Learning and Endogenous Business Cycles in a Standard Growth Model [1]

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Cyclical or chaotic competitive equilibria that do not exist under perfect foresight are shown to occur in a decentralized growth model under constant gain adaptive learning. This paper considers an economy populated by boundedly rational households making one-period ahead constant gain adaptive input price forecasts, and using simple expectation rules to predict long-run physical capital holdings and consumption. Under these hypotheses, lifetime decisions are derived as time unfolds, and analytical solutions to the representative household's problem exist for a standard class of preferences. Under various characteristics of the model's functional forms, competitive equilibrium trajectories under learning may exhibit opposite local stability properties depending whether the underlying information set accommodates all contemporary data. Calibrated to the U.S. economy, the model with boundedly rational households may exhibit endogenous business cycles around the permanent regime which is a saddle point under perfect foresight.

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1 Introduction

Decentralized Neoclassical growth models rely on the assumptions that all economic agents have complete as well as perfect knowledge of their lifetime environment, and are endowed with unlimited computing skills. As results, solutions to the households' intertemporal planning problems in these models are always carried out and derived once and for all at the beginning of their lifetime. If the planning horizon faced be these agents is long or infinite, then computing their lifetime decisions and analyzing the resulting competitive equilibrium trajectories of the economy are extremely complex tasks. In standard growth models, analytical solutions to the households' planning problems cannot be derived for general specifications of preferences, and log-linearizing the optimality conditions around a permanent regime as in King, Plosser, and Rebelo (1988), has become a frequently used technique to approximate the competitive equilibrium trajectories.

The equilibrium dynamics of a perfectly competitive economy does not only depend on individuals' preferences, technologic and demographic factors, but also relies on how forward-looking agents form their expectations. Since any decisions made by firms and households are based on price and quantity forecasts, characteristics of their expectation functions and information sets have a significant influence on the actual path of the economy. In practice, economic agents derive their forecasts on the basis of their knowledge about the functioning of the economy, and observations of the available data on prices and quantities. Since knowledge is both incomplete and imperfect at the individual level, expectations are in reality not always fulfilled and are frequently revised as new information becomes available. As results, lifetime plans are not always carried out and derived all at once, but instead are set and revised period after period as time unfolds. This observation is also consistent with the fact that agents' limited computing abilities naturally prevent them from solving highly complex problems when making intertemporal plans.

This paper presents a decentralized production economy populated by identical and infinitely lived boundedly rational households. At any given time period, each of them derives both current and future

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expected consumption plans by solving a two-period approximation of his infinite lifetime planning problem he would be able to solve all at once if he was endowed with both perfect foresight and unlimited computing skills. This model is a discrete time decentralized version with exogenous labor supply and labor augmenting technical progress of the standard neoclassical growth framework developed by Cass (1965), and Koopmans (1965). For different specifications of the underlying information sets, I consider that each boundedly rational household uses both constant gain adaptive learning mechanisms to forecast next period's input prices, and simple forecasting rules consistent with the economy's growth path to predict his two-period ahead physical capital holdings and future consumption stream. As a result, his lifetime planning problem corresponds to an infinite succession of two-period optimization problems in which analytical solutions can be derived under large specifications of preferences. Such representation of the households' planning problems has been suggested by Leontief (1958), and analyzed in simple growth settings by Day (1969), Day and Lin (1992) for particular specifications of preferences and expectation functions. I consider in this paper a standard growth framework with general specifications of preferences and expectation functions, and characterized the local dynamic properties around the perfect foresight steady-state in term of the properties of the functional forms and the information sets. Then I calibrate the model to the U.S. economy and present numerical illustrations. For particular values for the coefficient of relative risk aversion, the model may generate complex attractors that do not exist under the perfect foresight hypothesis.

The paper is organized as follows. In section 2, I describe the model. In section 3, I present the household's problem under perfect foresight as well as the local dynamic properties of the competitive equilibrium trajectories. In section 4, I present the household's problem under bounded rationality and analyze the local competitive equilibrium trajectories. In section 5, I calibrate the model to the US economy and provide numerical illustrations. In section 6, I conclude the paper.

2 The Model

Let us consider a perfectly competitive production economy populated at time t for $t \in Z_+ \cup \{0\}$ by N_t identical and infinitely lived households who own all firms and production factors. From time t to $t+1$, each household rents to firms x_t units of physical capital at the real rental rate R_t as well as 1 unit of labor service at the real wage rate w_t, receives a fraction π_t of the firms' profits, and allocates his resulting total income between current consumption c_t and investment i_t. By introducing superscripts to denote the planning time, I consider that the representative household's rental income and resource constraint expected to be received in j period(s) from time t where both $t, j \in Z_+ \cup \{0\}$, are given by: $y_t^j = w_t^j + R_t^j x_t^j$ and $c_t^j + i_t^j = y_t^j + \pi_t^j$ respectively. Planned individual investment, physical capital holdings, and lifetime consumption stream, are assumed to be derived from a constrained utility maximizing problem based on expected future prices and quantities. The household's expected lifetime utility evaluated at time t is represented by a function U defined over an infinite stream of planned consumption starting at time t: $\left\{c_t^j\right\}_{j=0}^{\infty}$. I consider the usual time separable specification of expected lifetime preferences by assuming that U is an infinite weighted sum of instantaneous utility functions u:

$$U\left(c_t^0, c_t^1, c_t^2, ...\right) = \sum_{j=0}^{\infty} \beta^j u(c_t^j) \tag{1}$$

where β denotes the discount factor with $0 < \beta < 1$.

Assumption 1 *The instantaneous utility function u: $\mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, concave, twice continuously differentiable with $u'(c)$ homogenous of degree p* for $-1 < p < 0$, *guarantees in each period the normality of the consumption commodity, and satisfies Inada conditions.*

The relationship between the aggregate level of output Y, the utilized aggregate level of physical capital K, and the employed aggregate level of labor L is described by a constant returns to scale aggregate production function F:

$$Y_t = \theta F(K_t, B_t L_t) = \theta L_t F(k_t, B_t) \tag{2}$$

where θ represents the total factor productivity, B_t denotes a labor augmenting technical progress, and $k_t = K_t/L_t$ is the utilized physical capital per employed worker.

Assumption 2 *The production function F: $\mathbb{R}_+^2 \to \mathbb{R}_+$ is concave, twice continuously differentiable, positive, increasing, strictly concave, and satisfies Inada conditions.*

Population and labor augmenting technical progress are assumed to grow at constant and known geometric rates n and e respectively:

$$N_t = (1+n)N_{t-1} \tag{3}$$

$$B_t = (1+e)B_{t-1} \tag{4}$$

where N_0, B_0 are given. I consider that these two laws of motion are part of the individual knowledge. I assume that the capital stock used in the production process depreciates from any time period to the next at a constant and known fraction δ. If at every time t, households can borrow or lend consumption commodities to themselves at the real interest rate r_t, and under the hypothesis that all assets are perfect substitutes, then the real rental rate of physical capital is always equal to the real interest rate plus the depreciation rate: $R_t = r_t + \delta$. Under this non-arbitrage condition, households are indifferent between lending consumption commodities to themselves or renting capital stock to firms. If the real interest rate is positive: $r_t > 0$ for any $t \in Z_+ \cup \{0\}$, then the entire stock of physical capital owned by households will be rented to firms. The expected individual holdings of physical capital in $j+1$ period(s) ahead depends on the expected individual investment and undepreciated physical capital holdings in j period(s) ahead: $(1+n)x_t^{j+1} = i_t^j + (1-\delta)\,x_t^j$. Thus, the expected budget constraint of a household in j period(s) from time t may be rewritten as follows:

$$c_t^j + (1+n)x_t^{j+1} = w_t^j + \left(1 + R_t^j - \delta\right)x_t^j + \pi_t^j \tag{5}$$

I consider that the economy consists of a large number of identical firms using the same constant returns to scale production technology given by (2). At every time t, each of them derives its capital demand k_t^d, labor demand l_t^d, and output supply y_t^s to maximize its current profits: $\Pi_t = \theta F\left(k_t^d, B_t l_t^d\right) - R_t k_t^d - w_t l_t^d$. From the first-order necessary condition to the representative firm's time period optimization problem, each production factors is paid its marginal product: $R_t = \theta F_1\left(k_t^d, B_t l_t^d\right)$ and $w_t = \theta B_t F_2\left(k_t^d, B_t l_t^d\right)$ under perfect competition. Since technology exhibits constant returns to scale, each firm realizes zero profits: $\Pi_t = \pi_t = 0$ for $t \in Z_+ \cup \{0\}$. In equilibrium, the aggregate level of output, utilized physical capital stock, and employment are equal to their respective aggregate supply and demand: $Y_t = N_t(c_t + i_t) = Y_t^s$, $K_t = N_t x_t = K_t^d$, $L_t = N_t = L_t^d$. Therefore at any given time period, the market clearing real wage and real interest rate depend on the current state of the economy:

$$w_t = W(k_t, B_t) \tag{6}$$

$$r_t = \Gamma(k_t, B_t) \tag{7}$$

where k_0 and B_0 are given. Forecasting input prices implies knowing the functional forms for W, Γ: $\mathbb{R}_+^2 \to \mathbb{R}_+$, and the laws of motion for the state variables k_t, B_t. The aggregate physical capital stock at time t depends on both its last period's undepreciated level, and last period's aggregate investment: $K_{t+1} = I_t + (1-\delta)K_t$. In the competitive equilibrium, the law of motion for the physical stock per capita will be given by: $k_{t+1} = g(k_t, B_t)$.

Variables per effective amount of labor are denoted by $\widetilde{y}_t^j = y_t^j/B_{t+j}$, $\widetilde{x}_t^j = x_t^j/B_{t+j}$, $\widetilde{c}_t^j = c_t^j/B_{t+j}$, $\widetilde{i}_t^j = I_t^j/B_{t+j}$, $\widetilde{w}_t^j = w_t^j/B_{t+j}$, and are assumed to stay unchanged along with the real interest rate in the permanent regime: $\widetilde{y}_t = \overline{y}$, $\widetilde{x}_t = \overline{x}$, $\widetilde{c}_t = \overline{c}$, $\widetilde{i}_t = \overline{i}$, $\widetilde{w}_t = \overline{w}$, $\widetilde{r}_t = \overline{r}$ for $t \in Z_+ \cup \{0\}$.

