HETEROGENEITY, PROFITABILITY AND AUTOCORRELATIONS

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ABSTRACT. This paper contributes to the development of recent literature on the explanation power and calibration issue of heterogeneous asset pricing models by presenting a simple stochastic market fraction asset pricing model of two types of traders (fundamentalists and trend followers) under a market maker scenario. It seeks to explain aspects of financial market behaviour (such as market dominance, under and over-reaction, profitability and survivability) and to characterize various statistical properties (including autocorrelation structure) of the stochastic model by using the the dynamics of the underlying deterministic system, traders' behaviour and market fractions. Statistical analysis based on Monte Carlo simulations shows that the long-run behaviour and convergence of the market prices, long (short)-run profitability of the fundamental (trend following) trading strategy, survivability of chartists, and various under and over-reaction autocorrelation patterns of returns can be characterized by the stability and bifurcations of the underlying deterministic system. Our analysis underpins mechanism on various market behaviour (such as under/over-reactions), market dominance and stylized facts in high frequency financial markets.

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1. INTRODUCTION

Traditional economic and finance theory is based on the assumptions of investor homogeneity and the efficient market hypothesis. However, there is a growing dissatisfaction with models of asset price dynamics, based on the representative agent paradigm, as expressed for example by Kirman (1992), and the extreme informational assumptions of rational expectations. As a result, the literature has seen a rapidly increasing number of heterogeneous agents models. These models characterise the dynamics of financial asset prices; resulting from the interaction of heterogeneous agents having different attitudes to risk and having different expectations about the future evolution of prices.[1] For example, Brock and Hommes (1997, 1998) proposed a simple *Adaptive Belief System* to model economic and financial markets. Agents' decisions are base upon predictions of future values of endogenous variables whose actual values are determined by the equilibrium equations. A key aspect of these models is that they exhibit expectations feedback. Agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance as measured by the realized profits. The resulting dynamical system is nonlinear and, as Brock and Hommes (1998) show, capable of generating the entire *zoo* of complex behaviour from local stability to high order cycles and even chaos as various key parameters of the model change. It has been shown (e.g. Hommes (2002)) that such simple nonlinear adaptive models are capable of explaining important empirical observations, including fat tails, clustered volatility and long memory, of real financial series. The analysis of the stylized simple evolutionary adaptive system, and its numerical analysis provides insight into the connection between individual and market behaviour. Specifically, it provides insight into whether asset prices in real markets are driven only by news or, are at least in part, driven by market psychology.

The heterogeneous agents literature attempts to address two interested issues among many others. It attempts to explain various types of market behaviour, and to replicate the well documented empirical findings of actual financial markets, the stylized facts. Recent literature has demonstrated the ability to explain various types of market behaviour. However, in relation to stylized facts, there is a gap between the heterogeneous model and observed empirical findings.

[1]See, e.g., Arthur *et al.* (1997), Brock and Hommes (1997, 2002), Brock and LeBaron (1996), Bullard and Duffy (1999), Chen and Yeh (1997, 2002), Chiarella (1992), Chiarella *et al.* (2002), Chiarella and He (2001, 2002, 2003*c*), Dacorogna *et al.* (1995), Day and Huang (1990), De Long et al (1990), Farmer and Joshi (2002), Frankel and Froot (1987), Gaunersdorfer (2000), Hommes (2001, 2002), Iori (2002), LeBaron (2000, 2001, 2002), LeBaron *et al.* (1999), Lux (1995, 1997, 1998) and Lux and Marchesi (1999))

It is well known that most of the stylized facts can be observed only for high frequency data (e.g. daily) and not for low frequency data (e.g. yearly). However, two unrealistic assumptions underpin this literature.[2] The first is a risk-free rate of approximately 10 per-cent per trading period.[3] Given that this rate is crucial for model calibration in generating stylized facts[4], it is obviously unrealistic. Second, the unrealistic nature of the assumed trading period is problematic for the quantitative calibration to actual time series. As pointed out by LeBaron (2002), *'This (unrealistic trading period) is fine for early qualitative comparisons with stylized facts, but it is a problem for quantitative calibration to actual time series'*.

Another more important issue for various heterogeneous asset pricing models is the interplay of noise and deterministic dynamics. Given that deterministic models are simplified versions of realistic stochastic models and stability and bifurcation are the most powerful tools (among others) to investigate the dynamics of nonlinear system, it is interesting to know how deterministic properties influence the statistical properties, such as the existence and convergence of stationary process, and the autocorrelation (AC) structure of the corresponding stochastic system. In particular, we can ask if there is a connection between various AC patterns of the stochastic system and different types of bifurcations of the underlying deterministic skeleton. This has the potential to provide insights into the mechanisms of generating various AC patterns and stylized facts in financial markets. At present, the mathematical theory has not yet be able to achieve those tasks in general. Consequently, statistical analysis and Monte Carlo simulations is the approach adopted in this paper.

This paper builds upon the existent literature by incorporating a realistic trading period[5], which eliminates the unrealistic risk-free rate assumption, whilst also introducing market fractions of heterogeneous traders into a simple asset-pricing and wealth dynamics model. In this study this model is referred to as the Market Fraction (**MF**) Model. The model assumes three types of participants in the asset market. This including two groups of boundedly rational

[2]See, e.g., Arthur *et al.* (1997), Brock and Hommes (1997), Chen and Yeh (2002), Chiarella et al (2002), Chiarella and He (2002, 2003*c*), Iori (2002), LeBaron (2002), LeBaron *et al.* (1999), Levy *et al.* (1994)).

[3]Apart from $r_f = 1\%$ in Gaunersdorfer (2000) and LeBaron (2001) and $r_f = 0.04\%$ in Hommes (2002).

[4]In this literature, as risk-free rate of trading period decreases, demand on the risky asset increases. Consequently, the price of the risky asset become rather larger numbers resulting sometimes in break-down in theoretic analysis and overflows in numerical simulations. In addition, some of interesting dynamics disappear as the risk-free rate of trading period decreases to realistic level (e.g. (5/250)% per day given a risk-free rate of 5% p.a. and 250 trading days per year).

[5]In fact, the trading period of the model can be scaled to any level of trading frequency ranging from annually, monthly, weekly, to daily.

traders—fundamentalists (also called informed traders) and trend followers (also called less informed traders or chartists), and a market-maker. The MF model shows that long-run behaviour of asset prices, wealth accumulations of heterogeneous trading strategies and the autocorrelation structure of the stochastic system can be characterised by the dynamics of the underlying deterministic system, traders' behaviour and market fractions. In addition, statistical analysis based on Monte Carlo simulations show that the long-run behaviour and convergence of the market prices, long (short)-run profitability of the fundamental (trend following) trading strategy, survivability of chartists, and various under and over-reaction AC patterns of returns can be characterized by the stability and bifurcations of the underlying deterministic system. Our analysis gives us some insights into mechanism of various market behaviour (such as under/over-reactions), market dominance and stylized facts in high frequency financial markets.

This paper is organized as follows. Section 2 outlines a market fraction model of heterogeneous agents with the market clearing price set by a market maker, introduces the expectations function and learning mechanisms of the fundamentalists and trend followers, and derives a complete market fraction model on asset price and wealth dynamics. Price dynamics of the underlying deterministic model is examined in Section 3. Statistical analysis, based on Monte Carlo simulations, of the stochastic model is given in Section 4. By using the concept of random fixed point, we examine the long-run behavior and convergence of the market price to the fundamental price. By examining wealth accumulation, we analyze the profitability and survivability. By choosing different set of parameters near different types of bifurcation boundaries of the underlying deterministic system, we explore various under and over-reaction AC patterns. Section 5 concludes and all proofs and additional statistical results are included in the Appendixes.

2. Heterogeneous Beliefs, Market Fractions and Market-Maker

Both empirical (e.g. Taylor and Allen (1992)) and theoretical (e.g. Brock and Hommes (1997)) studies show that market fractions among different types of traders play an important role in financial markets. Empirical evidence from Taylor and Allen (1992) suggests that at least 90% of traders place some weight on technical analysis at one or more time horizons. In particular, traders rely more on technical analysis, as opposed to the fundamental analysis, at shorter horizons. As the length of time horizons increase, more traders rely on the fundamental rather than technical analysis. In addition, there is certain proportion of traders who do not change their strategies over all time horizons. Theoretically, study from Brock and Hommes (1997) shows that, when different groups of traders, such as fundamentalists and chartists, having different expectations about future prices and dividends compete between trading strategies and choose their strategy according to an evolutionary *fitness measure*, the corresponding deterministic system exhibits rational routes to randomness. The adaptive switching mechanism proposed by Brock and Hommes (1997) is an important elements of the adaptive belief model. It is based on both *fitness function* and discrete choice probability. In this paper, we take a simplified version of Brock and Hommes' framework. The MF model assumes that the market fractions among heterogeneous agents are fixed and are treated as fixed parameters. Apart from mathematical tractability, this simplification is motivated as follows. First, because of the amplifying effect of the exponential function used in the discrete choice probability, the market fractions become very sensitive to price changes and the fitness functions. Therefore, it is not very clear to see how different market fractions themselves do actually influence the market price. Secondly, when agents switch intensively, it becomes difficult to characterize market dominance, profitability and survivability when dealing with heterogeneous trading strategies.[6] Thirdly, different types of agents play different roles (such as the autocorrelation structure we discuss late) and it is important to understand their responsibility to certain dynamics. Such analysis becomes clear when we isolate the market fractions from switching. In doing so, we can examine explicitly the influence of the market fractions on the price behaviour.

The set up follows the standard discounted value asset pricing model with heterogeneous agents, which is closely related to the framework of Day and Huang (1990), Brock and Hommes (1997, 1998) and Chiarella and He (2002). However, the market clearing price is arrived at via a market maker scenario in line with Day and Huang (1990) and Chiarella and He (2003*c*) rather than the Walrasian scenario used in Brock and Hommes (1998). We focus on a simple case in which there are three classes of participants in the asset market: two groups of traders, fundamentalists and trend followers, and a market maker, as described in the following discussion.

2.1. **Market Fraction and Market Clearing Price under a Market Maker.** Consider an asset pricing model with one risky asset and one risk free asset. It is assumed that the risk free asset is perfectly elastically supplied at gross return of $R = 1 + r/K$, where r stands for a

[6]This analysis in turn leads to a justification on agents switching, which is discussed in Section 3.

constant risk-free rate per annual and K stands for the frequency of trading period per year. Typically, $K = 1, 12, 52$ and 250 for trading period of year, month, week and day, respectively. To calibrate the stylized facts observed from daily price movement in financial market, we select $K = 250$ in our following discussion.

