Pricing bonds in an incomplete market: Linear and Dynamic Programing approach

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Abstract

We consider a finite horizon discrete time model for bond market where bond prices are functions of the short rate process. We use a variant of the Ito's formula to decompose the bond price process into unique drift and martingale processes. We then apply the Girsanov's Theorem for finding a change of measure under which the discounted bond price processes are martingales, thereby implying the existence of an arbitrage-free bond market. We next show that under a particular martingale measure given by a specific form of the Radon Nikodym derivative, the bond price process of exponentially quadratic form reduces to the well known exponentially linear form. We further prove that the bond market is incomplete and and the set of martingale measures is not a singleton. The analytical formulation of all martingale measures is difficult to obtain. A finite discretization of the state space of the rate process and subsequent solution of a set of martingale equations generates the set of all martingale measures in an incomplete bond market. A suitable cost function is then minimized to obtain a particular martingale measure. Linear programming and Dynamic programming approaches for solving the minimization problem are discussed. Assuming compactness of the bond price process, we further prove the convergence of the optimal solution of the discretized problem to the optimal solution of the original problem.

1 Introduction

Interest rate modeling lays the foundation for pricing, hedging and trading the whole class of interest rate contingent securities from bonds to fixed income derivatives. The fundamental question is to model the law of the rate process under the risk neutral measure. The theory of bond pricing assumes the bond prices to be functions of the short rate process, and the affine class turns out to be the most commonly used one, primarily due to their easy analytical solutions. The momentum gained from its popularity and extensive research led analysts to migrate to the next obvious function on the hierarchy, namely the quadratic class. An explicit method of pricing for bond options for the quadratic class is given by Jamshidian [1]. Recently, Leippold and Wu [2] have constructed a quadratic class of bond prices and have shown their

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existence. First, these works deal with the continuous time formulation. Furthermore, in these recent works the bond prices are still considered exponentially quadratic function of a state vector, not of short rate. An introductory work on discrete-time affine bond pricing was presented by Backus, Foresi and Telmer [3]. However, the question of "Completeness" of the bond market was not addressed to while giving a pricing methodology. We formulate a version in which we discuss the nonlinearity in the bond prices by making the prices to be exponentially quadratic function of the rate process. In this paper we show the existence of an arbitrage-free market for the quadratic model and conclude that such markets are incomplete in most cases. Nevertheless, a trader can hedge any final payoff in the mean square deviation sense as opposed to hedge the payoff faithfully.

2 Discrete-time Itô's formula for decomposing the bond price process

Consider the interest rate model given by:

$$r_n - r_{n-1} = \mu(r_{n-1}) + \sigma(r_{n-1})\epsilon_n \tag{1}$$

where μ and σ are bounded polynomial functions of the short rate process and $\epsilon = (\epsilon_1, \epsilon_2, ...)$ is a sequence of independent standard normally distributed random variables, $\epsilon_n \sim N(0,1)$. We assume that the value of a bond which matures at time N is a function of the short rate. Let the price of a bond at time n be $F_n = F(r_n, n, N) := F(r_n)$ with the terminal condition $F_N = F(r_N, N, N) = 1$. Let $T(r_n)$ ($:= T_n$) be the discounted price of a bond at time n. We would like to discuss the properties of $T(r_p)$ where $T : R \to R$ be a function of class C^∞. We assume without loss of generality that T has a compact support, and so T, DT, D^2T are bounded. For $n > 0$, the discrete Itô's formula yields:

$$\begin{aligned}
&T(r_n)\\
&= T(r_0) + \sum_{p=1}^{n}[DT(r_{p-1})(r_p - r_{p-1})] + \sum_{p=1}^{n}[\frac{D^2}{2}T(r_{\eta_p})(r_p - r_{p-1})^2]\\
&= T(r_0) + \sum_{p=1}^{n}[DT(r_{p-1})(r_p - r_{p-1})] + \sum_{p=1}^{n}[\frac{D^2}{2}T(r_{p-1})(r_p - r_{p-1})^2]\\
&+ \sum_{p=1}^{n}[\frac{D^2}{2}[T(r_{\eta_p}) - T(r_{p-1})]](r_p - r_{p-1})^2
\end{aligned} \tag{2}$$

where $r_{\eta_p}(w) = r_{p-1}(w) + \theta_p(w)(r_p(w) - r_{p-1}(w))$ for some $\theta_p(w)$ satisfying $0 \le \theta_p < 1$, $w \in \Omega$. (In the continuous time Itô's formula, the last term in (2) becomes zero due to the continuity of the function T. However, in the discrete-time situation, the Itô's formula has a modification where the last term does not vanish.) Now,

$$\begin{aligned} T(r_n) &= T(r_0) + \sum_{p=1}^{n}[DT(r_{p-1})(\mu(r_{p-1}) + \sigma(r_{p-1}))] \\ &+ \sum_{p=1}^{n}[\frac{D^2}{2}T(r_{p-1} + \theta_p(r_p - r_{p-1}))(r_p - r_{p-1})^2] \\ &= T(r_0) + \sum_{p=1}^{n} DT(r_{p-1})\mu(r_{p-1}) + \sum_{p=1}^{n} DT(r_{p-1})\sigma(r_{p-1})\epsilon_p \\ &+ \sum_{p=1}^{n} \frac{D^2}{2}T(r_{p-1} + \theta_p(r_p - r_{p-1}))(r_p - r_{p-1})^2 \\ &=: T_I + T_{II} + T_{III} + T_{IV}. \end{aligned} \tag{3}$$

2.1 Decomposing the terms of the bond price process

Note that we have three types of terms - predictable drift term $(T_I + T_{II})$, a martingale (T_{III}) and an adapted term (T_{IV}). Since, T_{IV} has finite absolute first moments, *i.e.*, $E[|T_{IV}|] < \infty$ for $n \leq N$, we have by *Doob decomposition*,

$$\begin{aligned} T_{IV} &= A_n^1 \text{ (predictable term) } + M_n^1 \text{ (martingale term)} \\ \text{Also, } T_I + T_{II} &= A_n^2 \text{ (predictable term)}, T_{III} = M_n^2 \text{ (martingale)} \\ \text{Thus, } T_n = T(r_n) &= A_n^1 + M_n^1 + A_n^2 + M_n^2 \\ \text{Define, } A_n^1 + A_n^2 &:= A_n, \ \ \& \ \ M_n^1 + M_n^2 := M_n \\ \text{Thus, } T_n = T(r_n) &= A_n + M_n. \end{aligned}$$

2.2 The condition for no-arbitrage in the bond market

Let $\tilde{P}$ be locally absolutely continuous with P. Then there exists Radon-Nikodym derivative (density processes) Z_n with $E(\alpha_p|M_p - M_{p-1}||\Im_{p-1}) < \infty$, $\forall p, \alpha_n = \frac{Z_n}{Z_{n-1}} I_{\{Z_{n-1}>0\}}$ and $|M_n| < \infty$, $\forall n$ such that the sequence $\tilde{T}_n$ has a representation $\tilde{T}_n = \tilde{A}_n + \tilde{M}_n$ where

$$\tilde{A}_n = A_n + \sum_{p=1}^{n} E(\alpha_p \Delta M_p|\Im_{p-1}); \ \ \tilde{M}_n = M_n - \sum_{p=1}^{n} E(\alpha_p \Delta M_p|\Im_{p-1})$$

are the drift and martingale terms with respect to the new measure $\tilde{P}$. The no-arbitrage condition demands the discounted price processes to be martingales. This implies that under a change of measure the drift term should vanish and we are left with only the martingale term:

$$\tilde{A}_n = A_n + \sum_{p=1}^{n} E(\alpha_p \Delta M_p|\Im_{p-1}) = 0.$$

3 Exponentially quadratic bond price process

We consider that the price of a bond at time n, maturing at time $N(< \infty)$ is a function of the short rate and given as

$$F(r_n, n, N) = F(r_n) = \exp(-a_{n'} - b_{n'} r_n - c_{n'} r_n^2) \tag{4}$$

where $n' = N - n$. $F(r_N, N, N) = 1$ implies that $a_0 = b_0 = c_0 = 0$. We name this class of bond pricing models as Exponentially Quadratic class. Note that we let interest rates assume a continuous range of real values.

In an arbitrge-free bond market the conditional expectation (under a martingale measure) of the discounted value of the future bond price is equal to the current value of the bond:

$$e^{\sum_{k=0}^{n-1} -r_k} F(r_{n-1}) = \tilde{E}[e^{\sum_{k=0}^{n} -r_k} F(r_n)|\Im_{n-1}] \tag{5}$$

$$F(r_{n-1}) = E[\frac{Z_n}{Z_{n-1}} \exp(-r_n) F(r_n)|\Im_{n-1}]. \tag{6}$$

We are in search of a process Z_n such that (6) holds.

We guess $Z_n = \exp[-\sum_{k=1}^{n}(d_k\epsilon_k + \frac{d_k^2}{2})]$ where d_k is $\Im_{k-1}$ measurable. The variable d_k is also known as the market price of risk. This form of Z_n is motivated from the discrete version of the Girsanov's theorem. We can now show that Z_n is a valid density process; we do not present details here.