3 Perfect Foresight

In this section, I describe the lifetime constrained optimization problem of a representative household endowed with complete and perfect knowledge about his environment. Then I present the local stability properties of the perfect foresight competitive equilibrium trajectories around the non-trivial steady state.

3.1 The Household's Problem

If each household is endowed with complete and perfect knowledge about his environment, then he knows for any $t \in Z_+ \cup \{0\}$ the relationships between equilibrium input prices and state variables given by functions W, Γ, the actual law of motion for the capital stock per capita: $k_{t+1} = g(k_t, B_t)$, and both the non-arbitrage and zero-profit conditions: $R_t = r_t + \delta$, $\pi_t = 0$ respectively. As results, input price and state variable expectations made by the representative agent are always accurate: $w_t^j = w_{t+j}$, $r_t^j = r_{t+j}$, $k_t^j = k_{t+j}$, $B_t^j = B_{t+j}$ for any t, $j \in Z_+ \cup \{0\}$, and his lifetime consumption and capital holdings plans are always carried out: $c_t^j = c_{t+j}$, $x_t^j = x_{t+j}$ for any t, $j \in Z_+ \cup \{0\}$. The constrained lifetime utility maximizing problem of a representative perfect foresight household born at time t may be written as follows:

$$\begin{aligned} V(x_t, k_t, B_t) &= \underset{c_t, x_{t+1}}{Max} \{u(c_t) + \beta V(x_{t+1}, k_{t+1}, B_{t+1})\} \\ \text{s.t. } c_t + (1+n)x_{t+1} &= W(k_t, B_t) + (1 + \Gamma(k_t, B_t))\, x_t \\ k_{t+1} &= g(k_t, B_t) \\ B_{t+1} &= (1+e)B_t \\ &\text{given } x_t,\ k_t,\ B_t \end{aligned} \tag{8}$$

where V: $\mathbb{R}_+^3 \rightarrow \mathbb{R}$ denotes the value function. In equilibrium: $k_t = x_t$ for any $t \in Z_+ \cup \{0\}$. The formulation of the representative household's lifetime optimization problem given by (8) is standard in optimal growth literature. Since all future input prices and state variable levels are known, lifetime decisions are set once and for all at the beginning of time t. The first-order necessary condition for consumption expressed in per effective amount of labor may be written as:

$$u'(\widetilde{c}_{t+1}) = \frac{(1+n)}{\beta\,(1+r_{t+1})\,(1+e)^p} u'(\widetilde{c}_t) \tag{9}$$

At every time period, the utility maximizing levels for current consumption and next period's physical capital holdings solution to the household's lifetime optimization problem (8) are functions of the current realized values for the states variables:

$$c_t = c\,(x_t, k_t, B_t) \tag{10}$$

$$x_{t+1} = x\,(x_t, k_t, B_t) \tag{11}$$

Remark 1: *For a Cobb-Douglas production technology F, a log-linear instantaneous utility function u, and a full depreciation of physical capital: $\delta = 1$, the constrained lifetime utility maximizing problem (8) under perfect foresight has explicit solutions given by equations (10), (11).*

Remark 2: *If constrained lifetime utility maximizing problem (8) cannot be solved analytically, log-linearizing its first-order condition around a permanent regime is frequently used to approximate equations (10), (11); see King, Plosser and Rebelo (1988).*

3.2 Competitive Equilibrium Trajectories

A competitive equilibrium trajectory for this production economy populated with perfect foresight households is a sequence of factors prices: $\{w_t, r_t\}_{t=0}^{+\infty}$, consumption: $\{c_t\}_{t=0}^{\infty}$, physical capital holdings: $\{x_t\}_{t=0}^{\infty}$, input demands: $\left\{k_t^d, l_t^d\right\}_{t=1}^{\infty}$, and output supply: $\{y_t^s\}_{t=1}^{\infty}$ such that the following three conditions are satisfied at every time period t for $t \in Z_+ \cup \{0\}$: *i)* consumption $\{c_t\}_{t=0}^{\infty}$, and capital holdings: $\{x_t\}_{t=0}^{\infty}$ solve (8) given $\{w_t, r_t\}_{t=0}^{+\infty}$; *ii)* input demands: k_t^d, l_t^d, and output supply: y_t^s maximize firms' profits given w_t, r_t; *iii)* w_t, r_t clear the labor market and the physical capital market: $l_t^d = 1$, $k_t^d = x_t = k_t$ respectively.

At time $t+1$, the competitive equilibrium capital stock per effective amount of labor is a function of both its lagged level and last period's utility maximizing consumption per effective amount of labor:

$$\widetilde{k}_{t+1} = \frac{\theta f\left(\widetilde{k}_t\right) - \widetilde{c}_t + (1-\delta)\widetilde{k}_t}{(1+n)(1+e)} \tag{12}$$

where $\widetilde{y}_t = f\left(\widetilde{k}_t\right)$ with f: $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ denotes the production function (2) written in per effective amount of labor.

Proposition 1 *According to assumptions 1 and 2, the non-trivial steady state $(\overline{y}, \overline{c})$ is a saddle; see Azariadis 1993.*

Proof. The eigenvalues of the Jacobian matrix J of the system (9), (12) are the roots of the following characteristic polynomial: $Q(z) = z^2 - Tr\overline{J}z + Det\overline{J} = 0$, where both the trace and the determinant of the Jacobian matrix evaluated at the non-trivial steady state $(\overline{y}, \overline{c})$ are $Tr\overline{J} = 1 + \beta f''\left(\overline{k}\right) u'(\overline{c})/(1+n)^2(1+e)^{-p+1}u''(\overline{c}) + 1/\beta(1+e)^{p+1}$, $Det\overline{J} = 1/\beta(1+e)^{p+1}$ respectively. Under the assumption that $\beta(1+e)^{p+1} < 1$, then $Tr\overline{J} > 2$, and $Det\overline{J} > 1$. Since $Q(1) = -\beta f''\left(\overline{k}\right) u'(\overline{c})/(1+n)^2(1+e)^{-p+1}u''(\overline{c}) < 0$, $Q(-1) = 2 + 2/\beta(1+e)^{p+1} + \beta f''\left(\overline{k}\right) u'(\overline{c})/(1+n)^2(1+e)^{-p+1}u''(\overline{c}) + 1/\beta(1+e)^{p+1} > 0$ and $\left(Tr\overline{J}\right)^2 - 4Det\overline{J} \geq \left(1 - 1/\beta(1+e)^{p+1}\right)^2 > 0$, we can conclude that the Jacobian matrix of the system (9), (12) has a pair of positive real eigenvalues namely z_1, z_2 with $0 < z_1 < 1$ and $1 < z_2$. Therefore, the non-trivial steady state $(\overline{y}, \overline{c})$ is a saddle. ■

4 Bounded Rationality

In this section, I present the lifetime constrained optimization problem of a representative boundedly rational household and analyze the local stability properties of the competitive equilibrium trajectories around the non-trivial steady state under various characteristics of the functional forms and assumptions about the underlying information set.

4.1 The Household's Problem

Let us consider that households have incomplete knowledge about their environment in the sense that they do not know the actual law of motion for the physical capital stock per capita given by an equation of the form: $k_{t+1} = g(k_t, B_t)$, and the relationships described by equations (6), (7) between the market clearing input prices and the current state variables. At the beginning of every time t for $t \in Z_+ \cup \{0\}$, each household observes his current physical capital holdings: $x_t^0 = x_t$, the current state of the economy: (k_t, B_t), and sets current input price expectations to be equal to the prices announced by the 'Walrasian auctioneer': $w_t^0 = w_t$, $r_t^0 = r_t$. For each of these prices, every household evaluates his rental income $y_t^0 = y_t = w_t + R_t x_t$, derives his planned consumption for the current and the next period: (c_t^0, c_t^1) based on one-period ahead input price forecasts, and two-period ahead and up physical capital holdings and consumption expectations.

I consider that the representative boundedly rational household derives one-period ahead price forecasts using time invariant expectation functions defined over information sets including observations from time $t - M$ up to time $t - h$ with $h \in \{0, 1\}$ on the variable being predicted:

$$w_t^1 = \Psi\left(w_{t-h}, w_{t-1}, ..., w_{t-M}\right) \tag{13}$$

$$r_t^1 = \Phi\left(r_{t-h}, r_{t-1}, ..., r_{t-M}\right) \tag{14}$$

For $h = 0$, new expectations are formed for every different prices announced by the 'Walrasian auctioneer'. For $h = 1$, expectations are based on realized prices and are not revised for every different prices announced by the 'Walrasian auctioneer'. The parameter M denotes a fixed memory length, with $M \in \mathbb{Z}_+$ and $M \geq h$. I consider in the rest of the paper that individuals have unlimited memory: $M = t$. Therefore, the information set expands as time goes by and includes at time t a total number of $t + 1 - h$ observations on input prices.