Let P_t be the price (ex dividend) per share of the risky asset at time t and $\{D_t\}$ be the stochastic dividend process of the risky asset. Then the wealth of a typical investor-h at $t+1$ is given by

$$W_{h,t+1} = RW_{h,t} + [P_{t+1} + D_{t+1} - RP_t]z_{h,t}, \tag{2.1}$$

where $W_{h,t}$ and $z_{h,t}$ are the wealth and the number of shares of the risky asset purchased of investor-h at t, respectively. Let $E_{h,t}$ and $V_{h,t}$ be the *beliefs* of type h traders about the conditional expectation and variance of quantities at $t+1$ based on their information set at time t. Denote by R_{t+1} the excess capital gain on the risky asset at $t+1$, that is

$$R_{t+1} = P_{t+1} + D_{t+1} - R\,P_t. \tag{2.2}$$

Then it follows from (2.1) and (2.2) that

$$E_{h,t}(W_{t+1}) = RW_t + E_{h,t}(R_{t+1})z_{h,t}, \qquad V_{h,t}(W_{t+1}) = z_{h,t}^2 V_{h,t}(R_{t+1}), \tag{2.3}$$

where $z_{h,t}$ is the demand by agent h for the risky asset. Assume that traders have a constant absolute risk aversion (CARA) utility function with the risk aversion coefficient a_h for type h traders (that is $U_h(W) = -\exp(-a_h W)$) and their optimal demand on the risky asset $z_{h,t}$ are determined by maximizing their expected utility of the wealth. Then

$$z_{h,t} = \frac{E_{h,t}(R_{t+1})}{a_h V_{h,t}(R_{t+1})}. \tag{2.4}$$

Given the heterogeneity and the nature of asymmetric information among traders, we consider two most popular trading strategies corresponding to two types of boundedly rational traders—fundamentalists and trend followers, and their beliefs will be defined in the following discussion. Assume the market fraction of the fundamentalists and trend followers is n_1 and n_2 with risk aversion coefficient a_1 and a_2, respectively. Let $m = n_1 - n_2 \in [-1, 1]$. Obviously, $m = 1, -1$ corresponds to the case when all the traders are fundamentalists and trend followers, respectively. Assume zero supply of outside shares. Then, using (2.4), the aggregate excess demand per investor $z_{e,t}$ is given by

$$z_{e,t} \equiv n_1 z_{1,t} + n_2 z_{2,t} = \frac{1+m}{2} \frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + \frac{1-m}{2} \frac{E_{2,t}[R_{t+1}]}{a_2 V_{2,t}[R_{t+1}]}. \tag{2.5}$$

To complete the model, we assume that the market is cleared by a market maker. The role of the market maker is to take a long (when $z_{e,t} < 0$) or short (when $z_{e,t} > 0$) position so as to clear the market. At the end of period t, after the market maker has carried out all transactions, he or she adjusts the price for the next period in the direction of the observed excess demand. Let μ be the speed of price adjustment of the market maker (this can also be interpreted as the market aggregate risk tolerance). To capture unexpected market news or noise created by *noise traders*, we introduce a noisy demand term $\tilde{\delta}_t$ which is an IID normally distributed random variable[7] with $\tilde{\delta}_t \sim \mathcal{N}(0, \sigma_\delta^2)$. Based on those assumptions, the market price is determined by

$$P_{t+1} = P_t + \mu z_{e,t} + \tilde{\delta}_t.$$

Using (2.5), this becomes

$$P_{t+1} = P_t + \frac{\mu}{2}\left[(1+m)\frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + (1-m)\frac{E_{2,t}[R_{t+1}]}{a_1 V_{2,t}[R_{t+1}]}\right] + \tilde{\delta}_t. \tag{2.6}$$

It should be pointed out that the market maker behavior in this model is highly stylized. For instance, the inventory of the market maker built up as a result of the accumulation of various long and short positions is not considered. This could affect his or her behavior and the market maker price setting role in (2.6) could be a function of the inventory. Allowing μ to be a function of inventory would be one way to model such behavior. Such considerations are left to future research. Future research should also seek to explore the microfoundations of the coefficient μ. In the present paper it is best thought of as a market friction, and an aim of our analysis is to understand how this friction affects the market dynamics.

2.2. **Fundamentalists.** Denote by $F_t = \{P_t, P_{t-1}, \cdots; D_t, D_{t-1}, \cdots\}$ the common information set formed at time t. We assume that, apart from the common information set, the fundamentalists have *superior* information on the fundamental value, P_t^*, of the risky asset and they also realize the existence of non-fundamental traders, such as trend followers introduced in

[7]In this paper, we assume a constant volatility noisy demand and the volatility is related to an average fundamental price level. This noisy demand may also depend on the market price. Theoretically, how the price dynamics are influenced by adding different noisy demand is still a difficult problem. Here, we focus on the constant volatility noisy demand case and use Monte Carlo simulations and statistical analysis to gain some insights into this problem.

the following discussion. They believe that the stock price may be driven away from the fundamental value in short-run, but it will eventually converge to the fundamental value in long-run. The speed of the convergence measures their confidence level on the fundamental value. More precisely, we assume that the fundamental value follows a stationary random walk process

$$P^*_{t+1} = P^*_t[1 + \sigma_\epsilon \tilde{\epsilon}_t], \qquad \tilde{\epsilon}_t \sim \mathcal{N}(0, 1), \qquad \sigma_\epsilon \geq 0, \qquad P^*_0 = \bar{P} > 0, \tag{2.7}$$

where $\tilde{\epsilon}_t$ is independent of the noisy demand process $\tilde{\delta}_t$. This specification ensures that neither fat tails nor volatility clustering are brought about by the exogenous news arrival process. Hence, emergence of any autocorrelation pattern of the return of the risky asset in our late discussion would be driven by the trading process itself, rather than news. We assume the conditional mean and variance of the fundamental traders follow

$$E_{1,t}(P_{t+1}) = P_t + \alpha(P^*_{t+1} - P_t), \qquad V_{1,t}(P_{t+1}) = \sigma_1^2, \tag{2.8}$$

where σ_1^2 stands for a constant variance on the price. Here parameter $\alpha \in [0, 1]$ is the speed of price adjustment of the fundamentalist toward the fundamental value. It measures their confidence level on the fundamental value. In particular, for $\alpha = 1$, the fundamental traders are fully confident about the fundamental value and adjust their expected price at next period instantaneously to the fundamental value. For $\alpha = 0$, the fundamentalists become naive traders. In general, the fundamental traders believe that markets are efficient and prices converge to the fundamental value. An increase (decrease) in α indicates that the fundamental traders have high (low) confidence on their estimated fundamental value, leading to a quick (slow) adjustment of their expected price towards the fundamental price.

2.3. **Trend followers.** Unlike the fundamentalists, trend followers are technical traders who believe the future price change can be predicted from various patterns or trends generated from history price. The trend followers are assumed to extrapolate the latest observed price change over a long-run sample mean price and to adjust their variance estimate accordingly. More precisely, their conditional mean and variance are assumed to follow

$$E_{2,t}(P_{t+1}) = P_t + \gamma(P_t - u_t), \qquad V_{2,t}(P_{t+1}) = \sigma_1^2 + b_2 v_t, \tag{2.9}$$

where $\gamma, b_2 \geq 0$ are constants, and u_t and v_t are sample mean and variance, respectively, which may follow some learning processes. Parameter γ measures the extrapolation rate and high (low) values of γ correspond to strong (weak) extrapolation from the trend followers. The coefficient b_2 measures the influence of the sample variance on the conditional variance estimated by the trend followers who believe more volatile price movement. Various learning schemes[8] can be used to estimate the sample mean u_t and variance v_t. In this paper we assume that

$$u_t = \delta u_{t-1} + (1-\delta)P_t, \tag{2.10}$$

$$v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2, \tag{2.11}$$

where $\delta \in [0.1]$ is a constant. This process on the sample mean and variance is a limiting process of *geometric decay process* when the memory lag length tends to infinity.[9] Basically, a geometric decay probability process $(1-\delta)\{1, \delta, \delta^2, \cdots\}$ is associated to the history prices $\{P_t, P_{t-1}, P_{t-2}, \cdots\}$. Parameter δ measures the geometric decay rate. For $\delta = 0$, the sample mean $u_t = P_t$, which is the latest observed price, while $\delta = 0.1, 0.5, 0.95$ and 0.999 gives a half life of 0.43 day, 1 day, 2.5 weeks and 2.7 years, respectively. The selection of this process is two folds. First, traders tend to put high weight to the most recent prices and less weight to the more remote prices when they estimate the sample mean and variance. Secondly, we believe that this geometric decay process may contribute to certain autocorrelation patterns, even the long memory feature observed in real financial markets. In addition, it has mathematical advantage of tractability.

2.4. **The Complete Stochastic Model.** To simplify our analysis, we assume that the dividend process D_t follows $D_t \sim \mathcal{N}(\bar{D}, \sigma_D^2)$, the expected long-run fundamental value $\bar{P} = (R-1)\bar{D}$, and the unconditional variances of price and dividend over the trading period are related by

[8]For related studies on heterogeneous learning and asset pricing models with heterogeneous agents who's conditional mean and variance follow various learning processes, we refer to Chiarella and He (2003*a*, 2003*b*).
[9]See Chiarella *et. al.* (2005) for the proof.

