International Capital Flows in a World of Greater Financial Integration

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Preliminary and Incomplete

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Abstract

International capital flows have increased dramatically since the 1980s, with much of the increase being due to trade in equity and debt markets. Such developments are often attributed to the increased integration of world financial markets. We present a model that allows us to examine how greater integration in world financial markets affects the structure of asset ownership and the behavior of international capital flows. Our model predicts that international capital flows are large (in absolute value) and very volatile during the early stages of financial integration when international asset trading is concentrated in bonds. As integration progresses and households gain access to world equity markets, the size and volatility of international bond flows fall dramatically but continue to exceed the size and volatility of international equity flows. We also find that variations in the equity risk premia account for almost all of the international portfolio flows in bonds and equities. We argue that both effects arise naturally as a result of increased risk sharing facilitated by greater financial integration. The paper also makes a methodological contribution to the literature on dynamic general equilibrium asset-pricing. We present a new technique for solving a dynamic general equilibrium model with production, portfolio choice and incomplete markets.

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Introduction

International capital flows have increased dramatically since the 1980s. During the 1990s gross capital flows between industrial countries rose by 300 per cent, while trade flows increased by 63 percent and real GDP by a comparatively modest 26 percent. Much of the increase in capital flows is due to trade in equity and debt markets, with the result that the international pattern of asset ownership looks very different today than it did a decade ago. These developments are often attributed to the increased integration of world financial markets. Easier access to foreign financial markets, so the story goes, has led to the changing pattern of asset ownership as investors have sought to realize the benefits from international diversification. It is much less clear how the growth in the size and volatility of capital flows fits into this story. If the benefits of diversification were well-known, the integration of debt and equity markets should have been accompanied by a short period of large capital flows as investors re-allocated their portfolios towards foreign debt and equity. After this adjustment period is over, there seems little reason to suspect that international portfolio flows will be either large or volatile. With this perspective, the prolonged increase in the size and volatility of capital flows we observe suggests that the adjustment to greater financial integration is taking a very long time, or that integration has little to do with the recent behavior of capital flows.

In this paper we present a model that allows us to examine how greater integration in world financial markets affects the structure of asset ownership and the behavior of international capital flows. We use the model to address two main questions: (i) How is the size and volatility of international capital flows affected by greater financial integration in world debt and equity markets? (ii) What factors drive international portfolio flows, and does their influence change with the degree of integration? To the best of our knowledge, neither question has been addressed in the international finance literature.

The model we present captures the effects of financial integration in the simplest possible way. We consider a symmetric two-country model with production for traded and non-traded goods. Firms in both the traded and non-traded sectors issue equity on domestic stock markets. We examine the impact of financial integration in this world by considering three configurations: Financial Autarky FA, Partial Integration PI, and Full Financial Integration FI. Under FA, households only have access to the domestic stock market and so can only hold their wealth in the form of the equity of domestic firms producing traded and non-traded goods. The equilibrium in this economy serves as a benchmark for gauging the effects of financial integration. Under PI, we open a world bond market. Now households can allocate their wealth between domestic equity and international bonds. This configuration roughly corresponds the state of world financial markets before the mid-1980's where bonds are the main medium for international financial transactions. The third configuration, FI, corresponds to the current state of world financial markets. Under FI, households have access to international bonds, equity issued by domestic firms, and equity issued by foreign firms producing traded goods.

Two aspects of our model deserve special note. First, in all three market configurations we consider, international risk-sharing among households is less than perfect. In other words, we only consider international capital flows in equilibria where markets are incomplete. As we move from the FA to PI and then to FI configurations of the model, the degree of risk-sharing increases, but households never have access to a rich enough array of financial assets to make markets complete. We view this as an important feature of the model. There is ample evidence that incomplete risk-sharing persists even with the high degree of financial integration we see today. This observation precludes us from characterizing our FI configuration as an equilibrium with complete markets.[2]

The second important feature of the model concerns information. The equilibria we study are derived under the assumption that all households and firms have access to the same information regarding the current state of the world economy. While this common-knowledge assumption is standard in international macro models, it does have important implications for the role played by international capital flows. Specifically, capital flows in our model do not result from differences of opinion concerning the future returns or risks associated with different assets. As such, capital flows do not convey any information to firms and households that is unavailable from other sources. We do not view this common-knowledge framework as necessarily the correct one for analyzing capital flows. Nevertheless we adopt it here to establish a theoretical benchmark for how greater financial integration affects capital flows when information about risks and returns is common-knowledge. By contrast, Evans and Lyons (2004) present a model where information about the state of the economy is dispersed internationally, and as a result capital flows convey information that is not available elsewhere. That paper does not undertake the task of analyzing the effects of increased financial integration.

Our analysis is related to three major strands of research. The first strand studies the effects of financial liberalization on capital flows and returns. Examples of theoretical research with this focus include Martin and Rey (2002) and Bacchetta and van Wincoop (1998), while empirical assessments can be found in Bekaert, Harvey and Lumsdaine (2002a,b), Henry (2000a,b), Bekaert and Harvey (1995, 2000) and many others. The second strand of research focuses on the joint determination of capital flows and equity returns. Representative papers in this area include Bohn and Tesar (1996), Froot and Teo (2004), Stulz (1999), and Froot, O'Connell and Seasholes (1998). Hau and Rey (2002, 2004) extend the analysis of equity return-capital flow interaction to include the real exchange rate. The third strand of the literature studies the macroeconomic implications of financial integration. Baxter and Crucini (1994) and Heathcote and Perri (2003) compare the equilibrium of models with restricted asset trade against an equilibrium with complete markets. The comparative approach adopted by these papers is closest to the methodology we adopt, but our model does not equate financial integration with complete markets. An alternative view of integration is that it reduces the frictions that inhibit asset trade. Examples of this approach include Buch and Pierdzioch (2003), Sutherland (1998), and Senay (1998).

Although the model we develop has a relatively simple structure, several technical problems need to be solved in order to find the equilibrium associated with any of our market configurations. The first of these problems concerns portfolio choice. We interpret increased financial integration as giving households a wider array of assets in which to hold their wealth. How households choose to allocate their wealth among these assets is key to understanding how financial integration affects international capital flows, so there is no way to side-step portfolio allocation decisions. We model the portfolio problem as part of the intertemporal optimization problem of the households allowing for the fact that returns do not follow i.i.d. processes in equilibrium. The second problem relates to market incompleteness. Since markets are incomplete in all the configurations we study, we cannot find the equilibrium allocations by solving an appropriate planning problem. Instead, the equilibrium allocations must be established by directly checking the market clearing conditions implied by the decisions of households and firms. This paper uses a new solution methodol-

[2] Complete risk sharing has implication for the international correlations in consumption that odds with the empirical evidence reported by Backus and Smith (1993), Kollman (1995) and many others.

ogy to compute equilibrium allocations and prices in this decentralized setting. The methodology also incorporates the complications of portfolio choice in an intertemporal setting. The third problem concerns non-stationarity. In the equilibria we study, temporary productivity shocks have permanent effects on a number of state-variables. This general feature of models with incomplete markets arises because the shocks permanently affect the distribution of wealth. Recognizing this aspect of our model, the solution method provides us with equilibrium dynamics for the economy in a large neighborhood of a specified initial wealth distribution.[3]

A comparison of the equilibria associated with our three market configurations provides us with several striking results. First, in the PI configuration where all international asset trading takes place via the bond market, international capital flows are large (in absolute value) and very volatile. Second, when households gain access to foreign equity markets, the size and volatility of international bond flows falls dramatically. Third, the size and volatility of bond flows remains above the size and volatility of equity portfolio flows under FI. The standard deviation of quarterly bond flows measured relative to GDP is approximately 1.6 percent, while the corresponding value for equity is 0.88 precent, a figure that is roughly comparable to estimates from the data. Thus, the high volatility of capital flows we observed is consistent with a high degree of financial integration. Our fourth main finding concerns the factors driving capital flows. In our model, variations in the equity risk premia (i.e., expected excess returns on equity relative to the risk free rate) account for almost all of the international portfolio flows in bonds and equities. Changes in the risk premia arise endogenously as productivity shocks affect the distribution of wealth, with the result that households are continually adjusting their portfolios. Although these portfolio adjustments are small, their implications for international capital flows are large relative to GDP. Overall, the results from our model indicate that greater financial integration could indeed be responsible for the large and variable international capital flows we observe in the real world.

The paper is organized as follows. The next section documents how the international ownership of assets and the behavior of capital flows has evolved over the past thirty years. The model is presented in Section 3. Section 4 describes the solution to the model. Our comparison of the equilibria under the three market configurations is presented in Section 5. We examine the robustness of our results in Section 6. Section 7 concludes.

1 The Globalization of Financial Markets

The large increase in international capital flows represents one of the most striking developments in the world economy over the past thirty years. In recent years, the rise in international capital flows has been particularly dramatic. IMF data indicates that gross capital flows between industrialized countries (the sum of absolute value of capital inflows and outflows) expanded 300 percent between 1991 and 2000. Much of this increase was attributable to the rise in foreign direct investment and portfolio equity flows, which both rose by roughly 600 percent. By contrast, gross bond flows increased by a comparatively modest 100 percent. The expansion in all these flows vastly exceeds the growth in the real economy or the growth in international trade. During 1991-2000 period, real GDP in industrialized countries increased by 26 percent,

[3] For the results we study, the neighborhood is large enough to cover the dynamics of the economy for 100 years.

and international trade rose by 63 percent[4]. So while the growth in international trade is often cited as indicating greater interdependence between national economies, the growth in international capital flows suggests that the integration of world financial markets has proceeded even more rapidly.

Greater financial integration is manifested in both asset holdings and capital flows. Figures 1a and 1b show how the scale and composition of foreign asset holdings have changed between 1976 and 2003. US ownership of foreign equity, bonds and capital (accumulated FDI) is plotted in Figure 1a, while foreign ownership of US corporate bonds, equity, capital and Treasury securities are shown in Figure 1b. All the series are shown as a fraction of US GDP. Before the mid-1980s, capital accounted for the majority of foreign assets held by US residents, followed by bonds. US ownership of foreign equity was below 1% of GDP. The size and composition of these asset holdings began to change in the mid-1980s when the fraction of foreign equity surpassed bonds. Thereafter, US ownership of foreign equity increased rapidly peeking at roughly 22 percent of GDP in 1999. US ownership of foreign capital and bonds also increased during this period but to a lesser extent. In short, foreign equities have become a much more important component of US financial wealth in the last decade or so. Foreign ownership of US assets has also risen significantly. As Figure 1b shows, foreign ownership of corporate bonds, equity and capital have steadily increased as a fraction of US GDP over the past thirty years. By 2003, foreign ownership of debt, equity and capital totalled 45 percent of US GDP.

Figure 1a. U.S.-owned assets abroad, %GDP

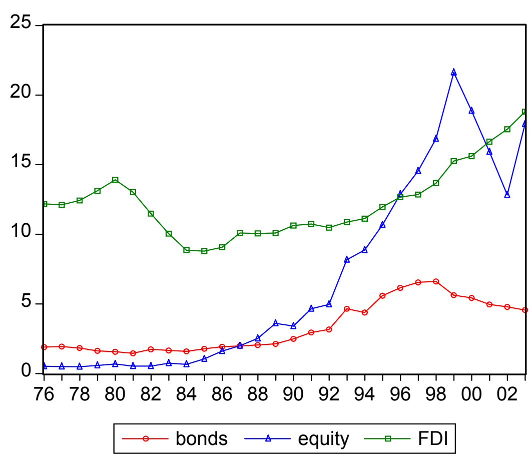
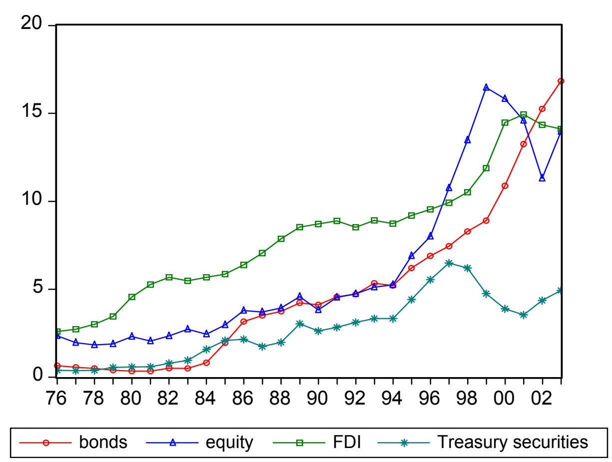


Figure 1b. Foreign-owned assets in US, %GDP



The pattern of asset ownership depicted in Figures 1a and 1b is consistent with increased international portfolio diversification by both US and non-US residents. More precisely, the plots show changes in ownership similar to those that would be necessary to reap the benefits of diversification. This is most evident in the pattern of equity holdings. Foreign ownership of equities has been at historically high levels over the past five years.

[4]Trade volume is calculated as exports plus imports.

Figure 2a. US portfolio investment, outflows, %GDP

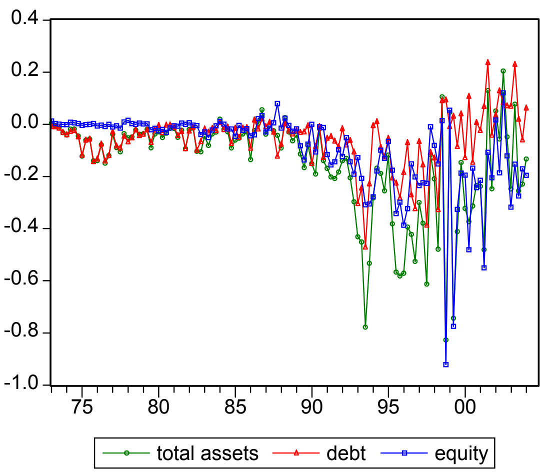
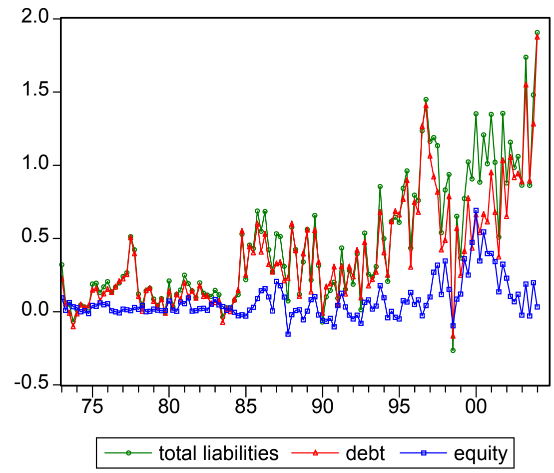


Figure 2b. US portfolio investment, inflows, %GDP



The change in asset ownership has been accompanied by a marked change in international capital flows. Figures 2a and 2b plot the quarterly capital flows associated with transactions in US assets and liabilities as a fraction of GDP. Negative outflows represent US net purchases of foreign assets, while positive inflows represent foreign net purchases of US assets. Two features of these plots stand out. First, capital flows were a small fraction of GDP before the mid-1980s. On average, annual gross capital flows accounted for only 1 percent of US GDP until the mid 1980s, but by 2003 amounted to almost 6 percent of GDP. Second, the volatility of capital inflows and outflows increased markedly in the 1990s. This is most clearly seen in Figures 3a and 3b where we plot the standard deviation of the capital flows over a rolling window of 58 quarters. The increased volatility of equity outflows is particularly noticeable: between 1987 and 2004 volatility increased eleven-fold as a fraction of GDP.

Figure 3a. Volatility of portfolio investment, outflows %GDP

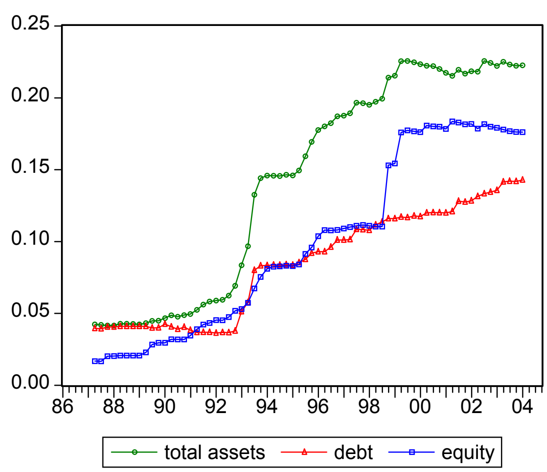
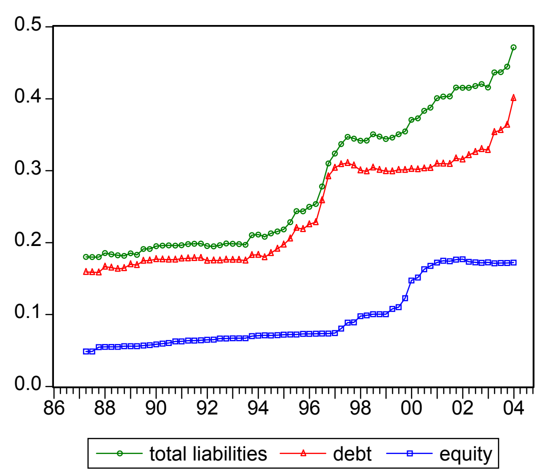


Figure 3b. Volatility of portfolio investment, inflows %GDP



We will focus on the three outstanding features of the data in our analysis below: the increase in (i) ownership of foreign equities, (ii) the size of portfolio flows, and (iii) the volatility of portfolio flows. In particular, we will investigate whether all three features of the data arise as natural consequences of greater integration in world financial markets.

2 The Model

We consider a world economy consisting of two identical countries, called HOME (H) and FOREIGN (F). Each country is populated by a continuum of identical households who supply their labor inelastically to domestic firms in the traded and non-traded goods sectors. Firms in both sectors are perfectly competitive, and issue equity that is traded on the domestic stock market. Our model is designed to study how the degree of financial integration affects international capital flows and returns. For this purpose, we focus on three equilibria. First we consider the benchmark case of financial autarky. In this environment, households allocate their portfolios between equity in domestic firms producing traded and non-traded goods. Second, we consider a world with partial integration where households allocate their portfolios between domestic equity and an international bond. Finally, we allow for integration of equity markets. Here we allow households to hold shares issued by foreign traded-good firms as well as domestic equities and the international bond. This case amounts to full financial integration in our model because households have no reason to hold the equity issued by foreign firms producing non-traded goods. This is not to say that markets are complete. In all three cases, the array of assets available to households is insufficient to provide complete risk-sharing.

Below we first describe the production of traded and non-traded goods. Next we present the consumption, saving and portfolio choice problem facing households. Finally, we characterize the market clearing conditions that apply under different degrees of financial market integration.