Assumption 3 *The expectation functions Ψ, Φ: $\mathbb{R}_+^{1-h+M} \rightarrow \mathbb{R}_+$ are continuously differentiable and satisfy the following conditions at the permanent regime $(\overline{w}, \overline{r})$: $\overline{w} = \Psi\left(\overline{w}, ..., \overline{w}\right)$ and $\overline{r} = \Phi\left(\overline{r}, ..., \overline{r}\right)$.*

I consider that the representative household derives his two-period ahead physical capital holdings and future consumption stream from simple time invariant expectation rules consistent with the economy's balanced growth path:

$$x_t^j = (1+e)x_t^{j-1} \tag{15}$$

$$c_t^j = (1+e)c_t^{j-1} \tag{16}$$

where $j \in Z_+ - \{1\}$. The lifetime utility function (1) defined over an infinite planned stream of consumption satisfying forecasting rule (16) is denoted by v: $\mathbb{R}_+^2 \to \mathbb{R}$, which can be written as:

$$v\left(c_t^0, c_t^1\right) = u\left(c_t^0\right) + \sum_{j=1}^{\infty} \beta^j u\left((1+e)^{j-1} c_t^1\right) \tag{17}$$

Proposition 2 *Under a constant relative risk aversion instantaneous utility function u with a coefficient of relative risk aversion $\gamma \in R_+ - \{1\}$, and the assumption that $\beta(1+e)^{1-\gamma} < 1$, the expected lifetime utility function (17) can be written as: $v\left(c_t^0, c_t^1\right) = u\left(c_t^0\right) + \varphi u\left(c_t^1\right) + \xi$ where $\varphi = \beta / \left[1 - \beta(1+e)^{1-\gamma}\right]$ and $\xi = -\beta^2\left(1-(1+e)^{1-\gamma}\right) / \left[(1-\beta)(1-\gamma)\left(1-\beta(1+e)^{1-\gamma}\right)\right]$.*

Proof. For a constant relative risk aversion specification of the households' instantaneous utility function: $u(c) = \left(c^{1-\gamma}-1\right)/(1-\gamma)$, the second term of the lifetime utility function (17) can be rewritten as: $\beta \sum_{j=1}^{\infty}\left[c_t^{1-\gamma}\left(\beta(1+e)^{1-\gamma}\right)^{j-1} - \beta^{j-1}\right]/(1-\gamma)$. For $\beta(1+e)^{1-\gamma} < 1$, this geometric series converges to $\beta c_t^{1-\gamma} / \left[\left(1-\beta(1+e)^{1-\gamma}\right)(1-\gamma)\right] - \beta / [(1-\beta)(1-\gamma)]$. Therefore, we can easily show that equation (17) can be written as: $v\left(c_t^0, c_t^1\right) = u\left(c_t^0\right) + \varphi u\left(c_t^1\right) + \xi$, where $\varphi = \beta / \left[1-\beta(1+e)^{1-\gamma}\right]$, and $\xi = -\beta^2\left(1-(1+e)^{1-\gamma}\right) / \left[(1-\beta)(1-\gamma)\left(1-\beta(1+e)^{1-\gamma}\right)\right]$. ■

The expected budget constraint for the current period is given by equation (5) for $j = 0$ and $w_t^0 = w_t$, $r_t^0 = r_t$. The next period's expected budget constraint is obtained by plugging the forecasting rule (15) into equation (5) for $j = 1$:

$$c_t^1 = w_t^1 + \left(R_t^1 - \delta - e - n - ne\right) x_t^1 + \pi_t^1 \tag{18}$$

Under bounded rationality, the representative household's infinite horizon lifetime utility maximizing problem consists of a succession of two-period constrained optimization problems in which planned consumption plans for the current and the next period are derived at every time period t for $t \in Z_+ \cup \{0\}$ by maximizing (17) subject to (5) for $j = 0$ and $w_t^0 = w_t$, $r_t^0 = r_t$, (18), the zero profit conditions: $\pi_t^0 = \pi_t^1 = 0$, along with the current and the next period's expected non-arbitrage conditions: $R_t = r_t + \delta$, $R_t^1 = r_t^1 + \delta$:

$$\begin{aligned} \nu(x_t, k_t, B_t) = & \underset{c_t^0, c_t^1}{ArgMax}\ v\left(c_t^0, c_t^1\right) \\ \text{s.t. } & c_t^0 + (1+n)x_t^1 = w_t + (1+r_t)\, x_t \\ & c_t^1 = w_t^1 + \left(r_t^1 - e - n - ne\right) x_t^1 \\ & \text{given } x_t,\ w_t,\ r_t,\ w_t^1,\ r_t^1 \end{aligned} \tag{19}$$

where v: $\mathbb{R}_+^3 \to \mathbb{R}$ denotes the expected value function. Because there is no uncertainty about the present once the equilibrium prices have been found by the 'Walrasian auctioneer', consumption plans for the current period are always carried out: $c_t^0 = c_t$. Therefore: $x_t^1 = x_{t+1}$ for $t \in Z_+ \cup \{0\}$, and the solution to the household's optimization problem (19) can be written as:

$$c_t = c\left(x_t, w_t, r_t, w_t^1, r_t^1\right) \tag{20}$$

$$c_t^1 = c^1\left(x_t, w_t, r_t, w_t^1, r_t^1\right) \tag{21}$$

$$x_{t+1} = x\left(x_t, w_t, r_t, w_t^1, r_t^1\right) \tag{22}$$

where both the market clearing real wage and the market clearing real interest rate depend on the current state of the economy according to equations (6) and (7). In equilibrium, the individual physical capital holdings coincides with the physical capital per capita in the economy: $k_t = x_t$ for any $t \in Z_+ \cup \{0\}$. In this framework, future consumption and physical capital holdings plans may not always be carried out: $c_t^j \neq c_{t+j}$, $x_t^j \neq x_{t+j}$ for $j \in Z_+$, $t \in Z_+ \cup \{0\}$, price expectations may be wrong: $w_t^j \neq w_{t+j}$, $r_t^j \neq x_{t+j}$ for $j \in Z_+$, $t \in Z_+ \cup \{0\}$, and expectations may be revised as time goes by: $c_t^j \neq c_{t+i}^{j-i}$, $x_t^j \neq x_{t+j}^{j-i}$ for $j \geq i \geq 1$ where i, j, $t \in Z_+ \cup \{0\}$.

Proposition 3 *If the model under perfect foresight has a unique non-trivial steady state, and assumption 3 is satisfied, then the model with boundedly rational households has the same unique steady state.*

Proof. Under perfect foresight, the optimality condition at time t for consumption per effective amount of labor satisfies equation (9). From the first-order necessary conditions associated with the constrained optimization problem (19), the optimality condition at time t for actual and planned consumption per effective amount of labor satisfies: $u'(\widetilde{c}_t)/u'(\widetilde{c}_t^1) = \beta(1+e)^p \rho_t^1 / \left(1-\beta(1+e)^{p+1}\right)$ where $\rho_t^1 = \left(r_t^1 - n - e - ne\right)/(1+n)$. According to assumption 3, those two optimality conditions imply the same steady level for capital stock per effective amount of labor. ■

4.2 Competitive Equilibrium Trajectories under Constant Gain Learning

An intertemporal competitive equilibrium for a production economy populated with boundedly rational households is a sequence of factors prices: $\{w_t, r_t\}_{t=0}^{+\infty}$, lifetime consumption: $\{c_t\}_{t=0}^{\infty}$, beliefs: $\left\{w_t^1, r_t^1, c_t^1\right\}_{t=1}^{\infty}$, capital holdings: $\{x_t\}_{t=0}^{\infty}$, input demands: $\left\{k_t^d, l_t^d\right\}_{t=1}^{\infty}$, and output supply: $\{y_t^s\}_{t=1}^{\infty}$ such that the following four conditions are satisfied at every time period t for $t \in Z_+ \cup \{0\}$: *i)* current consumption and expected next period's consumption and physical capital holdings c_t, c_t^1, x_{t+1} solve (19); *ii)* given w_t, r_t, input demands: k_t^d, l_t^d, and output supply: y_t^s maximize firms' profits; *iii)* w_t, r_t clear the labor market and the physical capital market: $l_t^d = 1$, $k_t^d = x_t = k_t$ respectively; *iv)* given $\{w_{t-i}, r_{t-i}\}_{i=h}^{M}$, one-period ahead price forecasts: w_t^1, r_t^1 are derived from the expectation functions Ψ, Φ respectively.

In the competitive equilibrium, the capital stock per effective amount of labor at time $t+1$ is a function of its lagged value and one-period ahead input price expectations formed at time t:

$$\widetilde{k}_{t+1} = G\left(\widetilde{k}_t, \widetilde{w}_t^1, r_t^1\right) \tag{23}$$

where $\overline{G}_1$, $\overline{G}_2$, $\overline{G}_3$ denote the partial derivatives of function G: $\mathbb{R}_+^3 \to \mathbb{R}$ evaluated in the steady state $(\overline{k}, \overline{w}, \overline{r})$ with $\overline{G}_1 > 0$, $\overline{G}_2 < 0$ according to the normality of the consumption commodity stated by assumption 1, and $\overline{G}_3 > 0$ under the standard hypothesis that the substitution effect outweighs the income effect. Let us consider that one-period ahead input price expectation functions (13), (14) are derived from simple constant gain adaptive expectation schemes denoted by weighted averages between the last observation and prediction:

$$\widetilde{w}_t^1 = \widetilde{w}_{t-1}^1 + \lambda\left(\widetilde{w}_{t-h} - \widetilde{w}_{t-1}^1\right) \tag{24}$$

$$r_t^1 = r_{t-1}^1 + \mu\left(r_{t-h} - r_{t-1}^1\right) \tag{25}$$

where λ, $\mu \in (0,1]$ can be interpreted as speed of adjustment parameters. For $h = 0$, the second terms of equations (24), (25) represent last period's forecast errors. These two expectation schemes imply that each new observation is as important as the previous in making next period's forecasts. In the limit case of "fast" learning: $\lambda = \mu = 1$, constant gain adaptive schemes (24), (25) correspond to the situation in which households have naive expectations: $\widetilde{w}_t^1 = \widetilde{w}_t$, $r_t^1 = r_t$ for $t \in Z_+ \cup \{0\}$. The competitive equilibrium trajectories under constant gain adaptive learning are described by a three-dimensional system in $\widetilde{k}_t$, $\widetilde{w}_t^1$, r_t^1 given by difference equations (23), (24), (25). To simplify the local dynamic analysis of the model, I consider that parameters λ, μ are identical and equal to η with $\eta \in (0,1]$. I characterize in the rest of this section the local dynamic properties of the model around its non-trivial steady state. Depending whether households' information sets used in making forecasts include the input prices announced at the beginning of the time period by the 'Walrasian auctioneer': $h \in \{0,1\}$, I characterize the eigenvalues of the Jacobian matrix of the system (23), (24), (25) evaluated at $(\overline{k}, \overline{w}, \overline{r})$ under various possible values for $\overline{G}_1$, a, η where $\overline{G}_1 > 0$, $a \equiv -(1-\alpha)\overline{G}_2 f'\left(\overline{k}\right) - \overline{G}_3 f''\left(\overline{k}\right) > 0$ and $\eta \in (0,1]$.