$\sigma_D^2 = q\sigma_1^2$.[10] Based on assumptions (2.8)-(2.9),

$$E_{1,t}(R_{t+1}) = P_t + \alpha(P_{t+1}^* - P_t) + \bar{D} - R\,P_t = \alpha(P_{t+1}^* - P_t) - (R-1)(P_t - \bar{P}),$$
$$V_{1,t}(R_{t+1}) = (1+q)\sigma_1^2$$

and hence the optimal demand for the fundamentalist is given by

$$z_{1,t} = \frac{1}{a_1(1+q)\sigma_1^2}[\alpha(P_{t+1}^* - P_t) - (R-1)(P_t - \bar{P})]. \tag{2.12}$$

In particular, when $P_t^* = \bar{P}$,

$$z_{1,t} = \frac{(\alpha + R - 1)(\bar{P} - P_t)}{a_1(1+q)\sigma_1^2}. \tag{2.13}$$

Similarly, from (2.9), (using $\bar{D} = (R-1)\bar{P}$)

$$E_{2,t}(R_{t+1}) = P_t + \gamma(P_t - u_t) + \bar{D} - R\,P_t = \gamma(P_t - u_t) - (R-1)(P_t - \bar{P}),$$
$$V_{2,t}(R_{t+1}) = \sigma_1^2(1 + q + b\,v_t),$$

where $b = b_2/\sigma_1^2$. Hence the optimal demand of the trend followers is given by

$$z_{2,t} = \frac{\gamma(P_t - u_t) - (R-1)(P_t - \bar{P})}{a_2\sigma_1^2(1 + q + b\,v_t)}. \tag{2.14}$$

Subsisting (2.12) and (2.14) into (2.6), the price dynamics under a market maker is determined by the following 4-dimensionally stochastic difference system (**SDS** hereafter)

$$\begin{cases} P_{t+1} = P_t + \dfrac{\mu}{2}\Bigg[\dfrac{1+m}{a_1(1+q)\sigma_1^2}[\alpha(P_{t+1}^* - P_t) - (R-1)(P_t - \bar{P})] \\ \qquad\qquad + (1-m)\dfrac{\gamma(P_t - u_t) - (R-1)(P_t - \bar{P})}{a_2\sigma_1^2(1 + q + b\,v_t)}\Bigg] + \tilde{\delta}_t, \\ u_t = \delta u_{t-1} + (1-\delta)P_t, \\ v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2, \\ P_{t+1}^* = P_t^*[1 + \sigma_\epsilon \tilde{\epsilon}_t]. \end{cases} \tag{2.15}$$

[10] In this paper, we choose $\sigma_1^2 = (\bar{P}\sigma)^2/K$ and $q = r^2$. This can be justified as follows. Let $\sigma\bar{P}$ be the annual volatility of P_t and $\bar{D}_t = rP_t$ be the annual dividend. Then the annual variance of the dividend $\bar{\sigma}_D^2 = r^2(\bar{P}\sigma)^2$. Therefore $\sigma_D^2 = \bar{\sigma}_D^2/K = r^2(\bar{P}\sigma)^2/K = r^2\sigma_1^2$. For all numerical simulations in this paper, we choose $\bar{P} = \$100, r = 5\%$ p.a. $\sigma = 20\%$ p.a., $K = 250$. Correspondingly, $R = 1 + 0.05/250 = 1.0002, \sigma_1^2 = (100 \times 0.2)^2/250 = 8/5$ and $\sigma_D^2 = 1/250$.

2.5. **Wealth Dynamics and Shares.** Traders' wealth in general follow some growing processes. To be able to measure the wealth dynamics among different trading strategies, to examine the market dominance and price behaviour, we introduce two wealth measures. The first measures the absolute level of the wealth share (or proportion) of the representative agent from each type, called the *absolute wealth share* for short, which is defined by

$$w_{1,t} = \frac{W_{1,t}}{W_{1,t} + W_{2,t}}, \qquad w_{2,t} = \frac{W_{2,t}}{W_{2,t} + W_{2,t}}, \tag{2.16}$$

where $W_{1,t}$ and $W_{2,t}$ are the wealth at time t of the representative trader of the fundamentalists and trend followers, respectively. This measure can be used to measure the evolutionary performance or profitability of the two trading strategies and a high $w_{1,t}$ ($w_{2,t}$) indicates profitability of the fundamentalists (trend followers). The second measures the overall market wealth share, called the *market wealth share* for short, of different trading strategy and it is defined as market fraction weighted average of the absolute wealth proportions,

$$\bar{w}_{1,t} = \frac{(1+m)W_{1,t}}{(1+m)W_{1,t} + (1-m)W_{2,t}}, \qquad \bar{w}_{2,t} = \frac{(1-m)W_{2,t}}{(1+m)W_{1,t} + (1-m)W_{2,t}}. \tag{2.17}$$

A high market wealth share $\bar{w}_{1,t}$ ($\bar{w}_{2,t}$) indicates market dominance of the fundamentalists (trend followers) with respect to the overall market wealth. Let $V_{1,t} = 1/W_{1,t}$ and $V_{2,t} = 1/W_{2,t}$. Then it follows from (2.1) that

$$V_{1,t+1} = \frac{V_{1,t}}{R + R_{t+1} z_{1,t} V_{1,t}}, \qquad V_{2,t+1} = \frac{V_{2,t}}{R + R_{t+1} z_{1,t} V_{1,t}}.$$

Note that

$$\begin{aligned} \frac{V_{1,t}}{V_{1,t} + V_{2,t}} &= \frac{1/W_{1,t}}{1/W_{1,t} + 1/W_{2,t}} = \frac{W_{2,t}}{W_{1,t} + W_{2,t}}, \\ \frac{V_{2,t}}{V_{1,t} + V_{2,t}} &= \frac{1/W_{2,t}}{1/W_{1,t} + 1/W_{2,t}} = \frac{W_{1,t}}{W_{1,t} + W_{2,t}} \end{aligned}$$

and therefore the absolute wealth shares are determined by

$$w_{1,t} = \frac{V_{2,t}}{V_{1,t} + V_{2,t}}, \qquad w_{2,t} = \frac{V_{1,t}}{V_{1,t} + V_{2,t}} \tag{2.18}$$

and the market wealth shares are governed by

$$\bar{w}_{1,t} = \frac{(1+m)V_{2,t}}{(1+m)V_{2,t} + (1-m)V_{1,t}}, \qquad \bar{w}_{2,t} = \frac{(1-m)V_{1,t}}{(1+m)V_{2,t} + (1-m)V_{1,t}}. \tag{2.19}$$

For these wealth measures, it is difficult to obtain explicitly closed form expressions in terms of (stationary) state variables. In this paper, we use the auxiliary functions $(V_{1,t}, V_{2,t})$ and numerical simulations to study the wealth dynamics of the fundamentalists and trend followers and the market impact of the two differential trading strategies.

It has been widely accepted that stability and bifurcation theory is a powerful tool in the study of asset-pricing dynamics (see, for example, Day and Huang (1990), Brock and Hommes (1997, 1998) and Chiarella and He (2002, 2003*c*)). However, how the stability and various types of bifurcation of the underlying deterministic system affect the nature of the stochastic system, including stationarity, distribution and statistic properties of returns, is not very clear at the current stage. Although the techniques discussed in Arnold (1998) may be useful in this regard, mathematical analysis of nonlinear stochastic dynamical system is still difficult in general. In this paper, we consider first the corresponding deterministic skeleton of the stochastic model by assuming that the fundamental price is given by its long-run value $P_t^* = \bar{P}$ and there is no demand shocks, i.e. $\sigma_\delta = \sigma_\epsilon = 0$. We then conduct stochastic analysis of the stochastic model through Monte Carlo simulation.

3. Dynamics of the Deterministic Model

When the long run fundamental price is a constant and there is no noisy demand, the 4-dimensionally stochastic system (2.15) reduces to the following 3-dimensionally deterministic difference system (**DDS** hereafter)

$$\begin{cases} P_{t+1} = P_t + \mu \dfrac{1+m}{2}\left[\dfrac{(1-\alpha-R)(P_t-\bar{P})}{a_1(1+q)\sigma_1^2}\right] + \dfrac{1-m}{2}\left[\dfrac{\gamma(P_t-u_t)-(R-1)(P_t-\bar{P})}{a_2\sigma_1^2(1+q+b\,v_t)}\right], \\ u_t = \delta u_{t-1} + (1-\delta)P_t, \\ v_t = \delta v_{t-1} + \delta(1-\delta)(P_t-u_{t-1})^2. \end{cases} \tag{3.1}$$

The following result on the existence and uniqueness of steady state of the deterministic system is obtained.

Proposition 3.1. *For DDS (3.1),* $(P_t, u_t, v_t) = (\bar{P}, \bar{P}, 0)$ *is the unique steady state.*

Proof. See Appendix A.1. □

We call this unique steady state as the fundamental steady state. In the following discussion, we focus on the stability and bifurcation of the fundamental steady state of the deterministic model. We first examine two special cases $m = 1$ and $m = -1$ before we deal with the general case $m \in (-1, 1)$.

3.1. **The case $m = 1$.** In this case, the following result on the global stability and bifurcation is obtained.

Proposition 3.2. *For DDS (3.1), if all the traders are fundamentalists, i.e. $m = 1$, then the fundamental price $\bar{P}$ is globally asymptotically stable if and only if*

$$0 < \mu < \mu_{0,1} \equiv \frac{2a_1(1+q)\sigma_1^2}{(R+\alpha-1)}. \tag{3.2}$$

In addition, $\mu = \mu_{0,1}$ leads to a flip bifurcation with $\lambda = -1$, where

$$\lambda = 1 - \mu \frac{R+\alpha-1}{a_1(1+q)\sigma_1^2}. \tag{3.3}$$

Proof. See Appendix A.2. □

The stability region of the fundamental price $\bar{P}$ is plotted in (α, μ) plane in Fig.A.1 in Appendix A.2, where $\mu_{0,1}(1) = [2a_1(1+q)\sigma_1^2]/R$ for $\alpha = 1$ and $\mu_{0,1}(0) = [2a_1(1+q)\sigma_1^2]/(R-1)$ for $\alpha = 0$. Along the flip bifurcation boundary, μ decreases as α increases. It follows from Proposition 3.2 that the stability of the fundamental steady state is independent of the price adjustment of the fundamentalists when the market maker is under-reacted (i.e. $\mu \leq \mu_{0,1}(1)$). However, when $\mu > \mu_{0,1}(1)$, the stability of the steady state can be maintained only when the reactions of the fundamentalists and the market maker are balanced. Numerical simulations indicate that the over-reaction from either the market maker or the fundamentalists can push the price to explode (through the flip bifurcation).

3.2. **The case $m = -1$.** Similarly, we obtain the following stability and bifurcation result when all traders are trend followers.

Proposition 3.3. *For DDS (3.1), if all the traders are trend followers (that is $m = -1$), then*

(1) *for $\delta = 0$, the fundamental steady state is globally asymptotically stable if and only if $0 < \mu < Q/(R-1)$, where $Q = 2a_2(1+q)\sigma_1^2$. In addition, a flip bifurcation occurs along the boundary $\mu = Q/(R-1)$;*

(2) *for* $\delta \in (0,1)$, *the fundamental steady state is stable for*

$$0 < \mu < \begin{cases} \bar{\mu}_1 & 0 \leq \gamma \leq \bar{\gamma}_0 \\ \bar{\mu}_2, & \bar{\gamma}_0 \leq \gamma, \end{cases}$$

where

$$\bar{\mu}_1 = \frac{Q}{(R-1) - \gamma 2\delta/(1+\delta)}, \quad \bar{\mu}_2 = \frac{(1-\delta)Q}{2\delta[\gamma - (R-1)]}, \quad \bar{\gamma}_0 = (R-1)\frac{(1+\delta)^2}{4\delta}.$$

In addition, a flip bifurcation occurs along the boundary $\mu = \bar{\mu}_1$ *for* $0 < \gamma \leq \bar{\gamma}_0$ *and a Hopf bifurcation occurs along the boundary* $\mu = \bar{\mu}_2$ *for* $\gamma \geq \bar{\gamma}_0$.

