tag{3.14}$$

It follows that (2.7) holds: $U^q_X = (1-q)U^q_{\bar{X}} U^q_{\underline{X}}$. In other words, we can factorize the EPV–operator U^q_X in terms of the EPV–operators $U^q_{\bar{X}}$ and $U^q_{\underline{X}}$. Indeed, if we apply both sides of (2.7) to e^{zx}, we obtain identity on the strength of (3.14). By linearity, (2.7) extends to linear combinations of exponential functions, and more generally, to wide classes of functions which can be represented as integrals of exponentials w.r.t. parameter z.

Under an additional very weak regularity condition on the density p (piecewise continuous p are allowed), it is proved in [7] that if g is measurable and satisfies the growth condition (3.5), then for $\tau^+ = \tau_h^+$, (2.6) holds. It is possible to prove (2.6) without additional conditions on p.

If the probability density p is given by exponential polynomials on the positive and negative half-axis, then all the functions in (3.14) are rational functions. The calculation of the factors $\kappa_q^\pm(z)$ reduces to the calculation of roots of the numerator of the rational function $1 - qM(z)$, that is, of a polynomial. To obtain $\kappa_q^+(z)$, one collects all the factors in the numerator and denominator of the rational function $(1-q)/(1-qM(z))$, which do not vanish in the half-plane $\Re z \geq 0$ (the real part of z is non-negative), and $\kappa_q^-(z)$ contains the factors which do not vanish in the half-plane $\Re z \leq 0$ (see, e.g., [17] and [7]. In many cases (including the examples above), all the roots are real. Now the same argument as at the end of Section 2 shows that the action of operators $U^q_{\bar{X}}$ and $U^q_{\underline{X}}$ is given by simple integral operators. As the result, we obtain an effective procedure for the calculation of $U^q_{\bar{X}} g(x)$ and $U^q_{\underline{X}} g(x)$. In this section, we consider the case of the probability density (3.8); for the general case, see Appendix B.

First, we calculate the the moment-generating function $M(z)$, find the roots of the equation $1 - qM(z) = 0$ (see (3.10)), and represent $\kappa_q^+(z)$ and $\kappa_q^-(z)$ as the sums of constants and simple fractions:

$$\kappa_q^+(z) = \frac{(\lambda^+ - z)\beta^+}{\lambda^+(\beta^+ - z)} = \frac{\beta^+}{\lambda^+} + \frac{\beta^+(\lambda^+ - \beta^+)}{\lambda^+(\beta^+ - z)}, \tag{3.15}$$

and

$$\kappa_q^-(z) = \frac{(\lambda^- - z)\beta^-}{\lambda^+(\beta^- - z)} = \frac{\beta^-}{\lambda^-} + \frac{\beta^-(\lambda^- - \beta^-)}{\lambda^-(\beta^- - z)}. \tag{3.16}$$

Using (3.13), (3.15), (3.16) and the same considerations as in the derivation of (3.12), we obtain for a continuous g satisfying (3.5)

$$(1-q)U^q_{\bar{X}}g(x) = \frac{\beta^+}{\lambda^+}g(x) + \frac{\beta^+(\lambda^+-\beta^+)}{\lambda^+}\int_0^{+\infty} e^{-\beta^+ y}g(x+y)dy, \qquad (3.17)$$

and for a continuous g satisfying (3.6), we calculate

$$(1-q)U^q_{\underline{X}}g(x) = \frac{\beta^-}{\lambda^-}g(x) - \frac{\beta^-(\lambda^--\beta^-)}{\lambda^-}\int_{-\infty}^{0} e^{-\beta^- y}g(x+y)dy. \qquad (3.18)$$

Under condition (3.7), the roots $\beta^\pm$ are outside $[\sigma^-,\sigma^+]$, therefore (3.5) and (3.6) ensure the convergence of the integrals in (3.17) and (3.18).

Now we return to the investment problem of Subsection 2.4. From (3.13), we find

$$U^q_{\underline{X}}g(x) = (1-q)^{-1}\kappa^-_q(1)qM(1)Ge^x - C.$$

Set

$$u(x) = \mathbf{1}_{[h,+\infty)}(x)(U^q_{\underline{X}}g)(x) = \mathbf{1}_{[h,+\infty)}(x)[(1-q)^{-1}\kappa^-_q(1)qM(1)Ge^x - C],$$

and calculate the value $V^+(x;h) = E^x[q^{\tau^+}U^q_{\underline{X}}g(X_{\tau^+})]$ of the investment project at time 0 (provided h is chosen as the investment threshold). Using (2.6) and (3.17), we find for $x<h$:

$$\begin{aligned} V^+(x;h) &= (1-q)(U^q_{\bar{X}}u)(x)\\ &= \frac{\beta^+(\lambda^+-\beta^+)}{\lambda^+}\int_{x+y>h} e^{-\beta^+ y}[(1-q)^{-1}\kappa^-_q(1)qM(1)Ge^{x+y} - C]dy\\ &= \frac{\beta^+(\lambda^+-\beta^+)}{\lambda^+}\left[\frac{\kappa^-_q(1)qM(1)G}{1-q}\int_{h-x}^{+\infty} e^{(1-\beta^+)y+x}dy - \int_{h-x}^{+\infty} Ce^{-\beta^+ y}dy\right]\\ &= \frac{\beta^+(\lambda^+-\beta^+)}{\lambda^+}e^{\beta^+(x-h)}\left[\frac{e^h\kappa^-_q(1)qM(1)G}{(1-q)(\beta^+-1)} - \frac{C}{\beta^+}\right]. \qquad (3.19)\end{aligned}$$

3.3 Timing an investment

In the case of the stream $g(x) = e^x qM(1)G - (1-q)C$, (2.11) assumes the form

$$(1-q)^{-1}e^x\kappa^-_q(1)qM(1)G - C = 0, \qquad (3.20)$$

and the investment threshold is

$$e^{h^*} = \frac{(1-q)C}{\kappa^-_q(1)qM(1)G}. \qquad (3.21)$$

After h^* is found, the EPV of the project can be calculated with the help of (2.6). In particular, if the probability density is given by (3.8), we can use (3.19) with $h=h^*$, and substituting (3.21), find

$$V^+(x;h^*) = C\frac{(\lambda^+-\beta^+)e^{\beta^+(x-h^*)}}{(\beta^+-1)\lambda^+}, \quad x<h^*. \qquad (3.22)$$

3.4 Timing an exit

Consider a special case when the profit function in Subsection 2.4.2 is given as $f(X_t) = Ge^{X_t}$. Then $g(x)$ is a linear combination of exponentials, so that we can calculate $U^q_{\bar{X}} g(x)$ using (3.13):

$$U^q_{\bar{X}} g(x) = q^2 C - (1-q)^{-1} \kappa^+_q(1) q M(1) G e^x,$$

and find the optimal disinvestment threshold from (2.15):

$$e^{h_*} = \frac{q^2(1-q)C}{\kappa^+_q(1) q M(1) G}. \tag{3.23}$$

After h_* is found from (3.23), we can calculate the expected present value of the gains from scrapping the production unit at time 0, $V^-(x; h_*)$, using (2.8) and (3.23). Similarly to (3.22), for $x > h_*$,

$$V^-(x; h_*) = Cq^2 \frac{(\lambda^- - \beta^-) e^{\beta^-(x-h_*)}}{(1-\beta^-)\lambda^-}.$$

Finally, the value of the firm is the sum of the EPV of the perpetual stream of profits plus the option value of scrapping: for $x > h_*$,

$$V(x) = \frac{e^x G}{1 - qM(1)} + Cq^2 \frac{(\lambda^- - \beta^-) e^{\beta^-(x-h_*)}}{(1-\beta^-)\lambda^-}.$$

4 Incremental capital expansion

In this Section, we assume that the production function depends only on capital, and that $G(K)$ is differentiable, concave, and satisfies the Inada conditions. A similar situation was considered in [1] for a two-period model of partially reversible investment, in [14] for the geometric Brownian motion model, and in [4] for Lévy processes. At each time period t, the firm receives $e^{X_t} G(K_t)$ from the sales of its product, and, should it decide to increase the capital stock, suffers the installation cost $C \cdot (K_{t+1} - K_t)$. The firm's objective is to chose the optimal investment strategy $\mathcal{K} = \{K_{t+1}(K_t, X_t)\}_{t \ge 1}, K_0 = K, X_0 = x$, which maximizes the NPV of the firm:

$$V(K, x) = \sup_{\mathcal{K}} E^x \left[\sum_{t \ge 0} q^t (e^{X_t} G(K_t) - C(K_{t+1} - K_t)) \right]. \tag{4.1}$$

Here we treat the current log price x and capital stock K as state variables, and $\mathcal{K}$ as a sequence of control variables. Due to irreversibility of investment, $K_{t+1} \ge K_t, \ \forall t$.

To ensure that firm's value (4.1) is bounded, we impose a resource constraint: there exists $\bar{K} < \infty$, such that $K_t \le \bar{K}, \ \forall t$. The resource constraint, condition (3.1), and properties of the production function ensure that the value function (4.1) is well defined.