3.1 Pricing bonds in the arbitrage-free situation

We denote our interest rate model to have a form given by:

$$r_n - r_{n-1} = \mu(r_{n-1}) + \sigma(r_{n-1})\epsilon_n. \tag{7}$$

For pricing zero coupon bonds in an arbitrage-free situation we compute the values of $a_{n'}, b_{n'}$ and $c_{n'}$ such that the bond prices follow (6). Thus,

$$\begin{aligned}
&\tilde{E}[F(r_n)|\Im_{n-1}] \\
&= E[e^{-\frac{d_n^2}{2} - a_{n'} - b_{n'}\mu(r_{n-1}) - b_{n'}r_{n-1} - c_{n'}\mu(r_{n-1})^2 - c_{n'}r_{n-1}^2 - 2\mu(r_{n-1})r_{n-1}} \\
&e^{[-b_{n'}\sigma(r_{n-1}) - 2\mu(r_{n-1})\sigma(r_{n-1})c_{n'} - 2c_{n'}\sigma(r_{n-1})r_{n-1} - d_n]\epsilon_n + \sigma(r_{n-1})^2\epsilon_n^2}|\Im_{n-1}] \\
&= e^{r_{n-1}} F(r_{n-1})
\end{aligned} \tag{8}$$

We denote

$$- a_{n'} - b_{n'}\mu(r_{n-1}) - c_{n'}\mu(r_{n-1})^2 - c_{n'}r_{n-1}^2 - (2\mu(r_{n-1}) + b_{n'})r_{n-1} := K_n$$
$$- b_{n'}\sigma(r_{n-1}) - 2\mu(r_{n-1})\sigma(r_{n-1})c_{n'} - 2c_{n'}\sigma(r_{n-1})r_{n-1} := L_n$$

Further by using $E[\exp(a\epsilon + b\epsilon^2)] = \exp(\frac{a^2}{2(1-2b)})(1-2b)^{-1/2}$ for (8) we arrive at our Pricing Equation under the Arbitrage-free condition:

$$\begin{aligned}
-2(1 + 2c_{n'}\sigma(r_{n-1})^2)\ln((1 + 2c_{n'}\sigma(r_{n-1})^2)^{1/2}F(r_{n-1})) \\
-2c_{n'}\sigma(r_{n-1})^2 d_n^2 - 2L_n d_n + L_n^2 + 2K_n + 4K_n c_{n'}\sigma(r_{n-1})^2 = 0
\end{aligned} \tag{9}$$

3.2 Computing the coefficients of the price processes with Vasicek's interest rate model

For the Vasicek's model we have: $r_n - r_{n-1} = k_1 + (k_2 - 1)r_{n-1} + \sigma\epsilon_n$

where k_1, σ and k_2 are constants. The Pricing Equation (9) gives us the following recursions:

$$\begin{aligned}
c_{n'-1} &= \frac{2c_{n'}k_2^2}{2(1+2c_{n'}\sigma^2)} \\
b_{n'-1} &= \frac{2b_{n'}k_2 + 4c_{n'}k_1k_2 + 2(1+2c_{n'}\sigma^2) - 4d_nk_2\sigma c_{n'}}{2(1+2c_{n'}\sigma^2)} \\
a_{n'-1} &= \frac{J_{n'}}{2(1+2c_{n'}\sigma^2)}
\end{aligned} \tag{10}$$

$$\begin{aligned}
\text{where } J_{n'} = &-2c_{n'}\sigma^2 d_n^2 + 2d_nb_{n'}\sigma + 4d_n\sigma k_1c_{n'} + b_{n'}^2\sigma^2 - 2a_{n'} \\
&- 2b_{n'}k_1 - 2c_{n'}k_1^2 - 4c_{n'}\sigma^2 a_{n'} - (1+2c_{n'}\sigma^2)\ln(1+2c_{n'}\sigma^2)
\end{aligned}$$

with $a_0 = b_0 = c_0 = 0$.

The co-efficients $a_{n'}, b_{n'}$ & $c_{n'}$ computed recursively in the above manner will thus lead to an Arbitrage-free bond market.

Estimation of the parameters of the Vasicek's model

The data are the monthly estimates of annualized continuously-compounded zero coupon US government bond yields computed by Mc Culloch and Kwon (1993), January 1952 to February 1991 (470 observations). We choose the parameter values to approximate some of the salient features of the bond yields. We consider our *estimation horizon* to be evenly distributed with monthly observations from January 1952 to December 1981. By considering the modeling period as a month we actually consider the continuously-compounded zero coupon yield of a bond with 1 month maturity as the prevailing short rate.

Properties of US Government 1-month maturity bond yields:

Mean	St. Dev	Skewness	Kurtosis	Autocorrelation
4.65081	3.06	1.434	2.239	0.976

$\frac{k_1}{1-k_2}$ is the unconditional mean of the short rate process and we set it equal to the sample mean of the one-month yield as given in the table above. The factor 1200 converts an annual percentage rate to a monthly rate.

$$\frac{k_1}{1-k_2} = \frac{4.65081}{1200} = 0.003876$$

The mean reversion parameter k_2 is the first autocorrelation of the short rate and hence $k_2 = 0.976$. Thus, $k_1 = 0.000093024$. We choose the volatility parameter σ to equate the unconditional variance of the short rate process:

$$\frac{\sigma^2}{1-k_2^2} = (\frac{3.06}{1200})^2.$$

With $k_2 = 0.976$, the implied value of $\sigma = 0.0005553$.

3.3 Computing coefficients for Cox-Ingersoll-Ross's interest model

We take the CIR model given by,

$$r_n = (1-\phi)\theta + \phi r_{n-1} + \sigma\sqrt{r_{n-1}}\epsilon_n.$$

An important point to make at this moment is that the market price of risk d_n in this case is taken to be proportional to $\sqrt{r_{n-1}}$. It is evident that a simple additional adjustment to the form of the market price of risk helps us to get affine bond prices. Now considering $d_n = \lambda_n\sqrt{r_{n-1}}$, for an arbitrage-free market, we compute the co-efficients recursively:

$$a_{n'-1} = a_{n'} + b_{n'}(1-\phi)\theta;\ b_{n'-1} = 1 + b_{n'}\phi - \frac{b_{n'}^2\sigma^2}{2} - 2b_{n'}\sigma\lambda_n \tag{11}$$

with $a_0 = b_0 = c_{n'} = 0$.

We thus see that the bond prices under the CIR model also reduces to the affine model. However, it would even have not retained the affine structure, had we not introduced the dependence of d_n on r_n. Thus, there is a strong relation between the market price of risk and the instantaneous rate process for various term structure models.

Estimation of the parameters of the CIR model

We set the autocorrelation parameter ϕ equal to the autocorrelation of the short rate: $\phi = 0.976$. We set θ equal to the mean short rate, which implies $\theta = 0.003876$. We choose σ to reproduce the variance of the short rate:

$$\frac{\theta\sigma^2}{1-\phi^2} = (\frac{3.06}{1200})^2,$$

which implies $\sigma = 0.00627$.

3.4 Reduction to the linear structure from the quadratic model

We observe that the terminal condition $c_0 = 0$ makes all subsequent values of $c_{n'} = 0$ thereby reducing the exponentially quadratic model to the well known exponentially linear structure. This leads to the following two propositions.

Proposition 1 *Assume that the price of a bond is a function of the short rate process having the Vasicek's structure. Let the price of a zero coupon bond $F(r_n, n, N)$, having maturity date N at time n ($\leq N < \infty$) be given by,*

$$F(r_n, n, N) = F(r_n) = \exp(-a_{n'} - b_{n'}r_n - c_{n'}r_n^2), \quad \textit{where } n' = N - n.$$

Then, under a martingale measure given by $Z_n = \exp[\sum_{p=1}^{n}(-\frac{d_p^2}{2} - \epsilon_p d_p)]$, we have $c_{n'} \equiv 0$, $\forall n'$.

Proposition 2 *Assume that the price of a bond is a function of the short rate process where the "drift" and "diffusion" terms of the short rate process are polynomial functions of the short rate itself. Let the price of a zero coupon bond $F(r_n, n, N)$, having maturity date N at time n ($\leq N < \infty$) be given by,*

$$F(r_n, n, N) = F(r_n) = \exp(-a_{n'} - b_{n'}r_n - c_{n'}r_n^2), \quad \textit{where } n' = N - n.$$

Then, under a martingale measure given by $Z_n = \exp[\sum_{p=1}^{n}(-\frac{d_p^2}{2} - \epsilon_p d_p)]$, we have $c_{n'} \equiv 0$, $\forall n'$.

4 Incompleteness of the Bond Market

We present here a proof for the incompleteness of our discrete-time bond market model. This suggests that the set of martingale measures is not a singleton. It further motivates the use of Linear and Dynamic programming for finding the Minimum Variance Martingale Measure.