2.1 Production

The traded goods sector in each country is populated by a continuum of identical firms. Each firm owns its own capital and issues equity on the domestic stock market. Period t production by a representative firm in the traded goods sector of the H country is

$$Y_t^{\mathrm{T}} = Z_t^{\mathrm{T}} K_t^{\theta}, \tag{1}$$

with $\theta > 0$, where K_t denotes the stock of physical capital at the start of the period, and Z_t^{T} is the exogenous state of productivity. The output of traded goods in the F country, $\hat{Y}_t^{\mathrm{T}}$, is given by an identical production function using foreign capital $\hat{K}_t$, and productivity $\hat{Z}_t^{\mathrm{T}}$. Hereafter we use "ˆ" to denote foreign variables. The traded goods produced by H and F firms are identical and can be costlessly transported between countries. Under these conditions, the law of one price must prevail for traded goods to eliminate arbitrage opportunities.

At the beginning of each period, traded goods firms observe the current state of productivity, and then decide how to allocate output between consumption and investment goods. Output allocated to consumption is supplied competitively to domestic and foreign households and the proceeds are used to finance dividend payments to the owner's of the firm's equity. Output allocated to investment adds to the stock of physical capital available for production next period. We assume that firms allocate output to maximize the value of the firm to its shareholders.

Let P_t^{T} denote the ex-dividend price of a share in the representative H firm producing traded-goods at the end of period t, and let D_t^{T} be the dividend per share paid during period t. P_t^{T} and D_t^{T} are measured in terms of H traded goods. We normalize the number of shares issued by the representative traded-good firm to one so the value of the firm at the start of period t is $P_t^{\mathrm{T}} + D_t^{\mathrm{T}}$. H firms allocate output to investment, I_t, by solving

$$\max_{I_t} \left(D_t^{\mathrm{T}} + P_t^{\mathrm{T}} \right), \tag{2}$$

subject to

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + I_t, \\ D_t^{\mathrm{T}} &= Z_t^{\mathrm{T}} K_t^{\theta} - I_t, \end{aligned}$$

where $\delta > 0$ is the depreciation rate on physical capital. The representative firm in the F traded goods sector choose investment $\hat{I}_t$ to solve an analogous problem. Notice that firms do not have the option of financing additional investment through the issuance of additional equity or corporate debt. Additional investment can only be undertaken at the expense of current dividends.

The production of non-traded goods does not require any capital. The output of non-traded goods by representative firms in countries H and F is given by

$$Y_t^{\mathrm{N}} = \kappa Z_t^{\mathrm{N}}, \tag{3a}$$
$$\hat{Y}_t^{\mathrm{N}} = \kappa \hat{Z}_t^{\mathrm{N}}, \tag{3b}$$

where $\kappa > 0$ is a constant. Z_t^{N} and $\hat{Z}_t^{\mathrm{N}}$ denote the period-t state of non-traded goods productivity in countries H and F respectively. The output of non-traded goods can only be consumed by domestic households. The resulting proceeds are then distributed in the form of dividends to owners of equity. As above, we normalize the number of shares issued by the representative firms to unity, so period t dividends for H firms are $D_t^{\mathrm{N}} = Y_t^{\mathrm{N}}$, and for F firms are $\hat{D}_t^{\mathrm{N}} = \hat{Y}_t^{\mathrm{N}}$. We denote the ex-dividend price of a share in the representative H and F firm, measured in terms of non-traded goods, as P_t^{N} and $\hat{P}_t^{\mathrm{N}}$ respectively.

Productivity in the traded and non-traded good sectors is governed by an exogenous productivity process. In particular, we assume that the vector $z_t \equiv [\ln Z_t^{\mathrm{T}}, \ln \hat{Z}_t^{\mathrm{T}}, \ln Z_t^{\mathrm{N}}, \ln \hat{Z}_t^{\mathrm{N}}]'$ follows an AR(1) process:

$$z_t = a z_{t-1} + e_t, \tag{4}$$

where e_t is a (4×1) vector of i.i.d. normally distributed, mean zero shocks with covariance Ω_e.

2.2 Households

Each country is populated by a continuum of households who have identical preferences over the consumption of traded and non-traded goods. The preferences of a representative household in country H are given by

$$\mathbb{U}_t = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i U(C^{\mathrm{T}}_{t+i}, C^{\mathrm{N}}_{t+i}), \tag{5}$$

where $0 < \beta < 1$ is the discount factor, and $U(.)$ is a concave sub-utility function defined over the consumption of traded and non-traded goods, C^{T}_t and C^{N}_t :

$$U(C^{\mathrm{T}}, C^{\mathrm{N}}) = \frac{1}{\phi} \ln \left[\lambda_{\mathrm{T}}^{1-\phi} \left(C^{\mathrm{T}}_t\right)^{\phi} + \lambda_{\mathrm{N}}^{1-\phi} \left(C^{\mathrm{N}}_t\right)^{\phi} \right],$$

with $\phi < 1$. λ_{T} and λ_{N} are the weights the household assigns to tradable and non-tradable consumption respectively. The elasticity of substitution between tradable and non-tradable consumption is $(1-\phi)^{-1} > 0$. Preferences for households in country F are similarly defined in terms of foreign consumption of tradables and non-tradables, $\hat{C}^{\mathrm{T}}_t$ and $\hat{C}^{\mathrm{N}}_t$.

The array of financial assets available to households differs according to the degree of financial integration. Under financial autarky (FA), households can hold their wealth in the form of equity issued by domestic firms in the traded and non-traded goods sectors. Under partial integration (PI), households can hold internationally traded bonds in addition to their domestic equity holdings. The third case we consider is that of full integration (FI). Here households can hold domestic equity, international bonds and equity issued by firms in the foreign traded-goods sector.

The household budget constraint associated with each of these different financial structures can be written in a simple common form. In the case of the representative H household, we write

$$W_{t+1} = R^{\mathrm{P}}_{t+1} \left(W_t - C^{\mathrm{T}}_t - Q^{\mathrm{N}}_t C^{\mathrm{N}}_t \right), \tag{6}$$

where Q^{N}_t is the relative price of H non-tradables in terms of tradables. R^{P}_{t+1} is the (gross) return on wealth between period t and $t+1$, where wealth, W_t, is measured in terms of tradables. The return on wealth depends on how the household allocates wealth across the available array of financial assets, and on the realized return on those assets. In the FI case, the return is given by

$$R^{\mathrm{P}}_{t+1} = R_t + \alpha^{\mathrm{T}}_t (R^{\mathrm{T}}_{t+1} - R_t) + \alpha^{\hat{\mathrm{T}}}_t (R^{\hat{\mathrm{T}}}_{t+1} - R_t) + \alpha^{\mathrm{N}}_t (R^{\mathrm{N}}_{t+1} - R_t), \tag{7}$$

where; R_t is the return on bonds, R^{T}_{t+1} and $R^{\hat{\mathrm{T}}}_{t+1}$ are the returns on H and F tradable equity, and R^{N}_{t+1} is the return on H non-tradable equity. The fraction of wealth held in H and F tradable equity and H non-tradable equity are $\alpha^{\mathrm{T}}_t, \alpha^{\hat{\mathrm{T}}}_t$ and α^{N}_t respectively. In the PI case, H households cannot hold F tradable equity, so $\alpha^{\hat{\mathrm{T}}}_t = 0$. Under FA, households can only hold domestic equity so $\alpha^{\hat{\mathrm{T}}}_t = 0$ and $\alpha^{\mathrm{T}}_t + \alpha^{\mathrm{N}}_t = 1$.

The budget constraint for F households is similarly represented by

$$\hat{W}_{t+1} = \hat{R}^{\mathrm{P}}_{t+1} (\hat{W}_t - \hat{C}^{\mathrm{T}}_t - \hat{Q}^{\mathrm{N}}_t \hat{C}^{\mathrm{N}}_t), \tag{8}$$

with

$$\hat{R}^{\mathrm{P}}_{t+1} = R_t + \hat{\alpha}^{\mathrm{T}}_t(\hat{R}^{\mathrm{T}}_{t+1} - R_t) + \hat{\alpha}^{\hat{\mathrm{T}}}_t(\hat{R}^{\hat{\mathrm{T}}}_{t+1} - R_t) + \hat{\alpha}^{\hat{\mathrm{N}}}_t(\hat{R}^{\hat{\mathrm{N}}}_{t+1} - R_t), \tag{9}$$

where $\hat{R}^{\mathrm{T}}_{t+1}$, and $\hat{R}^{\hat{\mathrm{T}}}_{t+1}$ denote the return on H and F tradable equity, and $\hat{R}^{\hat{\mathrm{N}}}_{t+1}$ is the return on F non-tradable equity. Although these returns are also measured in terms of tradeables, they can differ from the returns available to H households. In particular, the returns on non-tradable equity received by F households, $\hat{R}^{\hat{\mathrm{N}}}_{t+1}$, will in general differ from the returns received by H households because the assets are not internationally traded. Arbitrage will equalize returns in other cases. In particular, if bonds are traded, the interest received by H and F households must be the same as (7) and (9) show. Similarly, arbitrage will equalize the returns on tradable equity in the case of partial and full integration so that $R^{\mathrm{T}}_{t+1} = \hat{R}^{\mathrm{T}}_{t+1}$ and $R^{\hat{\mathrm{T}}}_{t+1} = \hat{R}^{\hat{\mathrm{T}}}_{t+1}$.

2.3 Market Clearing

The market clearing requirements of the model are most easily stated if we normalize the national populations to unity, as well as the population of firms in the tradeable and non-tradeable sectors. Output and consumption of traded and non-traded goods can now be represented by the output and consumption of representative households and firms. In particular, the market clearing conditions in the non-tradeable sector of each country are given by

$$C^{\mathrm{N}}_t = Y^{\mathrm{N}}_t, \qquad \text{and} \qquad \hat{C}^{\mathrm{N}}_t = \hat{Y}^{\mathrm{N}}_t. \tag{10}$$

Recall that firms in the non-traded sector pay dividends to their shareholders with the proceeds from the sale of non-tradeables to households. Thus, market clearing in the non-traded sector also implies that

$$D^{\mathrm{N}}_t = Y^{\mathrm{N}}_t, \qquad \text{and} \qquad \hat{D}^{\mathrm{N}}_t = \hat{Y}^{\mathrm{N}}_t. \tag{11}$$

The market clearing conditions in the tradeable goods market are equally straightforward. Recall that the traded goods produced by H and F firms are identical and can be costlessly transported between countries. Market clearing therefore requires that the world demand for tradables equals world output less the amount allocated to investment:

$$C^{\mathrm{T}}_t + \hat{C}^{\mathrm{T}}_t = Y^{\mathrm{T}}_t + \hat{Y}^{\mathrm{T}}_t - I_t - \hat{I}_t. \tag{12}$$

Next, we turn to market clearing in financial markets. Let A^{T}_t, $A^{\hat{\mathrm{T}}}_t$and A^{N}_t denote the number of shares of H tradeable, F tradeable and H non-tradeable firms held by H households between the end of periods t and $t+1$. F household share holdings in H tradeable, F tradeable and F non-tradeable firms are represented by $\hat{A}^{\mathrm{T}}_t$, $\hat{A}^{\hat{\mathrm{T}}}_t$ and $\hat{A}^{\hat{\mathrm{N}}}_t$. H and F household holdings of bonds between the end of periods t and $t+1$ are denoted by B_t and $\hat{B}_t$. Household demand for equity and bonds are determined by their optimal choice of portfolio shares (i.e., α^{T}_t, $\alpha^{\hat{\mathrm{T}}}_t$ and α^{N}_t for H households, and $\hat{\alpha}^{\mathrm{T}}_t$, $\hat{\alpha}^{\hat{\mathrm{T}}}_t$ and $\hat{\alpha}^{\hat{\mathrm{N}}}_t$ for F households) described below. We assume that bonds are in zero net supply. We also normalized the number of outstanding shares issued by firms in each sector to unity.

The market clearing conditions in financial markets vary according to the degree of financial integration.

Under financial autarky, households can only hold the equity issued by domestically located firms, so the equity market clearing conditions are

$$\text{HOME:} \qquad 1 = A_t^{\mathrm{T}}, \qquad 0 = A_t^{\hat{\mathrm{T}}}, \qquad \text{and} \qquad 1 = A_t^{\mathrm{N}}, \tag{13a}$$

$$\text{FOREIGN:} \qquad 0 = \hat{A}_t^{\mathrm{T}}, \qquad 1 = \hat{A}_t^{\hat{\mathrm{T}}}, \qquad \text{and} \qquad 1 = \hat{A}_t^{\hat{\mathrm{N}}}, \tag{13b}$$

while bond market clearing requires that

$$0 = B_t, \qquad \text{and} \qquad 0 = \hat{B}_t. \tag{14}$$

Notice that FA rules out the possibility of international borrowing or lending, so neither country can run at positive or negative trade balance. Domestic consumption of tradeables must therefore equal the fraction of tradeable output not allocated to investment. Hence, market clearing under FA also implies that

$$D_t^{\mathrm{T}} = C_t^{\mathrm{T}}, \qquad \text{and} \qquad \hat{D}_t^{\mathrm{T}} = \hat{C}_t^{\mathrm{T}}. \tag{15}$$

Under partial integration, households can hold bonds in addition to domestic equity holdings. In this case, equity market clearing requires the conditions in (13), but the bond market clearing condition becomes

$$0 = B_t + \hat{B}_t. \tag{16}$$

The bond market can now act as the medium for international borrowing and lending, so there is no longer a balanced trade requirement restricting dividends. Instead, the goods market clearing condition in (12) implies that

$$D_t^{\mathrm{T}} + \hat{D}_t^{\mathrm{T}} = C_t^{\mathrm{T}} + \hat{C}_t^{\mathrm{T}}. \tag{17}$$

Under full integration, households have access to domestic equity, international bonds and foreign equity issued by firms in the tradeable sector. In this case market clearing in equity markets requires that

$$\text{TRADEABLE} \quad : \qquad 1 = A_t^{\mathrm{T}} + \hat{A}_t^{\mathrm{T}}, \qquad \text{and} \qquad 1 = A_t^{\hat{\mathrm{T}}} + \hat{A}_t^{\hat{\mathrm{T}}}, \tag{18a}$$

$$\text{NON-TRADEABLE} \quad : \qquad 1 = A_t^{\mathrm{N}}, \qquad \text{and} \qquad 1 = \hat{A}_t^{\hat{\mathrm{N}}}. \tag{18b}$$

Market clearing in the bond market continues to require condition (16) so tradeable dividends satisfy (17). In the full integration case, international borrowing and lending takes place via trade in international bonds and the equity of H and F firms producing tradeable goods.

3 Equilibrium

An equilibrium in our world comprises a set of asset prices and relative goods prices that clear markets given the state of productivity, the optimal investment decisions of firms producing tradeable goods, and the optimal consumption, savings and portfolios decisions of households. Since markets are incomplete under all three levels of financial integration we consider, an equilibrium can only be found by solving the firm and households' problems for a conjectured set of equilibrium price processes, and then checking that resulting decisions are indeed consistent with market clearing. In this section, we first characterize the solutions to the optimization problems facing households and firms. We then describe a procedure for finding the equilibrium price processes.

3.1 Consumption, Portfolio and Dividend Choices

Consider the problem facing a H household under full financial integration. In this case the H household chooses consumption of tradeable and non-tradeable goods, C_t^{T} and C_t^{N}, and portfolio shares for equity in H and F firms producing tradeables and H firms producing non-tradeables, α_t^{T}, $\alpha_t^{\hat{\mathrm{T}}}$ and α_t^{N}, to maximize expected utility (5) subject to (6) and (7) given current equity prices P_t^{T} $P_t^{\hat{\mathrm{T}}}$ and P_t^{N}, the interest rate on bonds R_t, and the relative price of non-tradeables Q_t^{N}. The first order conditions for this problem are

$$\begin{aligned} Q_t^{\mathrm{N}} &= \frac{\partial U/\partial C_t^{\mathrm{N}}}{\partial U/\partial C_t^{\mathrm{T}}}, && (19a)\\ 1 &= \mathbb{E}_t\left[M_{t+1}R_{t+1}^{\mathrm{T}}\right], && (19b)\\ 1 &= \mathbb{E}_t\left[M_{t+1}R_{t+1}^{\mathrm{N}}\right], && (19c)\\ 1 &= \mathbb{E}_t\left[M_{t+1}R_t\right], && (19d)\\ 1 &= \mathbb{E}_t\left[M_{t+1}R_{t+1}^{\hat{\mathrm{T}}}\right], && (19e) \end{aligned}$$

where $M_{t+1} \equiv \beta\left(\partial U/\partial C_{t+1}^{\mathrm{T}}\right)/\left(\partial U/\partial C_t^{\mathrm{T}}\right)$ is the discounted intertemporal marginal rate of substitution (IMRS) between the consumption of tradeables in period t and period $t+1$. Condition (19a) equates the relative price of non-tradeables to the marginal rate of substitution between the consumption of tradeables and non-tradeables. Under financial autarky, consumption and portfolio decisions are completely characterized by (19a) - (19c). When households are given access to international bonds, there is an extra dimension to the portfolio choice problem facing households so (19d) is added to the set of first order conditions. Under full financial integration, all the conditions in (19) are needed to characterize optimal household behavior at H country.