Formally, the manager has to choose both the timing and the size of the capital expansion. However, it is well-known (see, for example, [14]) that for each level of the capital stock, it is only necessary to decide when to invest. The manager's problem is equivalent to finding the boundary (the investment threshold), $h(K;C)$, between two regions in the state variable space (K,x): inaction and action ones. For all pairs (K,x) belonging to the inaction region, it is optimal to keep the capital stock unchanged. In the action region, investment becomes optimal. To derive the equation for the investment boundary, suppose first that every new investment can be made in chunks of capital, ΔK, only[4]. In this case, the firm has to suffer the cost $C\Delta K$, and the EPV of the revenue gain due to this investment can be represented in the form of the EPV of the stream $g(X_t) = qM(1)(G(K+\Delta K) - G(K))e^{X_t} - (1-q)C\Delta K$. On the strength of the result of Subsection 3.3, the optimal exercise boundary is determined from the equation $U^q_{\underline{X}} g(h) = 0$, which can be written as

$$(1-q)^{-1} qM(1)(G(K+\Delta K) - G(K))\kappa_q^-(1)e^h = C\Delta K. \tag{4.2}$$

Dividing by ΔK in (4.2) and passing to the limit, we obtain the equation for the optimal threshold, $h^* = h^*(K)$:

$$qM(1)\kappa_q^-(1)G'(K)e^h = C(1-q). \tag{4.3}$$

Equivalently, the optimal exercise price is

$$e^{h^*} = e^{h^*(K)} = \frac{C(1-q)}{qM(1)\kappa_q^-(1)G'(K)}. \tag{4.4}$$

The rigorous justification of this limiting argument can be made exactly as in the continuous time model in [4]. Let $h = h(K;\Delta)$ be the solution to (4.2). Then the option value associated with the increase of the capital by ΔK, at the price level e^x, is

$$(1-q)U^q_{\bar{X}} \mathbf{1}_{[h,+\infty)}(x) \left(\frac{qM(1)}{1-q}(G(K+\Delta K) - G(K))\kappa_q^-(1)e^x - C\Delta K \right).$$

As $\Delta K \to 0$, we have $h = h(K;\Delta) \to h^*(K)$; therefore, dividing by ΔK and passing to the limit, we obtain the formula for the derivative of the option value of future investment opportunities w.r.t. K:

$$V_K^{\text{opt}}(K,x) = (1-q)U^q_{\bar{X}} \mathbf{1}_{[h^*,+\infty)}(x) \left(\frac{qM(1)G'(K)}{1-q}\kappa_q^-(1)e^x - C \right). \tag{4.5}$$

Substituting C from (4.4) into (4.5) and using the definition of $\kappa_q^-(1)$, we obtain

$$V_K^{\text{opt}}(K,x) = (1-q)E\left[\sum_{t=1}^{\infty} q^t G'(K)e^{\underline{X}_t} \mid X_0 = 0\right] E^x \left[\sum_{t=0}^{\infty} q^t (e^{\bar{X}_t} - e^{h^*})_+\right].$$

[4]The authors are indebted for this simplifying trick to Mike Harrison; our initial proof was more involved.

The last formula factors out the contributions of the infimum and supremum price processes to the marginal option value of capital. The first expectation on the RHS decreases if the probability of downward jumps in prices increases, and the second expectation increases if the probability of positive jumps in prices increases. Hence the marginal option value of capital increases in downward uncertainty and decreases in upward uncertainty. The overall effect of uncertainty is ambiguous.

Now the marginal, or shadow, value of capital is given by

$$V_K(K,x) = \frac{e^x G'(K)}{1 - qM(1)} - V_K^{\mathrm{opt}}(K,x). \tag{4.6}$$

Equation (4.6) expresses the marginal value of capital as the difference of two components. The first one is the expected present value of the marginal returns to capital, given that the capital stock remains constant at the level K in the future. The second component is the marginal option value of the future investment opportunities. This value is subtracted because investing extinguishes the option. A similar result was obtained in [1] for a two-period model of partially reversible investment.

Exactly the same calculations which lead to (3.22) allow us to derive from (4.5)

$$V_K^{\mathrm{opt}}(K,x) = C\frac{(\lambda^+ - \beta^+)e^{\beta^+(x-h^*)}}{(\beta^+ - 1)\lambda^+}. \tag{4.7}$$

Substituting (4.4) into the last equation, we derive

$$V_K^{\mathrm{opt}}(K,x) = C\frac{\lambda^+ - \beta^+}{(\beta^+ - 1)\lambda^+}\left(\frac{\kappa_q^-(1)qM(1)}{(1-q)C}\right)^{\beta^+} e^{\beta^+ x} G'(K)^{\beta^+}. \tag{4.8}$$

Integrating (4.8) w.r.t. capital, we obtain

$$V^{\mathrm{opt}}(K,x) = C\frac{\lambda^+ - \beta^+}{(\beta^+ - 1)\lambda^+}\left(\frac{\kappa_q^-(1)qM(1)}{(1-q)C}\right)^{\beta^+} e^{\beta^+ x}\int_K^{\bar{K}} G'(k)^{\beta^+}\,dk.$$