4.1 Proof for Incompleteness

To show that the proposed discrete-time model is incomplete, we will prove that it is impossible to find a replicating portfolio for an $\Im_N$-measurable contingent claim (An European Option in this case)[1].

Consider a (B, F)-market formed by a bank account B and a bond F. We first construct a portfolio with the *Value process* V and a *strategy* $\pi = (\beta, \gamma)$. Observe, that $\beta = (\beta_n)_{n\geq 0}$ and $\gamma = (\gamma_n)_{n\geq 0}$ are predictable processes (By definition).

The value process is given by,

$$V_n = \beta_n B_n + \gamma_n F_n$$

Further, for a *self-financed* portfolio we have,

$$\beta_n = \frac{V_{n-1} - \gamma_n F_{n-1}}{B_{n-1}}$$

Thus, we have,

$$V_n = (V_{n-1} - \gamma_n F_{n-1})\frac{B_n}{B_{n-1}} + \gamma_n F_n$$

so that this holds for both (ω_{n-1}, ω_n) and $(\omega_{n-1}, \hat{\omega_n})$.

$$V_n(\omega_{n-1}, \omega_n) = (V_{n-1} - \gamma_n F_{n-1})(\omega_{n-1})\frac{B_n}{B_{n-1}} + \gamma_n(\omega_{n-1})F_n(\omega_{n-1}, \omega_n).$$

$$V_n(\omega_{n-1}, \hat{\omega_n}) = (V_{n-1} - \gamma_n F_{n-1})(\omega_{n-1})\frac{B_n}{B_{n-1}} + \gamma_n(\omega_{n-1})F_n(\omega_{n-1}, \hat{\omega_n}).$$

At each ω_{n-1},

$$\gamma_n(\omega_{n-1}) = \frac{V_n(\omega_{n-1}, \omega_n) - V_n(\omega_{n-1}, \hat{\omega_n})}{F_n(\omega_{n-1}, \omega_n) - F_n(\omega_{n-1}, \hat{\omega_n})}$$

Recall that for the Europen call option, a replicating portfolio satisfies $V_N = (F_N - K)^+$, and under this constraint the portfolio will not satisfy the previous equation for $n = N$.

In particular, $\gamma_N(\omega_{N-1}) = 0$ since for some (ω_{n-1}, ω_n),

$$F_{N-1}e^{r_N(\omega_N)} - K \leq 0, F_{N-1}e^{r_N(\hat{\omega_N})} - K \leq 0,$$

but also $\gamma_N(\omega_{N-1}) = 1$ since for some (ω_{n-1}, ω_n)

$$F_{N-1}e^{r_N(\omega_N)} - K > 0, F_{N-1}e^{r_N(\hat{\omega_N})} - K > 0,$$

with the impossibility that $0 = \gamma_N(\omega_{N-1}) = 1$. Thus, for the discrete-time exponentially quadratic model has no replicating portfolio.

[1] The proof is a bit unorthodox in nature since we prove Incompleteness by showing the impossibility of constructing a replicating portfolio. An orthodox way of showing incompleteness would be to show the analytical formulation of a martingale measure other than the Esscher Transformation that satisfies our pricing methodology. We however, were unable to analytically construct one.

4.2 The EMM set

It is well known that in the case of a complete bond market situation the set of martingale measures is a singleton. We will restrict our attention to martingale measures such that the Radon-Nikodym derivative ($Z_n = \frac{dQ}{dP}|\Im_n, 0 \leq n \leq N$) is a P-square integrable martingale, *i.e.*, $\sup_{n\in[0,N]} E[Z_n^2] < \infty$. Recall that we have assumed that the price process $F(r_n)$ follow $F(r_n) = e^{x_n+y_n\epsilon_n+g_n\epsilon_n^2}, \forall n \leq N$. Let M be the set of all martingale measures equivalent to the original subjective measure. We next find one such measure given by the *Esscher* transformation.

Proposition 3 *The set Ψ of equivalent martingale measures given by the Esscher transform is the set of probability measures Q^d such that $\frac{dQ^d}{dP}|\Im_n = Z_n^d$ where*

$$Z_n = \exp(\sum_{p=1}^{n}[-\frac{a_p^2 y_p^2}{2(1-2a_p g_p)} + a_p y_p \epsilon_p + a_p g_p \epsilon_p^2)] \prod_{p=1}^{n} \sqrt{1-2a_p g_p}.$$

We thus point out that the martingale measure $\tilde{P}$, is not unique in general (the measure corresponding to the Esscher transformation is just one of the many possible martingale measures). Further, the different possible solutions of a_n's which satisfy (3.2) for a given set of $x_n s$, $y_n s$ and $g_n s$. We thus explore other conditions [8] which would help us to next select a unique pricing strategy for the bond issuer.

Definition 1 *Let Q be an equivalent martingale measure with respect to the subjective measure P. The* minimal variance criteria *for an EMM is defined by:*

$$V(Q^*, P) = \inf_{Q\in\Psi} V(Q,P) = \inf_{Q\in\Psi} Var_Q[e^{\sum_{k=0}^{N} -r_k} - F_0] \tag{12}$$

where Q^* is the minimum variance Martingale Measure (MVMM).

We have,

$$V(Q,P) = Var_Q[e^{\sum_{k=0}^{N} -r_k} - F_0] = E_Q[(e^{\sum_{k=0}^{N} -r_k} - F_0)^2] + (E_Q[e^{\sum_{k=0}^{N} -r_k} - F_0])^2 \tag{13}$$

$$= E_Q[(e^{\sum_{k=0}^{N} -r_k} - F_0)^2] \tag{14}$$

Remark 1 *Note that the cost function is linear in Z since $E_Q[(e^{\sum_{k=0}^{N} -r_k} - F_0)^2] = E[Z_N(e^{\sum_{k=0}^{N} -r_k} - F_0)^2]$. In fact $E_Q[f(e^{\sum_{k=0}^{N} -r_k})]$ is linear in Z for any well behaved f since $E_Q[f(e^{\sum_{k=0}^{N} -r_k})] = E[Z_N f(e^{\sum_{k=0}^{N} -r_k})]$. We will discuss the solution of such cost function through the LP and DP method in later sections.*

5 Construction of Radon-Nikodym derivatives

The construction of a martingale measure based on the *conditional Esscher transformation* gives us only *one* particular measure, although the class of martingale measures equivalent to the subjective measure P can be more rich. We discuss the construction of martingale measures in the following section. We first characterize the density process $Z = (Z_n)_{n\geq 1}$. Let $Q <<^{loc} P$. The density process $Z_n = \frac{d\tilde{P}_n}{dP_n}$ is then a non-negative $(P, (\Im_n))$-martingale with $E[Z_n] = 1$. We would next like to characterize the set of densities such that the discounted price processes are martingales. For $E[|e^{\sum_{k=1}^{p}-r_k}F(r_p)|] < \infty, \forall p, 0 \leq p \leq N$, the no-arbitrage condition gives:

$$E_Q[e^{\sum_{k=0}^{n}-r_k}F(r_n)|\Im_{n-1}] = e^{\sum_{k=0}^{n-1}-r_k}F(r_{n-1}), \tag{15}$$

$$E_Q[e^{-r_n}F(r_n)|\Im_{n-1}] = F(r_{n-1}),$$

$$E[\frac{Z_n}{Z_{n-1}}e^{-r_n}F(r_n)|\Im_{n-1}] = F(r_{n-1}),$$

$$\int_A Z_n e^{-r_n}F(r_n)dP = \int_A Z_{n-1}F(r_{n-1})dP, \text{ for each } A \in \Im_{n-1}. \tag{16}$$

The set of martingale measures is thus characterized by the non-negative density processes that are P-martingales with expectation *one* and satisfying (16). However, it is difficult to guess a functional form for the density process from the above conditions.