It is important to note that all the returns in (19) are measured in terms of tradeables. In particular, the return on the equity of firms producing tradeable goods in the H and F counties are

$$R_{t+1}^{\mathrm{T}} = \left(P_{t+1}^{\mathrm{T}} + D_{t+1}^{\mathrm{T}}\right)/P_t^{\mathrm{T}}, \qquad \text{and} \qquad R_{t+1}^{\hat{\mathrm{T}}} = \left(\hat{P}_{t+1}^{\mathrm{T}} + \hat{D}_{t+1}^{\mathrm{T}}\right)/\hat{P}_t^{\mathrm{T}}. \tag{20}$$

Because the law of one price applies to tradeable goods, these equations also define the return FOREIGN households receive on their equity holdings in H and F firms producing tradeable goods. In other words, $\hat{R}_{t+1}^{\mathrm{T}} = R_{t+1}^{\mathrm{T}}$ and $\hat{R}_{t+1}^{\hat{\mathrm{T}}} = R_{t+1}^{\hat{\mathrm{T}}}$. The law of one price similarly implies that the return on bonds R_t is the same for all households. By contrast, the returns on equity producing non-tradeable goods differ. In particular, the return on equity for H households is

$$R_{t+1}^{\mathrm{N}} = \left\{\left(P_{t+1}^{\mathrm{N}} + D_{t+1}^{\mathrm{N}}\right)/P_t^{\mathrm{N}}\right\}\left\{Q_{t+1}^{\mathrm{N}}/Q_t^{\mathrm{N}}\right\}, \tag{21}$$

while for F households the return is

$$\hat{R}^{\hat{\mathrm{N}}}_{t+1} = \left\{ \left(\hat{P}^{\mathrm{N}}_{t+1} + \hat{D}^{\mathrm{N}}_{t+1} \right) / \hat{P}^{\mathrm{N}}_{t} \right\} \left\{ \hat{Q}^{\mathrm{N}}_{t+1} / \hat{Q}^{\mathrm{N}}_{t} \right\} \tag{22}$$

where Q^{N}_t is the relative price of non-tradeables in country F. R^{N}_{t+1} and $\hat{R}^{\hat{\mathrm{N}}}_{t+1}$ will differ from each other in our model for two reasons. First, international productivity differentials in the non-tradeable sectors will create differences in returns measured in terms of non-tradeables. These differences will affect returns via the first term on the right hand side of (21) and (22). Second, international differences in the dynamics of relative prices Q^{N}_t and $\hat{Q}^{\mathrm{N}}_t$ will affect returns via the second term in each equation. These differences arise quite naturally in equilibrium as the result of productivity shocks in either the tradeable or non-tradeable sectors. Variations in the relative prices of non-traded goods also drive the real exchange rate, which is defined as the ratio of price indices in the two countries:

$$Q_t = \left\{ \frac{\lambda_T + \lambda_N \left(Q^{\mathrm{N}}_t \right)^{\frac{\phi}{\phi-1}}}{\lambda_T + \lambda_N (\hat{Q}^{\mathrm{N}}_t)^{\frac{\phi}{\phi-1}}} \right\}^{\frac{\phi-1}{\phi}} . \tag{23}$$

The returns on equity shown in (20) - (22) are functions of equity prices, the relative price of non-tradeables, and the dividends paid by firms. The requirements of market clearing and our specification for the production of non-traded goods implies that dividends D^{N}_{t+1} and $\hat{D}^{\mathrm{N}}_{t+1}$ are exogenous. By contrast, the dividends paid by firms producing tradeable goods are determined optimally. Recall that H firms choose real investment I_t in period t to maximize the current value of the firm, $D^{\mathrm{T}}_t + P^{\mathrm{T}}_t$. Combining (19b) with the definition of returns R^{T}_{t+1} in (20) implies that $P^{\mathrm{T}}_t = \mathbb{E}_t \left[M_{t+1} \left(P^{\mathrm{T}}_{t+1} + D^{\mathrm{T}}_{t+1} \right) \right]$. This equation identifies the price a H household would pay for equity in the firm (after period $-\ t$ dividends have been paid). Using this expression to substitute for P^{T}_t in the H firm's investment problem (2) gives the following first order condition:

$$1 = \mathbb{E}_t \left[M_{t+1} \left(\theta Z^{\mathrm{T}}_{t+1} \left(K_{t+1} \right)^{\theta-1} + (1-\delta) \right) \right]. \tag{24}$$

This condition implicitly identifies the optimal level of dividends in period t because next period's capital depends on current capital, productivity and dividend payments: $K_{t+1} = (1-\delta)K_t + Z^{\mathrm{T}}_t K^{\theta}_t - D^{\mathrm{T}}_t$. Dividends on the equity of F firms producing tradeable goods is similarly determined by

$$1 = \mathbb{E}_t \left[\hat{M}_{t+1} \left(\theta \hat{Z}^{\mathrm{T}}_{t+1} (\hat{K}_{t+1})^{\theta-1} + (1-\delta) \right) \right], \tag{25}$$

where $\hat{M}_{t+1}$ is the IMRS for tradeable goods in country F, and $\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \hat{Z}^{\mathrm{T}}_t \hat{K}^{\theta}_t - \hat{D}^{\mathrm{T}}_t$.

The dividend policies implied by (24) and (25) maximize the value of each firm from the perspective of *domestic* shareholders. For example, the stream of dividends $\{D^{\mathrm{T}}_t\}$ implied by (24) maximizes the value of H firm producing traded goods for households in country H because the firm uses M_{t+1} to value future dividends. This is an innocuous assumption under financial autarky and partial integration because domestic households must hold all the firm's equity. Under full integration, however, foreign households have the opportunity to hold the firm's equity so the firm's dividend policy need not maximize the value of equity to all shareholders. In particular, since markets are incomplete even under full integration, the IMRS for H and F households will differ, so F households holding domestic equity will generally prefer a different dividend stream from the one implied by (24). In short, the dividend streams implied by (24) and (25) incorporate a form of home bias because they focus exclusively on the interests of domestic shareholders.

We can now summarize the equilibrium actions of firms and households. At the beginning of period t, firms in the traded-goods sector observe the new level of productivity and decide on the amount of real investment to undertake. This decision determines dividend payments D_t^{T} and $\hat{D}_t^{\mathrm{T}}$, as a function of existing productivity, physical capital and expectations regarding future productivity and the IMRS of domestic shareholders. Firms in the non-tradeable sectors have no real investment decision to make so in equilibrium D_t^{N} and $\hat{D}_t^{\mathrm{N}}$ depend only on current productivity. At the same time, households begin period t with a portfolio of financial assets. Under financial autarky the menu of assets is restricted to domestic equities, under partial integration households may hold domestic equities and bonds, and under full financial integration the menu may contain domestic equity, foreign equity and bonds. Households receive dividend payments from firms according to the composition of their portfolios. They then make consumption and new portfolio decisions based on the market clearing relative price for non-tradeables, and the market-clearing (ex-dividend) prices for equity. The first-order conditions in (19) implicitly identify the decisions made by H households. The decisions made by F households are characterized by an analogous set of equations. The portfolio shares determined in this manner will depend on household expectations concerning future returns and the IMRS. As equations (20) - (22) show, equity returns are a function of current equity prices and future dividends and prices, so expectations regarding the latter will be important for determining how households choose portfolios in period t. Current and future consumption decisions also affect period $-\ t$ portfolios shares through the IMRS. Household demand for financial assets in period t follows from decisions on consumption and the portfolio shares in a straightforward manner. In the case of full financial integration, the demand for each asset from H and F households are

$$\begin{array}{lll}
 & \text{HOME} & \text{FOREIGN} \\
\text{H TRADEABLE EQUITY:} & A_t^{\mathrm{T}} = \alpha_t^{\mathrm{T}} W_t^{\mathrm{C}} / P_t^{\mathrm{T}}, & \hat{A}_t^{\mathrm{T}} = \hat{\alpha}_t^{\mathrm{T}} \hat{W}_t^{\mathrm{C}} / P_t^{\mathrm{T}}, \\
\text{F TRADEABLE EQUITY:} & A_t^{\hat{\mathrm{T}}} = \alpha_t^{\hat{\mathrm{T}}} W_t^{\mathrm{C}} / \hat{P}_t^{\mathrm{T}}, & \hat{A}_t^{\hat{\mathrm{T}}} = \hat{\alpha}_t^{\hat{\mathrm{T}}} \hat{W}_t^{\mathrm{C}} / \hat{P}_t^{\mathrm{T}}, \\
\text{NON-TRADEABLE EQUITY:} & A_t^{\mathrm{N}} = \alpha_t^{\mathrm{N}} W_t^{\mathrm{C}} / Q_t^{\mathrm{N}} P_t^{\mathrm{N}}, & \hat{A}_t^{\mathrm{N}} = \hat{\alpha}_t^{\mathrm{N}} \hat{W}_t^{\mathrm{C}} / \hat{Q}_t^{\mathrm{N}} \hat{P}_t^{\mathrm{N}}, \\
\text{BONDS} & B_t = \alpha_t^{\mathrm{B}} W_t^{\mathrm{C}} R_t, & \hat{B}_t = \hat{\alpha}_t^{\mathrm{B}} \hat{W}_t^{\mathrm{C}} R_t,
\end{array} \tag{26}$$

where $W_t^{\mathrm{C}} \equiv W_t - C_t^{\mathrm{T}} - Q_t^{\mathrm{N}} C_t^{\mathrm{N}}$ and $\hat{W}_t^{\mathrm{C}} \equiv \hat{W}_t - \hat{C}_t^{\mathrm{T}} - \hat{Q}_t^{\mathrm{N}} \hat{C}_t^{\mathrm{N}}$ denote period $-\ t$ wealth net of consumption expenditure and $\alpha_t^{\mathrm{B}} \equiv 1 - \alpha_t^{\mathrm{T}} - \alpha_t^{\hat{\mathrm{T}}} - \alpha_t^{\mathrm{N}}$, $\hat{\alpha}_t^{\mathrm{B}} \equiv 1 - \hat{\alpha}_t^{\mathrm{T}} - \hat{\alpha}_t^{\hat{\mathrm{T}}} - \hat{\alpha}_t^{\mathrm{N}}$. Equation (26) shows that asset demands depend on expected future returns and risk via optimally chosen portfolio shares, α_t, accumulated net wealth W_t^{C} and $\hat{W}_t^{\mathrm{C}}$, and current asset prices (i.e., P_t^{T}, $\hat{P}_t^{\mathrm{T}}, P_t^{\mathrm{N}}$ and $\hat{P}_t^{\mathrm{N}}$ for equity, and $1/R_t$ for bonds). All of these factors vary in the equilibria we study.

3.2 Equilibrium Dynamics

Finding an equilibrium in this model is conceptually straightforward. All that is required are the time series processes for equity prices $\{P_t^{\mathrm{T}}, \hat{P}_t^{\mathrm{T}}, P_t^{\mathrm{N}}$ and $\hat{P}_t^{\mathrm{N}}\}$, the relative prices of non-tradeables $\{Q_t^{\mathrm{N}}$ and $\hat{Q}_t^{\mathrm{N}}\}$, and interest rate on bonds R_t, that clear markets given the optimal behavior of firms and households. Finding these time series in practice is complicated by the need to completely characterize how firms and households behave. When markets are complete, this complication can be circumvented by finding the equilibrium allocations as the solution of an appropriate social planning problem and then deriving the price and interest rates processes that support these allocations when decision–making is decentralized. This solution method is inapplicable in our model. When markets are incomplete, as they under financial autarky, partial integration, and full financial integration, there is no way to formulate a social planning problem that will provide the equilibrium allocation of the decentralized market economy. To solve the model, we must therefore characterize the optimal behavior of firms and households for a wide class of price and interest rate processes, and then use the implied allocations in conjunction with the market clearing conditions to find the particular set of price and interest rate processes that clear markets. We implement this solution procedure as follows.

Our first step is to conjecture the form of the vector of state variables that characterize the equilibrium dynamics of the economy. For this purpose, let $k_t \equiv \ln(K_t/K)$ and $\hat{k}_t = \ln\left(\hat{K}_t/K\right)$ where K is the steady state capital stock for firms producing tradeable goods. Our conjecture for the state vector is given by

$$x_t = [z_t, k_t, \hat{k}_t, w_t, \hat{w}_t]',$$

where $w_t \equiv \ln(W_t/W_0)$, $\hat{w}_t \equiv \ln(\hat{W}_t/\hat{W}_0)$ and $z_t \equiv [\ln Z_t^{\mathrm{T}}, \ln \hat{Z}_t^{\mathrm{T}}, \ln Z_t^{\mathrm{N}}, \ln \hat{Z}_t^{\mathrm{N}}]'$. Our conjecture for x_t contains the current state of productivity, the capital stocks in the H and F traded-goods sectors relative to their steady state levels, and the wealth of H and F households relative to their initial levels W_0 and $\hat{W}_0$. All eight variables are needed to characterize the optimal period–t decisions of firms and households, and period–t market clearing prices.

The next step is to characterize the dynamics of x_t. Nonlinearities in our model make it impossible to describe the dynamics of x_t using just its own lagged values. When households face portfolio choice problems, wealth in period t will depend on the first and second moments of returns conditioned on period–$t-1$ information. In general, these moments will be high order polynomials in the elements of x_{t-1} (e.g., $w_{t-1}^2, \hat{w}_{t-1}^2, w_{t-1}\hat{w}_{t-1}, ...w_{t-1}^3,$), so elements of x_t will depend on not just x_{t-1} but also elements in $x_{t-1}x_{t-1}'$ and so on. We consider an approximate solution to the model that ignores the impact of third and higher order terms. Under this assumption, we conjecture that the dynamics of the economy can be summarized by

$$X_{t+1} = \mathbb{A}X_t + U_{t+1}, \tag{27}$$

where $X_{t+1} \equiv [\begin{array}{ccc} 1 & x_{t+1} & \tilde{x}_{t+1} \end{array}]'$, $\tilde{x}_{t+1} \equiv vec\left(x_{t+1}x_{t+1}'\right)$ and U_{t+1} is a vector of shocks with $\mathbb{E}\left[U_{t+1}|X_t\right] = 0$, and $\mathbb{E}\left[U_{t+1}U_{t+1}'|X_t\right] = \mathbb{S}(X_t)$. Equation (27) describes the approximate dynamics of the augmented state vector X_t that contains a constant, the original state vector x_t and all the cross-products of x_t in $\tilde{x}_t$. Notice that X_{t+1} depends linearly on lagged X_t so forecasting future states of the economy is straightforward: $E\left[X_{t+1}|X_t\right] = \mathbb{A}X_t$. Since firms and households based their period–t decisions on expectations concerning variables in $t+1$, this aspect of (27) is useful when checking the optimality of decision-making (see below). Equation (27) also introduces conditional heteroskedasticity into the state variables via the $\mathbb{S}(.)$ function. Heteroskedasticity arises endogenously in our model if households change the composition of their portfolios, so our conjecture for the equilibrium dynamics of the state variables must allow the covariance of U_{t+1} to vary with elements of X_t.

The final step is to find the elements of the $\mathbb{A}$ matrix and the covariance function $\mathbb{S}(.)$ implied by the equilibrium of the model. Some elements of $\mathbb{A}$ and $\mathbb{S}(.)$ are simple functions of the model's parameters, others depend on the decisions made by households and firms. To find these elements, we use the method of undetermined coefficients. Specifically, we posit that the log dividend, log consumption and portfolio shares in period t can be written as particular linear functions of the augmented state vector, X_t. With these functions we can then characterize the dynamics of capital and wealth from period to period, and hence fill in all the unknown elements of the $\mathbb{A}$ matrix and the covariance function $\mathbb{S}(.)$. We also use the assumed form of period–t decisions in conjunction with the market clearing conditions to derive expressions for equilibrium equity prices, relative prices and the interest rate as log linear functions of X_t. Lastly, we verify that the assumed form of the period–t decisions are consistent with the firm and household first order conditions given the equilibrium price and return dynamics implied by (27). Appendix A contains a detailed description of this procedure.

Two further aspects of this solution procedure deserve emphasis. First, it does not make any assumption about the stationarity of individual state variables. In the calibrated version of the model we examine below, productivity is assumed to follow a stationary process, but capital and wealth are free for follow unit root processes in equilibrium. This turns out to be a useful feature of the procedure. As we discuss in detail below, there are good economic reasons for transitory shocks to productivity to have permanent effects on equilibrium wealth in our model. So a solution procedure that imposed stationarity on wealth would be inappropriate. Our procedure allows for these permanent wealth effects but in a limited manner. The limitation arises from the second important aspect of our procedure, namely it use of (27). This equation approximates the equilibrium dynamics of the economy under the assumption that terms involving third and higher order powers of the state variables have negligible impact on the elements of x_t. This is reasonable along a sample path where all the elements of x_t are small. However, our specification for x_t contains the log deviation of household wealth from its initial level, $w_t \equiv \ln(W_t/W_0)$, and $\hat{w}_t \equiv \ln(\hat{W}_t/\hat{W}_0)$, so a sequence of transitory productivity shocks could push w_t and $\hat{w}_t$ permanently far from zero. At this point the dynamics of X_t are poorly approximated by (27) and our characterization of the equilibrium would be unreliable. In this sense (27) approximates the dynamics of the economy in a neighborhood near the initial distribution of wealth. We are cognizant of this fact when studying the equilibrium dynamics below. In particular, when simulating the model we check that the sample paths for wealth and capital remain in a neighborhood of their initial distributions so that third order terms are unimportant.

Our solutions to the model use the parameter values summarized in Table 1. We assume that household preferences and firm technologies are symmetric across the two countries, and calibrate the model for a period equalling one quarter. The value for ϕ is chosen to set the intratemporal elasticity of substitution between tradable and non–tradeable at 0.74, consistent with the value in Corsetti, Dedola and Leduc (2003). The share parameters for traded and non-traded goods, λ_{T} and λ_{N} are both set to 0.5, and the discount factor $\beta = 0.99$. On the production side, we set the capital share in tradeable production θ to 0.36, and the depreciation rate δ to 0.02. These values are consistent with the estimates in Backus, Kehoe and Kydland (1995). The only other parameters in the model govern the productivity process. In our benchmark calibration we assume that each of the four productivity processes (i.e. $\ln Z_t^{\text{T}}, \ln \hat{Z}_t^{\text{T}}, \ln Z_t^{\text{N}}$, and $\ln \hat{Z}_t^{\text{N}}$) follow AR(1) processes with independent shocks. The AR(1) coefficients in the processes for tradeable–goods

Table 1: Model Parameters

Preferences	β	λ_{T}	λ_{N}	$1/(1-\phi)$
	0.99	0.5	0.5	0.74
Production	θ	δ		
	0.36	0.02		
Productivity	a_{ii}^{T}	a_{ii}^{N}	Ω_e	
	0.78	0.99	0.0001	

productivity, $\ln Z_t^{\mathrm{T}}$ and $\ln \hat{Z}_t^{\mathrm{T}}$, are 0.78, while the coefficients for non-tradeable productivity, $\ln Z_t^{\mathrm{N}}$, and $\ln \hat{Z}_t^{\mathrm{N}}$, are 0.99. Shocks to all four productivity process have a variance of 0.0001. This specification implies that all shocks have persistent but temporary affects on productivity. Any permanent effects they have on other variables must arise endogenously from the structure of the model.