The value of the firm is $V(K,x) = e^x G(K)/(1-qM(1)) + V^{\mathrm{opt}}(K,x)$. Suppose that the capital stock available for investment, $\bar{K}$, is very large, and function $G'(K)^{\beta^+}$ is integrable on $[1,+\infty)$. Then we can obtain a simpler formula by replacing the upper limit $\bar{K}$ with $+\infty$. In the case of Cobb-Douglas production function $G(K) = dK^\theta$, we have $G'(K) = d\theta K^{\theta-1}$, therefore the integrals converge iff $(\theta-1)\beta^+ < -1$, or, equivalently, $\beta^+ > 1/(1-\theta)$. If this condition is satisfied, the value of the firm for $x < h^*(K)$ is

$$\begin{aligned} V(K,x) \;&=\; \frac{e^x dK^\theta}{1-qM(1)} \\ &\quad + \frac{\lambda^+ - \beta^+}{(\beta^+ - 1)\lambda^+}\left(\frac{\kappa_q^-(1)qM(1)}{(1-q)}\right)^{\beta^+} \frac{C^{1-\beta^+} e^{\beta^+ x} K^{1-\beta^+(1-\theta)}}{(d\theta)^{\beta^+}(\beta^+(1-\theta)-1)}. \end{aligned}$$

One of the standing assumptions in corporate finance is that the value of a firm follows a (geometric) Brownian motion. Our solution clearly shows that the firm's value is not described even by a Markovian process.

Notice that the proof of (4.3) in [4] was based on the reduction to the case of the perpetual American call, and therefore the generalization for more general dependence on the stochastic factor was not possible. Here the result holds for any continuous increasing revenue flow $R(K, x)$, and the formula for the optimal investment threshold obtains in the form:

$$U^q_{\underline{X}} qTR_K(K, h) = C, \tag{4.9}$$

where the EPV–operator $U^q_{\underline{X}}$ and transition operator T act w.r.t. the second argument (we have qTR_K instead of R_K because the revenues will start to accrue the period after the investment is made). For instance, if the firm faces the operational cost $a + bKe^{X_t/2}$, then the revenue flow is $R(K_t, X_t) = e^{X_t}G(K_t) - a - bKe^{X_t/2}$, and instead of (4.4), we now have

$$(1-q)^{-1}[qM(1)\kappa^-_q(1)G'(K)e^h - bqM(1/2)\kappa^-_q(1/2)e^{h/2}] - C = 0. \tag{4.10}$$

The function on the LHS in (4.10) changes sign only once, and therefore the solution to equation (4.10) gives the optimal investment threshold.

A Proof of optimality of h^* and h_*

On the strength of Lemma on p.1364 in [11], to prove the optimality of the solution $V(h^*; x)$, it suffices to check the following two conditions:

$$V(h^*; x) \geq \max\{G(x), 0\}, \quad \forall\ x, \tag{A.1}$$

and

$$(I - qT)V(h^*; x) \geq 0, \quad \forall\ x. \tag{A.2}$$

By our choice of h^*, $w(x) \geq 0$ for all $x \geq h^*$, therefore $V(h^*; x) = (1 - q)U^q_{\bar{X}}\mathbf{1}_{[h^*,+\infty)}w(x) \geq 0$ for all x. Using (2.6), we represent $V(h^*; x)$ in the form

$$\begin{aligned} V(h^*; x) &= (1-q)U^q_{\bar{X}}U^q_{\underline{X}}g(x) - (1-q)(U^q_{\bar{X}}\mathbf{1}_{(-\infty,h^*)}w)(x) \\ &= U^q_{X}g(x) - v(h^*; x) = G(x) - v(h^*; x), \end{aligned}$$

where $v(h^*; x) = (1-q)(U^q_{\bar{X}}\mathbf{1}_{(-\infty,h^*)}w)(x)$. Due to the choice of h^*, we have $w(x) < 0\ \forall\, x < h^*$, so that $v(h^*; x) \leq 0$ for all x, and hence $V(h^*; x) \geq G(x)$ for all x. We conclude that (A.1) holds.

Under a very weak regularity assumption on p, it is proved in [7] that for any h,

$$V(h; x) = qTV(h; x), \qquad x < h. \tag{A.3}$$

Using more elaborate arguments as in [6] and [7] in continuous time, one can prove that (A.3) holds if the probability density p exists. Thus, (A.2) holds on $(-\infty, h^*)$, and it remains to verify (A.2) for $x \geq h^*$. Introduce $W(x) = (I - qT)V(h^*; x)$. Using (2.2) and (2.7), we obtain

$$\begin{aligned} W(x) &= (I - qT)(1-q)U^q_{\bar{X}} \mathbf{1}_{[h^*,+\infty)} U^q_{\underline{X}} g(x) \\ &= (U^q_{\underline{X}})^{-1} \mathbf{1}_{[h^*,+\infty)} U^q_{\underline{X}} g(x), \end{aligned} \tag{A.4}$$

and also

$$\begin{aligned} W(x) &= (I - qT)(1-q)U^q_{\bar{X}} \mathbf{1}_{[h^*,+\infty)} U^q_{\underline{X}} g(x) \\ &= (I - qT)(1-q)U^q_{\bar{X}} U^q_{\underline{X}} g(x) - (I - qT)(1-q)U^q_{\bar{X}} \mathbf{1}_{(-\infty,h^*)} U^q_{\underline{X}} g(x) \\ &= g(x) + (-I + qT)(1-q)U^q_{\bar{X}} \mathbf{1}_{(-\infty,h^*)} U^q_{\underline{X}} g(x). \end{aligned}$$