Consider, a process $h = (h_n)_{n\geq 1} = e^{\sum_{k=0}^{n}-r_k}F(r_n)$ such that $E[|h_n|] < \infty, \forall n, 1 \leq n \leq N$. For h to be a martingale under a measure $Q \sim P$ it should satisfy:

$$E[Z_n h_n|\Im_{n-1}] = Z_{n-1}h_{n-1}, \tag{17}$$

$$\int_A Z_n h_n dP = \int_A Z_{n-1}h_{n-1}dP, \text{ for each } A \in \Im_{n-1}, \tag{18}$$

$$\int_A (Z_n h_n - Z_{n-1}h_{n-1})dP = 0, \text{ for each } A \in \Im_{n-1}, \tag{19}$$

$$\int_A Z_{n-1}(\frac{Z_n}{Z_{n-1}}h_n - h_{n-1})dP = 0, \text{ for each } A \in \Im_{n-1} \tag{20}$$

5.1 The finite state space model

We discretize the sample space of possible values of the short rate process r. We have assumed the discounted price process h to be a function of the process r. In practice, we have to assume that r has a finite/countable state space as against our earlier assumption of continuous state space. Let $\Omega = \{w^1, w^2, w^3, ..., w^m\}$, thereby restricting our state space to cardinality of m. Thus, (17) translates to:

$$E[Z_n h_n|\Im_{n-1}] = Z_{n-1}h_{n-1}, \tag{21}$$

$$\sum_{k=1}^{m} Z_n^k h_n^k P(h_n = h_n^k|h_1, h_2, ..., h_{n-1}) = Z_{n-1}h_{n-1}. \tag{22}$$

For $n = 1$,

$$\sum_{k=1}^{m} Z_1^k h_1^k P(h_1 = h_n^k|h_0) = Z_0 h_0, \tag{23}$$

$$\sum_{k=1}^{m} Z_1^k h_1^k P(h_1 = h_n^k) = Z_0 h_0. \tag{24}$$

Also note that we have the normalizing condition, $E[Z_1] = 1$:

$$\sum_{k=1}^{m} Z_1^k P(Z_1 = Z_1^k) = 1 \tag{25}$$

and the property that Z_n is a P-martingale:

$$E[Z_n|\Im_{n-1}] = Z_{n-1}. \tag{26}$$

Note that (24), (25) and (26) are the governing equations from which the density process at $n = 1$ has to calculated. We now show the calculations for $n = 2$.

$$\text{For } n = 2, \qquad \sum_{k=1}^{m} Z_2^k h_2^k P(h_2 = h_n^k | h_1) = Z_1 h_1, \tag{27}$$

$$\text{with the additional constrains, } \sum_{k=1}^{m} Z_1^k P(Z_2 = Z_2^k) = 1 \tag{28}$$

$$\text{and } E[Z_2|\Im_1] = Z_1, \tag{29}$$

$$E[Z_2] = 1. \tag{30}$$

5.2 The two states case, m = 2

We define the two states of the process r as r^1 and r^2 and h^1 and h^2 as the respective discounted prices. We associate probabilities $P(h_n = h_n^1) = p$ and $P(h_n = h_n^2) = q$.

$$\text{For } n = 1, \qquad \sum_{k=1}^{2} Z_1^k h_1^k P(h_1 = h_n^k | h_0) = h_0, \tag{31}$$

$$Z_1^1 h_1^1 P(h_1 = h_n^1) + Z_1^2 h_1^2 P(h_1 = h_n^2) = h_0, \tag{32}$$

with the additional constraint,

$$Z_1^1 P(Z_1 = Z_1^1) + Z_1^2 P(Z_1 = Z_1^2) = 1. \tag{33}$$

Assume $h_1^1 > h_0 > h_1^2 > -1$. From (32) and (33) we get:

$$Z_1^1 = \frac{h_0 - h_1^2}{h_1^1 - h_1^2} \frac{1}{P(h_1 = h_n^1)} \tag{34}$$

$$Z_1^2 = \frac{h_1^1 - h_0}{h_1^1 - h_1^2} \frac{1}{P(h_1 = h_n^2)}. \tag{35}$$

Observe finally that the values are **unique** and hence the martingale measure given by (34) & (35) is the only one. The above model is an extension of the CRR model associated with stocks.

5.3 The Incomplete Bond Market

We observe that equations (21) and (26) are the only equations that are to be satisfied for a price process h_n to be a martingale. Further, it suggests that for any model which has more than two possible states for the price process at any time t is *incomplete*, since the values of corresponding Radon Nikodym derivatives are not *unique* as in the CRR model.

Since, the choice of martingale measure is not unique and any martingale measure doesn't ensure faithful replication of final payoff, from now onwards, we mean square deviation as criteria to choose a martingale measure from the set of measures available.

We thus are interested in finding the value of Z which minimizes

$$E_Q[(e^{\sum_{k=0}^{N} -r_k} - F_0)^2] = E[Z_N(h_N - h_0)^2].$$

5.4 The m state model

$$\begin{aligned}
\text{minimize} \quad & J_{N,\pi}(Z_0) = \sum_{k=0}^{N-1} \{E[Z_{k+1}(h_{k+1}-h_0)^2 - Z_k(h_k-h_0)^2]\} && (36)\\
\text{subject to} \quad & E[Z_n h_n|\Im_{n-1}] = Z_{n-1}h_{n-1},\ \forall n, 0 \le n \le N, && (37)\\
& E[Z_n|\Im_{n-1}] = Z_{n-1},\ \forall n, 0 \le n \le N, && (38)\\
& E[Z_n] = 1,\ \forall n, 0 \le n \le N, && (39)\\
& Z_n > 0,\ \forall n, 0 \le n \le N. && (40)
\end{aligned}$$

Let the number of states be m and let p_i be the probability of occurrence of i^{th} state. The problem then simplifies to:

For n = 1, we have

$$\begin{aligned}
\text{minimize} \quad & J_{1,\pi}(Z_0) = \sum_{i=1}^{m} p_i Z_1^i (h_1^i - h_0)^2 && (41)\\
\text{subject to} \quad & \sum_{i=1}^{m} p_i Z_1^i h_1^i = h_0, && (42)\\
& \sum_{i=1}^{m} p_i Z_1^i = Z_0 = 1, && (43)\\
& Z_1^i > 0, \forall i, i = 1, .., m. && (44)
\end{aligned}$$

For n = 2, we have

$$\text{minimize} \quad J_{2,\pi}(Z_0) = \{\sum_{j=1}^{m}\sum_{i=1}^{m}[p_i p_j Z_2^{ij}(h_2^j - h_0)^2] - \sum_{i=1}^{m} p_i Z_1^i (h_1^i - h_0)^2\} + \sum_{i=1}^{m} p_i Z_1^i (h_1^i - h_0)^2 \qquad (45)$$

$$\begin{aligned}
\text{subject to} \quad & \sum_{j=1}^{m} p_i Z_2^{ij} h_2^j = Z_1^i h_1^i, \forall i, i = 1, 2, ...m, && (46)\\
& \sum_{j=1}^{m} p_j Z_2^{ij} = Z_1^i, \forall i, i = 1, 2, ...m, && (47)\\
& \sum_{i=1}^{m} p_j Z_1^i h^i = h_0, && (48)\\
& \sum_{i=1}^{m} p_j Z_1^i = Z_0 = 1, && (49)\\
& \sum_{j=1}^{m}\sum_{i=1}^{m} p_i p_j Z_2^{ij} = 1, && (50)\\
& Z_1^i > 0, \forall i, i = 1, .., m \ \ \& Z_2^{ij} > 0, \forall i, \forall j, i = 1, .., m, j = 1, .., m. && (51)
\end{aligned}$$

Observe that the (50) is a linear combination of (47) .

For n = t, we have

$$\text{minimize} \quad J_{t,\pi}(Z_0) = \sum_{n=1}^{t}\{(\prod_{k=1}^{n}\sum_{i^k=1}^{m})[(\prod_{k=1}^{n}p_{i^k})Z_n^{i^1i^2\ldots i^n}(h_n^{i^n}-h_0)^2] - (\prod_{k=1}^{n-1}\sum_{i^k=1}^{m})[(\prod_{k=1}^{n-1}p_{i^k})Z_{n-1}^{i^1i^2\ldots i^{n-1}}(h_{n-1}^{i^{n-1}}-h_0)^2]\} \tag{52}$$

$$\text{subject to} \quad \sum_{i^k=1}^{m} p_{i^k}Z_k^{i^1i^2\ldots i^k}h_k^{i^k} = Z_{k-1}^{i^1i^2\ldots i^{k-1}}h_{k-1}^{i^{k-1}}, \forall i^k, i^k = 1,2\ldots., m, \forall k, k = 1,2\ldots, t, \tag{53}$$

$$\sum_{i^k=1}^{m} p_{i^k}Z_k^{i^1i^2\ldots i^k} = Z_{k-1}^{i^1i^2\ldots i^{k-1}}, \forall i^k, i^k = 1,2\ldots., m, \forall k, k = 1,2\ldots, t, \tag{54}$$

$$(\prod_{k=1}^{n}\sum_{i^k=1}^{m})[(\prod_{k=1}^{n}p_{i^k})Z_n^{i^1i^2\ldots i^n}] = 1, \forall n, n = 1,2,.., t, \tag{55}$$

$$Z_k^{i^1i^2\ldots i^k} > 0, \forall i^k, i^k = 1,2\ldots., m, \forall k, k = 1,2\ldots, t. \tag{56}$$

Observe that (55) is a linear combination of (54) .