Proof. See Appendix A.3. □

The local stability regions and bifurcation boundaries are indicated in Fig. A.2 (a) for $\delta = 0$ and (b) for $\delta \in (0,1)$ in Appendix A.3, where $\bar{\gamma}_2 = (1+\delta)(R-1)/(2\delta)$ is obtained by letting $\bar{\mu}_2 = Q/(R-1)$. Given that $R = 1 + r/K$ is very close to 1, the value of μ along the flip boundary is very large and $\bar{\gamma}_o$ is close to 0. This implies that, for $\delta = 0$, the fundamental price is stable for a wide range of μ. Foe $\delta \in (0,1)$, the stability region is mainly bounded by the Hopf bifurcation boundary. Along the Hopf boundary, μ decreases as γ increases, implying that the stability of the steady state is maintained when the speed of the market maker and the extrapolation of the trend followers are balanced. Numerical simulations indicate that, near the bifurcation boundary, price either converges periodically to the fundamental value or oscillates regularly or irregularly. In addition, the Hopf bifurcation boundary shifts to the left when δ increases. This implies that the steady state is stabilizing when more weights are given to the most recent prices.

3.3. **The general case $m \in (-1,1)$.** We now consider the complete market fraction model DDS with both fundamentalists and trend followers by assuming $m \in (-1,1)$. Let $a = a_2/a_1$ be the ratio of the absolute risk aversion coefficients. It turns out that the stability and bifurcation of the fundamental steady state are different from the previous two special cases and they are determined jointly by the geometric decay rate and extrapolation rate of the trend followers, the speed of the price adjustment of the fundamentalists towards the fundamental steady state, and the speed of adjustment of the market maker towards the market aggregate demand.

Proposition 3.4. *For DDS (3.1) with* $m \in (-1,1)$,

(1) *if* $\delta = 0$, *the fundamental steady state is stable for* $0 < \mu < \mu^*$, *where*

$$\mu^* = \frac{2Q}{(R-1)(1-m) + a(R+\alpha-1)(1+m)}.$$

In addition, a flip bifurcation occurs along the boundary $\mu = \mu^*$ *with* $\alpha \in [0,1]$*;*

(2) *if* $\delta \in (0,1)$, *the fundamental steady state is stable for*

$$0 < \mu < \begin{cases} \mu_1 & 0 \le \gamma \le \gamma_0 \\ \mu_2, & \gamma_0 \le \gamma, \end{cases}$$

where

$$\mu_1 = \frac{1+\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma_2-\gamma}, \qquad \mu_2 = \frac{1-\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma-\gamma_1},$$

$$\gamma_1 = (R-1) + a(R+\alpha-1)\frac{1+m}{1-m}, \qquad \gamma_0 = \frac{(1+\delta)^2}{4\delta}\gamma_1, \qquad \gamma_2 = \frac{1+\delta}{2\delta}\gamma_1.$$

In addition, a flip bifurcation occurs along the boundary $\mu = \mu_1$ *for* $0 < \gamma \le \gamma_0$ *and a Hopf bifurcation occurs along the boundary* $\mu = \mu_2$ *for* $\gamma \ge \gamma_0$.

Proof. See Appendix A.3. □

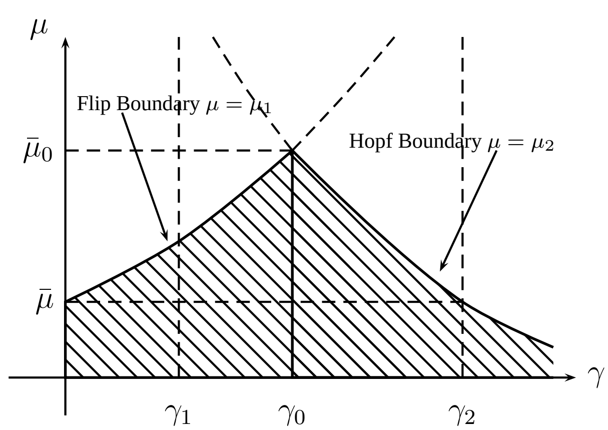


FIGURE 3.1. Stability region and bifurcation boundaries for $m \in (-1,1)$ and $\delta \in (0,1)$.

The model with the fundamentalists only can be treated as a degenerated case of the complete model with $\delta = 0$. For $\delta \in (0,1)$, the fundamental steady state becomes unstable through either flip or Hopf bifurcation, indicated in Fig.3.1, where

$$\bar{\mu}_0 = \frac{2}{1-\delta}\bar{\mu}, \qquad \bar{\mu} = \frac{2Q}{(R-1)(1-m)+a(R+\alpha-1)(1+m)}.$$

Variations of the stability regions and their bifurcation boundaries characterise different impacts of different types of trader on the market price behaviour, summarised as follows.

- *The market fraction* has a great impact on the shape of the stability region and its boundaries. It can be verified that γ_1, γ_0, γ_2 and μ_1, μ_2 increase as m increases. This observation has two implications: (i) the locals stability region in parameters (γ, μ) is enlarged as the fraction of the fundamentalists increases and this indicates a stabilizing effect of the fundamentalists; (ii) the flip (Hopf) bifurcation boundary becomes dominant as the fraction of the fundamentalists (trend followers) increases, correspondingly, the market price displays different behaviour near the bifurcation boundaries. Numerical simulations of the nonlinear system (3.1) show that price becomes explosive near the flip bifurcation boundary, but converges to either periodic or quasi-periodic cycles near the Hopf bifurcation boundary.
- *The speed of price adjustment of the fundamentalists towards the fundamental value* has an impact that is negatively correlated to the market fraction. This observation comes from the fact that, as α increases, γ_1 and hence γ_0 and γ_2 decreases. In other word, an increase (decrease) of the fundamentalists fraction is equivalent to a decrease (increase) of the price adjustment speed of the fundamentalists toward the fundamental value.
- *The memory decay rate* of the trend followers has a similar impact on the price behaviour as the speed of the price adjustment of the fundamentalists does. This is because that, as δ decreases, both γ_0 and γ_2 increase. In particular, as $\delta \to 0$, $\gamma_0, \gamma_2 \to +\infty$ and the stability and bifurcation is then characterised by the model with the fundamentalists only. On the other hand, as $\delta \to 1$, both γ_0 and γ_2 tend to γ_1 whilst $\bar{\mu}_0$ tends to infinity and the stability and bifurcation are then characterised by the model with the trend followers only. In addition, $\bar{\mu}_o$ increases as δ decreases, implying the steady state is stabilizing as trend followers put more weights on the more recent prices.
- *The risk aversion coefficients* have different impact on the price bahaviour, depending on the relative risk aversion ratio. Note that, $\bar{\mu}$ and hence $\bar{\mu}_0$ increase for $a = a_2/a_1 < a^*$ and decrease for $a = a_2/a_1 > a^*$, where $a^* = (R-1)/(R+\alpha-1) \in (1-1/R, 1]$.

Hence the local stability region is enlarged (reduced) when the trend followers are less (more) risk averse than the fundamentalists in the sense of $a_2 < a^* a_1$ ($a_2 > a^* a_1$).

Overall, in terms of the local stability and bifurcation of the fundamental steady state, a similar effect happens for either high (low) geometric decay rate, or high (low) market fraction of the trend followers, or high (low) speed of the price adjustment of the fundamentalists towards the fundamental value. This observation make us concentrate our statistical analysis of the stochastic model (2.15) on m (the market fraction) and α (the speed of the price adjustment of the fundamentalists toward the fundamental value). Numerical simulations (not reported here) for the deterministic system (3.1) show that: (i) the market prices converge to the fundamental value for the parameters located insider the local stability region; (ii) near the flip bifurcation boundary, prices are explode and near the Hopf bifurcation boundary, prices converge to either periodic or quasi-periodic price cycles (as we move away from the Hopf boundary, more complicated price dynamics can be generated, but this is not the focus of this paper.); (iii) there is no significant difference between the average wealth shares of two types of investors.[11]

4. Statistic Analysis of the Stochastic Model

In this section, by using numerical simulations, we examine various aspects of the price dynamics of the stochastic heterogeneous asset pricing model (2.15) where both the noisy fundamental price and noisy demand processes are presented. The analysis is conducted by establishing a connection on the price dynamics between SDS (2.15) and its underlying DDS (3.1). In so doing, we are able to obtain some theoretical insights into the generating mechanism of various statistic properties, including those econometric properties and stylised facts observed in high frequency financial time series.

Our analysis is conducted as follows. As a benchmark, we first review briefly the so-called stylized facts based on both S&P500 and AOI (Australian All Ordinary Index). Secondly, we use the concept of random fixed point to examine the convergence of the market temporal equilibrium price and its long-run behaviour. It is found that the convergence of asset prices of SDS (2.15) to the random fixed point is related to the stability of the fixed point of DDS (3.1).

[11]There is no difference when prices converge to the fundamental value. However, when prices converge to cycles, the trend followers can accumulate more wealth share (at most of 2% over 5,000 trading days, about 20 years). Overall, for the deterministic system, the fundamentalists cannot accumulate more wealth than the trend followers and both survive in the market.

Thirdly, we use Monte Carlo simulations to conduct statistical analysis and test on the convergence of the market prices to the fundamental price. It is commonly believed that the market price is mean-reverting to the fundamental price in long-run, but it can deviate from the fundamental price in short-run. We analyze market conditions under which this is true. Fourthly, by analysing wealth accumulation, we examine the profitability and survivability of differential trading strategies. Our analysis shows long-run (short-run) profitability of the fundamental (trend following) strategy and long-run survivability of the trend followers. Finally, by examining the autocorrelation (AC) structure of (relative) returns near different types of bifurcations, we study the generating mechanism of different AC patterns. Most of our results are very intuitive and can be explained by various behaviour aspects of the model, including the mean reverting of the fundamentalists, the extrapolation of the trend followers, the speed of price adjustment of the market maker, and the market dominance. The statistical analysis and test are based on Monte Carlo simulations.