But if a function u vanishes outside $(-\infty, h)$, then

$$U^q_{\bar{X}} u(x) = E\left[\sum_{t=0}^{\infty} q^t u(x + \bar{X}_t)\right] = 0, \quad \forall\ x \geq h,$$

as well. Therefore, for $x \geq h^*$,

$$W(x) = g(x) + q(1-q)TU^q_{\bar{X}} \mathbf{1}_{(-\infty,h^*)} U^q_{\underline{X}} g(x). \tag{A.5}$$

By our standing assumption, g is non-decreasing, hence from (A.5), $W(x)$ is non-decreasing on $[h^*, +\infty)$. We apply $U^q_{\underline{X}}$ to (A.4) and obtain

$$\mathbf{1}_{[h^*,+\infty)}(x) U^q_{\underline{X}} g(x) = U^q_{\underline{X}} W(x) = E\left[\sum_{t=0}^{\infty} q^t W(x + \underline{X}_t)\right]. \tag{A.6}$$

Suppose that $W(h^*) < 0$. Then there exists $h_1 > h^*$ such that $W(x) \leq 0$ for all $x \in (h^*, h_1)$. It follows that for the same x, the RHS in (A.6) is non-positive. But for these x, the LHS is positive by the very definition of h^*. Hence, our assumption $W(h^*) < 0$ is false, and since W is non-decreasing on $[h^*, +\infty)$, the condition (A.2) follows, and the proof of optimality is finished.

The verification of the optimality conditions (A.1)–(A.2) for the exit problem is quite similar to the proof above: just replace $\bar{X}$, $\underline{X}$, $U^q_{\bar{X}}, U^q_{\underline{X}}$ and $[h^*, +\infty)$ with $\underline{X}, \bar{X}, U^q_{\bar{X}}, U^q_{\underline{X}}$ and $(-\infty, h_*]$, respectively.

B Transition densities given by exponential polynomials

B.1 The case of three exponentials

In this Subsection, we demonstrate how to obtain the transition density of a desired shape. The density (3.8) has a kink (and maximum) at the origin. If we want to have a smooth p (and allow for the maximum to be not at the origin), we need to use more than two exponential functions. Suppose that we want to model a density which has the maximum on the positive half-axis. Then we use one exponential on the negative half-axis, and two on the positive one:

$$(c_1^+\lambda_1^+e^{-\lambda_1^+x} - c_2^+\lambda_2^+e^{-\lambda_2^+x})\mathbf{1}_{[0,+\infty)}(x) - c^-\lambda^-e^{-\lambda^-x}\mathbf{1}_{(-\infty,0]}(x), \tag{B.1}$$

where c_1^+, c_2^+ and c^- are positive, and $\lambda^- < 0 < \lambda_1^+ < \lambda_2^+$. Later in this Subsection, we show that for any choice of $\lambda^- < 0 < \lambda_1^+ < \lambda_2^+$, equation (B.1) with c^-, c_1^+, c_2^+ given by simple formulas (B.6) defines a probability density, which has the maximum on the positive half-axis. See Figure 1 for an example. Similarly, one can construct a 3-parameter family of probability densities which have the maximum on the negative half-axis. Should one wish to have a smooth probability density which has the maximum at the origin, one must use two exponential functions on each half-line or exponential polynomials of the form $(ax+b)e^{\gamma x}$.

The moment generating function of the probability density (B.1) is

$$M(z) = \frac{c_1^+\lambda_1^+}{\lambda_1^+ - z} - \frac{c_2^+\lambda_2^+}{\lambda_2^+ - z} + \frac{c^-\lambda^-}{\lambda^- - z}.$$

At the end of this Subsection we will show that the *fundamental rational function* $1 - qM(z) = 0$ has 3 roots, all of which real. Call these roots β^-, β_1^+ and β_2^+. We have

$$\lambda^- < \beta^- < 0 < \beta_1^+ < \lambda_1^+ < \lambda_2^+ < \beta_2^+. \tag{B.2}$$