6 LP formulation

For n = t, we have

$$\text{minimize} \quad J_{t,\pi}(Z_0) = \sum_{n=1}^{t}\{(\prod_{k=1}^{n}\sum_{i^k=1}^{m})[(\prod_{k=1}^{n}p_{i^k})Z_n^{i^1i^2\ldots i^n}(h_n^{i^n}-h_0)^2] - (\prod_{k=1}^{n-1}\sum_{i^k=1}^{m})[(\prod_{k=1}^{n-1}p_{i^k})Z_{n-1}^{i^1i^2\ldots i^{n-1}}(h_{n-1}^{i^{n-1}}-h_0)^2]\} \tag{57}$$

$$\text{subject to} \quad \sum_{i^k=1}^{m} p_{i^k}Z_k^{i^1i^2\ldots i^k}h_k^{i^k} = Z_{k-1}^{i^1i^2\ldots i^{k-1}}h_{k-1}^{i^{k-1}}, \forall i^k, i^k = 1,2\ldots., m, \forall k, k = 1,2\ldots, t, \tag{58}$$

$$\sum_{i^k=1}^{m} p_{i^k}Z_k^{i^1i^2\ldots i^k} = Z_{k-1}^{i^1i^2\ldots i^{k-1}}, \forall i^k, i^k = 1,2\ldots., m, \forall k, k = 1,2\ldots, t, \tag{59}$$

$$(\prod_{k=1}^{n}\sum_{i^k=1}^{m})[(\prod_{k=1}^{n}p_{i^k})Z_n^{i^1i^2\ldots i^n}] = 1, \forall n, n = 1,2,.., t, \tag{60}$$

$$Z_k^{i^1i^2\ldots i^k} \geq \delta, \forall i^k, i^k = 1,2\ldots., m, \forall k, k = 1,2\ldots, t. \tag{61}$$

For a given δ, the problem can be solved by Linear Programming. However, the number of variables to be found at time t is very large ($= \frac{m(m^t-1)}{m-1}$) and hence the method becomes computationally tedious. Observe that (60) is a linear combination of (59) $i^k = 1,2\ldots., m, \forall k, k = 1,2\ldots, t-1$.

7 DP formulation

We explore the dynamic programming approach to find the *minimal variance martingale measure* in a finite horizon market model. The *stochastic optimal control problem* is characterized by:

1. An underlying *discrete-time dynamic system*, and
2. a *cost function* that is additive over time.

The stochastic optimal control problem can be written as:

$$\begin{aligned}
\text{minimize} \qquad & J_{N,\pi}(Z_0) = \sum_{k=0}^{N-1} \{E[Z_{k+1}(h_{k+1}-h_0)^2 - Z_k(h_k-h_0)^2]\} \\
\text{subject to} \qquad & E[Z_n h_n | \Im_{n-1}] = Z_{n-1}h_{n-1},\ \forall n, 0 \le n \le N \\
& E[Z_n | \Im_{n-1}] = Z_{n-1},\ \forall n, 0 \le n \le N \\
& E[Z_n] = 1,\ \forall n, 0 \le n \le N \\
& Z_n > 0,\ \forall n, 0 \le n \le N.
\end{aligned}$$

We first analyze the case where the discounted price process h can assume three values (3 state space problem). Let $p, q \& r$ be the respective probabilities of associated with the three values. The control problem can be written as:

For n = 1, we have

$$\begin{aligned}
\text{minimize} \qquad & J_{1,\pi}(Z_0) = pZ_1^1(h_1^1-h_0)^2 + qZ_1^2(h_1^2-h_0)^2 + rZ_1^3(h_1^3-h_0)^2 \\
\text{subject to} \qquad & pZ_1^1h^1 + qZ_1^2h^2 + rZ_1^3h^3 = h_0 \\
& pZ_1^1 + qZ_1^2 + rZ_1^3 = 1
\end{aligned}$$

For n = 2, we have

$$\begin{aligned}
\text{minimize} \qquad J_{2,\pi}(Z_0) = & [ppZ_2^{11}(h_2^1-h_0)^2 - Z_1^1(h_1^1-h_0)^2] + [pqZ_2^{21}(h_2^1-h_0)^2 - Z_1^2(h_1^2-h_0)^2] \\
& + [prZ_2^{31}(h_2^1-h_0)^2 - Z_1^3(h_1^3-h_0)^2] + [qpZ_2^{12}(h_2^2-h_0)^2 - Z_1^1(h_1^1-h_0)^2] \\
& + [qqZ_2^{22}(h_2^2-h_0)^2 - Z_1^2(h_1^2-h_0)^2] + [qrZ_2^{23}(h_2^2-h_0)^2 - Z_1^3(h_1^3-h_0)^2] \\
& + [rpZ_2^{31}(h_2^3-h_0)^2 - Z_1^{(}h_1^1-h_0)^2] + [rqZ_2^{32}(h_2^3-h_0)^2 - Z_1^2(h_1^2-h_0)^2] \\
& + [rrZ_2^{33}(h_2^3-h_0)^2 - Z_1^3(h_1^3-h_0)^2] + pZ_1^1(h_1^1-h_0)^2 \\
& + qZ_1^2(h_1^2-h_0)^2 + rZ_1^3(h_1^3-h_0)^2
\end{aligned}$$

$$\begin{aligned}
\text{subject to} \qquad & pZ_2^{11}h_2^1 + qZ_2^{12}h_2^2 + rZ_2^{13}h_2^3 = Z_1^1h_1^1 \\
& pZ_2^{21}h_2^1 + qZ_2^{22}h_2^2 + rZ_2^{23}h_2^3 = Z_1^2h_1^2 \\
& pZ_2^{31}h_2^1 + qZ_2^{32}h_2^2 + rZ_2^{33}h_2^3 = Z_1^3h_1^3 \\
& pZ_2^{11} + qZ_2^{12} + rZ_2^{13} = Z_1^1 \\
& pZ_2^{21} + qZ_2^{22} + rZ_2^{23} = Z_1^2 \\
& pZ_2^{31} + qZ_2^{32} + rZ_2^{33} = Z_1^3 \\
& pZ_1^1h_1^1 + qZ_1^2h_1^2 + rZ_1^3h_1^3 = h_0 \\
& pZ_1^1 + qZ_1^2 + rZ_1^3 = 1 \\
& ppZ_2^{11} + pqZ_2^{21} + prZ_2^{31} + qpZ_2^{12} + qqZ_2^{22} + qrZ_2^{23} + rpZ_2^{31} + rqZ_2^{32} + rrZ_2^{33} = 1
\end{aligned}$$

Observe that the set of possible values of Z is uncountable when the state space of the rate process is greater than or equal to 3. Since our state parameter Z is also the control parameter, uncountability of the control parameter results in uncountability of the state parameter. In particular, it is necessary that admissible policies consist of Borel measurable functions μ_k. An alternate solution to the uncountable state space problem may be resolved by discretizing the space and considering only a finite number of elements of that set. The deterministic **finite-state** optimal problem is equivalent to the *shortest path* problem the optimal solution of which can be easily found by the forward DP algorithm.

7.1 Formulation of the 3 state DP Problem

We will first discuss the above case where the state space of the rate process is restricted to three states. We consider a part of Z_1 as the state variable denoted by Z_1^s and the other part as control variable.

For n = 1, we have

$$\text{minimize} \qquad J_{1,\pi}(Z_0) = pZ_1^1(h_1^1 - h_0)^2 + qZ_1^2(h_1^2 - h_0)^2 + rZ_1^3(h_1^3 - h_0)^2$$

$$\text{subject to} \begin{bmatrix} ph_1^1 & qh_1^2 \\ p & q \end{bmatrix} \begin{bmatrix} Z_1^1 \\ Z_1^2 \end{bmatrix} = \begin{bmatrix} Z_0 h_0 - rZ_1^3 h_1^3 \\ Z_0 - rZ_1^3 \end{bmatrix}$$

$$\begin{bmatrix} Z_1^1 \\ Z_1^2 \end{bmatrix} = \begin{bmatrix} ph_1^1 & qh_1^2 \\ p & q \end{bmatrix}^{-1} \begin{bmatrix} Z_0 h_0 - rZ_1^3 h_1^3 \\ Z_0 - rZ_1^3 \end{bmatrix}$$

$$Z_1^s = \begin{bmatrix} Z_1^1 \\ Z_1^2 \end{bmatrix} = \begin{bmatrix} ph_1^1 & qh_1^2 \\ p & q \end{bmatrix}^{-1} \begin{bmatrix} h_0 & -rh_1^3 \\ 1 & -r \end{bmatrix} \begin{bmatrix} Z_0 \\ Z_1^3 \end{bmatrix}$$

$$Z_1^s = \begin{bmatrix} Z_1^1 \\ Z_1^2 \end{bmatrix} = A^{-1} \begin{bmatrix} h_0 & -rh_1^3 \\ 1 & -r \end{bmatrix} \begin{bmatrix} Z_0 \\ Z_1^3 \end{bmatrix}$$

$$Z_1^s = f(Z_0, u_0, h_1) \text{ where } u_0 = Z_1^3$$

The control parameter $Z_1^3 \in (0, \frac{1}{r})$. The state parameter Z_1 is thus a function of the state at a previous time Z_0 and the control parameter Z_1^3, and the stochastic variable h_1. The problem can hence be formulated in the DP format provided the inverse of A exists. The inverse of A exists when $h_1^1 \neq h_1^2$. As we have already assumed a much stronger condition *i.e.*, $h_1^1 \neq h_1^2 \neq h_1^3$, we can formulate it as a DP problem.