4.1. **Financial Time Series and Stylized Facts.** Recent research on heterogeneous asset pricing models are aimed to explain various market behaviour and to replicate the econometric properties and stylized facts of financial time series. As a benchmark, we include time series plots on prices and returns for both S&P500 and AOI (Australian All Ordinary Index) from Aug. 10, 1993 to July 24, 2002 and the corresponding density distributions, autocorrelation coefficients (ACs) and statistics of the returns in Appendix B. All (high-frequency) financial time series share some common facts, the so called stylised facts, including excess volatility (relative to the dividends and underlying cash flows), volatility clustering (high/low fluctuations are followed by high/low fluctuations), skewness (either negative or positive) and excess kurtosis (comparing to the normally distributed returns), long rang dependence (insignificant ACs of returns, but significant and decaying ACs for absolute and squared returns), etc. For a comprehensive discussion of stylized facts characterizing financial time series, we refer to Pagan (1996) and Lux (2004).

Recent structure models on asset pricing and heterogeneous beliefs have shown relatively well understood mechanism of generating volatility clustering, skewness and excess kurtosis. However, it is less clear on the mechanism of generating long-rang dependence.[12] In addition,

[12]See Lux (2004) for a recent survey on possible mechanisms generating long rang dependence, including coexistence of multiple attractors and multiplicative noise process.

there is lack of statistical analysis and test on those mechanisms. Our statistic analysis in this paper is based on Monte Carlo simulations, hoping to establish a connection between various AC patterns of the SDS and the bifurcation of the underlying DDS. Such connection is necessary to understanding the mechanism of generating stylised facts, to replicating econometric properties of financial time series, and to calibrating the model to financial data.

In the following discussion, we choose the annual volatility of the fundamental price to be 20% (hence $\sigma_\epsilon = (20/\sqrt{K})\%$ with $K = 250$) and the volatility of the noisy demand $\sigma_\delta = 1$, which is about 1% of the average fundamental price level $\bar{P} = \$100$. For all of the Monte Carlo simulation, without mention, we run 1,000 simulations over 6,000 time periods and discard the first 1,000 time periods to wash out the initial noise effect. Each simulation generates two independent sets of random numbers, one is for the fundamental price and the other is for the noisy demand. The draws are i.i.d. across 1,000 simulations, but the same sets of draws are used for different scenario with different sets of parameters.

4.2. **Random Fixed Point and Long-Run Behaviour.** One of the primary objectives of this paper is to analyse the long run behaviour of SDS (2.15). For DDS (3.1), the long-run behaviour is characterised by either stable fixed points or various attractors examined in the previous section. For stochastic dynamic system, the long-run behaviour is often characterised by stationarity and invariant probability distribution. As pointed in Bohm and Chiarella (2005), this view does not provide information about stationary solutions generated by the stochastic difference system and cannot supply any information about the stability of a stationary solution.

The theory of random dynamical system (e.g. Arnold (1998)) provides the appropriate concepts and tools to analyze sample paths and investigate their limiting behaviour. The central concept is that of a *random fixed point*[13] and its asymptotic stability, which are generalisations of the deterministic fixed point and its stability. Intuitively, a random fixed point corresponds to a stationary solution of a stochastic difference system like (2.15) and the asymptotic stability implies that sample paths converge to the random fixed point point wise for all initial conditions of the system. We are interested in the existence and stability of a random fixed point of SDS (2.15) when the deterministic fixed point of DDS (3.1) is asymptotical stable. However, since

[13]We refer to Arnold (1998) for mathematical definitions of random dynamical systems and of stable random fixed points and Bohm and Chiarella (2005) for economical applications to asset pricing with heterogeneous mean variance preferences.

SDS (2.15) is nonlinear, a general theory on the existence and stability of a random fixed point is not yet available and we conduct our analysis by numerical simulations.

For illustration, we choose parameters as follows

$$\gamma = 2.1, \quad \delta = 0.85, \quad \mu = 0.43, \quad m = 0, \quad w_{1,0} = 0.5 \quad \text{and} \quad \alpha = 1, 0.5, 0.1, 0. \qquad (4.1)$$

For DDS (3.1) with the set of parameters (4.1), applying Proposition 3.4 implies that the fundamental value is locally asymptotically stable for $\alpha = 1, 0.5, 0.1$ and unstable for $\alpha = 0$. Our numerical simulations show that this is also true for the nonlinear system (3.1).

For the parameter set (4.1), Fig.4.1 shows the price dynamics of the corresponding SDS (2.15) with four different values of $\alpha = 1, 0.5, 0.1, 0$ and (arbitrarily) different initial conditions but with a fixed set of noisy fundamental value and demand processes. It is found that, for $\alpha = 1, 0.5$ and 0.1, respectively, there exists a random fixed point and prices with different conditions converge to the fixed random point in long run. In fact, the convergence only takes about 50, 100 and 400 time periods for $\alpha = 1, 0.5$ and 0.1, respectively. However, there is no such stable random fixed point for $\alpha = 0$ and prices with different initial conditions lead to different random sample paths (In fact, the sample paths are shifted by different initial conditions.). This is a surprising result—the stability of fixed point of both the deterministic and stochastic systems is same for the same parameter set (4.1). In fact, this result holds for other selections of parameters (as long as the solutions of DDS (3.1) do not explode). Theoretically, how the stability of the deterministic system and the corresponding stochastic system are related is a difficult problem in general.[14]

4.3. **Convergence of Market Price to the Fundamental Value.** We now turn to the relation between the market price and the fundamental price. It is commonly believed that the market price is mean-reverting to the fundamental price in long-run, but it can deviate from the fundamental price in short-run. The following discussion indicates that this is true under certain market conditions.

[14]It is well known from the stochastic differential equation literature (e.g. see the examples in Mao (1997), pages 135-141) that, for continuous differential equations, adding noise can have double-edged effect on the stability—it can either stabilize or destabilize the steady state of the differential equations. For our SDS (2.15), numerical simulations show that adding a small (large) noise can stabilizing (destabilize) the price dynamics when parameters are near the flip bifurcation boundary of the DDS (3.1).

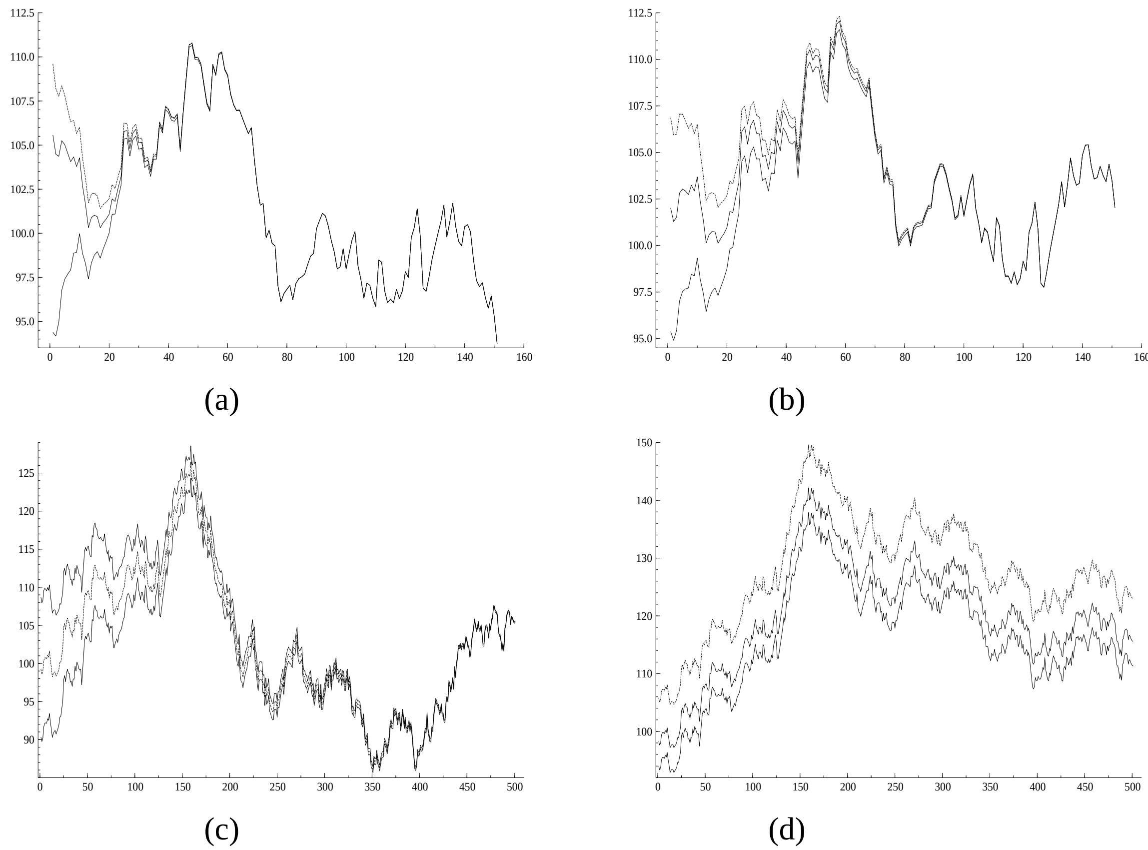


FIGURE 4.1. Price convergence with α=1 (a); 0.5 (b); 0.1 (c); and 0 (d) for different initial conditions.

As we known from the local stability analysis of DDS (3.1) that an increase in α has a similar effect to an increase in m. Parameter α measures confidence level of the fundamentalists on their estimated fundamental value P_t^*. The previous discussion illustrates that, for fixed $m = 0$, as α increases (i.e. as the fundamentalists become more confident on their estimated fundamental price), the speed of convergence of the market price to the random fixed point increases. When price of DDS (3.1) is stable, it converges to the fixed point corresponding to the constant fundamental value $\bar{P}$. For SDS (2.15), it is interesting to know how the stable random fixed point is related to the fundamental value process.

To illustrate, for parameter set (4.1), the averaged time series of the difference of market and fundamental prices $P_t - P_t^*$ based on Monte Carlo simulations are reported in Fig. 4.2. It shows that, as α increases, the deviation of the market price from the fundamental price decreases. That is, as the fundamentalists become more confident on their estimated fundamental price, the deviation of market price from the fundamental price are reduced.