Clearly, $1/(1 - qM(z))$ can be represented in the form

$$\frac{1}{1 - qM(z)} = \frac{a_1^+}{\beta_1^+ - z} + \frac{a_2^+}{\beta_2^+ - z} + \frac{a^-}{\beta^- - z}, \tag{B.3}$$

where $a_{+,j} = 1/(qM'(\beta_j^+)),\ j = 1, 2$, and $a^- = 1/(qM'(\beta^-))$, and therefore,

$$\begin{aligned} U_X^q g(x) &= a_1^+ \int_0^\infty e^{-\beta_1^+ y} g(x+y)dy \qquad \text{(B.4)} \\ &+ a_2^+ \int_0^\infty e^{-\beta_2^+ y} g(x+y)dy - a^- \int_{-\infty}^0 e^{-\beta^- y} g(x+y)dy. \end{aligned}$$

Similarly, we can consider a probability density given by linear combinations of two or more exponents on each of the half-axis. If we use two exponentials for each, we have two roots $\beta_j^\pm$, $j = 1, 2$, of the "characteristic equation" $1 - qM(z) = 0$ on each half-axis, and (B.2)–(B.4) change in the straightforward manner. One can also use more than two exponentials on each axis, and obtain more elaborate probability densities.

Now we show that any choice $\lambda^- < 0 < \lambda_1^+ < \lambda_2^+$ defines a probability density. Three conditions: $\int_{-\infty}^{+\infty} p(x)dx = 1$, p is continuous at 0, and p is smooth at 0, give a linear system of three equations

$$\begin{aligned} c_1^+ - c_2^+ + c^- &= 1, \\ c_1^+\lambda_1^+ - c_2^+\lambda_2^+ + c^-\lambda^- &= 0, \\ c_1^+(\lambda_1^+)^2 - c_2^+(\lambda_2^+)^2 + c^-(\lambda^-)^2 &= 0. \end{aligned} \tag{B.5}$$

Using Cramer's rule, it is easy to find a unique solution (c_1^+, c_2^+, c^-) to (B.5) for any $\lambda^- < \lambda_1^+ < \lambda_2^+$:

$$\begin{aligned} c_1^+ &= \frac{-\lambda^-\lambda_2^+}{(\lambda_2^+ - \lambda_1^+)(\lambda_1^+ - \lambda^-)}, \\ c_2^+ &= \frac{-\lambda^-\lambda_1^+}{(\lambda_2^+ - \lambda_1^+)(\lambda_2^+ - \lambda^-)}, \\ c^- &= \frac{\lambda_1^+\lambda_2^+}{(\lambda_1^+ - \lambda^-)(\lambda_2^+ - \lambda^-)}. \end{aligned} \tag{B.6}$$

It is easily seen that c_1^+, c_2^+ and c^- are positive, and that p is positive as well.

The roots of $1 - qM(z)$ are found as follows. Clearly, $1 - qM(z)$ has 3 roots at most. As $z \to \lambda^- + 0$, $1 - qM(z) \to -\infty$, and the same holds as $z \to \lambda_1^+ - 0$, and as $z \to \lambda_2^+ + 0$. Under condition $q \in (0, 1)$, $1 - qM(0) = 1 - q > 0$, and $1 - qM(+\infty) = 1 > 0$. Hence, on each of the intervals $(\lambda^-, 0)$ $(0, \lambda_1^+)$, and $(\lambda_2^+, +\infty)$, $1 - qM(z)$ changes sign. Therefore, on each of these three intervals, there is exactly one root, which we have called β^-, β_1^+ and β_2^+, respectively.

B.2 General scheme for the computation of $U^q_{\bar{X}}$ and $U^q_{\underline{X}}$

Step 1. Calculate the moment-generating function $M(z)$, and consider the rational function $1 - qM(z)$. Find the roots of the denominator, $\lambda_j^\pm$, and the numerator, $\beta_j^\pm$, with their multiplicities (sign "+" for the roots on the positive axis, sign "-" for the ones on the negative axis).

Step 2. Define

$$\kappa_q^+(z) = \prod_j \frac{\lambda_j^+ - z}{\lambda_j^+} \prod_k \frac{\beta_k^+}{\beta_k^+ - z}, \tag{B.7}$$

$$\kappa_q^-(z) = \prod_j \frac{\lambda_j^- - z}{\lambda_j^-} \prod_k \frac{\beta_k^-}{\beta_k^- - z}. \tag{B.8}$$

Step 3. If all the roots $\beta_k^\pm$ are simple, we represent $\kappa_q^+(z)$ and $\kappa_q^-(z)$ in the form

$$\kappa_q^+(z) = \kappa_q^+(\infty) + \sum_k \frac{a_k^+}{\beta_k^+ - z}, \quad \kappa_q^-(z) = \kappa_q^-(\infty) - \sum_k \frac{a_k^-}{\beta_k^- - z}, \tag{B.9}$$

where

$$a_k^+ = \prod_j \frac{\lambda_j^+ - \beta_k^+}{\lambda_j^+} \beta_k^+ \prod_{l \neq k} \frac{\beta_l^+}{\beta_l^+ - \beta_k^+}, \quad a_k^- = -\prod_j \frac{\lambda_j^- - \beta_k^-}{\lambda_j^-} \beta_k^- \prod_{l \neq k} \frac{\beta_l^-}{\beta_l^- - \beta_k^-}.$$

The case of multiple roots can be treated similarly.