For n = 2, we have

$$\text{minimize} \qquad J_{2,\pi}(Z_0)$$

$$\text{subject to} \begin{bmatrix} ph_2^1 & qh_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & ph_2^1 & qh_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & ph_2^1 & qh_2^2 \\ p & q & 0 & 0 & 0 & 0 \\ 0 & 0 & p & q & 0 & 0 \\ 0 & 0 & 0 & 0 & p & q \end{bmatrix} \begin{bmatrix} Z_2^{11} \\ Z_2^{12} \\ Z_2^{21} \\ Z_2^{22} \\ Z_2^{31} \\ Z_2^{32} \end{bmatrix} = \begin{bmatrix} Z_1^1 h_2^1 - rZ_2^{13} h_2^3 \\ Z_1^2 h_2^2 - rZ_2^{23} h_2^3 \\ Z_1^3 h_2^3 - rZ_2^{33} h_2^3 \\ Z_1^1 - rZ_2^{13} \\ Z_1^2 - rZ_2^{23} \\ Z_1^3 - rZ_2^{33} \end{bmatrix}$$

We consider a part of Z_2 as the state variable denoted by Z_2^s and the other part as control variable.

$$Z_2^s = \begin{bmatrix} Z_2^{11} \\ Z_2^{12} \\ Z_2^{21} \\ Z_2^{22} \\ Z_2^{31} \\ Z_2^{32} \end{bmatrix} = \begin{bmatrix} ph_2^1 & qh_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & ph_2^1 & qh_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & ph_2^1 & qh_2^2 \\ p & q & 0 & 0 & 0 & 0 \\ 0 & 0 & p & q & 0 & 0 \\ 0 & 0 & 0 & 0 & p & q \end{bmatrix}^{-1} \begin{bmatrix} Z_1^1 h_2^1 - rZ_2^{13} h_2^3 \\ Z_1^2 h_2^2 - rZ_2^{23} h_2^3 \\ Z_1^3 h_2^3 - rZ_2^{33} h_2^3 \\ Z_1^1 - rZ_2^{13} \\ Z_1^2 - rZ_2^{23} \\ Z_1^3 - rZ_2^{33} \end{bmatrix}$$

$$Z_2^s = g(Z_1, u_1, h_2) \text{ where } u_1 = \{Z_2^{13}, Z_2^{23}, Z_2^{33}\}$$

Thus the problem can be formulated as a DP problem given that

$$p^3 q^3 (h_2^1 - h_2^2)^3 \neq 0,$$

which is assured as $h_2^1 \neq h_2^2 \neq h_2^3$. The control parameter $u_1 = \{Z_2^{13} \in (0, \frac{1}{pr}), Z_2^{23} \in (0, \frac{1}{qr}), Z_2^{33} \in (0, \frac{1}{rr})\}$.

n	Equations	Unknowns	Variables to be fixed	Condition for DP
1	2	3	1	$pq(h_2^1 - h_2^2) \neq 0$
2	6	9	3	$p^3q^3(h_2^1 - h_2^2)^3 \neq 0$
3	18	27	9	$p^9q^9(h_2^1 - h_2^2)^9 \neq 0$
4	54	81	27	$p^{27}q^{27}(h_2^1 - h_2^2)^{27} \neq 0$
k	$3^{k-1} \times 2$	3^k	3^{k-1}	$p^{3^{k-1}}q^{3^{k-1}}(h_2^1 - h_2^2)^{3^{k-1}} \neq 0$

7.2 m state DP problem

We consider a part of Z_t as the state variable denoted by Z_t^s and the other part as control variable. We first discuss the discretization of the control space.

7.2.1 Discretizing the Control Space

$$Z_t = f(Z_{t-1}, u_{t-1}),$$
$$\text{where } u_{t-1} = \{Z_t^{i^1 i^2 \ldots i^{t-1} i^t} : i^k = 1, 2, .., m, k = 1, 2, .., t-1 \,\&\, i^t = 3, ..., m\},$$
$$Z_t^{i^1 i^2 \ldots i^{t-1} i^t} \in (0, \frac{1}{p_{i^1} p_{i^1} \ldots p_{i^{t-1}} p_{i^{t-1}}}),\ \ i^k = 1, 2, .., m, k = 1, 2, .., t-1 \,\&\, i^t = 3, ..., m.$$

The minimization problem can easily be solved by the Backward DP algorithm if we have a finite control state space. The control space associated with each control parameter $Z_t^{i^1 i^2 \ldots i^{t-1} i^t}$ can thus be partitioned into g discrete points $\forall i^k = 1, 2, .., m, k = 1, 2, .., t-1 \,\&\, i^t = 3, ..., m$ and the Minimum variance measure according to the quadratic cost criterion can be found.

For n = t, we have

minimize $J_{t,\pi}(Z_0) = \sum_{n=1}^{t} \{ (\prod_{k=1}^{n} \sum_{i^k=1}^{m}) [(\prod_{k=1}^{n} p_{i^k}) Z_n^{i^1 i^2 \ldots i^n} (h_n^{i^n} - h_0)^2 - (\prod_{k=1}^{n-1} p_{i^k}) Z_{n-1}^{i^1 i^2 \ldots i^{n-1}} (h_{n-1}^{i^{n-1}} - h_0)^2] \}$

subject to

$$
Z_t^k = \begin{bmatrix} Z_t^{11\ldots1} \\ Z_t^{11\ldots2} \\ Z_t^{21\ldots1} \\ Z_t^{21\ldots2} \\ .. \\ .. \\ Z_t^{m1\ldots1} \\ Z_t^{m1\ldots2} \\ .. \\ .. \\ Z_t^{12\ldots1} \\ Z_t^{12\ldots2} \\ Z_t^{22\ldots1} \\ Z_t^{22\ldots2} \\ .. \\ .. \\ Z_t^{mm\ldots1} \\ Z_t^{mm\ldots2} \end{bmatrix} = \begin{bmatrix} p_1h_t^1 & p_2h_t^2 & 0 & 0 & .. & .. & 0 & 0 \\ 0 & 0 & p_1h_t^1 & p_2h_t^2 & 0 & 0 & 0 & 0 \\ .... & & & & & & & \\ .... & & & & & & & \\ 0 & 0 & .. & .. & 0 & 0 & p_1h_t^1 & p_2h_t^2 \\ p_1 & p_2 & 0 & 0 & .. & .. & 0 & 0 \\ 0 & 0 & p_1 & p_2 & 0 & 0 & 0 & 0 \\ .... & & & & & & & \\ .... & & & & & & & \\ 0 & 0 & .. & .. & 0 & 0 & p_1 & p_2 \end{bmatrix}^{-1} \begin{bmatrix} Z_{t-1}^{1..1}h_{t-1}^1 - \sum_{i=3}^m p_i Z_t^{1..1i} h_t^i \\ Z_{t-1}^{1..2}h_{t-1}^2 - \sum_{i=3}^m p_i Z_t^{1..2i} h_t^i \\ .. \\ .. \\ Z_{t-1}^{1..m}h_{t-1}^m - \sum_{i=3}^m p_i Z_t^{1..mi} h_t^i \\ Z_{t-1}^{1.11}h_{t-1}^1 - \sum_{i=3}^m p_i Z_t^{1.11i} h_t^i \\ Z_{t-1}^{1.21}h_{t-1}^1 - \sum_{i=3}^m p_i Z_t^{1.21i} h_t^i \\ .. \\ .. \\ Z_{t-1}^{1.m1}h_{t-1}^1 - \sum_{i=3}^m p_i Z_t^{1.m1i} h_t^i \\ .. \\ .. \\ Z_{t-1}^{m\ldots m}h_{t-1}^m - \sum_{i=3}^m p_i Z_t^{m..mi} h_t^i \\ Z_{t-1}^{1..1} - \sum_{i=3}^m p_i Z_t^{1..1i} \\ Z_{t-1}^{1..2} - \sum_{i=3}^m p_i Z_t^{1..2i} \\ .. \\ .. \\ Z_{t-1}^{1..m} - \sum_{i=3}^m p_i Z_t^{1..mi} \\ Z_{t-1}^{1.11} - \sum_{i=3}^m p_i Z_t^{1.11i} \\ Z_{t-1}^{1.21} - \sum_{i=3}^m p_i Z_t^{1.21i} \\ .. \\ .. \\ Z_{t-1}^{1.m1} - \sum_{i=3}^m p_i Z_t^{1.m1i} \\ .. \\ .. \\ Z_{t-1}^{m\ldots m} - \sum_{i=3}^m p_i Z_t^{m..mi} \end{bmatrix}
$$

$$Z_t^s = [A]_{2m^{t-1} \times 2m^{t-1}} [B]_{2m^{t-1} \times 1},$$
$$Z_t^s = f(Z_{t-1}, u_{t-1}, h_t).$$

$$
\text{Let, } A = \begin{bmatrix} p_1h_t^1 & p_2h_t^2 & 0 & 0 & .. & .. & 0 & 0 \\ 0 & 0 & p_1h_t^1 & p_2h_t^2 & 0 & 0 & 0 & 0 \\ .... & & & & & & & \\ .... & & & & & & & \\ 0 & 0 & .. & .. & 0 & 0 & p_1h_t^1 & p_2h_t^2 \\ p_1 & p_2 & 0 & 0 & .. & .. & 0 & 0 \\ 0 & 0 & p_1 & p_2 & 0 & 0 & 0 & 0 \\ .... & & & & & & & \\ .... & & & & & & & \\ 0 & 0 & .. & .. & 0 & 0 & p_1 & p_2 \end{bmatrix}_{2m^{t-1} \times 2m^{t-1}} .
$$