A statistical analysis is conducted by using Monte Carlo simulations for the given set of parameters (4.1) with four different values of α. The average prices, returns, absolute wealth shares

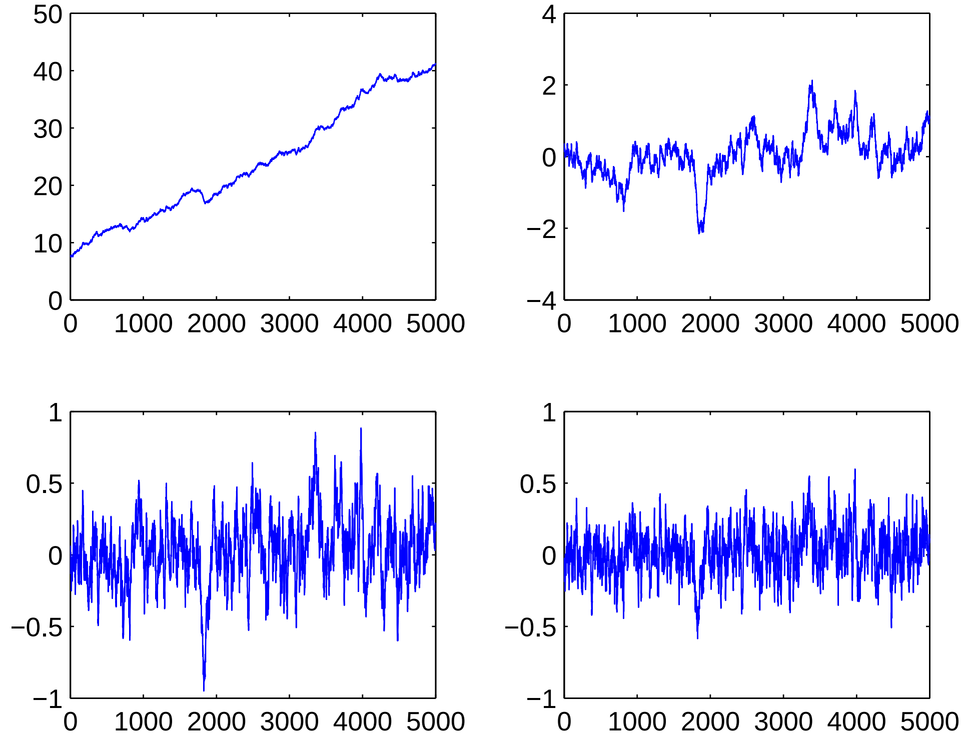


FIGURE 4.2. Time series of price difference $P_t - P_t^*$ with α=0 (top left); 0.1 (top right); 0.5 (second left); and 1 (second right).

of the fundamentalists are reported in Fig. 4.4. Because of $m = 0$, the absolute and market wealth shares are the same. The resulting Wald statistics to detect the differences between market prices and fundamental prices are reported in Table 4.1. The null hypothesis is specified as, respectively,

- Case 1: $H_0 : P_t = P_t^*, t = 1000, 2000, ..., 5000;$
- Case 2, $H_0 : P_t = P_t^*, t = 3000, 3500, 4000, ..., 5000;$
- Case 3, $H_0 : P_t = P_t^*, t = 4000, 4100, 4200, ..., 5000;$
- Case 4, $H_0 : P_t = P_t^*, t = 4000, 4050, 4100, ..., 5000;$
- Case 5, $H_0 : P_t = P_t^*, t = 4901, 4902, 4903..., 5000$, which refers to the last one hundred periods;
- Case 6, $H_0 : P_t = P_t^*, t = 4951, 4952, ..., 5000$, which refers to the last fifty periods.

Noticed that the critical values corresponding to above test statistics come from the χ_2 distribution with degree of freedom 5, 5, 11, 21, 100, and 50, respectively, at 5% significant level. We see that for $\alpha = 0$, all of the null hypothesis are strongly rejected at 5% significant level. For $\alpha = 0.5$ and 1, all of the null hypothesis can not be rejected at 5% significant level. We also see that when α increasing, the resulting Wald statistics decreasing (except Case 5 with $\alpha = 1$). This confirms that when α increasing, i.e. when the fundamentalists become more confident on the fundamental price, the differences between prices and fundamental prices become smaller.

TABLE 4.1. Wald test statistics for P_t and P_t^*.

	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1$	Critical value
Case 1	100.585	13.289	5.225	3.698	11.071
Case2	99.817	13.964	6.782	4.358	11.071
Case 3	121.761	24.971	16.041	10.840	19.675
Case 4	148.690	38.038	23.836	19.190	32.671
Case 5	293.963	105.226	99.618	103.299	124.342
Case 6	177.573	50.970	45.043	43.052	67.505

As we know that an increase in α has similar effect to an increase of the market fraction of the fundamentalists. The above statistic analysis thus implies that, as the fundamentalists dominate the market (as m increases), the market prices follow the fundamental prices closely. Trend extrapolation of the trend followers can drive the market price away from the fundamental price. This result is very intuitive.

4.4. **Wealth Accumulation, Profitability and Survivability.** It is commonly believed that irrational traders (such as the trend followers in our model) may do better than rational traders (such as the fundamentalists) over a short-run, but over a long-run, irrational traders will be driven out of the market and rational traders will be the only survivors over a long-run. We now justify this common belief by analyzing the wealth dynamics of our heterogeneous market fraction model in which traders do not change their beliefs over time periods. Consequently, we examine profitability and survivability of both types of trading strategies. Two situations are considered in the following discussion.

In the first case, we choose parameter set (4.1) by fixing market fraction m and varying α. For each set of parameters, we run one simulation over 20,000 time periods in order to see possible limiting behaviours. Fig. 4.3 demonstrates the absolute wealth share accumulations of the fundamentalists with $\alpha = 1, 0.5, 0.1, 0$ and keeping all the other conditions the same. It shows that (i) trend followers survive in long-run for $\alpha = 1, 0.5$ and 0.1 in the sense that their absolute wealth share does not vanish, although they accumulate less wealth shares over the time period; (ii) the trend followers are doing better then the fundamentalists when $\alpha = 0$; (iii) the profitability of the fundamentalists improves as α increases (i.e. as they become more confident on their estimated fundamental value). These results are further confirmed when we run Monte

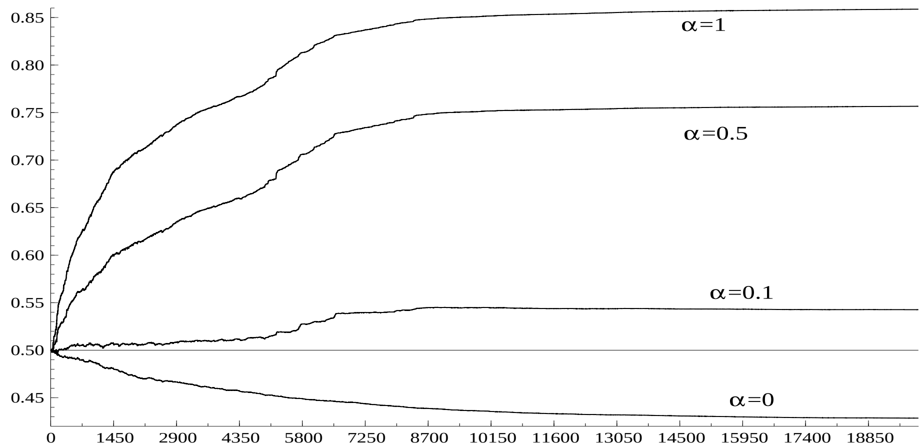


FIGURE 4.3. Time series of the absolute wealth accumulation of the fundamentalists $w_{1,t}$ with $\alpha = 1, 0.5, 0.1$ and 0.

Carlo simulations, the results are given in Fig. 4.4. For each value of α, we plot the average market price (left column), return (middle column) and absolute wealth share accumulation (right column) [15] of the fundamentalists for four values of α in Fig. 4.4. For $\alpha = 0$, the absolute wealth share of the fundamentalists is dropped from 50% to about 43%, while for $\alpha = 0.1, 0.5$ and 1, it is increased from 50% to 55%, 76% and 86%, respectively.

Given that both α and m have similar impact on the local stability of the deterministic system, we can demonstrate that they play similar role in terms of wealth accumulation. Again, by running one simulation over 20,000 time periods, Fig. 4.5 shows the absolute wealth share accumulations of the fundamentalists for four different values of $m = -0.95, 0$ and 0.5 with $\alpha = 0.5, \gamma = 2, \mu = 0.5, \delta = 0.85, w_{1,0} = 0.5$. In this case, the fundamentalists form their conditional expectation by taking average of the latest market price and fundamental price. In all four cases, (i) the fundamentalists accumulate more wealth share than the trend followers in long-run (an increase from 50% to about 70-75%), however, the trend followers survive in long-run and they can even accumulate more wealth share in short-run when they dominate the market (this is the case when $m = -0.95$, which corresponds to 97.5% of trend followers and 2.5% of the fundamentalists); (ii) the profitability of the fundamentalists improves as m increases (i.e. as the market fraction of the fundamentalists increases). Essentially, we have

[15] The initial wealth share for both types of traders are equal $w_{1,0} = 0.5$. Because of $m = 0$, both the absolute and market wealth shares are the same.