The case where $h = 0$

In the situation in which one-period ahead constant gain adaptive input price forecasts include the beginning of the period announcements made by the 'Walrasian Auctioneer', the eigenvalues denoted by z_i for $i = 1, 2, 3$ of the corresponding Jacobian matrix are the roots of the following characteristic polynomial: $Q(z) = z^3 - Tr(\eta)z^2 + \omega(\eta)z - Det(\eta) = 0$ whose coefficients are the trace: $Tr(\eta) = 2(1-\eta)+\overline{G}_1-\eta a \lesseqgtr 0$, the sum of the principal minors of order two: $\omega(\eta) = (1-\eta)\left[1 + 2\overline{G}_1 - \eta(1 + a)\right] \lesseqgtr 0$, and the determinant: $Det(\eta) = \overline{G}_1(1-\eta)^2 > 0$ of the Jacobian matrix with $dTr(\eta)/d\eta < 0$, $d\omega(\eta)/d\eta \lesseqgtr 0$, $dDet(\eta)/d\eta < 0$. Since one eigenvalue corresponds to the weight affected last period's predictions: $z_1 = 1 - \eta$, the other two: z_2, z_3 are solution to the characteristic polynomial rewitten as follows: $Q(z) = (z - 1 + \eta)\left[z^2 + \varepsilon(\eta)z + \kappa(\eta)\right] = 0$ whose coefficients are linear with respect to η and are given by $\varepsilon(\eta) = -(1 - \eta) - \overline{G}_1 + \eta a \lesseqgtr 0$, $\kappa(\eta) = \overline{G}_1(1 - \eta) > 0$ with $d\varepsilon(\eta)/d\eta > 0$, $d\kappa(\eta)/d\eta < 0$. Therefore $Q(1) = 0$ and $Q(-1) = 0$ correspond to equations $1 + \varepsilon + \kappa = 0$, $1 - \varepsilon + \kappa = 0$ respectively. In the rest of this section, I discuss the qualitatively local stability of the system (23), (24), (25) by locating the line segment M_0M_1 in the (ε, κ) plane where $M_\eta = (\varepsilon(\eta), \kappa(\eta))$ and $\eta \in (0, 1]$. For $\eta = 0$, the point $M_0 = (-1 - \overline{G}_1, \overline{G}_1)$ is located on the straight line associated with equation $1 + \varepsilon + \kappa = 0$ and satisfying $Q(1) = 0$. For $\eta = 1$, the point $M_1 = (a - \overline{G}_1, 0)$ lies on the horizontal axis.

Proposition 4 *Under constant gain learning with $h = 0$ and $0 < \overline{G}_1 \leq 1$, the following two configurations occur; i) if $a < 1 + \overline{G}_1$, then the non-trivial steady state is locally stable for any $\eta \in (0, 1]$; ii) if $a > 1 + \overline{G}_1$, then the non-trivial steady state is locally stable for any $\eta \in (0, \widehat{\eta}_F)$, undergoes a Flip bifurcation at $\widehat{\eta}_F = 2(1 + \overline{G}_1)/(1 + a + \overline{G}_1)$, and becomes locally unstable for any $\eta \in (\widehat{\eta}_F, 1]$.*

Proof. If $0 < \overline{G}_1 \leq 1$, then M_0 lies either anywhere between the points $(-1, 0)$ and $(-2, 1)$ or corresponds to the point $(-2, 1)$ on the line segment associated with equation $1 + \varepsilon + \kappa = 0$.

i) If $a < 1 + \overline{G}_1$, then M_1 lies on the horizontal axis anywhere between the points $(1, 0)$ and $(-1, 0)$. Therefore the line segment M_0M_1 lies above the straight line corresponding to the equation $1+\varepsilon+\kappa = 0$, above the straight line corresponding to the equation $1 - \varepsilon + \kappa = 0$, and ends on the horizontal axis. For any $\eta \in (0, 1]$, eigenvalues z_2, z_3 are either complex conjugate with modulus less than 1, or real in the interval $(-1, 1)$ with identical sign. Since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a sink (see figure A1)

ii) If $a > 1+\overline{G}_1$, then M_1 lies on the horizontal axis anywhere to the left of the point $(1, 0)$. Therefore the line segment M_0M_1 lies above the straight line corresponding to the equation $1 + \varepsilon + \kappa = 0$, crosses the straight line corresponding to the equation $1 - \varepsilon + \kappa = 0$ at $\eta = \widehat{\eta}_F$, and ends up on the horizontal axis. For any $\eta \in (0, \widehat{\eta}_F)$, eigenvalues z_2, z_3 are either complex conjugate with modulus less than 1, or real with the same sign in the interval $(-1, 1)$ and since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a sink. At $\widehat{\eta}_F = 2(1 + \overline{G}_1)/(1 + a + \overline{G}_1)$, eigenvalues z_2, z_3 are real with one equals to -1, and the other in the interval $(-1, 0)$. For any $\eta \in (\widehat{\eta}_F, 1]$, eigenvalues z_2, z_3 are real with identical sign: one in the interval $(-1, 1)$, and the other in the interval $(-\infty, -1)$. Since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a saddle (see figure A4) ■

Proposition 5 *Under constant gain learning with $h = 0$ and $1 < \overline{G}_1 < 1 + a$, the following three configurations occur, i) if $a < 1 + \overline{G}_1$, then the non-trivial steady state is locally unstable for any $\eta \in (0, \widehat{\eta}_H)$, undergoes a Hopf-Neimark bifurcation at $\widehat{\eta}_H = -(1 - \overline{G}_1)/\overline{G}_1$ and becomes locally stable for any $\eta \in (\widehat{\eta}_H, 1]$; ii) If $1 + \overline{G}_1 < a < \overline{G}_1 + \chi$ with $\chi = -(1 + 3\overline{G}_1)/(1 - \overline{G}_1)$, then the non-trivial steady state is locally unstable for any $\eta \in (0, \widehat{\eta}_H)$, undergoes a Hopf-Neimark bifurcation at $\widehat{\eta}_H = -(1 - \overline{G}_1)/\overline{G}_1$; becomes locally stable for any $\eta \in (\widehat{\eta}_H, \widehat{\eta}_F)$, undergoes a Flip bifurcation at $\widehat{\eta}_F = 2(1 + \overline{G}_1)/(1 + a + \overline{G}_1)$, and becomes locally unstable for any $\eta \in (\widehat{\eta}_F, 1]$; iii) if $\overline{G}_1 + \chi < a$ with $\chi = -(1 + 3\overline{G}_1)/(1 - \overline{G}_1)$, then the non-trivial steady state is locally unstable for any $\eta \in (0, 1]$.*

Proof. If $1 < \overline{G}_1 < 1 + a$, then M_0 lies either anywhere between the points $(-2, 1)$ and $(-2 - a, 1 + a)$ on the line segment associated with the equation $1 + \varepsilon + \kappa = 0$.

i) If $a < 1 + \overline{G}_1$, then M_1 lies on the horizontal axis anywhere between the points $(-1, 0)$ and $(1, 0)$. Therefore the line segment M_0M_1 lies above the straight line corresponding to the equation $1+\varepsilon+\kappa = 0$, crosses the horizontal line associated with equation $\kappa = 1$ at $\eta = \widehat{\eta}_H$, and ends on the horizontal axis above the straight line associated with equation $1 - \varepsilon + \kappa = 0$. For any $\eta \in (0, \widehat{\eta}_H)$, eigenvalues z_2,

z_3 are either complex conjugate with modulus greater than 1, or real with identical sign either in the interval $(-\infty, -1)$ or $(1, \infty)$. Since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a saddle. At $\eta_H = -(1 - \overline{G}_1)/\overline{G}_1$, eigenvalues z_2, z_3 are complex conjugate with modulus equals to 1. For any $\eta \in (\widehat{\eta}_H, 1]$, eigenvalues z_2, z_3 are complex conjugate with modulus less than 1, or real with identical sign in the interval $(-1, 1)$ and since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a sink, (see figure A6).

ii) If $1 + \overline{G}_1 < a < \overline{G}_1 + \chi$, then M_1 lies on the horizontal axis anywhere between the points $(1, 0)$ and $(\chi, 0)$ with $\chi = -(1 + 3\overline{G}_1)/(1 - \overline{G}_1)$. Therefore the line segment M_0M_1 lies above the straight line corresponding to the equation $1 + \varepsilon + \kappa = 0$, crosses both the horizontal line associated with equation $\kappa = 1$ at $\eta = \eta_H$ and the straight line associated with equation $1 - \varepsilon + \kappa = 0$ at $\eta = \widehat{\eta}_F$, then ends on the horizontal axis. For any $\eta \in (0, \widehat{\eta}_H)$, eigenvalues z_2, z_3 are either complex conjugate with modulus greater than 1, or real with identical sign either in the interval $(-\infty, -1)$ or $(1, \infty)$. Since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a saddle. At $\widehat{\eta}_H = -(1 - \overline{G}_1)/\overline{G}_1$, eigenvalues z_2, z_3 are complex conjugate with modulus equals to 1. For any $\eta \in (\widehat{\eta}_H, \widehat{\eta}_F)$, eigenvalues z_2, z_3 are complex conjugate with modulus less than 1, or real in the interval $(-1, 1)$ and since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a sink. At $\widehat{\eta}_F = 2(1 + \overline{G}_1)/(1 + a + \overline{G}_1)$, eigenvalues z_2, z_3 are real with identical sign one equals to -1, and the other in the interval $(-1, 0)$. For any $\eta \in (\widehat{\eta}_F, 1]$, eigenvalues z_2, z_3 are real with identical sign: one in the interval $(-1, 0)$, and the other in the interval $(-\infty, -1)$. Since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a saddle (see figure A9).

iii) If $a > \overline{G}_1 + \chi$, then M_1 lies on the horizontal axis anywhere to the left of point $(\chi, 0)$ with $\chi = -(1 + 3\overline{G}_1)/(1 - \overline{G}_1)$. Therefore the line segment M_0M_1 lies above the straight line corresponding to the equation $1 + \varepsilon + \kappa = 0$, crosses to the left of the point $(2, 1)$ both the straight line associated with equation $1 - \varepsilon + \kappa = 0$ at $\eta = \widehat{\eta}_F$ and the horizontal line associated with equation $\kappa = 1$ at $\eta = \widehat{\eta}_H$ where $\widehat{\eta}_F < \widehat{\eta}_H$. For any $\eta \in (0, \widehat{\eta}_F)$, eigenvalues z_2, z_3 are either complex conjugate with modulus greater than 1, or real with identical sign either in the intervals $(-\infty, -1)$ or $(1, \infty)$. Since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a saddle. At $\widehat{\eta}_F = 2(1 + \overline{G}_1)/(1 + a + \overline{G}_1)$, eigenvalues z_2, z_3 are real with one equals to -1, and the other in the interval $(-\infty, -1)$. For any $\eta \in (\widehat{\eta}_F, \widehat{\eta}_H)$, eigenvalues z_2, z_3 are real and outside the unit circle either in the intervals $(-\infty, -1)$ or $(1, \infty)$. Since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a saddle. At $\widehat{\eta}_H = -(1 - \overline{G}_1)/\overline{G}_1$, eigenvalues z_2, z_3 are real with one equals to -1, and the other in the interval $(-\infty, -1)$. For any $\eta \in (\eta_H, 1]$, eigenvalues z_2, z_3 are real with identical sign: one in the interval $(-1, 1)$ and the other one in the intervals $(-\infty, -1)$ or $(1, \infty)$. Since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a saddle. (see figure A10). ■

Proposition 6 *Under constant gain learning with* $h = 0$ *and* $\overline{G}_1 \geq 1 + a$, *then the non-trivial steady state is locally unstable for any* $\eta \in (0, 1]$.