4 Results

4.1 Simulations

We analyze the implication of our model for the behavior of capital flows as follows. First, we find the parameters of the state process in (27) that characterize the equilibrium under financial autarky FA, partial integration PI, and full financial integration FI. As part of this process, we also find the equilibrium relation between the augmented state vector X_t and the other endogenous variables in the model including the asset holdings of households (i.e., $A_t^{\mathrm{T}}, A_t^{\hat{\mathrm{T}}}, A_t^{\mathrm{N}}$ and B_t for H households and $\hat{A}_t^{\mathrm{T}}, \hat{A}_t^{\hat{\mathrm{T}}}, \hat{A}_t^{\mathrm{N}}$ and $\hat{B}_t$ for F households). Next, we simulate the equilibrium dynamics of X_t over 500 quarters for each financial configuration $\{$FA, PI, FI$\}$. For these simulations we assume that wealth is equally distributed between H and F households in period 0. The first 100 realizations of these simulations are then discarded to eliminate the influence of the initial wealth distribution, leaving us with a sample spanning 100 years. The innovations to equilibrium wealth are small enough to keep H and F wealth close to its initial levels over this span so the approximation error in (27) remains very small. The statistics we report below are derived from 100 simulations for each financial configuration and so are based on 10,000 years of simulated quarterly data in the neighborhood of the initial wealth distribution.

We begin our comparison of the three financial configurations by focusing of the volatility of returns. Table 2 reports the standard deviations of realized returns computed from our model simulations. Column (i) reports volatility under financial autarky. Here we see that the model produces far less volatility in bond and equity returns than we observe in the world. This is not surprising given our very standard specification for productivity, production and preferences. We do note, however, that the relative volatility of returns is roughly in accordance with reality: equity are much more volatility than the risk free rate, and foreign exchange returns, $\Delta q_t \equiv \ln Q_t - \ln Q_{t-1}$, are an order of magnitude more volatile than equity. Note also that the volatility of the return on equity in firms producing non-tradeable goods is almost twice that of the return on firms producing tradeables.

Table 2: Return Volatility, (annual % std. dev.)

	FA (i)	PI (ii)	PI-FA difference (iii)	FI (iv)	FI-PI difference (v)
Risk free rate r_t	0.15%	0.11%	-33.94%	0.11%	0.014%
Tradeable equity r_t^{T}	0.57%	0.44%	-27.17%	0.44%	-0.01%
Non-tradeable equity r_t^{N}	1.09%	0.95%	-14.64%	0.92%	-2.77%
Portfolio returns r_t^{P}	0.80%	0.64%	-21.29%	0.62%	-4.63%
Foreign exchange Δq_t	2.64%	3.56%	29.66%	3.55%	-0.124%

Columns (ii) - (v) of Table 2 show how the volatility of returns change as the degree of financial integration increases. Column (ii) reports the standard deviation of returns for the case of partial integration where households can trade international bonds. Column (iii) shows how the volatility changes relative to the case of Financial Autarky. Opening trade in international bonds reduces the volatility of equity returns and the risk-free rate by approximately one third, while the volatility of foreign exchange returns rises by roughly one third. The volatility of returns changes further when households are given access to foreign tradeable equity. As column (v) shows, the largest changes occur in the volatility of portfolio equity returns, which falls by 5% relative to its level under partial integration. Return on non-tradeable equity decreases by roughly 3%.

Next we turn to the behavior of capital flows. Table 3 compares the behavior of the bond and equity flows between countries in the PI and FI configurations. The bond flows are computed as $\frac{1}{R_t}B_t - B_{t-1}(= -\frac{1}{R_t}\hat{B}_t + \hat{B}_{t-1})$ and equity flows as $P_t^{\mathrm{T}}\Delta A_t^{\mathrm{T}}$ from the equilibrium portfolio shares and wealth as shown in (26). We study their behavior measured relative to GDP, $(B_t - R_tB_{t-1})/R_tY_t^{\mathrm{GDP}}$ and $P_t^{\mathrm{T}}\Delta A_t^{\mathrm{T}}/Y_t^{\mathrm{GDP}}$, where $Y_t^{\mathrm{GDP}} = Y_t^{\mathrm{T}} + Q_t^{\mathrm{N}}Y_t^{\mathrm{N}}$. Column (i) of Table 3 shows that bond flows are extremely volatile under partial financial integration. In this configuration, bonds serve two purposes. First, they allow households to share risks. Second, they provide the only medium through which international borrowing and lending takes place. Under financial autarky, the cross-country correlation in marginal utility is zero because the productivity shocks hitting each sector are independent. Under partial integration, this correlation rises to 0.52, so the creation of an international bond market facilitates a lot of risk-sharing. Columns (ii) and (iii) show how the volatility of capital flows is affected by opening both bond and foreign equity markets. It is important to remember that these flows are computed from the equilibrium in which full financial integration is established and do not include any of the adjustment flows that would accompany the opening of foreign equity markets. Two features stand out from the table. First, bond flows are much less volatile than they were under PI. Second, bond and equity flows display different degrees of volatility, with bond flows being almost twice as large and as volatile as equity flows.

As one would expect, the degree of risk-sharing in this configuration is higher than in the PI case - the cross-country correlation in marginal utility is 0.67. Access to foreign equity allows households to share more risk, but markets are still incomplete.

What lies behind the changing behavior of capital flows depicted in Table 3? To address this question

Table 3: International Portfolio Flows

	Partial Integration PI	Full Integration FI	
	bonds	bonds	equity
	(i)	(ii)	(iii)
std.	6.06%	1.64%	0.88%
min	-29.23%	-6.53%	-3.96%
max	30.99%	6.20%	4.17%
mean abs	4.83%	1.31%	0.70%

we must return to the structure of the model. Using (26) we can write the equity flow as

$$\begin{aligned} P_t^{\mathrm{T}} \Delta A_t^{\mathrm{T}} &= \alpha_t^{\mathrm{T}} W_t^{\mathrm{C}} - \alpha_{t-1}^{\mathrm{T}} W_{t-1}^{\mathrm{C}} \frac{P_t^{\mathrm{T}}}{P_{t-1}^{\mathrm{T}}}, \\ &= \Delta\alpha_t^{\mathrm{T}} W_t^{\mathrm{C}} + \left[\alpha_{t-1}^{\mathrm{T}} \Delta W_t^{\mathrm{C}} - \left(\frac{P_t^{\mathrm{T}}}{P_{t-1}^{\mathrm{T}}} - 1 \right) W_{t-1}^{\mathrm{C}} \alpha_{t-1}^{\mathrm{T}} \right]. \end{aligned} \tag{28}$$

The first term in the second line captures investor's desire to change shares due to changes in expected returns and risk. Bohn and Tesar's (1996) name this term the "return chasing" component of the portfolio flow. The second term reflects investor's intention to acquire or sell off some of the asset when her wealth changes or when there are some capital gains or losses on the existing portfolio. This term is called "portfolio rebalancing" component. Bond flows can be decomposed in a similar manner:

$$\frac{1}{R_t} B_t - B_{t-1} = \Delta\alpha_t^{\mathrm{B}} W_t^{\mathrm{C}} + \left[\alpha_{t-1}^{\mathrm{B}} \Delta W_t^{\mathrm{C}} - (R_{t-1} - 1) W_{t-1}^{\mathrm{C}} \alpha_{t-1}^{\mathrm{B}} \right]. \tag{29}$$

Table 4 reports the contribution each component makes to the variance of the bond and equity flows. Under PI, variations in the return chasing component are the main source of volatility in bond flows. Portfolio rebalancing, identified by the second term in (29), plays an insignificant role. This is not a surprising result. The statistics in Table 4 are based on the equilibrium dynamics of the economy in the neighborhood of an initial symmetric wealth distribution, so the bond position of households at the beginning of each period is typically a small fraction of total wealth. Under these circumstances $\alpha_{t-1}^{\mathrm{B}} \cong 0$, so the second term in (29) makes a negligible contribution to bond flows.

Table 4: Variance Decomposition of International Portfolio Flows

	Partial Integration PI		Full Integration FI	
	Return Chasing	Portfolio Rebalancing	Return Chasing	Portfolio Rebalancing
	(i)	(ii)	(iii)	(iv)
Equity	–	–	2.38	-1.38
Bonds	0.993	0.007	0.991	0.009

Table 4 also shows that both the "return chasing" and "portfolio rebalancing" components contributed to international equity flows under FI. Households increase the share of tradeable equity in their portfolios in response to shocks that increase the price of tradeable equity so the variations in the two components are negatively correlated. The figures of 2.38 and -1.38 in columns (iii) and (iv) mean that a unit of positive equity flow results from an increase of 2.38 units in the "return chasing" component and a 1.38 fall in the "portfolio rebalancing" component.[5] Rebalancing plays a more important role in equity flows because households begin each period with approximately 50 percent of their wealth in tradeable equity, which is evenly split between stock issued by domestic and foreign firms.

The decline in the size and volatility of capital flows in responce to integration predicted by the model also leads us to ask questions about the appropriate measure of financial integration. Conventional measures rely on the volumes of cross-border capital flows to capture the extent of integration. On the contrary, our model provides some evidence that such relation is not necessarily linear and capital flows may even decline in responce to integration. Other commonly used measures of integration are based on the law of one price arguments and use cross-markets premia to capture the degree to which different markets are integrated, in particular, how closely these markets can price identical assets. The estimates of the price-based measures of integration are often contaminated by the aggregation bias that occurs when price indices rather then single asset prices are used to compare different markets. The advantage of our theoretical framework consists of providing us with a direct measure of the pricing kernel under each integration scenario, which in turn allows us to construct theoretical tradeable depositary receipts (DRs) and obtain the return differentials between these DRs and the underlying equity shares at H and F equity markets. The results from such calculations are presented in Table 5. These results suggest that financial integration is well captured by the dynamics of return differentials. In particular, as integration progresses, volatility of return differentials declines.

Table 5. T **Equity Return Differentials (annual, %)**

	Autarky FA	Partial Integration PI	Full Integration FI
mean	-0.002%	0.000%	0.000%
median	0.003%	0.000%	0.000%
std.	0.811%	0.013%	0.002%
min	-3.360%	-0.255%	-0.010%
max	3.597%	0.229%	0.010%
mean abs	0.647%	0.007%	0.002%

4.1.1 Integration and the volatility of capital flows

Our simulations reveal a distinct relation between the degree of financial integration and the volatility of international capital flows. During the early stages of integration, characterized here by the move from the FA to PI equilibrium, the volatility of international capital flows rises significantly. Then, as integration

[5] More precisely, equation (28) implies that

$$V\left(P_t^{\mathrm{T}}\Delta A_t^{\mathrm{T}}\right) = CV\left(P_t^{\mathrm{T}}\Delta A_t^{\mathrm{T}}, \Delta\alpha_t^{\mathrm{T}} W_t^{\mathrm{C}}\right) + CV(P_t^{\mathrm{T}}\Delta A_t^{\mathrm{T}}, \alpha_{t-1}^{\mathrm{T}}\Delta W_t^{\mathrm{C}} - [(P_t^{\mathrm{T}}/P_{t-1}^{\mathrm{T}}) - 1]W_{t-1}^{\mathrm{C}}\alpha_{t-1}^{\mathrm{T}})$$

and Table 4 reports the first and second terms on the right as a fraction of $V\left(P_t^{\mathrm{T}}\Delta A_t^{\mathrm{T}}\right)$.

proceeds further, volatility declines. In short, our model predicts that the volatility in international capital flows should follow a humped-shaped pattern as financial integration increases. Our model also predicts that international capital flows should be facilitated by trade in both bonds and equity at high levels of financial integration. In this section, we shall argue that both results arise as a natural consequence of greater risk sharing.

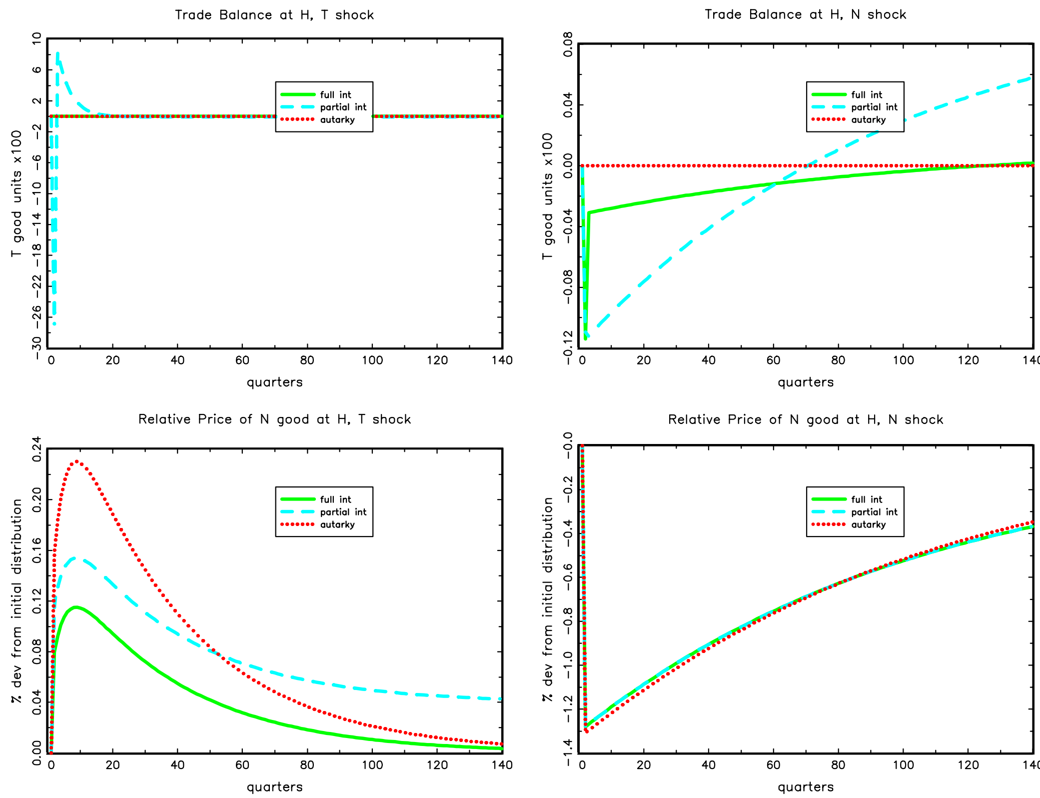


Figure 4a. Real effects of productivity shocks

We begin our explanation on the real side of the economy. In particular, let us consider how a positive productivity shock to domestic firms producing traded goods affects real output and consumption in both countries under our three market configurations. The effects on the trade balance and the relative price of tradeables are shown in the left hand panels of Figure 4a. Recall that productivity shocks only have temporary affects on the marginal product of capital. Thus, a positive productivity shock in the domestic traded-goods sector will induce an immediate one-period rise in real investment as firms in that sector take advantage of the temporarily high marginal product of capital. In short, there is an investment boom in the domestic tradeable goods sector. Because the equity issued by these firms represents a claim on the future dividend stream sustained by the firm's capital stock, one effect of the investment boom is to increase the equilibrium price of tradeable equity P_t^{T}. Under FA, this capital gain raises the wealth of H households so the domestic demand for both tradeable and non-tradeable goods increase. While increased domestic output can accommodate the rise in demand for tradeables, there is no change in the output of non-tradeables, so the relative price of non-tradeables, Q_t^{N}, must rise to clear domestic goods markets.

A similar adjustment pattern occurs under PI. The capital gain enjoyed by H households again translates into increased demand for tradeables and non-tradeables, but now the demand for tradeables can be accommodated by both H and F firms producing tradeables. As a result, the productivity shock is accompanied by a trade deficit in the H country and a smaller rise in Q_t^{N} than under FA. Once the investment boom is over, the domestic supply of tradeables available for consumption rises sharply above domestic consumption. From this point on, the H country runs a trade surplus under PI and FI until the foreign debt incurred during the investment boom is paid off.[6]

In the case of FI, the increase in P_t^{T} represents a capital gain to both H and F households because everyone diversifies their international equity holdings (i.e., all households hold equity issued by H and F firms producing tradeable goods). As a result, the demand for tradeables and non-tradeables rise in both countries. At the same time, by taking a fully diversified positions in T equities, both countries can finance higher tradeable consumption without borrowing from abroad and running a trade deficit. The result is trade balance at every equilibrium point. Market clearing in the non-tradeables markets raises relative prices (i.e. Q_t^{N} and $\hat{Q}_t^{\mathrm{N}}$), but less then under PI.

The right hand panels of Figure 4a show the effects of positive productivity shock in the H non-tradeable sector. Once again, the shock produces a trade deficit under PI, but it is much smaller and persists for much longer than the deficit associated with productivity shocks in the tradeable sector. The reason for this difference arises from the absence of an investment boom. A positive productivity shock in H non-tradeables increases the supply of non-tradeable output available for domestic consumption. This has two equilibrium effects. First, it lowers the relative price of non-tradeables, Q_t^{N}, so that the H market for non-tradeables clears. This is clearly seen in the lower right hand panel of Figure 4a. Second, it raises the H demand for tradeables because tradeables and non-tradeables are complementary consumption goods under our parametrization. The result is a persistent trade deficit. Under FI, in contrast with the effects of T shock, productivity increase in N sector leads to a trade deficit. The size of the deficit on impact is comparable with that under PI, and likewise is financed by borrowing from abroad. However, the amount of such borrowing under FI is much larger as it is used to finance both consumption demand and purchases of a diversified portfolio of T equity shares. In constrast to PI, in the period immediately following the shock, trade deficit falls by more then 50% and slowly reverts back to zero from that new level.

To summarize, the trade balance dynamics displayed in Figure 4a are readily understood in terms of intertemporal consumption smoothing once we recognize that shocks to tradeable productivity induce domestic investment booms. In addition, these dynamics differ under the PI and FI configurations. When given a choice between international bonds and equity, households choose to take fully diversified positions in stocks allowing them to share country specific risks internationally. Then, depending on the productivity shock, bonds are either used to finance the purchases of equity, or become redundant. When equity is not available, bonds must be used to smooth consumption via borrowing and lending.

[6] The trade deficit arises because the rise in the risk free rate reduces investment by F firms producing traded goods by more than the increase in F demand for tradeables, so F tradeable firms export some of their output to H households.