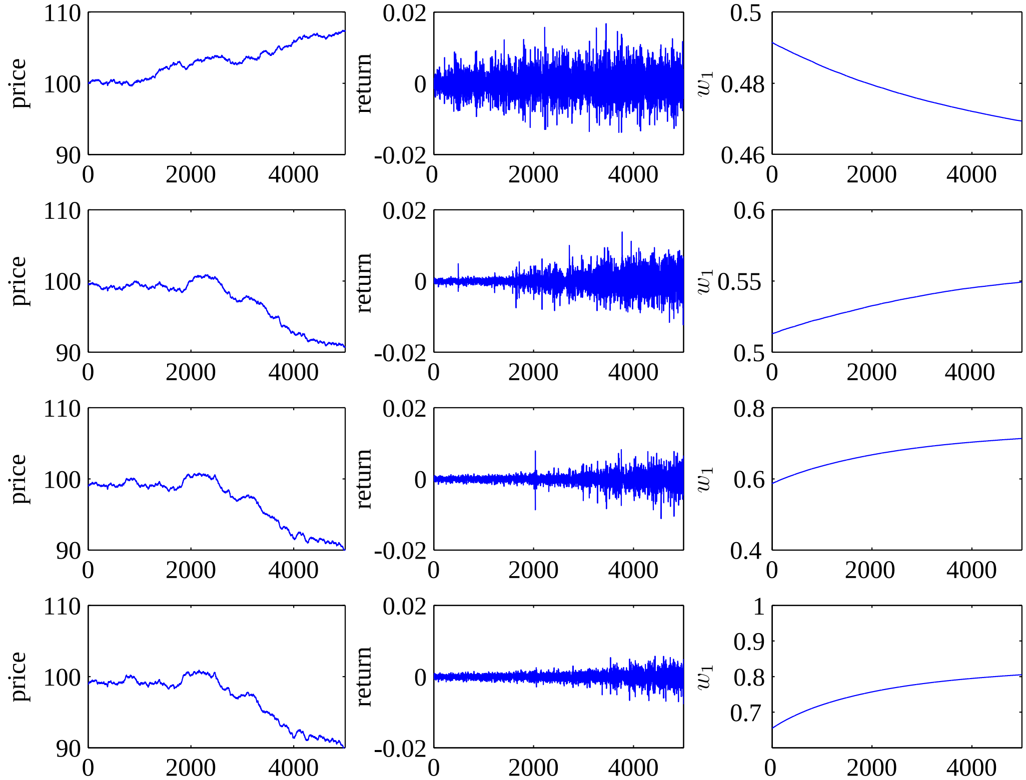


FIGURE 4.4. Average Monte Carlo time series of market prices, returns, absolute wealth share and market wealth share of the fundamentalists with $\alpha = 0, 0.1, 0.5$ and 1.

shown that both α and m play similar role on profitability (of the fundamentalists) and survivability (of the trend followers).[16]

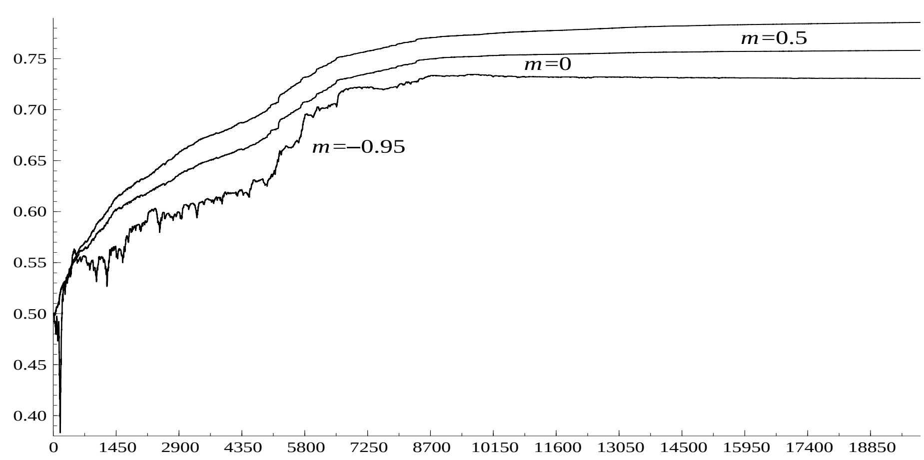


FIGURE 4.5. Time series of the absolute wealth accumulation of the fundamentalists $w_{1,t}$ with $m = -0.95, 0, 0.5$ and $\alpha = 0.5, \gamma = 2, \mu = 0.5, \delta = 0.85, w_{1,0} = 0.5$.

[16]Comparison of Fig.4.3 and Fig. 4.5 indicates that parameter α plays more important role on the wealth accumulation than parameter m does.

In the second case, we assume that the fundamentalists are naive traders (i.e. $\alpha = 0$ and $E_{1,t}(P_{t+1}) = P_t$). In this case the fundamental price plays no role on the conditional expectation formation of the fundamentalists. We choose

$$\alpha = 0, \quad \gamma = 1, \quad \mu = 0.4, \quad \delta = 0.85, \quad w_{1,0} = 0.5 \quad \text{and} \quad m = -1, -0.5, 0, 0.5, 1. \quad (4.2)$$

For each set of parameters, we run one simulation over 20,000 time periods such that the corresponding limiting behaviours become clear. Fig. 4.6 illustrates the absolute wealth share accumulations of the fundamentalists with different market fraction $m = -1, -0.5, 0, 0.5, 1$, and keeping all the other conditions the same. It shows that, overall, no one is doing significant better by accumulating significant wealth share than the others. However, different from the previous case, trend followers are doing slightly better by accumulating more wealth share, except $m = 1$. In addition, the profitability of the fundamentalists improves as m increases (i.e. as their market population share increases). These results are further confirmed when we run Monte Carlo simulations, the results are given in Fig. B.2 in Appendix B, which includes the average market price, return and absolute wealth share accumulation of the fundamentalists.[17] It is also interesting to see that the average market price increases, rather than decreases in the first case, stochastically. Given the naive expectation of the fundamentalists, this may due to the trend chasing activity of the trend followers.

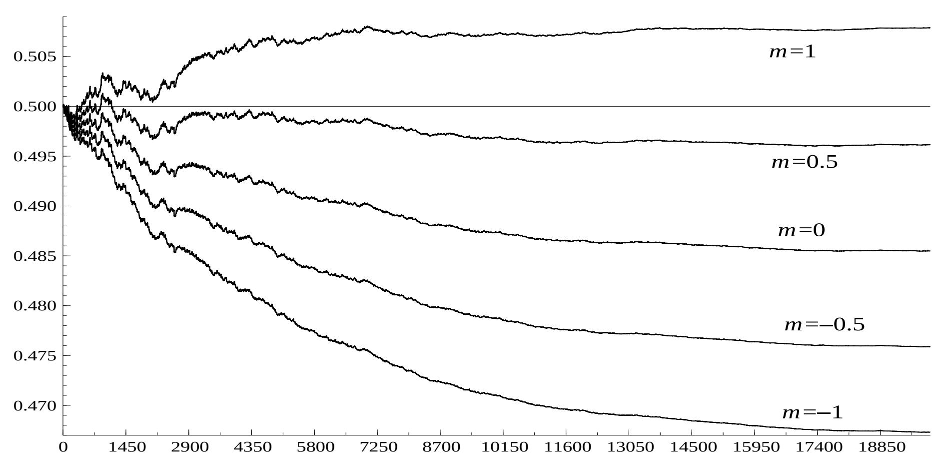


FIGURE 4.6. Time series of the absolute wealth accumulation of the fundamentalists $w_{1,t}$ with $m = -1, -0.5, 0, 0.5, 1$ and $\alpha = 0, \gamma = 1, \mu = 0.4, \delta = 0.85, w_{1,0} = 0.5$.

[17]The initial wealth share for both types of traders are equal $w_{1,0} = 0.5$. For different value of m, the market wealth shares are different.

The above analysis leads to the following implications on the profitability and survivability:

- Although the trend followers have no information on the fundamental value, they survive in long-run. This may due to the learning mechanism they engaged. In addition, they can do even better than the fundamentalists over short-run.
- The profitability of the fundamentalists improve as either they become more confident on their estimated fundamental value or they dominate the market.
- The trend followers are doing better by accumulating more wealth share when the fundamentalists become naive traders. In addition, their profitability improves as their market population share increases.

Overall, we have shown the long-run (short-run) profitability of the fundamental (trend following) trading strategy and long-run survivability of the trend following strategy. This result partially verifies a common belief that the chartists may do better in short-run, but market will be dominated by the fundamentalists in long-run. However, the chartists do survive in long-run and this may due to their learning. This result provides essentially incentive and justification on recent studies on heterogeneous asset pricing (e.g. Brock and Hommes (1997) and Chiarella and He (2002, 2003*c*)) in which traders switch their trading strategy based on certain fitness function from time to time.

4.5. **Bifurcations and Autocorrelation Patterns.** Understanding autocorrelation (AC) structure of returns plays an important role on the market efficiency and predictability. It is often a difficult task to understand the generating mechanism of various AC patterns, in particular those realistic patterns observed in financial time series. It is believed that the underlying deterministic dynamics of the stochastic system plays important role on the AC structure of the stochastic system. But how they are related is not clear. In the following discussion, we are trying to establish such connection by analyzing changes of autocorrelation (AC) structures of the stochastic returns when parameters change near the bifurcation boundaries of the underlying deterministic model. The analysis is conducted through Monte Carlo simulations. This analysis leads us to some insights into how particular AC patterns of the stochastic model are characterized by different types of bifurcation of the underlying deterministic system. In so ding, it helps us to understand the mechanism of generating realistic AC patterns.

From our discussion in the previous section, we know that the local stability region of the steady state is bounded by both flip and Hopf bifurcation boundaries in general. To see how the AC structure changes near different types of bifurcation boundary, we select two sets of parameters, denoted by (F1) and (H1), respectively,

(F1) $\alpha = 1, \gamma = 0.8, \mu = 5, \delta = 0.85, w_{1,0} = 0.5$ and $m = -0.8, -0.5, -0.3, 0$;

(H1) $\alpha = 1, \gamma = 2.1, \mu = 0.43, \delta = 0.85, w_{1,0} = 0.5$ and $m = -0.95, -0.5, 0, 0.5$.