Proof. If $\overline{G}_1 \geq 1 + a$, then M_0 lies either anywhere to the right of the point $(-2 - a, 1 + a)$ or at the point $(-2 - a, 1 + a)$ on the line segment associated with the equation $1 + \varepsilon + \kappa = 0$, and M_1 lies either anywhere to the right of the point $(-1, 0)$ or at the point $(-1, 0)$ on the horizontal axis. For $\overline{G}_1 > 1 + a$, the line segment M_0M_1 lies below the straight line corresponding to the equation $1 + \varepsilon + \kappa = 0$. Therefore eigenvalues z_2, z_3 are real with identical sign: one in the interval $(0, 1)$ and the other in the interval $(1, \infty)$. Since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a saddle. For $\overline{G}_1 = 1 + a$, the line segment M_0M_1 coincide with the straight line corresponding to the equation $1 + \varepsilon + \kappa = 0$. Therefore eigenvalues z_2, z_3 are real with one equals to 1, and the other in the interval $(1, \infty)$. Since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a saddle. (see figure A11). ■

Under the different restrictions imposed on the values of $\overline{G}_1$ and a when $h = 0$, figure A12 summarizes in the $\overline{G}_1 - a$ plane the local qualitative properties of the model around the non-trivial steady state.

The case where $h = 1$

In the situation in which one-period ahead constant gain adaptive input price forecasts do not include the beginning of the period announcements made by the 'Walrasian Auctioneer', the eigenvalues denoted by z_i for $i = 1, 2, 3$ of the corresponding Jacobian matrix are the roots of the following characteristic polynomial: $Q(z) = z^3 - Tr(\eta)z^2 + \omega(\eta)z - Det(\eta) = 0$ whose coefficients are the trace: $Tr(\eta) = 2(1 - \eta) + \overline{G}_1 > 0$, the sum of the principal minors of order two: $\omega(\eta) = (1 - \eta)\left[(1 - \eta) + 2\overline{G}_1\right] + \eta a > 0$, and the determinant: $Det(\eta) = (1 - \eta)\left[\eta a + (1 - \eta)\overline{G}_1\right] > 0$ of the Jacobian matrix with $dTr(\eta)/d\eta < 0$, $d\omega(\eta)/d\eta \lesseqgtr 0$, $dDet(\eta)/d\eta \lesseqgtr 0$. Since one eigenvalue $z_1 = 1 - \eta$, the other two: z_2, z_3 are solution to the characteristic polynomial rewritten as follows: $Q(z) = (z-1+\eta)\left[z^2 + \varepsilon(\eta)z + \kappa(\eta)\right] = 0$ whose coefficients are linear with respect to η and are given by $\varepsilon(\eta) = -(1-\eta) - \overline{G}_1 < 0$, $\kappa(\eta) = \eta a + (1-\eta)\overline{G}_1 > 0$ with $d\varepsilon(\eta)/d\eta > 0$, $d\kappa(\eta)/d\eta \lesseqgtr 0$. Therefore $Q(1) = 0$ and $Q(-1) = 0$ correspond to equations $1 + \varepsilon + \kappa = 0$, $1 - \varepsilon + \kappa = 0$ respectively. In the rest of this section, I discuss the qualitatively local stability of the system (23), (24), (25) by locating the line segment M_0M_1 in the (ε, κ) plane where $M_\eta = (\varepsilon(\eta), \kappa(\eta))$ and $\eta \in (0,1]$. For $\eta = 0$, the point $M_0 = (-1 - \overline{G}_1, \overline{G}_1)$ belongs to the straight line associated with equation $1 + \varepsilon + \kappa = 0$ and satisfying $Q(1) = 0$. For $\eta = 1$, the point $M_1 = (-\overline{G}_1, a)$ is located anywhere above the horizontal axis and to the left of the vertical axis.

Proposition 7 *Under constant gain learning with $h = 1$ and $0 < \overline{G}_1 \leq 1$, the following two configurations occur; i) if $a < 1$, then the non-trivial steady state is locally stable for any $\eta \in (0,1]$; ii) if $a > 1$, then the non-trivial steady state is locally stable for any $\eta \in (0, \widetilde{\eta}_H)$, undergoes a Hopf-Neimark bifurcation at $\widetilde{\eta}_H = (1 - \overline{G}_1)/(a - \overline{G}_1)$, and becomes locally unstable for any $\eta \in (\widetilde{\eta}_H, 1]$.*

Proof. If $0 < \overline{G}_1 \leq 1$, then M_0 lies either anywhere between the points $(-1,0)$ and $(-2,1)$ or corresponds to the point $(-2,1)$ on the line segment associated with equation $1 + \varepsilon + \kappa = 0$.

i) If $a < 1$, then M_1 lies below the horizontal line associated with $\kappa = 1$ east from M_0. For any $\eta \in (0,1]$, eigenvalues z_2, z_3 are either complex conjugate with modulus less than 1, or real with identical sign in the interval $(-1,1)$. Since $z_1 = 1 - \eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a sink. (see figure A2)

ii) If $a > 1$, then M_1 lies above the horizontal line associated with $\kappa = 1$ northeast from M_0. Therefore the line segment M_0M_1 lies above the straight line corresponding to the equation $1+\varepsilon+\kappa = 0$, and crosses the horizontal line $\kappa = 1$ at $\eta = \eta_H$. For any $\eta \in (0, \widetilde{\eta}_H)$, eigenvalues z_2, z_3 are either complex conjugate with modulus less than 1, or real with identical sign in the interval $(-1,1)$. Since $z_1 = 1 - \eta < 1$, then, $(\overline{k}, \overline{w}, \overline{r})$ is a sink. At $\widetilde{\eta}_H = (1 - \overline{G}_1)/(a - \overline{G}_1)$, eigenvalues z_2, z_3 are complex conjugate with modulus equals to 1. For any $\eta \in (\widetilde{\eta}_H, 1]$, eigenvalues z_2, z_3 are complex conjugate with modulus greater than 1. Since $z_1 = 1 - \eta < 1$, then, $(\overline{k}, \overline{w}, \overline{r})$ is a saddle. (see figure A5) ■

Proposition 8 *If $0 < \overline{G}_1 \leq 1$, local stability under constant gain learning when $h = 1$ implies local stability when $h = 0$. However the reverse is not true.*

Proof. If $0 < \overline{G}_1 \leq 1$, then local stability under constant gain learning when $h = 1$ requires according to proposition 7 that $a < 1$ implying that $a < 1 + \overline{G}_1$which is according to proposition 4 *i)* a necessary condition for local stability under constant gain learning when $h = 0$. However the reverse is not true since condition $a < 1 + \overline{G}_1$ may imply $a > 1$ which is according to proposition 7 a necessary condition for local instability under constant gain learning when $h = 1$. (see figures A2, A3) ■

Proposition 9 *If $0 < \overline{G}_1 \leq 1$, and $a > 1 + \overline{G}_1$, then the set of constant gain parameters η associated with local stability is larger for $h = 0$ than for $h = 1$.*

Proof. If $0 < \overline{G}_1 \leq 1$, and $a > 1 + \overline{G}_1$, local stability under constant gain learning requires $\eta \in (0, \widehat{\eta}_F)$ for $h = 0$, and $\eta \in (0, \widetilde{\eta}_H)$ for $h = 1$ where $\widehat{\eta}_F = 2(1 + \overline{G}_1)/(1 + a + \overline{G}_1)$ and $\widetilde{\eta}_H = (1 - \overline{G}_1)/(a - \overline{G}_1)$. Under above conditions on parameters $\overline{G}_1$, a, we can easily show that $\widehat{\eta}_F > \eta_H$. (see figure A4) ■

Proposition 10 *Under constant gain learning with $h = 1$ and $1 < \overline{G}_1 < 1 + a$, the following two configurations occur; i) if $a < 1$, then the non-trivial steady state is locally unstable for any $\eta \in (0, \widetilde{\eta}_H)$, undergoes a Hopf-Neimark bifurcation at $\widetilde{\eta}_H = (1 - \overline{G}_1)/(a - \overline{G}_1)$, and becomes locally stable for any $\eta \in (\widetilde{\eta}_H, 1]$; ii) if $a > 1$, then the non-trivial steady state is locally unstable stable for any $\eta \in (0,1]$.*

Proof. If $1 < \overline{G}_1 < 1 + a$, then M_0 lies anywhere between the points $(-2,1)$ and $(-2-a, 1+a)$ on the line segment associated with equation $1 + \varepsilon + \kappa = 0$.

i) If $a < 1$, then M_1 lies south east from point M_0. Therefore the line segment M_0M_1 lies above the straight line corresponding to the equation $1+\varepsilon+\kappa = 0$, and crosses the horizontal line $\kappa = 1$ at $\eta = \widetilde{\eta}_H$. For any $\eta \in (0, \widetilde{\eta}_H)$, eigenvalues z_2, z_3 are either complex conjugate with modulus greater than 1, or real with identical sign in the intervals $(-\infty, -1)$ or $(1, \infty)$. Since $z_1 = 1 - \eta < 1$, then, $(\overline{k}, \overline{w}, \overline{r})$ is a saddle. At $\widetilde{\eta}_H = (1 - \overline{G}_1)/(a - \overline{G}_1)$, eigenvalues z_2, z_3 are complex conjugate with modulus equals to 1. For any $\eta \in (\widetilde{\eta}_H, 1]$, eigenvalues z_2, z_3 are either complex conjugate with modulus less than 1, or real with identical sign in the interval $(-1,1)$. Since $z_1 = 1 - \eta < 1$, then, $(\overline{k}, \overline{w}, \overline{r})$ is a sink. (see figure A7) *ii)* If $a > 1$, then M_1 lies below the horizontal line associated with $\kappa = 1$ east from M_0. For any $\eta \in (0,1]$, eigenvalues z_2, z_3 are either complex conjugate with modulus greater than 1, or real with identical sign in the intervals $(-\infty,-1)$ or $(1,\infty)$. Since $z_1 = 1-\eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a saddle. (see figure A8). ■

Proposition 11 *If* $1 < \overline{G}_1 < 1+a$, *and* $a < 1$, *then the set of constant gain parameters* η *associated with local stability is larger for* $h = 0$ *than for* $h = 1$.