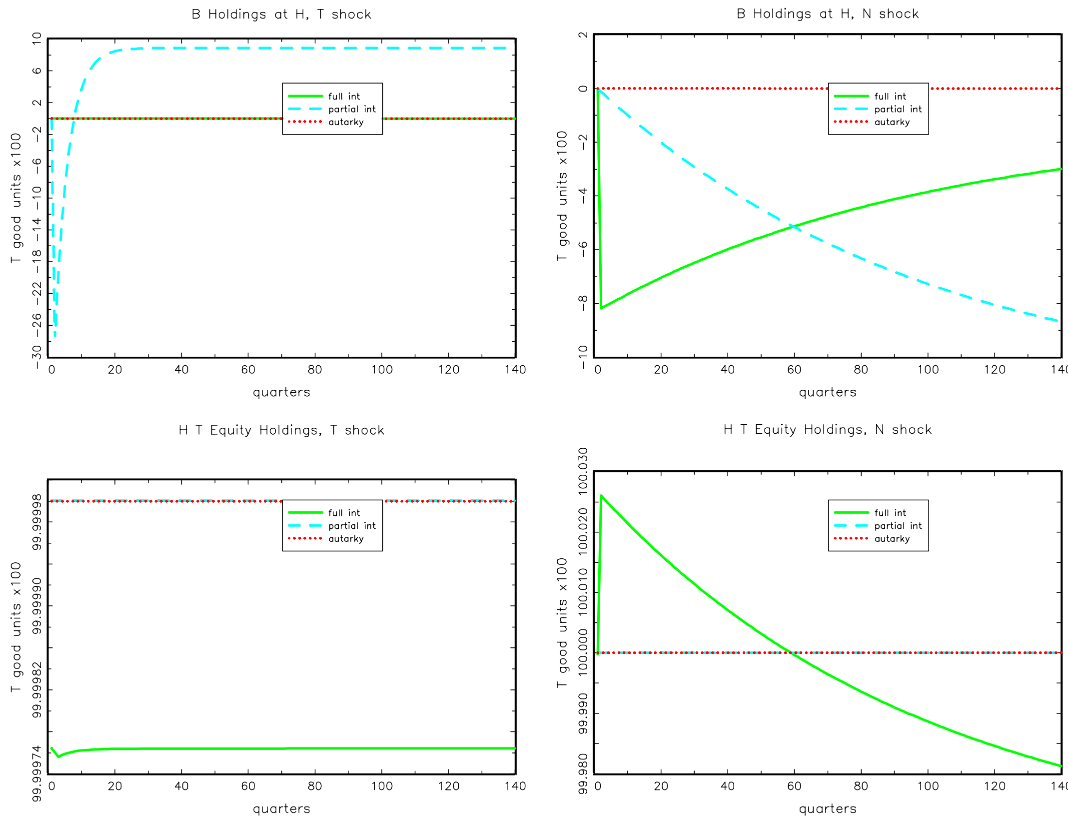


Figure 4b. Real effects of productivity shocks

We are now ready to think about capital flows, the dynamics of which are presented in Figure 4b. Under PI, capital flows only take place through the bond market and can be easily inferred from the dynamics of the trade balance. In particular, the sharp reversal in the trade balance immediately following a shock to productivity in the tradeable sector will be matched by a sharp outflow and then inflow of bonds into the H country. In contrast, productivity shocks in the non-tradeables induce a much smaller initial outflow that persists until the trade balance eventually moves into surplus. In sum, our model generates high volatility in capital flows under PI because shocks to tradeable productivity create short-lived investment booms that necessitate large changes in international bond holdings if households are to intertemporally smooth consumption.

Under FI the story is quite different. The left panel of Figure 4b tells us that the T productivity shocks do not induce any borrowing or lending as households are able to share the country specific risks using only the T equity markets. In effect, households choose to follow a buy-and-hold strategy for their diversified T equity portfolio and to passively consume T dividends every period. Such behavior is characterisitic of a complete market equilibrium. This is an expected results given that the available assets are sufficient to span the space of productivity shocks $\mathbf{z}_t$, when only T shocks are realized. On the contrary, N productivity changes can not be completely insured with the menu of available assets, which induces agents to adjust their non-contingent bond holdings. In particular, in the period of the shock, households borrow enough to finance increased demand for T consumption and T equities. As the effect of positive shock dies out, agents start selling off some of their equity holdings. These proceeds as well as dividend receipts on the remaining equity are used to finance still higher T consumption and to pay back the previous period debt and interest on that debt. The equity holdings are run down until the trade balance is restored.

What Figure 4b doesn't tell us is how international borrowing and lending is spit between bond and equity flows. More specifically, if investment booms account for the volatility of capital flows under PI, why does the standard deviation of international bond flows fall from roughly 6 to 1.6 percent of GDP when we move from the PI to FI equilibrium (see Table 3)?

To address this question we, first, evaluate the contribution of each asset to the variation in the trade balance and then study the portfolio decision facing households more formally. Table 6 reports the variance decomposition of the trade balance into three components corresponding to the asset flows used to finance that trade balance. Consistent with the results from impulse responces, under PI all variation in trade balance is attributable to shifts in bond holdings. Bond flows remain the largest contributor to the trade balance under FI as well, however, these changes are almost entirely offset by the variation in T equity position. For instance, one unit trade surplus is obtained by 18.4 unit bond inflows and 17.4 T equity outflow, all measured in units of T good.

Table 6: Variance Decomposition of Trade Balance

	Partial Integration PI (i)	Full Integration FI (ii)
Flows:		
Bonds, $\frac{1}{R_t^{\mathrm{T}}} B_t - B_{t-1}$	1	18.370868
T equity at H, $P_t^T \Delta A_t^{\mathrm{T}}$	–	-8.6874052
T equity abroad, $P_t^T \Delta A_t^{\hat{\mathrm{T}}}$	–	-8.6782443

To understand how the households choose to allocate their assets we must focus on the financial side of the model. Under FI, the optimal portfolio shares for H households are determined by the first order conditions in (19b) - (19e). These equations can be rewritten in log-linear form as

$$\mathbb{E}_t r_{t+1}^{\chi} - r_t + \frac{1}{2}\mathbb{V}_t\left(r_{t+1}^{\chi}\right) = -\mathbb{CV}_t\left(m_{t+1}, r_{t+1}^{\chi}\right), \tag{30}$$

where r_{t+1}^{χ} is the log return for equity $\chi = \{\mathrm{T}, \hat{\mathrm{T}}, \mathrm{N}\}$, r_t is the log risk free rate, and $m_{t+1} \equiv \ln M_{t+1}$ is the log IMRS (measured in terms of tradeables). $\mathbb{V}_t(.)$ and $\mathbb{CV}_t(.,.)$ denote the variance and covariance conditioned on period$-t$ information. Equation (30) says that under an optimal choice of portfolio, the expected excess log return on equity must equal minus the covariance of the log equity return with the log IMRS. We shall refer to the left hand side of (30) as the equity risk premium. In our model the term on the right hand side can be re-expressed as

$$\gamma_t \mathbb{CV}_t\left(c_{t+1}^{\mathrm{T}}, r_{t+1}^{\chi}\right) + (1-\gamma_t)\,\mathbb{CV}_t\left(q_{t+1}^{\mathrm{N}} + c_{t+1}^{\mathrm{N}}, r_{t+1}^{\chi}\right), \tag{31}$$

where

$$\gamma_t \equiv E_t \left[\left\{ 1 + \frac{\lambda_N}{\lambda_T} \left(Q_t^N \right)^{\frac{\phi}{\phi-1}} \right\}^{-1} \right].$$

Thus, (30) and (31) imply that the equity risk premium can be written as a weighed average of two covariances: the covariance between the log equity return and the log tradeable consumption c^{T}_{t+1}, and covariance between the log equity return and log non-tradeable consumption measured in terms of tradeables, $q^{\mathrm{N}}_{t+1}+c^{\mathrm{N}}_{t+1}$. In principle, both covariances can change as shocks hit the economy and so can induce variations in the equity premia. However, in practice most of the variation in the risk premia come through changes in γ_t. As figure 4a showed, productivity shocks have an immediate and long-lasting effects on the relative price of non-traded goods under FI, so they also induce variations in the equity risk premia via changes in γ_t.

Changes in the equity premia determine how households allocate the wealth between equities and bonds. This is easily demonstrated once we recognize that m_{t+1} is perfectly correlated with log wealth, w_{t+1}. Using this feature of the model, we can use the household's budget constraint to rewrite the right hand side of (30) for $\chi = \{\mathrm{T},\hat{\mathrm{T}},\mathrm{N}\}$. Re-arranging the resulting equations gives

$$\boldsymbol{\alpha}_t = \Sigma_t^{-1} \left(\mathbb{E}_t er_{t+1} + \tfrac{1}{2}\sigma_t^2 \right), \tag{32}$$

where $\boldsymbol{\alpha}_t$ is the vector of portfolio shares, $\boldsymbol{\alpha}'_t = [\ \alpha^{\mathrm{T}}_t \quad \alpha^{\hat{\mathrm{T}}}_t \quad \alpha^{\mathrm{N}}_t\ ]$, er_{t+1} is a vector of excess equity returns, $er'_{t+1} = [\ r^{\mathrm{T}}_{t+1} - r_t \quad r^{\hat{\mathrm{T}}}_{t+1} - r_t \quad r^{\mathrm{N}}_{t+1} - r_t\ ]$, Σ_t is the conditional covariance of er_{t+1}, and $\sigma_t^2 = diag\left(\Sigma_t\right)$. Notice that $\mathbb{E}_t er_{t+1} + \frac{1}{2}\sigma_t^2$ is just the vector of equity premia. Thus the variations in Q^{N}_t induced by productivity shocks change the equity premia and also the equilibrium portfolio shares of households. This is why the "return chasing" component is such an important component of both bond and equity flows under FI (see Table 4).

Finally, we summarize why the volatility of international bond flows falls as we move from the PI to the FI equilibrium. Under PI scenario, when economy is subject to a combination of T and N shocks, bonds serve as the means of financing investment boom occuring due to T shock and consumption boom resulting from a N productivity shock. When equity becomes available under FI configuration, it reduces the need for international borrowing and lending, thus lowering the necessary adjustment in bond flows and consequently their volatility.

4.1.2 Empirical evidence

Are the theoretical predictions for the size and volatility of capital flows in accordance with their empirical counterparts? To answer this questions we explore the relation between financial integration and volatilities of equity and bond flows along the time-series and cross-sectional dimensions. Some time-series properties of the capital flows have already been discussed at the beginning of the chapter. In particular, we showed that monthly US equity and bond flows and their volatilities have increased over time (see Figures 2a-b and 3a-b). We relate these changes to the degree of financial integration as measured by the volatility of return differentials.

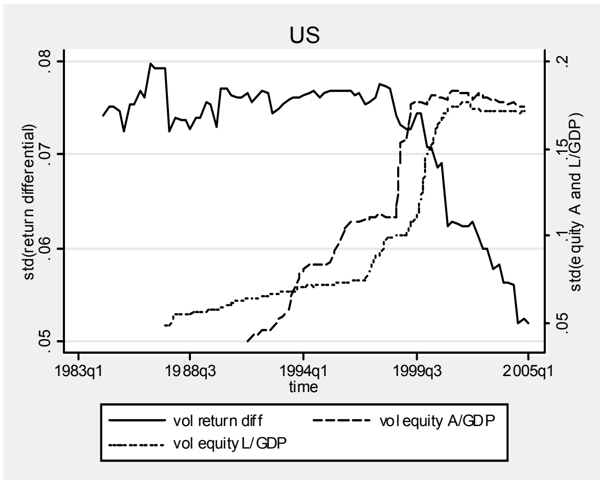


Figure 5a. US rolling window volatilities

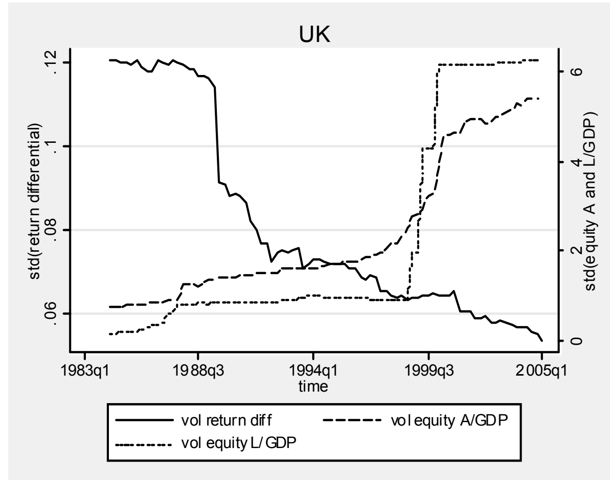


Figure 5b. UK rolling window volatilities

Figure 5a and 5b present some general patterns that emerge from the time-series data. For both US and UK the volatility of return differential as measured by the rolling wondow standard deviation over 58 quarters has declined, implying higher integration. At the same time, the volatility of equity inflows and outflows relative to GDP has increased. These results are robust to the size of the rolling wondow. Next, we study the relation between integration and capital flows for a cross-sector and a panel of 32 countries using monthly bilateral flows between US and a large number of foreign countries from US Treasury (TIC database). [TO BE COMPLETED]

One explanation for the dissimilarity of model predictions and the data could be the excess risk-sharing generated by the model under FI scenario. In particular, the model predictions of full diversification are highly implausible given the persistent equity home bias observed in the actual portfolio holdings of most developed countries. Our next task, therefore, consists of stufying the effects of integration on capital flow dynamics in the presence of home bias in equity portfolios. We accomplish this by departing from the symmetry assumption by endowing countries with unequal initial wealth, as well as using a more realistic process for productivity. The results are discussed in the robustness section.

4.2 Robustness

In this section we examine the robustness of our results along several dimensions. First, we examine how our results change if the countries are initially endowed with an unequal distribution of wealth. Second, we calibrate productivity process following Corsetti et al (2003). Third, we consider the effects of greater risk aversion.

[TO BE COMPLETED]

5 Conclusion

This paper develops a relatively simple model of international financial integration to study how the latter affects the structure of asset ownership and the behavior of international capital flows. We adopt a comparative approach to capture financial integration. In particular, we consider financial autraky, economy with a single non-contingent bond and an economy with bond and equity. Our findings suggest that at the early stages of financial integration international capital flows are large (in absolute value) and very volatile. When households gain access to foreign equity markets, the size and volatility of international bond flows falls dramatically, however remains higher than the size and volatility of equity portfolio flows. Using impulse responce analysis we show that such dynamics of capital flows are a direct consequence of international risk-sharing. When equity is not available bonds have to do all the heavy lifting associated with borrowing and lending. When equity becomes available, under a symmetric equilubrium households fully diversify their equity positions, thus reducing the volatility of bond flows. To study the factors driving capital flows we compute the variance decomposition of the flows and find that variations in the equity risk premia (i.e., expected excess returns on equity relative to the risk free rate) account for almost all of the international portfolio flows in bonds and equities. Overall, the results from our model indicate that financial integration could indeed be responsible for the large and variable international capital flows we observe in the real world.

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A Appendix: Solution method

This appendix describes the technique used to solve the model. Our starting point is an approximate solution method developed by Campbell, Chan and Viceira (2000) (CCV hereafter). They study an economy populated by an infinitely lived investor with Epstein-Zin preferences who faces a set of exogenous time-varying asset returns and chooses her optimal consumption and portfolio allocations every period. The optimal consumption and portfolio rules are obtained by log-linearizing the household's optimality conditions and solving the resulting system of linear equations using the empirically estimated laws of motion for asset returns, dividend-price ratios and other variables, taken to be state variables.

Our task is more complex since dynamics of returns are determined simultaneously with consumption and portfolio rules in our model. Therefore, we need to extend Campbell, Chan and Viceira (2000) technique to a decentralized general equilibrium setting in which returns are endogenously determined and the model dynamics are driven by the collection of exogenous stochastic technology disturbances.

The outline of our solution approach is the following. First, the non-stochastic steady state of the model is determined. Then we log-linearize the equations characterizing the equilibrium, while carefully choosing the linearization point. We posit the form of the law of motion for state variables, decision rules for control variables and prices. Using these conjectures, the system of equilibrium equations is expressed in terms of unknown coefficients and is solved numerically using the undetermined coefficients method.

Next, we describe these steps in more detail. For a formal discussion see Evans and Hnatkovska (2005).

A.1 Approximate model

In our model the equilibrium conditions include equations (6), (7) and (19a)-(19e) characterizing household's behaviour, equations (24) describing firm's problem at H country, a symmetric system of conditions for the F coutntry, a set of market clearing conditions in equations (10)-(12) and (16)-(18b), plus return definitions in (20)-(22). The system in summarized below.

$$
\begin{aligned}
R^p_{t+1} &= R_t + \alpha^{\mathrm{T}}_t \left(R^{\mathrm{T}}_{t+1} - R_t\right) + \alpha^{\hat{\mathrm{T}}}_t \left(R^{\hat{\mathrm{T}}}_{t+1} - R_t\right) + \alpha^{\mathrm{N}}_t \left(R^{\mathrm{N}}_{t+1} - R_t\right) \\
W_{t+1} &= R^p_{t+1}\left(W_t - C^{\mathrm{T}}_t - Q^{\mathrm{N}}_t C^{\mathrm{N}}_t\right) \\
Q^{\mathrm{N}}_t &= \left(\frac{\lambda_{\mathrm{N}}}{\lambda_{\mathrm{T}}}\right)^{1-\phi}\left(\frac{C^{\mathrm{N}}_t}{C^{\mathrm{T}}_t}\right)^{\phi-1} \qquad \text{(A1)} \\
1 &= \mathbb{E}_t\left[M_{t+1}R_t\right] \\
1 &= \mathbb{E}_t\left[M_{t+1}R^{\mathrm{T}}_{t+1}\right] \\
1 &= \mathbb{E}_t\left[M_{t+1}R^{\hat{\mathrm{T}}}_{t+1}\right] \\
1 &= \mathbb{E}_t\left[M_{t+1}R^{\mathrm{N}}_{t+1}\right] \\
M_{t+1} &= \beta\left(\frac{C^{\mathrm{T}}_t + Q^{\mathrm{N}}_t C^{\mathrm{N}}_t}{C^{\mathrm{T}}_{t+1} + Q^{\mathrm{N}}_{t+1} C^{\mathrm{N}}_{t+1}}\right) \qquad \text{(A2)} \\
K^{\mathrm{T}}_{t+1} &= (1-\delta)K^{\mathrm{T}}_t + e^{z^{\mathrm{T}}_t}K^{\mathrm{T}\theta}_t - D^{\mathrm{T}}_t \\
1 &= \mathbb{E}_t\left[M_{t+1}\left(\theta Z^{\mathrm{T}}_{t+1}\left(K_{t+1}\right)^{\theta-1} + (1-\delta)\right)\right] \\
C^{\mathrm{N}}_t &= Y^{\mathrm{N}}_t = D^{\mathrm{N}}_t \\
C^{\mathrm{T}}_t + \hat{C}^{\mathrm{T}}_t &= Y^{\mathrm{T}}_t + \hat{Y}^{\mathrm{T}}_t - I_t - \hat{I}_t.
\end{aligned}
$$

To solve the model, we first find its non-stochastic steady state. The system of equilibrium equations is then log-linearized as follows. The equations pertinent to the real side of the model are log-linearized up to the first order, while those related to the financial side - up to the second order. Real variables, such as capital, dividends, etc. are stationary around their unconditional mean (which is independent of the portfolio split). Individual's financial wealth is linearized around its initial distribution. Distributions of consumption and portfolio shares will be pin down endogenously by the model. All small letters denote log transformation of the corresponding variable, measured as deviation from the steady state level or its initial distribution. Foreign country equations have the same formulation, and hats are used to denote foreign variables.