A has a non-zero determinant since,

$$Determinant(A) = p_1^{m^{k-1}} p_2^{m^{k-1}} (h_t^1 - h_t^2)^{m^{k-1}} \neq 0 \ \text{ for } h_t^1 \neq h_t^2.$$

Thus, the above problem can be formulated as a DP problem.

n	Equations	Unknowns	Variables to be fixed	Condition for DP
1	2	m	m-2	$p_1p_2(h^1 - h^2) \neq 0$
2	$2m$	m^2	m(m-2)	$p_1^m p_2^m (h_t^1 - h_t^2)^m \neq 0$
3	$2m^2$	m^3	$m^2(m-2)$	$p_1^{m^2} p_2^{m^2} (h_t^1 - h_t^2)^{m^2} \neq 0$
4	$2m^3$	m^4	$m^3(m-2)$	$p_1^{m^3} p_2^{m^3} (h_t^1 - h_t^2)^{m^3} \neq 0$
k	$2m^{k-1}$	m^k	$m^{k-1}(m-2)$	$p_1^{m^{k-1}} p_2^{m^{k-1}} (h_t^1 - h_t^2)^{m^{k-1}} \neq 0$

8 Discretizing the Bond Price process

We have assumed that the price of a zero coupon bond is a function of the rate process: $h_n = e^{\sum_{k=0}^{n} -r_k} F(r_n)$ endowed with a probability density function. We consider finite partitions of the the *pdf* such that the discretized pdf converges to the the given pdf. We show the convergence of the discretization method for lognormal bond price process. Observe that the rate process is *stationary* under the *Vasicek's model* as shown below:

$$r_n - r_{n-1} = k_1 + (k_2 - 1)r_{n-1} + \sigma\epsilon_n$$

If $|k_2 - 1| < 1$ and $E|r_0| < \infty$, then

$$E[r_n] = k_1^n E[r_0] + \frac{k_1(1-k_2^n)}{1-k_2} \rightarrow \frac{k_1}{1-k_2} \text{ as n} \rightarrow \infty$$

and if $Var[r_0] < \infty$

$$Var[r_n] = k_2^{2n} Var[r_0] + \frac{\sigma^2(1-k_2^{2n})^2}{1-k_2^2} \rightarrow \frac{\sigma^2}{1-k_2^2};$$
$$Cov(r_n, r_{n-m}) \rightarrow \frac{\sigma^2 k_2^m}{1-k_2^2}.$$

Moreover, if the initial distribution (the distribution of r_0) is normal, i.e., $r_0 \sim N(\frac{k_1}{1-k_2}, \frac{\sigma^2}{1-k_2^2})$, then $r = (r_n)_{n\geq 0}$ is a *stationary* Gaussian sequence (both in wide and the strict sense) with

$$E[r_n] = \frac{k_1}{1-k_2}; Var[r_n] = \frac{\sigma^2}{1-k_2^2}.$$

8.1 Moment matching method

Representative values are assigned so that the first several moments of the distribution match the moments of the continuous distribution. Such approximations work because a *density function* is a limit of discrete mass functions. A distribution function is, in general, uniquely characterized by its moments. So, if we are progressively able to match all higher moments of a discrete distribution to the moments of a continuous distribution, then the two distributions converge in the limiting case. In general, n points of the discrete distribution can match $2n-1$ moments [4].

The intuition is that as more and more number of moments are matched, the neat structure of the optimization problems will lead to a closer and closer approximate solution that could be justified by variants of weak convergence results.

8.2 Calculating the Approximating Probabilities

We want to find a set of values and probabilities such that:

$$< x^k > := \int_{-\infty}^{\infty} x^k f(x)dx = \sum_{i=1}^{N} p_i x_i^k, \text{ for } k = 1, 2, ...$$

A discrete approximation with N probability-value pairs can match the first $(2N-1)$ moments exactly by finding p_i and x_i that satisfy the following equations:

$$
\begin{aligned}
p_1 + p_2 + p_3 + \ldots\ldots\ldots\ldots\ldots\ldots + p_N &= < x_0 >= 1, \\
p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots\ldots\ldots + p_N x_N &= < x > \\
p_1 x_1^2 + p_2 x_2^2 + p_3 x_3^2 + \ldots\ldots\ldots + p_N x_N^2 &= < x^2 >, \qquad (62) \\
&. \\
&. \\
&. \\
p_1 x_1^{2N-1} + p_2 x_2^{2N-1} + p_3 x_3^{2N-1} + \ldots\ldots + p_N x_N^{2N-1} &= < x^{2N-1} > . \qquad (63)
\end{aligned}
$$

There is a well-known method for solving these equations. First, define the polynomial:

$$\pi(x) = (x - x_1)(x - x_2)(x - x_3)\ldots.(x - x_N) = \sum_{k=0}^{N} c_k x^k.$$

It follows from this definition that $c_N = 1$ and $\pi(x_i) = 0$ for $i = 1, 2, .., N$. Taking the first N equations we multiply the first equation by c_0, the next by c_1, etc., and then add them to get:

$$\sum_{i=i}^{N} p_i \pi(x_i) = 0 = \sum_{k=0}^{N} c_0 x^k.$$

Now taking the second equation through $(N+1)$th equation and multiplying by the coefficients of the polynomial again we get:

$$\sum_{i=1} p_i x_i \pi(x_i) = 0 = \sum_{k=0}^{N} c_k x^{k+1}$$

This process is repeated N times to yield the following set of linear equations.

$$
\begin{aligned}
< x^0 > c_0 + < x > c_1 + < x^2 > c_2 + \ldots\ldots\ldots + < x^{N-1} > c_{N-1} &= - < x^N >, \\
< x > c_0 + < x^2 > c_1 + < x^3 > c_2 + \ldots\ldots\ldots + < x^N > c_{N-1} &= - < x^{N+1} >, \\
< x^2 > c_0 + < x^3 > c_1 + < x^4 > c_2 + \ldots\ldots\ldots + < x^{N+1} > c_{N-1} &= - < x^{N+2} >, \\
&. \\
&. \\
&. \\
< x^{N-1} > c_0 + < x^N > c_1 + < x^{N+1} > c_2 + \ldots\ldots\ldots + < x^{2N-2} > c_{N-1} &= - < x^{2N-1} >,
\end{aligned}
$$

These equations can be solved for the coefficients of the polynomial, and then the x_i can be determined by finding the zeroes of the polynomial. Finally, the p_i can be determined by substituting the x_i into the original set of equations for the moments of the approximate distributions. (The p_is can be determined without solving another set of N linear equations.) *It can be shown that if the original moments are finite and are derived from a probability distribution, this procedure must yield N real, distinct values, x_i, which all lie within the interval spanned by the original distribution. It must also produce positive probabilities.*

8.3 Error of Approximation

$$\text{minimize} \qquad J_{N,\pi}(Z_0) = E[Z_N(h_N - h_0)^2] = \int_A Z_N(h_N - h_0)^2 dP \tag{64}$$

$$\text{subject to} \qquad E[Z_n h_n | \Im_{n-1}] = Z_{n-1}h_{n-1},\ \forall n, 0 \leq n \leq N, \tag{65}$$

$$E[Z_n | \Im_{n-1}] = Z_{n-1},\ \forall n, 0 \leq n \leq N, \tag{66}$$

$$E[Z_n] = 1,\ \forall n, 0 \leq n \leq N, \tag{67}$$

$$Z_n > 0,\ \forall n, 0 \leq n \leq N. \tag{68}$$

The above problem can be written as:

$$\text{minimize} \qquad J_{N,\pi}(Z_0) = E[Z_N(h_N - h_0)^2] = \int_A Z_N(h_N - h_0)^2 dP \tag{69}$$

$$Z_N \in MR \subset [\delta, \infty), \tag{70}$$

where MR is the set of Radon Nikodym derivatives (RNDs) generating Equivalent Martingale Measures.

Remark 2 *The framework is similar to the Finite Horizon Problem by Ferretti [5], where an upper bound on the error estimate is calculated.*

Here, $Z_N(h_N - h_0)^2$ is the cost function, P is some continuous distribution function on R and $Z_N \in M$ is the feasible set (the set of Radon Nikodym derivatives that generate an Equivalent Martingale Measure). Denote by $Z^* = \arg\min_Z \int_A Z_N(h_N - h_0)^2 dP$, (We suppose for simplicity that it is unique, *Quadratic cost function*!). Let P_m be the approximate discrete distribution, *i.e.*, we minimize

$$J_{N,\pi}(Z_0)_m = \sum_{A_m} Z_N(h_N - h_0)^2 dP_m; Z_N \in MR_m.$$

We define the approximate error $e(J, J_m) := J(\arg\min_{Z_m} J_m) - J(\arg\min_Z J)$. Notice that $e(J, J_m) \geq 0$. The error however is difficult to calculate. It is in fact easier to find an upper bound.