Proof. If $1 < \overline{G}_1 < 1+a$, and $a < 1$, local stability under constant gain learning requires $\eta \in (\widehat{\eta}_H, 1]$ for $h = 0$, and $\eta \in (\widetilde{\eta}_H, 1]$ for $h = 1$ where $\widehat{\eta}_H = -(1-\overline{G}_1)/\overline{G}_1$ and $\widetilde{\eta}_H = (1-\overline{G}_1)/(a-\overline{G}_1)$. Under above conditions on parameters $\overline{G}_1$, a, we can easily show that $\widehat{\eta}_H < \widetilde{\eta}_H$. (see figure A7) ■

Proposition 12 *If* $1 < \overline{G}_1 < 1+a$, *and* $a < \overline{G}_1 + \chi$, *local instability for any* $\eta \in (0,1]$ *under constant gain learning when* $h = 1$ *may imply local stability for some values of the constant gain parameter* η *when* $h = 0$. *However the reverse is not true.*

Proof. If $1 < \overline{G}_1 < 1+a$, then local instability for any $\eta \in (0,1]$ under constant gain learning when $h = 1$ requires $a > 1$ according to proposition 10 *ii)*. If $a < \overline{G}_1 + \chi$ where $\chi > 1$, then the steady state under constant gain learning when $h = 0$ is local stable for $\eta \in (\widehat{\eta}_H, \widehat{\eta}_F)$ according to proposition 5. However the reverse is not true since local instability for any $\eta \in (0,1]$ under constant gain learning when $h = 0$ requires $\overline{G}_1 + \chi < a$ according to proposition 5 *iii)* which violates condition $a < 1$ of proposition 10 *ii)* for local stability under constant gain learning when $h = 1$. (see figures A6, A8, A9) ■

Proposition 13 *If* $1 < \overline{G}_1 < 1+a$, *and* $\overline{G}_1 + \chi < a$, *local instability for any* $\eta \in (0,1]$ *under constant gain learning when* $h = 0$ *implies local instability for any* $\eta \in (0,1]$ *when* $h = 0$.

Proof. If $1 < \overline{G}_1 < 1+a$, then local instability for any $\eta \in (0,1]$ under constant gain learning when $h = 0$ requires $\overline{G}_1 + \chi < a$ according to proposition 5 *iii)* which implies condition $a > 1$ of proposition 10 *ii)* for local instability under constant gain learning when $h = 1$. (see figure A10) ■

Proposition 14 *Under constant gain learning with* $h = 1$ *and* $\overline{G}_1 \geq 1+a$, *then the non-trivial steady state is locally unstable for any* $\eta \in (0,1]$.

Proof. If $\overline{G}_1 \geq 1+a$, then M_0 lies either anywhere between the points $(-2-a, 1+a)$ and $(-2-a, 1+a)$, or at the point $(-2-a, 1+a)$ on the line segment associated with the equation $1+\varepsilon+\kappa = 0$. The point M_1 lies anywhere between the horizontal axis and the straight line associated with the equation $1+\varepsilon+\kappa = 0$. For $\overline{G}_1 > 1+a$, the line segment M_0M_1 lies below the straight line corresponding to the equation $1+\varepsilon+\kappa = 0$. Therefore eigenvalues z_2, z_3 are real with identical sign: one in the interval $(0,1)$ and the other in the interval $(1,\infty)$. Since $z_1 = 1-\eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a saddle. For $\overline{G}_1 = 1+a$, the line segment M_0M_1 coincide with the straight line corresponding to the equation $1+\varepsilon+\kappa = 0$. Therefore eigenvalues z_2, z_3 are real with one equals to 1, and the other in the interval $(1,\infty)$. Since $z_1 = 1-\eta < 1$, then $(\overline{k}, \overline{w}, \overline{r})$ is a saddle. (see figure A11). ■

Remark 3: *If* $\overline{G}_1 \geq 1+a$, *then the non-trivial steady state is locally unstable under constant gain learning for any* $\eta \in (0,1]$ *and* $h \in \{0,1\}$.

Under the different restrictions imposed on the values of $\overline{G}_1$ and a when $h = 1$, figure A13 summarizes in the $\overline{G}_1 - a$ plane the local qualitative properties of the model around the non-trivial steady state.

5 Calibration and Simulations

In this section, I calibrate the model to the U.S. economy and illustrate some of the local stability results presented earlier by simulating the competitive equilibrium trajectories for different parameter specifications of the utility function and expectation functions.

I consider a Constant Relative Risk Aversion (CRRA) instantaneous utility function: $u(c_t) = (c_t^{1-\gamma} - 1)/(1-\gamma)$, where γ denotes a coefficient of relative risk aversion for $\gamma \in R_+ - \{1\}$ with $u(c_t) = \ln c_t$ for

$\gamma = 1$, and a Cobb-Douglas production technology: $F(K_t, E_t N_t) = K_t^{\alpha} (E_t N_t)^{1-\alpha}$ where α represents the capital's share with $0 < \alpha < 1$. In order to make the balanced growth path of the model consistent with the long-run characteristics for the U.S. Economy, I set the value for α to be equal to 0.4 which corresponds to the standard labor-income share of 60%. The estimates for the yearly rate of growth for the population and the labor augmenting technical progress over the sample period 1954-1992 are taken from Cooley and Prescott (1995) and are equal to 0.012 and 0.0156 respectively. Following Cooley and Prescott (1995), I use the law of motion for the capital stock and the first-order necessary condition for consumption both evaluated at the permanent regime to pin down the values for δ and β. Since the law of motion for the capital stock per capita in the steady state is given by: $(1+e)(1+n) = (1-\delta) + \overline{i}/\overline{k}$, and the yearly investment-capital ratio $\overline{i}/\overline{k}$ is equal to 0.076, then the resulting value for the yearly depreciation rate for physical capital δ is equal to 0.048 under above specifications for n and e. The first-order necessary condition for consumption per effective amount of labor evaluated in the steady state is: $(1+e)^{\gamma}(1+n) = \beta\left(1 - \delta + \alpha\overline{y}/\overline{k}\right)$. For a yearly capital-output ratio $\overline{k}/\overline{y}$ equals to 3.32, and a coefficient of relative risk aversion γ equals to 1, the optimality condition for consumption is satisfied in the steady state for a yearly discount factor β of 0.958. If we consider instead relative risk aversion coefficients of 0.02, 0.09, we get yearly discount factors of 0.944, 0.945 respectively. Table 1 in the appendix summarizes in quarterly terms all the parameter values considered for the simulations. For a CRRA specification of instantaneous preferences, the constrained utility maximizing problem under perfect foresight (8) cannot be solved analytically, and the competitive equilibrium trajectories of the model are derived by log-linearizing the households' first-order necessary condition around the permanent regime as in King, Plosser and Rebelo (1988). Under the bounded rationality hypothesis, the constrained planning problem (18) can be solved analytically for any $t \in Z_+ \cup \{0\}$, and the utility maximizing consumption plans per effective amount of labor are given by:

$$\widetilde{c}_t = \frac{\left(\varphi\rho_t^1\right)^{-1/\gamma}}{1 + \varphi^{-1/\gamma}\left(\rho_t^1\right)^{(\gamma-1)/\gamma}}\left[(1+e)\widetilde{w}_t^1 + \rho_t^1\left(\widetilde{w}_t + (1+r_t)\widetilde{x}_t\right)\right] \tag{26}$$

$$\widetilde{c}_t^1 = \frac{1}{1 + \varphi^{-1/\gamma}\left(\rho_t^1\right)^{(\gamma-1)/\gamma}}\left[\widetilde{w}_t^1 + \rho_t^1\left(\widetilde{w}_t + (1+r_t)\widetilde{x}_t\right)\right] \tag{27}$$

Using the parameter specifications of table 1, the model has a unique non-trivial steady state level of physical capital per effective amount of labor $\overline{k}$ equals to 75.4963, and the corresponding value for $\overline{G}_1$ is equal to 1 for $h \in \{0, 1\}$.