The return on the portfolio in (7) is a function of portfolio shares and asset returns and can be approximated following CCV as

$$r^p_{t+1} = r_t + \boldsymbol{\alpha}'_t er_{t+1} + \frac{1}{2}\boldsymbol{\alpha}'_t\left(\sigma^2_t - \Sigma_t\boldsymbol{\alpha}_t\right) \qquad \text{(A3)}$$

where where $\boldsymbol{\alpha}_t$ is the vector of portfolio shares, $\boldsymbol{\alpha}'_t = [\ \alpha^{\mathrm{T}}_t \quad \alpha^{\hat{\mathrm{T}}}_t \quad \alpha^{\mathrm{N}}_t\ ]$, er_{t+1} is a vector of excess equity returns, $er'_{t+1} = [\ r^{\mathrm{T}}_{t+1} - r_t \quad r^{\hat{\mathrm{T}}}_{t+1} - r_t \quad r^{\mathrm{N}}_{t+1} - r_t\ ]$, Σ_t is the conditional covariance of er_{t+1}, and $\sigma^2_t = diag\left(\Sigma_t\right)$. CCV show that this approximation holds exactly in continuous time and remains a good approximation in discrete formulation with short time intervals.

The budget constraint of the H household in (6) can be simplified when preferences are logarithmic. In particular, we conjecture that the optimal consumption-wealth ratio under log-utility preferences is constant and show this formally in the proposition.

Proposition 1 *Given the set of prices in the economies, optimal consumption rule of a household in each country consists of consuming a constant fraction of wealth every period,*

$$\begin{aligned} C_t^{\mathrm{T}*} + Q_t^{\mathrm{N}} C_t^{\mathrm{N}*} &= \mu W_t, \\ \hat{C}_t^{\mathrm{T}*} + \hat{Q}_t^{\mathrm{N}} \hat{C}_t^{\mathrm{N}*} &= \mu \hat{W}_t. \end{aligned}$$

This proportionality depends on the subjective discount factor β.

Proof of this proposition follows the derivation in Obstfeld and Rogoff (1996) and is presented in the Appendix B.

The simplified budget constraint in log-linearized form is

$$\Delta w_{t+1} = r_{t+1}^{p} + \ln \beta \tag{A4}$$

In the original paper, CCV work with a more general Epstein-Zin preferences and linearize the budget constraint around the unconditional mean of the log consumption-wealth ratio. Our assumption of logarithmic utility is, of course, restrictive and has direct implications for the portfolio and consumption-saving problems, but can be easily relaxed since the method presented here is general enough to handle more complex utility functions.

Euler equations are linearized using second-order Taylor series expansion and lognormality of asset returns:

$$\mathbb{E}_t r_{t+1}^{\chi} - r_t + \frac{1}{2}\mathbb{V}_t\left(r_{t+1}^{\chi}\right) = -\mathbb{CV}_t\left(m_{t+1}, r_{t+1}^{\chi}\right), \tag{A5}$$

where r_{t+1}^{χ} is the log return for equity $\chi = \{\mathrm{T}, \hat{\mathrm{T}}, \mathrm{N}\}$, r_t is the log risk free rate, and $m_{t+1} \equiv \ln M_{t+1}$ is the log IMRS (measured in terms of tradeables). $\mathbb{V}_t\left(.\right)$ and $\mathbb{CV}_t\left(.,.\right)$ denote the variance and covariance conditioned on period$-t$ information. Proposition 1 also allows to re-write the logarithm of the stochastic discount factor m_{t+1} as minus the growth rate of households' financial wealth

$$m_{t+1} = -\Delta w_{t+1}$$

Using this condition, the system (19b)-(19e) can be summarized is terms of portfolio shares $\boldsymbol{\alpha}_t$ and first and second moments of asset returns.

$$\mathbb{E}_t er_{t+1} = \Sigma_t \boldsymbol{\alpha}_t - \frac{1}{2}\sigma_t^2 \tag{A6}$$

Using equation (A6) budget constraint can be rewritten as:

$$\Delta w_{t+1} = r_t + \frac{1}{2}\boldsymbol{\alpha}_t' \Sigma_t \boldsymbol{\alpha}_t + \boldsymbol{\alpha}_t'\left(er_{t+1} - \mathbb{E}_t er_{t+1}\right) \tag{A7}$$

Given the portfolio allocations chosen by investors in period t, their financial wealth in period $t+1$ is a function of the unanticipated asset return realized during that period. In particular, each period agent's decisions on how to allocate their savings depends not only on their expected wealth, but also on the actual realizations of wealth.

This formulation proves to be an important feature of our model. Variations in the unexpected return dominate the variations in expected return, however, due to their nature, will be swamped away in any solution method that works with the model formulations of the form $\mathbb{E}_t\mathbf{f}\left(Y_{t+1}, Y_t, X_{t+1}, X_t\right) = 0$ and would make it inapplicable to our model. In order to account for the unexpected component of returns and, correspondingly, wealth, we must keep track of their second moments and include them explicitly as part of the model's equilibrium conditions.

A.2 Solution to the approximate model

The set of linearized equations characterizing equilibrium of the model can be written in a general form as

$$\begin{aligned} 0 &= \mathbf{f}\left(Y_{t+1}, Y_t, \mathcal{X}_{t+1}, \mathcal{X}_t\right) \\ \mathcal{X}_{t+1} &= h\left(\mathcal{X}_t, \mathbb{S}_u\left(\mathcal{X}_t\right)\right) + U_{t+1} \end{aligned} \tag{A8}$$

where

$\mathcal{X}_t$ - vector of predetermined variables (or states), such as exogenous productivity shocks, capital and wealth at the beginning of period t, as well as their polynomials of infinite order. State vector can include both exogeneous and endogenous predetermined variables. Y_t is a vector of non-predetermined variables at time t. It includes consumption, dividends, asset and goods prices. $h\left(.,.\right)$ is the law of motion for $\mathcal{X}$, while $\mathbf{f}$ contains equations characterizing the equilibrium.

Standard perturbation techniques assume U_{t+1} to be i.i.d with zero mean and constant variance. In this model, we show that U_{t+1} exhibits heteroskedasticity, which arises endogenously as an inherent feature of the model with incomplete markets.

The goal is to find a sequence of decision rules for Y_t, such that

$$Y_{t+1} = g\left(\mathcal{X}_{t+1}, \mathbb{S}_u\left(\mathcal{X}_t\right)\right)$$

In particular, Y must satisfy all the equilibrium conditions in (A8):

$$0 = \mathbb{F}\left(g\left(h\left(\mathcal{X}_t, \mathbb{S}_u\left(\mathcal{X}_t\right)\right) + U_{t+1}, \mathbb{S}_u\left(\mathcal{X}_t\right)\right), g\left(\mathcal{X}_t, \mathbb{S}_u\left(\mathcal{X}_t\right)\right), h\left(\mathcal{X}_t, \mathbb{S}_u\left(\mathcal{X}_t\right)\right) + U_{t+1}, \mathcal{X}_t\right) \qquad \forall t$$

One of the complexities of the portfolio choice models concerns the dimensionality of the state vector. In particular, when asset markets are incomplete and portfolio choice is endogenous, wealth becomes a state variable, which expands the state vector $\mathcal{X}$ infinitely. One of the tasks for solving such model, therefore, consists of characterizing a good approximation to the equilibrium dynamics of the finite subset of state variables X in $\mathcal{X}$.

State variables dynamics:

Decompose the finite vector of state variables as follows

$$X_t = \begin{bmatrix} 1 \\ x_t \\ \tilde{x}_t \end{bmatrix}$$

were x_t $(n \times 1)$ includes linear terms, such as exogenous productivity shocks, capital and wealth at the beginning of period t, while $\tilde{x}_t = vec\left(x_t x_t'\right)$ contains their squares and cross-products. In particular,

$$x_t = [z_t, k_t, \hat{k}_t, w_t, \hat{w}_t]'$$

where $z_t = \begin{bmatrix} \ln Z_t^{\mathrm{T}} & \ln \hat{Z}_t^{\mathrm{T}} & \ln Z_t^{\mathrm{N}} & \ln \hat{Z}_t^{\mathrm{N}} \end{bmatrix}'$, $w_t \equiv \ln\left(W_t/W_0\right), \hat{w}_t \equiv \ln\left(\hat{W}_t/\hat{W}_0\right), k_t \equiv \ln\left(K_t/K\right), \hat{k}_t \equiv \ln\left(\hat{K}_t/\hat{K}\right)$. $vec\left(\bullet\right)$ is a vectorization operator.

Following the method of undetermined coefficients, conjecture the law of motion for x_t:

$$x_{t+1} = \Phi_0 + (I - \Phi_1)x_t + \Phi_2 \tilde{x}_t + \varepsilon_{t+1} \tag{A9}$$

where Φ_0 is the $(n \times 1)$ vector of constants, Φ_1 is the $(n \times n)$ matrix of autoregressive coefficients and Φ_2 is the $\left(n \times n^2\right)$ matrix of coefficient on the second-order and cross-product terms. $vec\left(\cdot\right)$ is the vectorization operator. Innovations ε_{t+1} have zero conditional mean. Since wealth dynamics depend on the unexpected components of returns, which themselves are state dependent, we posit that the variance-covariance matrix of ε_{t+1} is a function of quadratic state variables and their cross-products

$$\begin{aligned} \mathbb{E}\left(\varepsilon_{t+1}|x_t\right) &= 0 \\ \mathbb{E}\left(\varepsilon_{t+1}\varepsilon_{t+1}'|x_t\right) &= \Omega(\tilde{x}_t) = \Omega_0 + \Omega_1 x_t x_t' \Omega_1' \end{aligned}$$

Dynamics of non-predetermined variabels:

Also conjecture the form of the optimal decision rules in Y_t, $g\left(.,.\right)$. Specifically, we posit that Y_t is some linear function of states X_t :

$$Y_{t+1} = \Pi X_{t+1} \tag{A10}$$

Specifically $Y_t = \begin{bmatrix} \alpha_t^{\mathrm{T}} & \alpha_t^{\hat{\mathrm{T}}} & \hat{\alpha}_t^{\mathrm{T}} & \hat{\alpha}_t^{\hat{\mathrm{T}}} & \alpha_t^{\mathrm{N}} & \hat{\alpha}_t^{\mathrm{N}} & p_t^{\mathrm{T}} & \hat{p}_t^{\mathrm{T}} & p_t^{\mathrm{N}} & \hat{p}_t^{\mathrm{N}} & q_t^{\mathrm{N}} & \hat{q}_t^{\mathrm{N}} & r_t & d_t^{\mathrm{T}} & \hat{d}_t^{\mathrm{T}} \end{bmatrix}_{15\times 1}'$ and includes optimal portfolio rules for T and N equity shares in both countries, optimal stock prices in the two sectors, relative prices of non-tradeable goods at H and F, risk free rate and T sector dividends. Matrix Π is a matrix of coefficients to be determined.

Then the law of motion for X_t, $h\left(.,.\right)$, can be formulated in terms of $\{\Phi_0, \Phi_1, \Phi_2\}$ using the Ito's Lemma.

First, vectorize the conditional variance of the error process:

$$vec\left(\Omega(\tilde{x}_t)\right) = \left[\begin{array}{ccc} \Sigma_0 & 0 & \Sigma_1 \end{array}\right] \left[\begin{array}{c} 1 \\ x_t \\ \tilde{x}_t \end{array}\right] = \Sigma \left[\begin{array}{c} 1 \\ x_t \\ \tilde{x}_t \end{array}\right] \tag{A11}$$

Second, apply Ito's Lemma to formulate the second-order approximate dynamics of x_{t+1} (see appendix B for detailed derivations):

$$\tilde{x}_{t+1} = \tfrac{1}{2}D\Sigma_0 + \left(\Phi_0 \otimes I\right) + \left(I \otimes \Phi_0\right) x_t + \left(I - \left(\Phi_1 \otimes I\right) - \left(I \otimes \Phi_1\right) + \tfrac{1}{2}D\Sigma_1\right) \tilde{x}_t + \left[\left(I \otimes x_t\right) + \left(x_t \otimes I\right)\right] \varepsilon_{t+1} \tag{A12}$$

The dynamics of X_t become:

$$\left[\begin{array}{c} 1 \\ x_{t+1} \\ \tilde{x}_{t+1} \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ \Phi_0 & I - \Phi_1 & \Phi_2 \\ \frac{1}{2}D\Sigma_0 & \left(\Phi_0 \otimes I\right) + \left(I \otimes \Phi_0\right) & I - \left(\Phi_1 \otimes I\right) - \left(I \otimes \Phi_1\right) + \frac{1}{2}D\Sigma_1 \end{array}\right] \left[\begin{array}{c} 1 \\ x_t \\ \tilde{x}_t \end{array}\right] + \left[\begin{array}{c} 0 \\ \varepsilon_{t+1} \\ \tilde{\varepsilon}_{t+1} \end{array}\right] \tag{A13}$$

which is equivalent to

$$X_{t+1} = \mathbb{A} X_t + U_{t+1}$$

where $X_t = \left[\begin{array}{ccc} 1 & x_t & \tilde{x}_t \end{array}\right]'$ defined before with the dimension $(k \times 1)$, where $k = 1 + n + n^2$, $\mathbb{A}$ is a $(k \times k)$ generalized matrix of coefficients that needs to be determined, and $U_{t+1} = \left[\begin{array}{ccc} 0 & \varepsilon_{t+1} & \tilde{\varepsilon}_{t+1} \end{array}\right]'$ is a new composite vector of innovations with the following properties:

$$\begin{aligned} \mathbb{E}\left(U_{t+1} | X_t\right) &= 0 \\ \mathbb{E}\left(U_{t+1} U_{t+1}' | X_t\right) &= \mathbb{S}\left(X_t\right) \end{aligned}$$

Note there are two sources of approximation error in the state dynamics. The first is induced by the discrete/continuous time shift. The second comes from ignoring the third-order terms $\tilde{x}_t x_t'$.

Several important properties of the posited process for state vector emerge. First, VAR(1) process that governs X_t is heteroskedastic. Heteroskedasticity in this case is endogeneous and arises as a natural feature of the model with incomplete asset markets. In particular, innovations to X_t, i.e. to wealth, depend on the optimal portfolio allocations and returns, which themselves are functions of the state vector X_t. We will show that this feature has direct implications for the time-varying nature of the equity risk-premium in our model.

Second, the size of the variance of shocks, as given by $\left[\begin{array}{ccc} \Sigma_0 & 0 & \Sigma_1 \end{array}\right] X_t$, affects the coefficients on the terms quadratic in the state vector. This contrasts with the results of some perturbation methods. In particular, Schmitt-Grohe and Uribe (2004) show that based on the Taylor series expansions of the decision rules and laws of motion, up to the second order, coefficients on the terms linear and quadratic in the state vector should be independent of the volatility of exogeneous shocks. This disparity is a direct consequence of the endogenous heteroskedasticity present in our model. Since the shocks to wealth are state dependent, they must affect the coefficients on X_t in the laws of motion. At the same time, our derivations for the necessary adjustment in the constant term are consistent with Schmitt-Grohe and Uribe (2004) findings. When second-order linearizations are used, constant terms in the law of motion for state variables must be corrected by $\frac{1}{2}D\Sigma_0$, which is comparable with their result.

Conditional heteroskedasticity.

To completely specify the system, we also need to characterize the distribution of the composite error term U_{t+1}. In particular, we show that its *conditional* variance-covariance matrix can be calculated as

$$\begin{aligned} \mathbb{E}\left(U_{t+1}U_{t+1}'|X_t\right) &= \mathbb{S}\left(X_t\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Omega\left(X_t\right) & \Gamma\left(X_t\right) \\ 0 & \Gamma\left(X_t\right)' & \Psi(X_t) \end{pmatrix} \\ \Omega(X_t) &= \Omega_0 + \Omega_1 x_t x_t' \Omega_1' \end{aligned} \tag{A14}$$

where Ω and Γ and Ψ are functions of coefficient matrices $\mathbb{A}, \Omega_0$ and Ω_1. In particular,

$$\begin{aligned} vec\left(\Gamma\left(X_t\right)\right) &= \Gamma_0 + \Gamma_1 x_t + \Gamma_2 \tilde{x}_t \\ \Gamma_0 &= -\frac{1}{2}\left(\mathbb{D}\Sigma_0 \otimes \Phi_0\right) vec(I) \\ \Gamma_1 &= -\left[\left(\Phi_0 \otimes I\right) + \left(I \otimes \Phi_0\right)\right] \otimes \Phi_0 + \frac{1}{2}\left(\mathbb{D}\Sigma_0 \otimes \left(I - \Phi_1\right)\right) \\ \Gamma_2 &= -\left[I - \left(\left(\Phi_1 \otimes I\right) + \left(I \otimes \Phi_1\right)\right) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right] \otimes \Phi_0 - \tfrac{1}{2}\left(\mathbb{D}\Sigma_0 \otimes \Phi_2\right) \\ &\quad - \left[\left(\Phi_0 \otimes I\right) + \left(I \otimes \Phi_0\right)\right] \otimes \left(I - \Phi_1\right) \end{aligned}$$

$$\begin{aligned} vec\left(\Gamma\left(X_t\right)'\right) &= \Lambda_0 + \Lambda_1 x_t + \Lambda_2 \tilde{x}_t \\ \Lambda_0 &= -\tfrac{1}{2}\left(\Phi_0 \otimes \mathbb{D}\Sigma_0\right) vec(I) \\ \Lambda_1 &= -\left(\Phi_0 \otimes \left[\left(\Phi_0 \otimes I\right) + \left(I \otimes \Phi_0\right)\right]\right) + \tfrac{1}{2}\left(\left(I - \Phi_1\right) \otimes \mathbb{D}\Sigma_0\right) \\ \Lambda_2 &= -\left(\Phi_0 \otimes \left[I - \left(\left(\Phi_1 \otimes I\right) + \left(I \otimes \Phi_1\right)\right) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right]\right) - \tfrac{1}{2}\left(\Phi_2 \otimes \mathbb{D}\Sigma_0\right) \\ &\quad - \left(\left(I - \Phi_1\right) \otimes \left[\left(\Phi_0 \otimes I\right) + \left(I \otimes \Phi_0\right)\right]\right) \end{aligned}$$

$$\begin{aligned} vec\left(\Psi\left(X_t\right)\right) &= \Psi_0 + \Psi_1 x_t + \Psi_2 \tilde{x}_t \\ \Psi_0 &= -\frac{1}{4}\left(\mathbb{D}\Sigma_0 \otimes \mathbb{D}\Sigma_0\right) vec(I) \\ \Psi_1 &= -\tfrac{1}{2}\left(\left[\left(\Phi_0 \otimes I\right) + \left(I \otimes \Phi_0\right)\right] \otimes \mathbb{D}\Sigma_0\right) - \tfrac{1}{2}\left(\mathbb{D}\Sigma_0 \otimes \left[\left(\Phi_0 \otimes I\right) + \left(I \otimes \Phi_0\right)\right]\right) \\ \Psi_2 &= -\frac{1}{2}\left[I - \left(\left(\Phi_1 \otimes I\right) + \left(I \otimes \Phi_1\right)\right) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right] \otimes \mathbb{D}\Sigma_0 \\ &\quad -\tfrac{1}{2}\left(\mathbb{D}\Sigma_0 \otimes \left[I - \left(\left(\Phi_1 \otimes I\right) + \left(I \otimes \Phi_1\right)\right) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right]\right) \\ &\quad - \left[\left(\Phi_0 \otimes I\right) + \left(I \otimes \Phi_0\right)\right] \otimes \left[\left(\Phi_0 \otimes I\right) + \left(I \otimes \Phi_0\right)\right] \end{aligned}$$

The derivations of these expressions are provided in Appendix B.