Proposition 4 $e(J, J_m) \leq 2\sup_Z |J - J_m|$.

Proof: Let $Z^* \arg\min J$ and $Z_m^* = \arg\min J_m$.
Set $\varepsilon = \sup_Z |J(Z) - J_m(Z)|$. Let $M = \{Z : J(Z) \leq J(Z^*) + 2\varepsilon\}$. Suppose that $Z_m^* \notin M$. Then,

$$J(Z^*) + 2\varepsilon \leq J(Z_m^*) \leq J_m(Z_m^*) + \varepsilon \leq J_m(Z^*) + \varepsilon \leq J(Z^*) + 2\varepsilon.$$

This contradiction establishes $Z_m^* \in M$, i.e.

$$e(J, J_m) = J(Z_m^*) - J(Z^*) \leq 2\varepsilon.$$

Remark 3 *The steady decrease of the error term for large values of m and subsequently to zero is discussed by Ferretti [5] for non-quadratic cost functions.*

9 Convergence of optimal solution in the one period problem

Let Z_m^* be the value of the Z_1 obtained by approximating the distribution of h_1 with m random variables and minimizing the $\sum_{k=1}^{m} Z_{1m}(h_{1m} - h_0)^2 p_m$ subject to martingale constraints. Let the probability measure Q_m^* be given by $Q_m^*(A) = \int_A Z_m^* dP_m, A \in \Im_m \subseteq \Im$. Assuming h_1 has a compact support, let Q_m^* converge weakly to Q^*, i.e., $Q_m^* \rightarrow^d Q^*$.

Remark 4 *Observe that $P_m \rightarrow^d P$ (by construction).*

Claim 1 *Q^* is an EMM.*

Proof: Q_m^* is absolutely continuous with respect to P_m and $Z_m^* = \frac{dQ_m^*}{dP_m}$ is a P_m martingale. Further, $E^{P_m}[Z_m^*] = 1$ makes Q_m^* a probability measure (specifically an equivalent martingale measure). Thus, we have $Q_m^* << P_m$.

Suppose, Q^* is not absolutely continuous with P.
Thus, for some $A_\phi \in \Im$, $\exists \rho > 0$, such that $\forall \phi > 0$, for $P(A_\phi) < \phi$, we have $Q^*(A_\phi) \geq \rho$. $Q_m^* \rightarrow^d Q^*$ and $P_m^* \rightarrow^d P^*$ further implies that $\exists m_0$, s.t. for all $m \geq m_0$, we have, for $P_m(A_\phi) < \frac{\phi}{2}$ implies $Q_m^*(A_\phi) \geq \frac{\rho}{2}, \forall \phi$ contradicting the absolute continuity of Q_m^* with respect to P_m. Hence $Q^* << P$.

Let $Z^* := \frac{dQ^*}{dP}$.
Moreover, we have, $E[Z^*] = 1$ and $E[Z^*|\Im_0] = E[Z^*] = 1$ making Q^* an equivalent martingale measure.

Claim 2 $E^{Q^*}[(h_1 - h_0)^2] \leq E^Q[(h_1 - h_0)^2], \forall Q \in M$.

Proof: For any Q_m, $Q_m \in M$, we have

$$E^{Q_m^*}[(h_1 - h_0)^2] \leq E^{Q_m}[(h_1 - h_0)^2].$$

We first consider the case when h_1 has a compact support. Taking limits on both sides,

$$\lim_{m\to\infty} E^{Q_m^*}[(h_1 - h_0)^2] \leq \lim_{m\to\infty} E^{Q_m}[(h_1 - h_0)^2];$$

$$\text{implies } E^{Q^*}[(h_1 - h_0)^2] \leq \lim_{m\to\infty} E^{Q}[(h_1 - h_0)^2].$$

10 Numerical Solution of the 'one' period problem

10.1 MVMM for Lognormal bond prices

We consider the bond price process h to be lognormal with parameters μ and σ. For a lognormal variable $X \sim LN(\mu, \sigma)$, the n-th moment is given by [6]:

$$E[x^n] = exp(n\mu + \frac{n^2\sigma^2}{2}).$$

An important observation while approximating a lognormal random variable is the moments grow exponentially for larger n's. Further, when solving for the co-efficients $c_0, c_1, c_2, ...$ by using

MATLAB for higher n's the values we get are highly unreliable. If A is a square matrix, $A\backslash B$ is roughly the same as $inv(A) * B$, except it is computed in a different way. If A is an n-by-n matrix and B is a column vector with n components, or a matrix with several such columns, then $X = A\backslash B$ is the solution to the equation AX = B computed by Gaussian elimination. A warning message gets printed if A is badly scaled or nearly singular.

The Table below shows the *minimum* value of the cost function $Z_1 E[(h_1 - h_0)^2]$ when the lognormal distribution is approximated with N random variables, thereby matching its first $2N - 1$ moments. We consider $Z_1 \geq \delta, \delta = 0.01$.

N	Minimum Cost
4	0.01023659729930
5	0.03260498418877
6	0.03755310515767
7	0.03419657511033
8	0.02676107361828
9	0.01738206877951
10	0.00719783570787

Remark 5 *Observe that the minimum cost steadily decreases after we increase the number of approximating random variables (N) beyond 6.*

The Radon Nikodym Derivative Z_1 corresponding to the MVMM for N = 7 is given by:

Z_1	Minimum Z_1
Z_1^1	0.01053570010757
Z_1^2	0.01000004744816
Z_1^3	0.01000000161230
Z_1^4	0.01000000031368
Z_1^5	0.01000000035239
Z_1^6	1.40434853801986
Z_1^7	9.74146229882888

Remark 6 *We observe that the values of* $Z_1^1, Z_1^2, Z_1^3, Z_1^4 \simeq 0.01$.

Remark 7 *The primal is infeasible for* $N = 3$. *This is because both* $E[Z_1 h_1|\Im_0] = Z_0 h_0 = 1$ *and* $E[Z_1] = 1$ *cannot be satisfied for* $Z_1 \geq \delta$. *In MATLAB, once the preprocessing has finished, the iterative part of the algorithm begins until the stopping criteria are met. If the residuals are growing instead of getting smaller, or the residuals are neither growing nor shrinking, then the problem is usually infeasible.*

10.2 MVMM for bond prices with triangular distribution

The probability distribution function f of the bond price process h is given by:

$$\begin{aligned} f(h) &= \frac{2(h-a)}{(b-a)(c-a)} \text{ for } a \leq h \leq c; \\ &= \frac{2(b-h)}{(b-a)(b-c)} \text{ for } c \leq h \leq b \end{aligned}$$

We consider $a = 0.01, b = 3, c = 1$.

The Table below shows the *minimum* value of the cost function $Z_1 E[(h_1 - h_0)^2]$ when the triangular distribution is approximated with N random variables, thereby matching its first $2N-1$ moments. We consider $Z_1 \geq \delta, \delta = 0.01$.

N	Minimum Cost
5	0.0939
6	0.0639
7	0.0304
8	0.0512
9	0.0183

Remark 8 *The minimum cost increases when the price process is approximated by 8 random variables. For higher of order discretization, calculating the random variables and their corresponding probabilities becomes computationally difficult. This is because the calculation involves inverting some nearly singular matrices in the moment matching method.*

11 Multi-period Problems

We show numerical solutions for the two and three period problems respectively. The following observations are for lognormal bond prices, the parameters of which were estimated from historical data. We can observe that the minimum cost decreases as we increase the number of random variables for approximating the bond price process.

11.1 Two period model

N	Minimum Cost
4	0.65995110654808
5	0.88906263588492
6	0.22928906886193
7	0.14774246209004
8	0.03428916742700
9	0.00626970588837
10	0.00369256412862

Remark 9 *Observe that the minimum cost steadily decreases after we increase the number of approximating random variables (N) beyond 5.*

11.2 Three period model

N	Minimum Cost
4	0.68609629492469
5	0.59977289393892
6	0.35861641706338
7	0.11466908485029
8	0.07380469227943
9	0.07828389730531
10	0.01118382079643

Remark 10 *Observe that the minimum cost steadily decreases after we increase the number of approximating random variables (N) except from step 8 to 9.*

12 Conclusions

The exponentially quadratic form reduces to the exponentially linear form in an arbitrage-free market given by a specific form of the Radon-Nikodym derivative. We have shown that a discrete-time bond market with uncountable state space for the short rate is Incomplete. Further, the analytical solution for Radon Nikodym derivatives were difficult to obtain and hence an approximation of the subjective measure by a discrete measure by Moment matching approach is used. The approximate distribution converges to the original distribution in the limiting case. Pricing of bonds can thus be seen as a minimization problem in an Incomplete market setup. Further, both the Linear Programming and Dynamic Programming approaches to solve the minimization problem were dicussed.

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