The case where $h = 0$

If the constant gain adaptive learning schemes (24), (25) include current input prices: $h = 0$, and for $\overline{G}_1 = 1$, then according to proposition 4, the model with boundedly rational households can only lose its stability through a Flip bifurcation. Using a total of 518,400 distinct ordered pairs $(1/\gamma, \eta) \in [1.1, 100] \otimes [0.1, 1]$, figure 14A illustrates in the parameter space $1/\gamma - \eta$ the local dynamic properties of the competitive equilibrium trajectories where $1/\gamma$ denotes the intertemporal elasticity of substitution between current and expected future consumption. The color codes used in computing this two-dimensional bifurcation diagram are defined in figure 15A. For each ordered pair $(1/\gamma, \eta)$ considered, figure 14A is obtained by simulating the model for a maximum of 10,000 iterates and deleting the first 1,000. Under the assumption that $\gamma = 0.02$, the corresponding value for a is equal to 4.56187, and the first Flip bifurcation occurs at $\widehat{\eta}_F = 0.609583$ according to proposition 4. For $\gamma = 0.02$, figure 16A illustrates the local qualitative behavior of the model under 475 distinct values of the gain parameter η ranging from 0.05 to 1. In this one-dimensional bifurcation diagram, the competitive equilibrium trajectory of physical capital per capita has been computed for a total of 1,000 iterates. The first 100 iterations have been discarded, and the 900 remaining have been plotted vertically just above the corresponding value of η represented along the horizontal axis. If $\eta < \widehat{\eta}_F$, the plotted horizontal segment reveals that the physical capital per effective amount of labor converges to the unique non-trivial stationary state $\overline{k}$. If $\eta \geq \widehat{\eta}_F$, the dynamic of physical capital per effective amount of labor alternates between cycles of every order or irregular behavior. At $\eta = 0.7$, the largest Lyapunov exponent we get after 40,000 iterates of the model is equal to 0.2549. A strictly greater than zero Lyapunov exponent diagnostics the presence of sensitive dependence on initial conditions in the dynamics. An attractor with this such property is said to be chaotic and it is illustrated in the $\widetilde{k}_t - \widetilde{k}_{t+1}$ space by figure A17 which has been computed for a total of $40,000$ iterates where the first 100 were discarded.

The case where $h = 1$

If the constant gain adaptive learning schemes (24), (25) do not include current input prices: $h = 1$, and $\overline{G}_1 = 1$, then according to proposition 7, the model with boundedly rational households can only lose its stability through a Hopf-Neimark bifurcation. Under the assumption that $\gamma = 0.09$, the corresponding value for a is equal to 1.01375, and the Hopf-Neimark bifurcation occurs at $\widehat{\eta}_H = 0$. For $\gamma = 0.09$, and $\eta = 0.7$, figure A18 illustrates in the $\widetilde{k}_t - \widetilde{k}_{t+1}$ spaces that the competitive equilibrium dynamics converges to an invariant closed curve for a total of $40,000$ iterates where the first 100 were discarded.

6 Conclusion

In a standard decentralized production economy characterized by a saddle point dynamics under perfect foresight, simple constant gain learning schemes may generate endogenous business cycles around the non-trivial steady-state. The perfect foresight hypothesis has been relaxed by introducing boundedly rational households forming one-period ahead input price forecasts from simple constant gain adaptive learning schemes, and predicting two-period ahead physical capital holdings and future consumption stream from expectation rules consistent with the economy's growth path. Under these assumptions, the representative households' planning problem can be written as a succession of two-period constrained optimization problems and analytical solutions can be found under a general class of preferences. For different specifications of preferences and technology captured by the relative values of $\overline{G}_1$ and a, the competitive equilibrium trajectories under constant gain adaptive learning schemes may exhibit distinct local stability properties depending on the size of the gain parameter η and the properties of the information sets. A given learning scheme can lead to opposite stability properties depending whether the underlying information sets used in making forecasts accommodate or not current input prices. If expectations take into account the latest announcement made by the 'Walrasian auctioneer': $h = 0$, then endogenous business cycles may occur through the Flip bifurcation theorem, or through the Hopf-Neimark bifurcation theorem, and though the Hopf-Neimark bifurcation theorem only when information about current prices is not used in making forecasts: $h = 1$. In the former case, the competitive equilibrium trajectories converges to the perfect foresight steady state for a broader range of constant gain parameters. Calibrated to the U.S. economy, and for low values of the coefficient of relative risk aversion, the model with boundedly rational households may exhibit limit cycles or chaotic competitive equilibrium trajectories that do not exist under the perfect foresight hypothesis.

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Appendix

α	e	n	δ	$\beta\ (\gamma = 1)$	$\beta\ (\gamma = 0.02)$	$\beta\ (\gamma = 0.09)$
0.4	0.00387	0.00298	0.01227	0.98946	0.98572	0.98598