With the conjectures for the dynamics of X_t and Y_t at hand, finding the equilibrium in our model reduces to solving the fixed-point problem in the recursive formulation:

$$\mathbf{F}\left(\Pi\mathbb{A}X_t + \Pi U_{t+1}, \Pi X_t, \mathbb{A}X_t + U_{t+1}, X_t\right) = 0$$

$\mathbf{F}$ consists of all the equilibrium conditions, including the restrictions on the second moments, implied by the model.

We use definitions of equity returns and budget constraint to illustrate the procedure for constructing $\mathbf{F}$. Linearized equity returns are obtained following Campbell and Shiller (). We will use π_i to denote row vector of Π corresponding to a non-predetermined variable i, while Π_i will refer to a collection of vectors. We also define h_i to be a selection vector of zeros and ones, such that it picks the i^{th} element out of the full state vector X_t.

Then the return on T equity is:

$$\begin{aligned} r^{\mathrm{T}}_{t+1} &= \beta p^{\mathrm{T}}_{t+1} + (1-\beta)d^{\mathrm{T}}_{t+1} - p^{\mathrm{T}}_t \\ &= \left[\pi^{\mathrm{T}}_a\mathbb{A} - \pi^{\mathrm{T}}_p\right]X_t + \pi^{\mathrm{T}}_a U_{t+1} \end{aligned} \tag{A15}$$

where $\pi^{\mathrm{T}}_a = \left(\beta\pi^{\mathrm{T}}_p + (1-\beta)\pi^{\mathrm{T}}_d\right)$. Similarly, return on N equity is

$$\begin{aligned} r^{\mathrm{N}}_{t+1} &= \Delta q^{\mathrm{N}}_{t+1} + \beta p^{\mathrm{N}}_{t+1} + (1-\beta)d^{\mathrm{N}}_{t+1} - p^{\mathrm{N}}_t \\ &= \left[\pi^{\mathrm{N}}_a\mathbb{A} - \pi^{\mathrm{N}}_q - \pi^{\mathrm{N}}_p\right]X_t + \pi^{\mathrm{N}}_a U_{t+1} \end{aligned} \tag{A16}$$

where $\pi^{\mathrm{N}}_a \equiv \left(\pi^{\mathrm{N}}_q + \beta\pi^{\mathrm{N}}_p + (1-\beta)\pi^{\mathrm{N}}_d\right)$. Linearized expression for the rental rate expresses capital return as a function of state variables z_t and k_t :

$$\begin{aligned} r^{\mathrm{K}}_{t+1} &\cong \psi z^{\mathrm{T}}_{t+1} - (1-\theta)\psi\left(k_{t+1} - k\right) \\ &= \psi\left[h^{\mathrm{T}}_z - (1-\theta)h_k\right]\mathbb{A}X_t + \psi h^{\mathrm{T}}_z U_{t+1} \end{aligned} \tag{A17}$$

where $\psi = 1 - \beta(1-\delta) < 1$.

The unanticipated components of returns can be obtained as

$$\begin{aligned} r^{\mathrm{T}}_{t+1} - \mathbb{E}_t r^{\mathrm{T}}_{t+1} &= \pi^{\mathrm{T}}_a u_{t+1} \\ r^{\mathrm{N}}_{t+1} - \mathbb{E}_t r^{\mathrm{N}}_{t+1} &= \pi^{\mathrm{N}}_a u_{t+1} \\ r^{\mathrm{K}}_{t+1} - \mathbb{E}_t r^{\mathrm{K}}_{t+1} &= \psi(z^{\mathrm{T}}_{t+1} - \mathbb{E}_t z^{\mathrm{T}}_{t+1}) = \psi h^{\mathrm{T}}_z u_{t+1} \end{aligned} \tag{A18}$$

At this point it is useful to describe the procedure developed to calculate the second moments of the state vector and its linear functions.

Definition 1 *The covariance between the two variables $m_t = \pi_m X_t$ and $n_t = \pi_n X_t$, both being some* *functions of the state vector X_t can be obtained as*

$$\mathbb{CV}_t\left(m_{t+1}, n_{t+1}\right) = \mathcal{A}\left(\pi_m, \pi_n\right) X_t$$

where vector $\mathcal{A}\left(\pi_m, \pi_n\right)$ collects together three sets of coefficients: $\mathcal{A}^0_{m,n}, \mathcal{A}^1_{m,n}, \mathcal{A}^2_{m,n}$ - coefficients corresponding to the constant, linear and quadratic terms in the state vector X_t respectively, and are functions of π_m and π_n and the parameters of variance-covariance matrix $\mathbb{S}(X_t)$.

$$\begin{aligned}
\mathcal{A}\left(\pi_m, \pi_n\right) &= \left[\begin{array}{ccc} \mathcal{A}^0_{m,n} & \mathcal{A}^1_{m,n} & \mathcal{A}^2_{m,n} \end{array}\right] \\
\mathcal{A}^0_{m,n} &= \left(\pi^1_n \otimes \pi^1_m\right)\Sigma_0 + \left(\pi^1_n \otimes \pi^2_m\right)\Lambda_0 + \left(\pi^2_n \otimes \pi^1_m\right)\Gamma_0 + \left(\pi^2_n \otimes \pi^2_m\right)\Psi_0 \\
\mathcal{A}^1_{m,n} &= \left(\pi^1_n \otimes \pi^2_m\right)\Lambda_1 + \left(\pi^2_n \otimes \pi^1_m\right)\Gamma_1 + \left(\pi^2_n \otimes \pi^2_m\right)\Psi_1 \\
\mathcal{A}^2_{m,n} &= \left(\pi^1_n \otimes \pi^1_m\right)\Sigma_1 + \left(\pi^1_n \otimes \pi^2_m\right)\Lambda_2 + \left(\pi^2_n \otimes \pi^1_m\right)\Gamma_2 + \left(\pi^2_n \otimes \pi^2_m\right)\Psi_2
\end{aligned}$$

$\left[\begin{array}{ccc} \pi^0_m & \pi^1_m & \pi^2_m \end{array}\right]$ *and* $\left[\begin{array}{ccc} \pi^0_n & \pi^1_n & \pi^2_n \end{array}\right]$ *are partitions of the vectors π_m and π_n, each part corresponding to a constant, linear or quadratic component of the state vector X_t.*
Proof can be found in appendix B.

With this notation, the expressions for conditional variances and covariances can be easily obtained. For instance, second moments of the returns are given by

$$\begin{aligned}
\mathbb{V}\left(r^{\{}_{t+1} | X_t\right) &= \left[\begin{array}{ccc} \mathcal{A}(\pi^{\mathrm{T}}_a, \pi^{\mathrm{T}}_a)X_t & \mathcal{A}(\pi^{\mathrm{T}}_a, \pi^{\hat{\mathrm{T}}}_a)X_t & \mathcal{A}(\pi^{\mathrm{T}}_a, \pi^{\mathrm{N}}_a)X_t \\ & \mathcal{A}(\pi^{\hat{\mathrm{T}}}_a, \pi^{\hat{\mathrm{T}}}_a)X_t & \mathcal{A}(\pi^{\hat{\mathrm{T}}}_a, \pi^{\mathrm{N}}_a)X_t \\ & & \mathcal{A}(\pi^{\mathrm{N}}_a, \pi^{\mathrm{N}}_a)X_t \end{array}\right] \\
&= \mathcal{A}\left(\Pi_a, \Pi'_a\right) X_t
\end{aligned}$$

where $r^{\{}_{t+1} = \left[\begin{array}{ccc} r^{\mathrm{T}}_{t+1} & r^{\hat{\mathrm{T}}}_{t+1} & r^{\mathrm{N}}_{t+1} \end{array}\right]'$ and $\Pi_a = \left[\begin{array}{ccc} \pi^{\mathrm{T}}_a & \pi^{\hat{\mathrm{T}}}_a & \pi^{\mathrm{N}}_a \end{array}\right]'$ is a matrix of coefficients on the unanticipated components of returns from (A18).

Definition 2 *Product of the two linear functions of the state vector X_t can itself be expressed as a linear function of X_t :*

$$\pi_m X_t X'_t \pi'_n = \mathcal{B}\left(\pi_m, \pi_n\right) X_t$$

where vector $\mathcal{B}\left(\pi_m, \pi_n\right)$ has analogous interpretation as in the previous definition.

$$\begin{aligned}
\mathcal{B}\left(\pi_m, \pi_n\right) &= \left[\begin{array}{ccc} \mathcal{B}^0_{m,n} & \mathcal{B}^1_{m,n} & \mathcal{B}^2_{m,n} \end{array}\right] \\
\mathcal{B}^0_{m,n} &= \left(\pi^0_n \otimes \pi^0_m\right) vec(I) = vec(\pi^0_n * \pi^0_m) \\
\mathcal{B}^1_{m,n} &= \left(\pi^0_n \otimes \pi^1_m\right) + \left(\pi^1_n \otimes \pi^0_m\right) \\
\mathcal{B}^2_{m,n} &= \left(\pi^0_n \otimes \pi^2_m\right) + \left(\pi^1_n \otimes \pi^1_m\right) + \left(\pi^2_n \otimes \pi^0_m\right)
\end{aligned}$$

Proof can be found in appendix B.

Summarize all portfolio shares in a vector

$$\alpha_t' = \begin{bmatrix} \pi_\alpha^{\mathrm{T}} \\ \pi_\alpha^{\hat{\mathrm{T}}} \\ \pi_\alpha^{\mathrm{N}} \end{bmatrix} X_t = \Pi_\alpha X_t$$

where Π_α extracts the rows of Π that correspond to portfolio shares in α_t.

With the definitions in place, budget constraint can be rewritten as a linear function of state vector X_t and a vector of innovations U_{t+1}, the coefficients on which depend on matrices $\Pi, \mathbb{A}, \Omega_0, \Omega_1$:

$$h_w \mathbb{A} X_t + h_w U_{t+1} = h_w X_t + \pi_r X_t + \frac{1}{2}\boldsymbol{\iota}' \left\{\mathcal{B}\left[\mathcal{B}\left(\Pi_\alpha, \Pi_\alpha'\right), \mathcal{A}\left(\Pi_a, \Pi_a'\right)\right] X_t\right\}' \boldsymbol{\iota} + \left(\Pi_\alpha X_t\right)' \left(\Pi_a U_{t+1}\right) \tag{A19}$$

Here Π_α is a matrix of coefficients in vector α_t' to be determined and $\boldsymbol{\iota}$ is a column vector of ones. Equating the coefficients on the left- and right-hand sides of (A19) yields the first set of restrictions on the equilibrium coefficient vectors in Π and matrix $\mathbb{A}$. Analogous set is obtained from the budget constraint for the foreign country. The last term on the right-hand side of equation (A19) is the source of endogenous heteroskedasticity in wealth dynamics.

A.3 Numerical procedure

Having derived all the building blocks we are now ready to describe the implementation of the numerical procedure developed to find the optimal matrix Π. Here is the policy iteration algorithm that we implement.

1. For the given set of parameter values we conjecture some initial values for policy matrix $\Pi^{(1)}$ and the coefficient matrices $\left\{\Phi_0^{(1)}, \Phi_1^{(1)}, \Phi_2^{(1)}\right\}$ governing the state vector dynamics. We also need to choose starting values for $\{\Omega_0, \Omega_1\}$ and arrange them into $[\Sigma]_i$ (the rows of Σ). Σ characterizes the heteroskedastic nature of the variance-covariance matrix of the state vector. We start with a homoskedastic guess:

$$\begin{aligned} [\Sigma]_i^{(1)} &= \begin{bmatrix} \sigma_e^2 & \mathbf{0}_{1\times(k-1)} \end{bmatrix}, \qquad i = \{V_t(z)\}, \\ [\Sigma]_i^{(1)} &= \begin{bmatrix} \rho\sigma_e^2 & \mathbf{0}_{1\times(k-1)} \end{bmatrix}, \qquad i = \{CV_t(z)\}, \\ [\Sigma]_i^{(1)} &= [\mathbf{0}_{1\times k}], \qquad i \neq \{V_t(z), CV_t(z)\} \end{aligned}$$

2. With these guesses we can construct a coefficient matrix $\mathbb{A}^{(1)}$ from (A13) and variance-covariance matrix $\mathbb{S}^{(1)}$ from (A14).

3. Form the value function

$$\mathbf{J}^1\left(\Pi^{(1)}\right) = \mathbf{F}\left(\Pi^{(1)}\mathbb{A}^{(1)}X_t + \Pi^{(1)}U_{t+1}, \Pi^{(1)}X_t, \mathbb{A}^{(1)}X_t + U_{t+1}, X_t\right)$$

4. For the given values of $\mathbb{A}$ and $\mathbb{S}$ find optimal $\Pi^{(2)}$ implied by solving $\mathbf{J}^1\left(\Pi^{(1)}\right) = 0$. Recall that $\mathbf{F}$ also includes restrictions on variance-covariance matrix Σ ($[\Sigma]_i$ being its rows) that needs to be checked for

consistency using the following set of definitions:

$$\begin{aligned}\Sigma_{ij\ (1\times k)} X_t &= \mathbb{CV}_t\left(h_i X_{t+1}, h_j X_{t+1}\right) \qquad \forall i = 1, ..., n \ \ \forall j = 1, ..., n \\ &= \mathcal{A}\left(h_i, h_j\right) X_t\end{aligned}$$

Since its initial values are set arbitrary, the new values in Π will tend to be different from $\Pi^{(1)}$. With $\Pi^{(2)}$ we return to step 2 and recompute matrices $\mathbb{A}$ and $\mathbb{S}$. Continue to iterate until the successive $\Pi^{(\tau)}$ have converged, given some convergence criteria.

B Appendix: Proofs and detailed derivations

B.1 Result 1

This subsection presents the proof of Proposition 1.

Proposition 1. *Given the set of prices in the economies, optimal consumption rule of a household in each country consists of consuming a constant fraction of wealth every period,*

$$\begin{aligned} C_t^{\mathrm{T}*} + Q_t^{\mathrm{N}} C_t^{\mathrm{N}*} &= \mu W_t, \\ \hat{C}_t^{\mathrm{T}*} + \hat{Q}_t^{\mathrm{N}} \hat{C}_t^{\mathrm{N}*} &= \mu \hat{W}_t. \end{aligned}$$

This proportionality depends on the subjective discount factor β.

Proof: ■ The proof consists of verifying that the guess above solves the households' problem given the equilibrium relative price Q^N. Recall the portfolio Euler equation:

$$1 = \beta \mathbb{E}_t \left[M_{t+1} R_{t+1}^p \right]$$

where M_{t+1} is the marginal rate of substitution. From the household's first-order conditions it can be shown that $M_{t+1} = \frac{C_t^T + Q_t^N C_t^N}{C_{t+1}^T + Q_{t+1}^N C_{t+1}^N}$, which allows to rewrite equation above as

$$\frac{1}{C_t^T + Q_t^N C_t^N} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}^T + Q_{t+1}^N C_{t+1}^N} R_{t+1}^p \right]$$

Now substitute the guess for the optimal consumption rule to obtain

$$\begin{aligned} \frac{1}{\mu W_t} &= \beta \mathbb{E}_t \left[\frac{1}{\mu W_{t+1}} R_{t+1}^p \right] \\ \frac{1}{\mu W_t} &= \beta \mathbb{E}_t \left[\frac{1}{\mu R_{t+1}^p \left(W_t - C_t^T + Q_t^N C_t^N \right)} R_{t+1}^p \right] \\ \frac{1}{\mu W_t} &= \beta \mathbb{E}_t \left[\frac{1}{\mu \left(1 - \mu\right) W_t} \right] \end{aligned}$$

which implies $\mu = (1 - \beta)$ thus confirming the guess that consumption constitutes a constant fraction of wealth.■

B.2 Result 2

In this subsection we provide detailed derivations of the result in (A12). We start with quadratic and cross-product terms, $\tilde{x}_t$ and approximate their laws of motion using Ito's lemma. In continuous time, the discrete process for x_{t+1} in (A9) becomes

$$dx_t = \left[\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t \right] dt + \Omega(\tilde{x}_t)^{1/2} dW_t$$

Then by Ito's Lemma:

$$
\begin{aligned}
dvec(x_t x_t') &= [(I \otimes x_t) + (x_t \otimes I)] \left([\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t] \, dt + \Omega(\tilde{x}_t)^{1/2} dW_t \right) \\
&+ \tfrac{1}{2} \left[(I \otimes U) \left(\frac{\partial x}{\partial x'} \otimes I \right) + \left(\frac{\partial x}{\partial x'} \otimes I \right) \right] d\,[x, x]_t \\
&= [(I \otimes x_t) + (x_t \otimes I)] \left([\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t] \, dt + \Omega(\tilde{x}_t)^{1/2} dW_t \right) \\
&+ \tfrac{1}{2} \left[\mathbb{U} \left(\frac{\partial x}{\partial x'} \otimes I \right) + \left(\frac{\partial x}{\partial x'} \otimes I \right) \right] vec \left\{ \Omega(\tilde{x}_t) \right\} dt \\
&= [(I \otimes x_t) + (x_t \otimes I)] \left([\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t] \, dt + \Omega(\tilde{x}_t)^{1/2} dW_t \right) + \tfrac{1}{2} \mathbb{D} vec \left\{ \Omega(\tilde{x}_t) \right\} dt \quad \text{(A20)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{D} &= \left[\mathbb{U} \left(\frac{\partial x}{\partial x'} \otimes I \right) + \left(\frac{\partial x}{\partial x'} \otimes I \right) \right] \\
\mathbb{U} &= \sum_r \sum_s E_{rs} \otimes E'_{r,s} \\
\frac{\partial x}{\partial x'} &= \begin{bmatrix} \frac{\partial x}{\partial x_1} & \frac{\partial x}{\partial x_2} & \cdots & \frac{\partial x}{\partial x_n} \end{bmatrix}_{n \times n}
\end{aligned}
$$

and $E_{r,s}$ is the elementary matrix which has a unity at the (r, s)th position and zero at all other elements. Law of motion for the quadratic states in (A20) can be rewritten in discrete time as

$$
\begin{aligned}
\tilde{x}_{t+1} &\cong \tilde{x}_t + [(I \otimes x_t) + (x_t \otimes I)] \, [\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t] + \tfrac{1}{2} \mathbb{D} vec \left(\Omega(\tilde{x}_t) \right) \\
&\quad + [(I \otimes x_t) + (x_t \otimes I)] \, \varepsilon_{t+1} \\
&\cong \tfrac{1}{2} \mathbb{D} \Sigma_0 + [(\Phi_0 \otimes I) + (I \otimes \Phi_0)] \, x_t + \left[I - (\Phi_1 \otimes I) - (I \otimes \Phi_1) + \tfrac{1}{2} \mathbb{D} \Sigma_1 \right] \tilde{x}_t + \tilde{\varepsilon}_{t+1}
\end{aligned}
$$

The last equality is obtained by using an expression for $vec\left(\Omega(\tilde{x}_t)\right)$ in (A11), where $\Sigma_0 = vec(\Omega_0)$ and $\Sigma_1 = \Omega_1 \otimes \Omega_1$, and by combining together the corresponding coefficients on a constant, linear and second-order terms. Note that $\tilde{x}_{t+1}$ is also a function of cubic terms, which we assumed to be negligibly small, such that $\tilde{x}_t x_t' \cong 0$. This assumption can be verified by further expanding the state vector to include the third-order terms and comparing the dynamics of the model under the two sets of solutions. The two should yield very similar results and thus we proceed with a more parsimonious state vector as shown below.