Step 4. For a continuous g satisfying (3.5), we can calculate

$$U^q_{\bar{X}} g(x) = \kappa_q^+(\infty) g(x) + \sum_k a_k^+ \int_0^{+\infty} e^{-\beta_k^+ y} g(x+y) dy, \tag{B.10}$$

and for a continuous g satisfying (3.6), we can find

$$U^q_{\underline{X}} g(x) = \kappa^-_q(\infty) g(x) + \sum_k a^-_k \int_{-\infty}^0 e^{-\beta^-_k y} g(x+y) dy. \tag{B.11}$$

Under condition (3.7), all the roots $\beta^\pm_k$ are outside $[\sigma^-, \sigma^+]$, therefore (3.5) and (3.6) ensure the convergence of the integrals in (B.10) and (B.11).

B.3 Value of investment project

By assuming that the probability density is given by exponential polynomials on each half-axis and using (2.6) and (B.10), we find for $x < h$:

$$\begin{aligned} V^+(x;h) &= (1-q)(U^q_{\bar{X}} u) \\ &= \sum_k a^+_k \left[\frac{\kappa^-_q(1) q M(1) G}{1-q} \int_{h-x}^{+\infty} e^{-\beta^+_k y + x + y} dy - C \int_{h-x}^{+\infty} e^{-\beta^+_k y} dy \right] \\ &= \sum_k a^+_k e^{\beta^+_k (x-h)} \left[\frac{e^h \kappa^-_q(1) q M(1) G}{(1-q)(\beta^+_k - 1)} - \frac{C}{\beta^+_k} \right]. \end{aligned} \tag{B.12}$$

If the threshold is chosen optimally, then we use (3.21) and find

$$V^+(x;h^*) = C \sum_k \frac{a^+_k e^{\beta^+_k (x-h^*)}}{(\beta^+_k - 1)\beta^+_k}. \tag{B.13}$$

B.4 Value of scrapping

After h_* is found from (3.23), we can calculate the expected present value of the gains from scrapping of the firm at time 0, $V^-(x;h_*)$, using (2.8) and (3.23). Similarly to (B.13), for $x > h_*$,

$$V^-(x;h_*) = Cq^2 \sum_k \frac{a^-_k e^{\beta^-_k (x-h_*)}}{(1-\beta^-_k)\beta^-_k}.$$

Finally, the value of the firm is the sum of the EPV of the perpetual stream of profits plus the option value of scrapping: for $x > h_*$,

$$V(x) = \frac{e^x G}{1 - qM(1)} + Cq^2 \sum_k \frac{a^-_k e^{\beta^-_k (x-h_*)}}{(1-\beta^-_k)\beta^-_k}.$$

B.5 Capital expansion

Exactly the same calculations which lead to (3.22) allow us to derive from (4.7)

$$V^{\text{opt}}_K(K,x) = C \sum_k \frac{a^+_k}{(\beta^+_k - 1)\beta^+_k} \left(\frac{\kappa^-_q(1) q M(1)}{(1-q)C} \right)^{\beta^+_k} e^{\beta^+_k x} G'(K)^{\beta^+_k}. \tag{B.14}$$

Integrating (B.14) w.r.t. K, we obtain

$$V^{\mathrm{opt}}(K,x) = C\sum_j \frac{a_j^+}{(\beta_j^+-1)\beta_j^+}\left(\frac{\kappa_q^-(1)qM(1)}{(1-q)C}\right)^{\beta_j^+} e^{\beta_j^+ x}\int_K^{\bar{K}} G'(k)^{\beta_j^+}dk.$$

Suppose that the available capital stock, $\bar{K}$, is very large, and the functions $G'(K)^{\beta_k^+}$ are integrable on $[1,+\infty)$. Then we can obtain a simpler formula by replacing the upper limit $\bar{K}$ with $+\infty$. In the case of Cobb-Douglas production function $G(K) = dK^\theta$, we have $G'(K) = d\theta K^{\theta-1}$, therefore the integrals converge iff $(\theta-1)\beta_k^+ < -1$ for all k. Let β_1^+ be the smallest positive root, then the equivalent condition is $\beta_1^+ > 1/(1-\theta)$. If this condition is satisfied, the option value of investment opportunities is

$$V^{\mathrm{opt}}(K,x) = \sum_k \frac{a_k^+}{(\beta_k^+-1)\beta_k^+}\left(\frac{\kappa_q^-(1)qM(1)}{(1-q)}\right)^{\beta_k^+} \frac{C^{1-\beta_k^+}e^{\beta_k^+x}K^{1-\beta_k^+(1-\theta)}}{(d\theta)^{\beta_k^+}(\beta_k^+(1-\theta)-1)}.$$