Table 1: Parameter Values

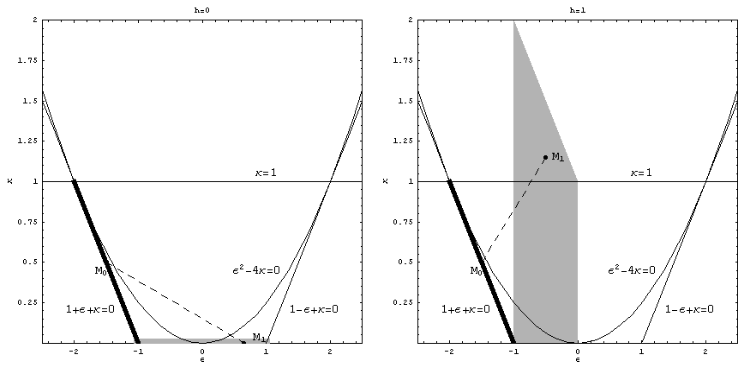

Figure A1: $0 < \overline{G}_1 \leq 1$ and $a < 1 + \overline{G}_1$

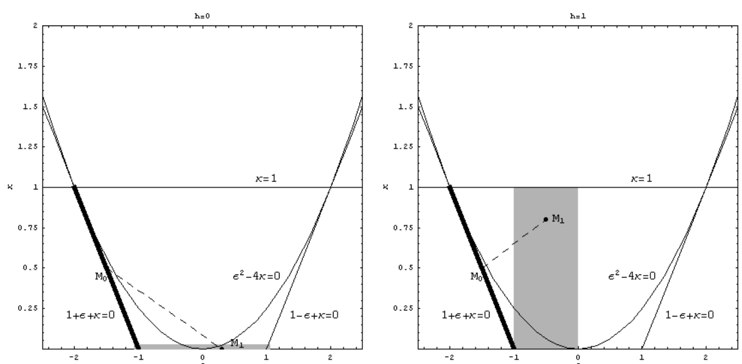

Figure A2: $0 < \overline{G}_1 \leq 1$ and $a < 1$

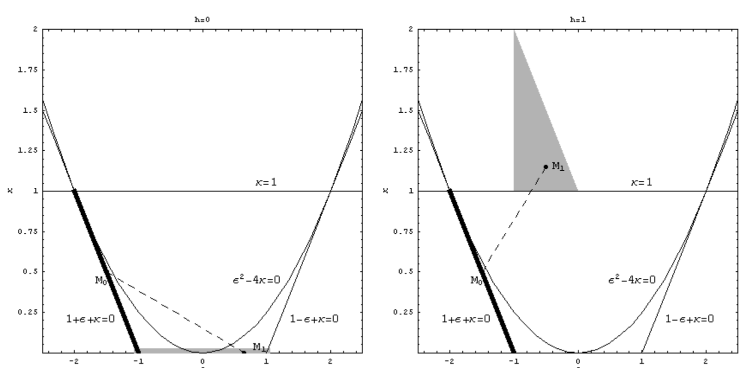

Figure A3: $0 < \overline{G}_1 \leq 1$ and $1 < a < 1 + \overline{G}_1$

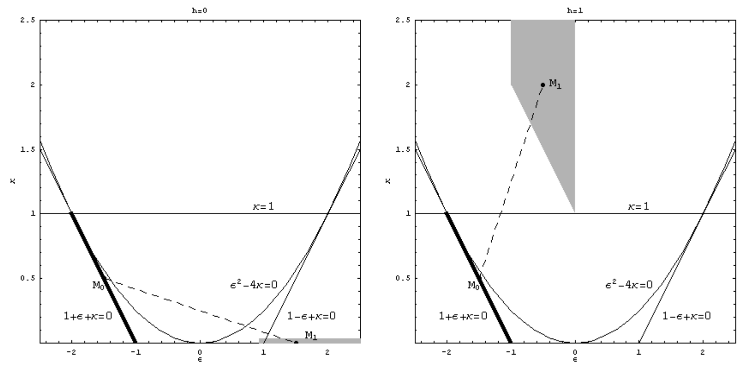

Figure A4: $0 < \overline{G}_1 \leq 1$ and $a > 1 + \overline{G}_1$

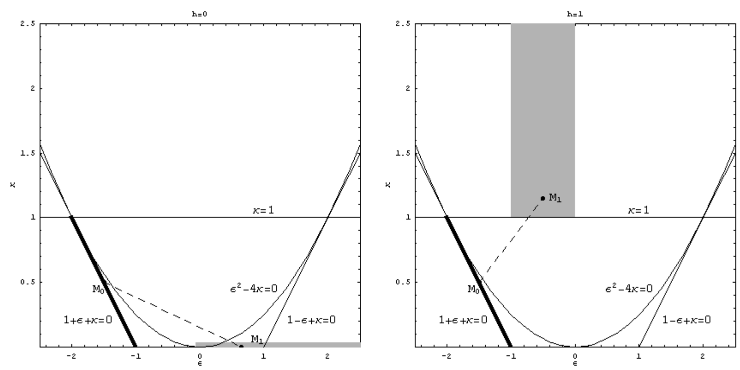

Figure A5: $0 < \overline{G}_1 \leq 1$ and $a > 1$

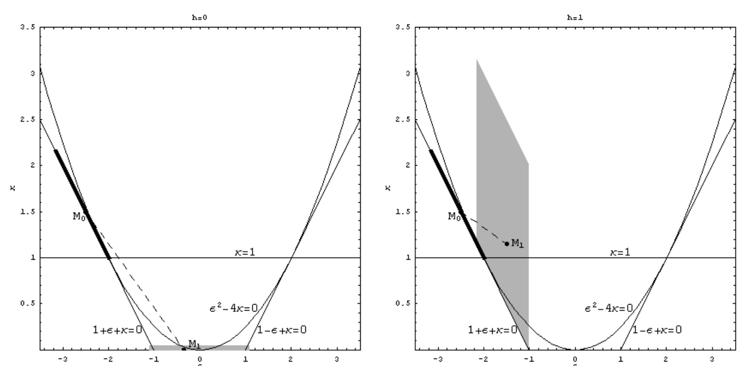

Figure A6: $1 < \overline{G}_1 < 1 + a$ and $a < 1 + \overline{G}_1$

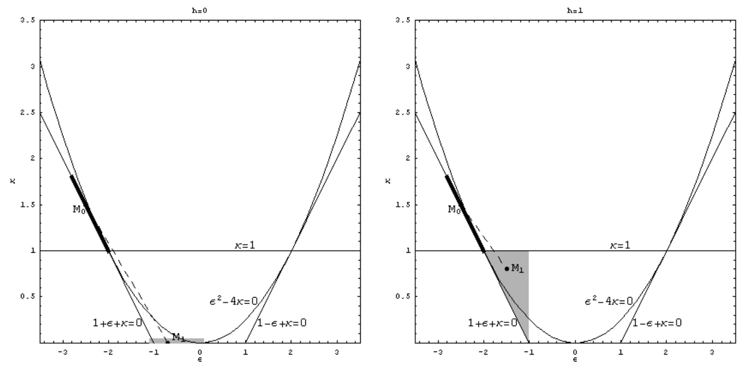

Figure A7: $1 < \overline{G}_1 < 1 + a$ and $a < 1$

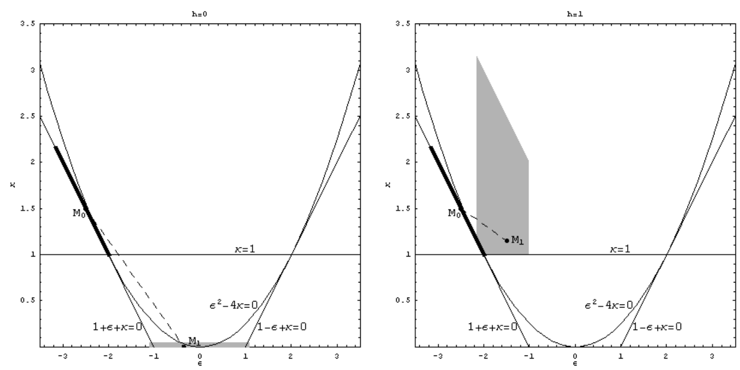

Figure A8: $1 < \overline{G}_1 < 1 + a$ and $1 < a < 1 + \overline{G}_1$

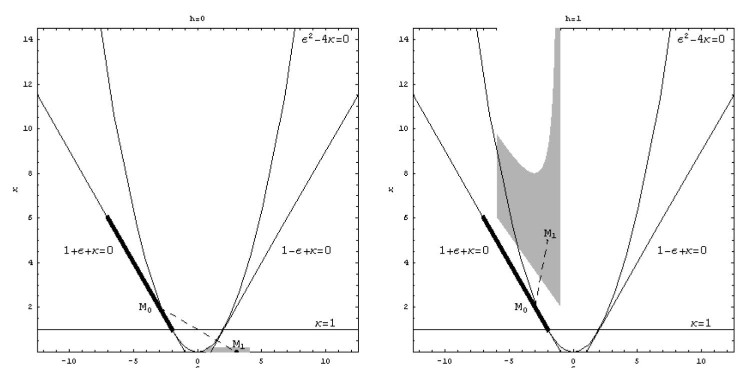

Figure A9: $1 < \overline{G}_1 < 1 + a$ and $1 + \overline{G}_1 < a < \overline{G}_1 + \chi$

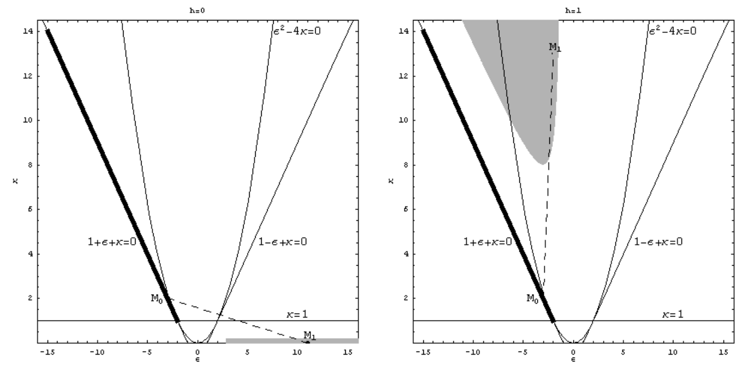

Figure A10: $1 < \overline{G}_1 < 1 + a$ and $\overline{G}_1 + \chi < a$

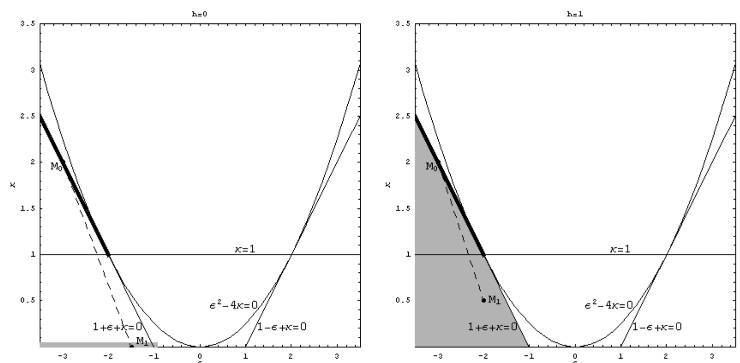

Figure A11: $1 + a \leq \overline{G}_1$

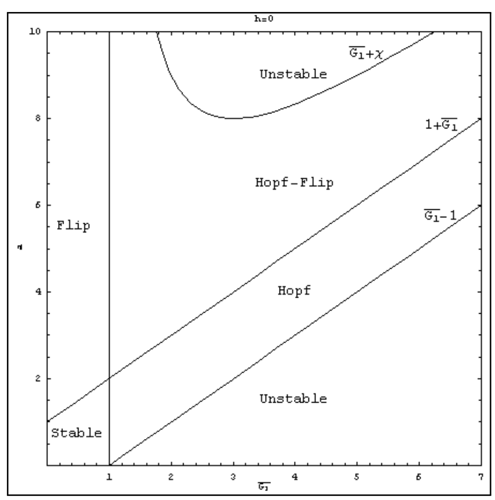

Figure A12: Bifurcation Boundaries I

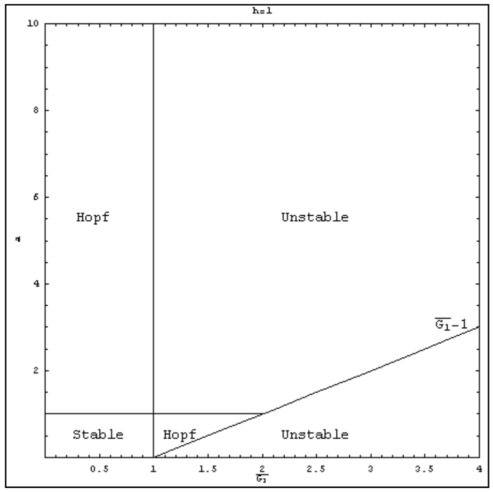

Figure A13: Bifurcation Boundaries II

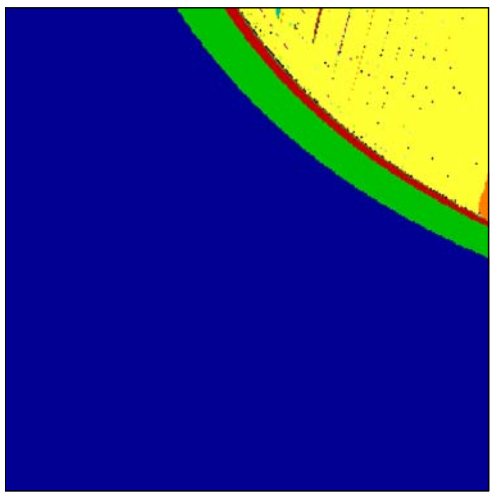
Figure A14: The Basin of Attraction in the Parameter Space $(1/\gamma, \mu)$

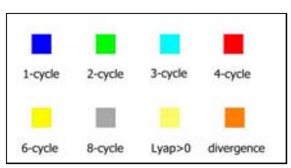

Figure A15: Color Codes

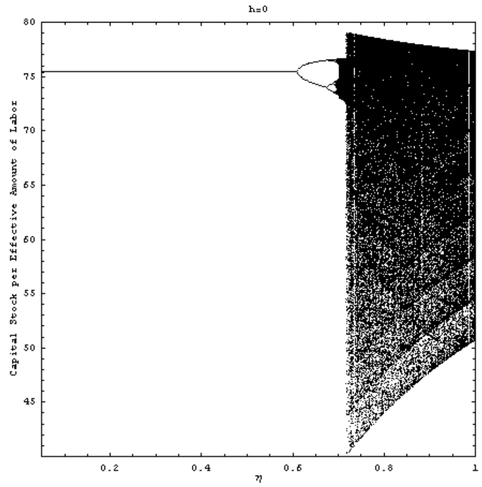

Figure A16: One-Dimensional Bifurcation Diagramm

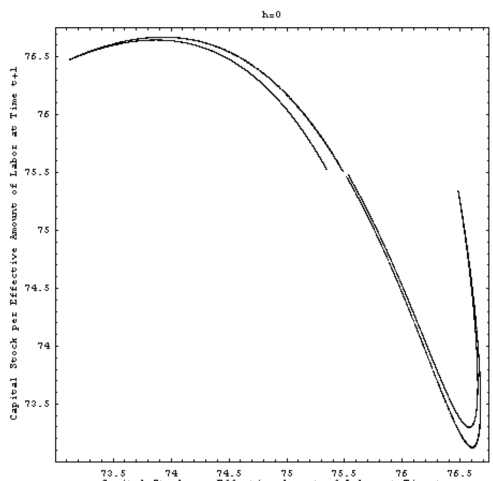

Figure A17: Chaotic Attractor

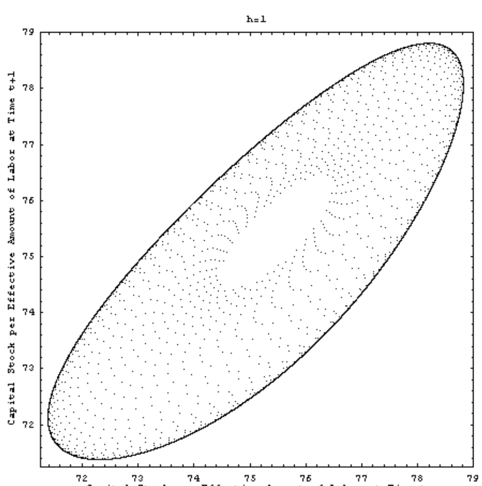

Figure A18: Invariant Closed Curve