B.3 Result 3

In this subsection we derive the conditional mean and *conditional* variance-covariance matrix of the composite error term in the state vector dynamics.

Recall $U_{t+1} = \left[\begin{array}{ccc} 0 & \varepsilon_{t+1} & \tilde{\varepsilon}_{t+1} \end{array} \right]'$. Then

$$\begin{aligned}
\mathbb{E}\left(U_{t+1}|X_t\right) &= 0 \\
\mathbb{V}\left(U_{t+1}|X_t\right) &= \mathbb{E}\left(U_{t+1}U'_{t+1}|X_t\right) \equiv \mathbb{S}\left(X_t\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Omega\left(X_t\right) & \Gamma\left(X_t\right) \\ 0 & \Gamma\left(X_t\right)' & \Psi(X_t) \end{pmatrix}
\end{aligned}$$

To evaluate the covariance matrix we apply the assumption $vec(x_{t+1}\tilde{x}'_{t+1}) \cong 0$ and define:

$$\begin{aligned}
\Gamma\left(X_t\right) &\equiv \mathbb{E}_t\varepsilon_{t+1}\tilde{\varepsilon}'_{t+1} \\
&= \mathbb{E}_t x_{t+1}\tilde{x}'_{t+1} - \mathbb{E}_t x_{t+1}\mathbb{E}_t\tilde{x}'_{t+1} \\
&= \mathbb{E}_t x_{t+1}\tilde{x}'_{t+1} - (\Phi_0 + (I-\Phi_1)x_t + \Phi_2\tilde{x}_t) \\
&\quad \times \left(\tfrac{1}{2}\Sigma'_0\mathbb{D}' + x'_t\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]' + \tilde{x}'_t\left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right]'\right) \\
&\cong -\Phi_0\left(\tfrac{1}{2}\Sigma'_0\mathbb{D}' + x'_t\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]' + \tilde{x}'_t\left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right]'\right) \\
&\quad -(I-\Phi_1)x_t\left(\tfrac{1}{2}\Sigma'_0\mathbb{D}' + x'_t\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]'\right) - \tfrac{1}{2}\Phi_2\tilde{x}_t\Sigma'_0\mathbb{D}' \\
&= -\frac{1}{2}\Phi_0\Sigma'_0\mathbb{D}' - \Phi_0 x'_t\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]' - \frac{1}{2}(I-\Phi_1)x_t\Sigma'_0\mathbb{D}' \\
&\quad -\Phi_0\tilde{x}'_t\left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right]' - (I-\Phi_1)x_t x'_t\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]' - \tfrac{1}{2}\Phi_2\tilde{x}_t\Sigma'_0\mathbb{D}'
\end{aligned}$$

Hence

$$\begin{aligned}
vec\left(\Gamma\left(X_t\right)\right) &= \Gamma_0 + \Gamma_1 x_t + \Gamma_2\tilde{x}_t \\
\Gamma_0 &= -\frac{1}{2}\left(\mathbb{D}\Sigma_0 \otimes \Phi_0\right)vec(I) \\
\Gamma_1 &= -\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right] \otimes \Phi_0 + \frac{1}{2}\left(\mathbb{D}\Sigma_0 \otimes (I-\Phi_1)\right) \\
\Gamma_2 &= -\left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right] \otimes \Phi_0 - \tfrac{1}{2}\left(\mathbb{D}\Sigma_0 \otimes \Phi_2\right) \\
&\quad -\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right] \otimes (I-\Phi_1)
\end{aligned}$$

Note also from above that

$$\begin{aligned}
\Gamma\left(X_t\right)' &= -\tfrac{1}{2}\mathbb{D}\Sigma_0\Phi'_0 - \left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]x_t\Phi'_0 - \Sigma_0 x'_t(I-\Phi_1)' \\
&\quad -\left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right]\tilde{x}_t\Phi'_0 - \left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]x_t x'_t(I-\Phi_1)' - \tfrac{1}{2}\mathbb{D}\Sigma_0\tilde{x}'_t\Phi'_2
\end{aligned}$$

So

$$\begin{aligned}
vec\left(\Gamma\left(X_t\right)'\right) &= \Lambda_0 + \Lambda_1 x_t + \Lambda_2 \tilde{x}_t \\
\Lambda_0 &= -\tfrac{1}{2}\left(\Phi_0 \otimes \mathbb{D}\Sigma_0\right) vec(I) \\
\Lambda_1 &= -\left(\Phi_0 \otimes \left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]\right) + \tfrac{1}{2}\left((I - \Phi_1) \otimes \mathbb{D}\Sigma_0\right) \\
\Lambda_2 &= -\left(\Phi_0 \otimes \left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right]\right) - \tfrac{1}{2}\left(\Phi_2 \otimes \mathbb{D}\Sigma_0\right) \\
&\quad - \left((I - \Phi_1) \otimes \left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]\right)
\end{aligned}$$

Next, consider the variance of $\tilde{\varepsilon}_{t+1}$

$$\begin{aligned}
\Psi\left(X_t\right) &\equiv \mathbb{E}_t \tilde{\varepsilon}_{t+1} \tilde{\varepsilon}'_{t+1} = \mathbb{E}_t \tilde{x}_{t+1} \tilde{x}'_{t+1} - \mathbb{E}_t \tilde{x}_{t+1} \mathbb{E}_t \tilde{x}'_{t+1} \\
&= \mathbb{E}_t \tilde{x}_{t+1} \tilde{x}'_{t+1} - \left(\tfrac{1}{2}\mathbb{D}\Sigma_0 + \left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right] x_t + \left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right] \tilde{x}_t\right) \\
&\quad \times \left(\tfrac{1}{2}\Sigma_0'\mathbb{D}' + x_t'\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]' + \tilde{x}_t'\left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right]'\right) \\
&\cong -\tfrac{1}{2}\mathbb{D}\Sigma_0\left(\tfrac{1}{2}\Sigma_0'\mathbb{D}' + x_t'\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]' + \tilde{x}_t'\left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right]'\right) \\
&\quad - \left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right] x_t \left(\tfrac{1}{2}\Sigma_0'\mathbb{D}' + x_t'\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]'\right) \\
&\quad - \left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right] \tilde{x}_t \tfrac{1}{2}\Sigma_0'\mathbb{D}' \\
&= -\frac{1}{4}\mathbb{D}\Sigma_0\Sigma_0'\mathbb{D}' - \tfrac{1}{2}\mathbb{D}\Sigma_0 x_t'\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]' - \tfrac{1}{2}\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right] x_t \Sigma_0'\mathbb{D}' \\
&\quad -\frac{1}{2}\mathbb{D}\Sigma_0 \tilde{x}_t'\left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right]' - \left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right] x_t x_t' \left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]' \\
&\quad -\tfrac{1}{2}\left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right] \tilde{x}_t \Sigma_0'\mathbb{D}'
\end{aligned}$$

Hence

$$\begin{aligned}
vec\left(\Psi\left(X_t\right)\right) &= \Psi_0 + \Psi_1 x_t + \Psi_2 \tilde{x}_t \\
\Psi_0 &= -\frac{1}{4}\left(\mathbb{D}\Sigma_0 \otimes \mathbb{D}\Sigma_0\right) vec(I) \\
\Psi_1 &= -\tfrac{1}{2}\left(\left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right] \otimes \mathbb{D}\Sigma_0\right) - \tfrac{1}{2}\left(\mathbb{D}\Sigma_0 \otimes \left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]\right) \\
\Psi_2 &= -\frac{1}{2}\left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right] \otimes \mathbb{D}\Sigma_0 - \tfrac{1}{2}\left(\mathbb{D}\Sigma_0 \otimes \left[I - ((\Phi_1 \otimes I) + (I \otimes \Phi_1)) + \tfrac{1}{2}\mathbb{D}\Sigma_1\right]\right) \\
&\quad - \left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right] \otimes \left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right]
\end{aligned}$$

And the variance-covariance matrix of ε_{t+1} is known from before:

$$\begin{aligned}
\Omega(X_t) &= \mathbb{E}_t \varepsilon_{t+1} \varepsilon'_{t+1} = \Omega_0 + \Omega_1 x_t x_t' \Omega_1' \\
vec\left(\Omega(X_t)\right) &= \begin{bmatrix} \Sigma_0 & 0 & \Sigma_1 \end{bmatrix} X_t = \Sigma X_t
\end{aligned}$$

where $\Sigma_0 = vec(\Omega_0)$ and $\Sigma_1 = \Omega_1 \otimes \Omega_1$. Having determined $\Omega(X_t)$, $\Psi\left(X_t\right)$ and $\Gamma\left(X_t\right)$ we can construct the aggregate variance-covariance matrix $\mathbb{S}(X_t)$.

B.4 Result 4

In this subsection we describe the construction of functions $\mathcal{A}$ and $\mathcal{B}$, which were used to derive variances and covariances of the functions of state vector X_t. We also show how to simplify the products of the vectors involving the state vector X_t.

Let $m_t = \pi_m X_t$ and $n_t = \pi_n X_t$ for two variables m_t and n_t.We want to find the conditional *covariance* between the two:

$$\begin{aligned}
\mathbb{CV}_t\left(m_{t+1}, n_{t+1}\right) &= \left[\begin{array}{ccc} \pi_m^0 & \pi_m^1 & \pi_m^2 \end{array}\right] \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & \Omega(X_t) & \Gamma\left(X_t\right) \\ 0 & \Gamma\left(X_t\right)' & \Psi(X_t) \end{array}\right] \left[\begin{array}{c} \pi_n^{0\prime} \\ \pi_n^{1\prime} \\ \pi_n^{2\prime} \end{array}\right] \\
&= \left[\begin{array}{ccc} 0 & \pi_m^1 \Omega(X_t) + \pi_m^2 \Gamma\left(X_t\right)' & \pi_m^1 \Gamma\left(X_t\right) + \pi_m^2 \Psi(X_t) \end{array}\right] \left[\begin{array}{c} \pi_n^{0\prime} \\ \pi_n^{1\prime} \\ \pi_n^{2\prime} \end{array}\right] \\
&= \pi_m^1 \Omega(X_t) \pi_n^{1\prime} + \pi_m^2 \Gamma\left(X_t\right)' \pi_n^{1\prime} + \pi_m^1 \Gamma\left(X_t\right) \pi_n^{2\prime} + \pi_m^2 \Psi(X_t) \pi_n^{2\prime} \\
&= \left(\pi_n^1 \otimes \pi_m^1\right) vec\left(\Omega(X_t)\right) + \left(\pi_n^1 \otimes \pi_m^2\right) vec\left(\Gamma\left(X_t\right)'\right) \\
&\quad + \left(\pi_n^2 \otimes \pi_m^1\right) vec\left(\Gamma\left(X_t\right)\right) + \left(\pi_n^2 \otimes \pi_m^2\right) vec\left(\Psi\left(X_t\right)\right) \\
&= \left(\pi_n^1 \otimes \pi_m^1\right) \Sigma_0 + \left(\pi_n^1 \otimes \pi_m^2\right) \Lambda_0 + \left(\pi_n^2 \otimes \pi_m^1\right) \Gamma_0 + \left(\pi_n^2 \otimes \pi_m^2\right) \Psi_0 \\
&\quad + \left(\left(\pi_n^1 \otimes \pi_m^2\right) \Lambda_1 + \left(\pi_n^2 \otimes \pi_m^1\right) \Gamma_1 + \left(\pi_n^2 \otimes \pi_m^2\right) \Psi_1\right) x_t \\
&\quad + \left(\left(\pi_n^1 \otimes \pi_m^1\right) \Sigma_1 + \left(\pi_n^1 \otimes \pi_m^2\right) \Lambda_2 + \left(\pi_n^2 \otimes \pi_m^1\right) \Gamma_2 + \left(\pi_n^2 \otimes \pi_m^2\right) \Psi_2\right) \tilde{x}_t
\end{aligned}$$

So, to summarize,

$$\begin{aligned}
\mathbb{CV}_t\left(m_{t+1}, n_{t+1}\right) &= \mathcal{A}\left(\pi_m, \pi_n\right) X_t \\
\mathcal{A}\left(\pi_m, \pi_n\right) &= \left[\begin{array}{ccc} \mathcal{A}_{m,n}^0 & \mathcal{A}_{m,n}^1 & \mathcal{A}_{m,n}^2 \end{array}\right] \\
\mathcal{A}_{m,n}^0 &= \left(\pi_n^1 \otimes \pi_m^1\right) \Sigma_0 + \left(\pi_n^1 \otimes \pi_m^2\right) \Lambda_0 + \left(\pi_n^2 \otimes \pi_m^1\right) \Gamma_0 + \left(\pi_n^2 \otimes \pi_m^2\right) \Psi_0 \\
\mathcal{A}_{m,n}^1 &= \left(\pi_n^1 \otimes \pi_m^2\right) \Lambda_1 + \left(\pi_n^2 \otimes \pi_m^1\right) \Gamma_1 + \left(\pi_n^2 \otimes \pi_m^2\right) \Psi_1 \\
\mathcal{A}_{m,n}^2 &= \left(\pi_n^1 \otimes \pi_m^1\right) \Sigma_1 + \left(\pi_n^1 \otimes \pi_m^2\right) \Lambda_2 + \left(\pi_n^2 \otimes \pi_m^1\right) \Gamma_2 + \left(\pi_n^2 \otimes \pi_m^2\right) \Psi_2
\end{aligned}$$

To obtain the products of vectors involving the state vector X_t the following simplifications can be used:

$$\begin{aligned}
\pi_m X_t X_t' \pi_n' &= \mathcal{B}\left(\pi_m, \pi_n\right) X_t \\
\mathcal{B}\left(\pi_m, \pi_n\right) &= \left[\begin{array}{ccc} \mathcal{B}_{m,n}^0 & \mathcal{B}_{m,n}^1 & \mathcal{B}_{m,n}^2 \end{array}\right] \\
\mathcal{B}_{m,n}^0 &= \left(\pi_n^0 \otimes \pi_m^0\right) vec(I) = vec(\pi_n^0 * \pi_m^0) \\
\mathcal{B}_{m,n}^1 &= \left(\pi_n^0 \otimes \pi_m^1\right) + \left(\pi_n^1 \otimes \pi_m^0\right) \\
\mathcal{B}_{m,n}^2 &= \left(\pi_n^0 \otimes \pi_m^2\right) + \left(\pi_n^1 \otimes \pi_m^1\right) + \left(\pi_n^2 \otimes \pi_m^0\right)
\end{aligned}$$

Proof:

$$\begin{aligned}
\pi_m X_t X_t' \pi_n' &= \left[\begin{array}{ccc} \pi_m^0 & \pi_m^1 & \pi_m^2 \end{array}\right] \left[\begin{array}{ccc} 1 & x_t' & \tilde{x}_t' \\ x_t & x_t x_t' & 0 \\ \tilde{x}_t & 0 & 0 \end{array}\right] \left[\begin{array}{c} \pi_n^{0\prime} \\ \pi_n^{1\prime} \\ \pi_n^{2\prime} \end{array}\right] \\
&= \left[\begin{array}{ccc} \pi_m^0 + \pi_m^1 x_t + \pi_m^2 \tilde{x}_t & \pi_m^0 x_t' + \pi_m^1 x_t x_t' & \pi_m^0 \tilde{x}_t' \end{array}\right] \left[\begin{array}{c} \pi_n^{0\prime} \\ \pi_n^{1\prime} \\ \pi_n^{2\prime} \end{array}\right] \\
&= \left(\pi_m^0 + \pi_m^1 x_t + \pi_m^2 \tilde{x}_t\right) \pi_n^{0\prime} + \left(\pi_m^0 x_t' + \pi_m^1 x_t x_t'\right) \pi_n^{1\prime} + \pi_m^0 \tilde{x}_t' \pi_n^{2\prime} \\
&= \left(\pi_n^0 \otimes \pi_m^0\right) + \left(\pi_n^0 \otimes \pi_m^1\right) x_t + \left(\pi_n^0 \otimes \pi_m^2\right) \tilde{x}_t + \left(\pi_n^1 \otimes \pi_m^0\right) x_t \\
&\quad + \left(\pi_n^1 \otimes \pi_m^1\right) \tilde{x}_t + \left(\pi_n^2 \otimes \pi_m^0\right) \tilde{x}_t
\end{aligned}$$

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