C Expected waiting time

Assume that the spot log-price x is less than h^*, and consider the waiting time R_x till the investment is made. If the transition density is given by (3.8), then the expected waiting time is finite iff $\lambda^+ + \lambda^- < 0$, or, equivalently,

$$m \equiv E[X_1] \equiv 1/\lambda^- + 1/\lambda^+ > 0, \tag{C.1}$$

and if (C.1) holds, then

$$E[R_x] = \frac{1}{m}(h^* - x + 1/\lambda^+). \tag{C.2}$$

Condition (C.1) has a clear interpretation: the expected waiting time is finite iff the drift of the log-price, m, is positive, and if it is positive, then (C.2) says that the expected waiting time is inversely proportional to the drift. It is also proportional to the distance to the barrier *plus the positive term which is independent of the distance.*

To obtain the formula (C.2) for the expected waiting time, we need to calculate the limit in (2.17). Using (3.17), we obtain for $x < h^*$:

$$\begin{aligned} U^q_{\bar{X}}\mathbf{1}_{[h^*,+\infty)}(x) &= (1-q)^{-1}\frac{\beta^+(\lambda^+-\beta^+)}{\lambda^+}\int_0^{+\infty} e^{-\beta^+ y}\mathbf{1}_{[h^*,+\infty)}(x+y)dy \\ &= \frac{e^{-\beta^+(h^*-x)}(\lambda^+-\beta^+)}{(1-q)\lambda^+} = (1-q)^{-1}e^{-\beta^+(h^*-x)}(1-\beta^+/\lambda^+). \end{aligned}$$

Hence, we have to calculate the limit

$$\lim_{q\to 1-0} \frac{1 - e^{-\beta^+(q)(h^*-x)} + e^{-\beta^+(q)(h^*-x)}\beta^+(q)/\lambda^+}{1-q}.$$

Both terms are positive, and if $\beta^+(q)/(1-q)$ is unbounded as $q \to 1$, the limit is clearly infinite. From (3.10), we find that this possibility does not realize iff $\lambda^- + \lambda^+ < 0$. If $\lambda^- + \lambda^+ < 0$, we obtain

$$\beta^+(q) = \frac{\lambda^+\lambda^-}{\lambda^+ + \lambda^-}(1-q) + O((1-q)^2),$$

and (C.2) follows.

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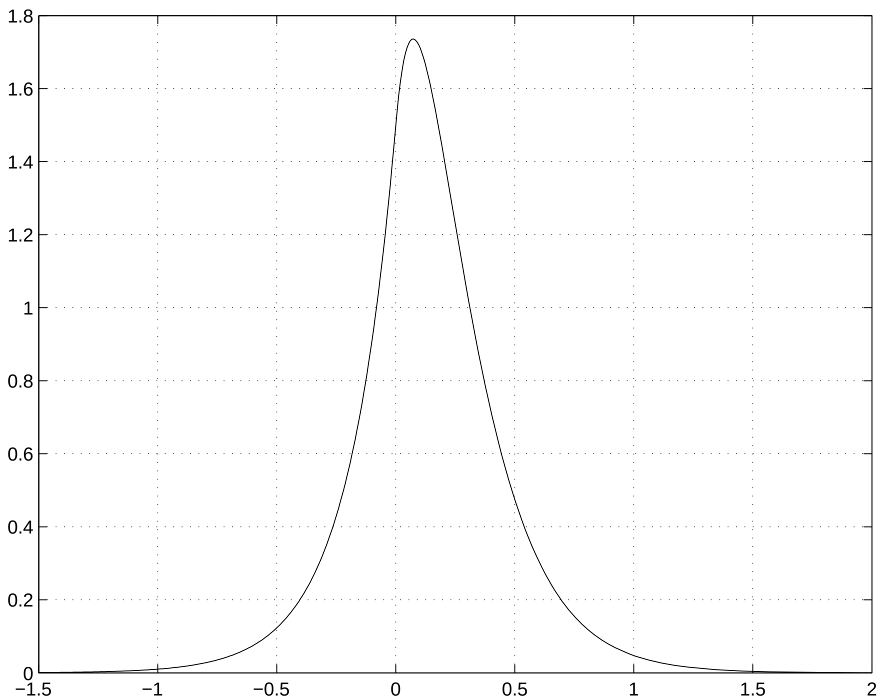

Figure 1: Density (B.1). Parameters: $\lambda^{-} = -5, \lambda_1^{+} = 5, \lambda_2^{+} = 7.5$