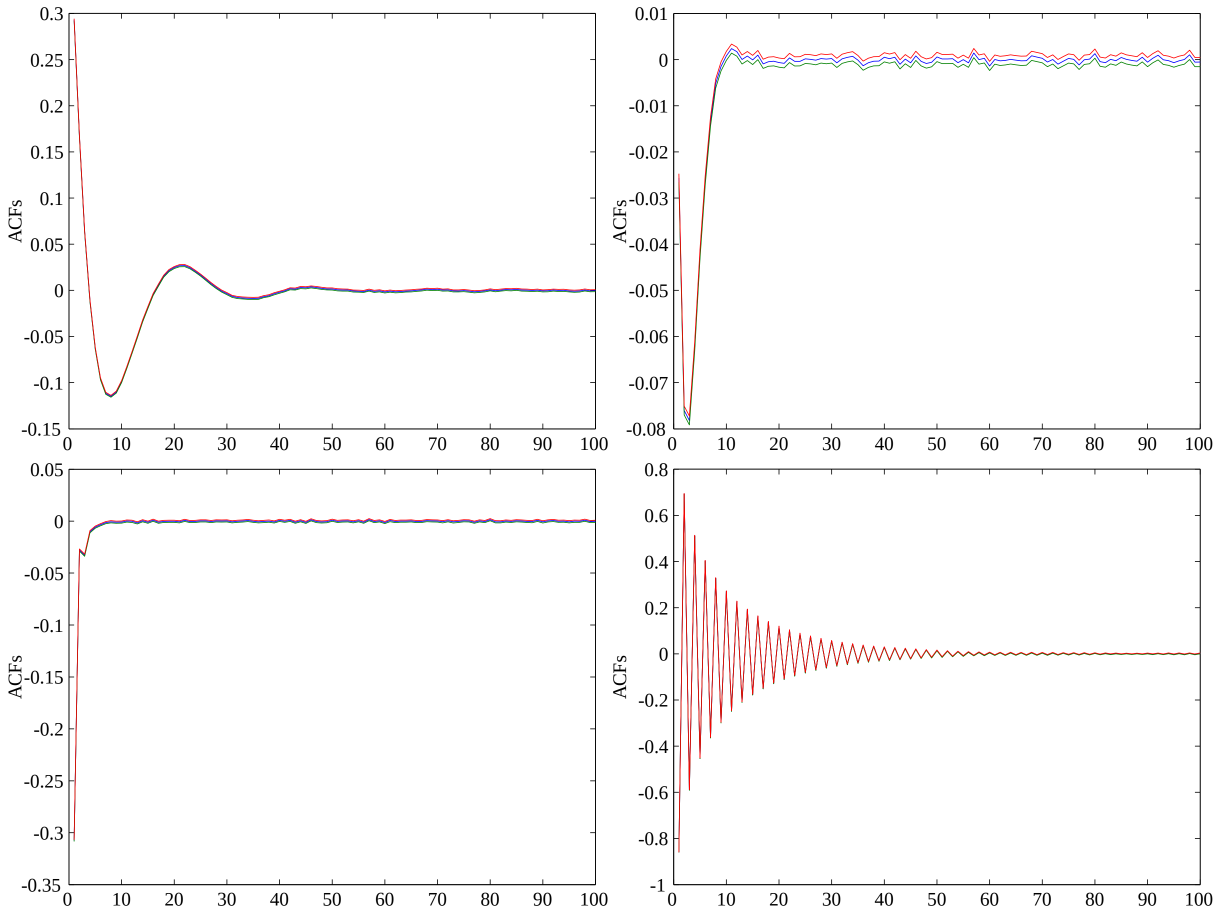


FIGURE 4.7. Monte Carlo simulation on the average ACs of return for $m = -0.8, -0.5, 0.3, 0$ for parameter set $(F1)$.

For (F1) with different values of m, the steady state of DDS (3.1) is locally stable.[18] However, as m increases, we move closely to the flip boundary.[19] For (H1), there exists a Hopf bifurcation value $\bar{m} \in (0, 0.005)$ such that the steady state is locally stable for $m = 0.5 \geq \bar{m}$ and unstable for $m = -0.95, -0.5, 0 < \bar{m}$ through a Hopf bifurcation. As m decreases, we are moving closely to the Hopf bifurcation boundary initially, and then crossing over the boundary, and

[18]The solutions become exploded when parameters are near the flip bifurcation boundary and hence we only choose parameters from inside the stable region.

[19]This means that the difference between the given μ and the corresponding flip bifurcation value $\mu_1(m)$ becomes smaller as m increases. It is in this sense that an increase in m is destabilising the steady state.

then moving away from the boundary. Therefore, an increase in m is stabilizing the steady state. It is interesting to see that the market fraction has different stabilizing effect near different bifurcation boundary.

For SDS (2.15), Fig. 4.7 and Fig. 4.8 report the average ACs of relative return for four different values of m with parameter set (F1) and (H1), respectively. Tables B.2 and B.3 in Appendix B report the average ACs of returns over the first 100 lags, the number in the parentheses are standard errors, the number in the second row for each lag are the total number of ACs that are significantly (at 5% level) different from zero among 1,000 simulations. It is found that adding the noise demand does not change the nature of ACs of returns.[20]

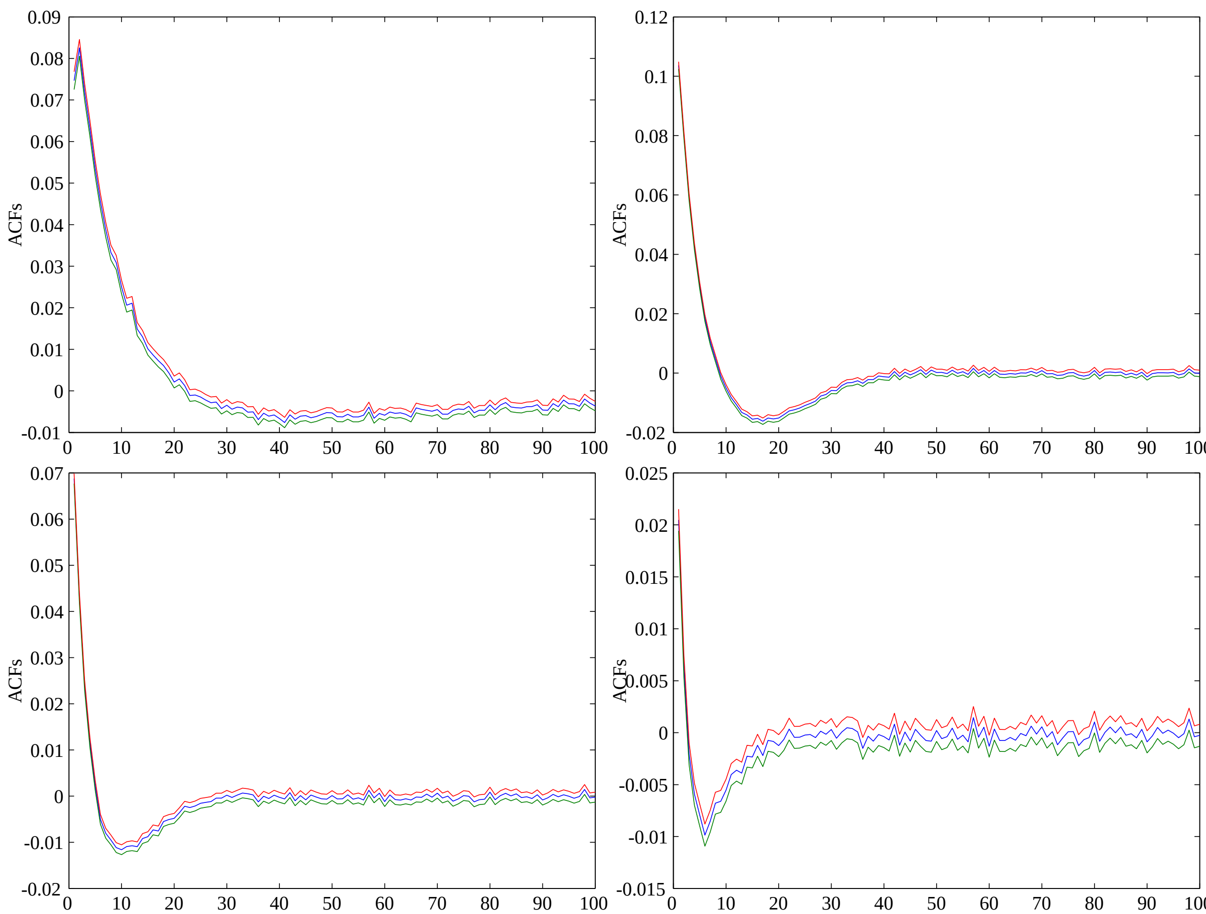


FIGURE 4.8. Monte Carlo simulation on the average ACs of return for $m = -0.95, -0.5, 0, 0.5$ for parameter set $(H1)$.

Given that there is no significant AC structure from the noisy returns of the fundamental values, the persistent AC patterns displayed in Figs. 4.7-4.8 indicate some connections between AC patterns of SDS (2.15) and the dynamics of the underlying DDS (3.1). For parameter set

[20]Noisy processes in our model do not change the qualitative nature of the AC of returns, however, they do change the AC patterns of the absolute and squared returns. This issue is addressed in our separate paper He and Li (2004).

(F1), the fundamental value of the underlying DDS (3.1) is locally stable and the AC structure of returns of SDS (2.15) changes as the parameters are moving closer to the flip bifurcation boundary. For the deterministic model, we know that an increase of m has a similar effect to an increase of α, the speed of price adjustment of the fundamentalists, or μ, the speed of price adjustment of the market maker. Corresponding to the case of $m = -0.8$ in Fig. 4.7, an **under and over-reaction** pattern[21] characterizing by an oscillatory decaying ACs with $AC(i) > 0$ for small lags followed by negative ACs for large lags is observed when the parameters are far away from the flip bifurcation boundary. Intuitively, this results from the constant price under-adjustment from either the fundamentalists or the market maker. As the parameters are moving toward the flip bifurcation boundary, such as the case of $m = -0.5, 0.3$ in Fig. 4.7, an **over-reaction** pattern characterized by an increasing AC with $AC(i) < 0$ for small lags i appears. As the parameters move closely to the flip boundary, such as when $m = 0$ in Fig. 4.7, this over-reaction pattern becomes a **strong over-reaction** pattern characterizing by an oscillating and decaying ACs which are negative for odd lags and positive for even lags. These results are very intuitive. When the market fractions of the fundamentalists are small, it is effectively equal to a slow price adjustment from either the fundamentalists or market maker, leading to under-reaction. As m increases, such adjustment becomes strong, leading to an over-reaction.[22]

Near the Hopf bifurcation boundary, the AC structure behaviours differently when parameters cross the Hopf boundary from unstable region to stable region, see Fig. 4.8. For small m, say $m = -0.95, -0.5$, the steady state of the deterministic model is unstable and it bifurcates to either periodic or quasi-periodic cycles. For the stochastic model, a **strong under-reaction** AC pattern characterizing by significantly decaying positive $AC(i)$ for small lags i and insignificantly negative $AC(i)$ for large lags i, as illustrated in Fig. 4.8 for $m = -0.95$.[23] As m increases, say to $m = -0.5$ and 0, the strong under-reaction pattern is replaced by an over-reaction pattern. As m increases further, say to $m = 0.5$ in Fig. 4.8, the steady state of the deterministic model becomes stable and the AC structure of the stochastic return reduces to an insignificant under-reaction pattern.

[21]This means a short-run under-reaction and long-run over-reaction.

[22]Based on this observation, one can see that both the fundamentalists and market maker need to react to the market price at *right way* in order to generate insignificant AC patterns observed in financial markets. Essentially, this is the mechanism we are using to characterising the long range dependence in our separate paper He and Li (2004).

[23]The AC structure discussed here are actually combined outcomes of the under-reacting trend followers and over-reacting fundamentalists. This leads price to be under-reacted for short lags, over-reacted for medium lags, and mean-reverted for long lags.

The above discussion is based on $\alpha = 1$ (i.e. the fundamentalists are fully confident about their estimated fundamental value). Similar result are observed for $\alpha < 1$ (when the fundamentalists are not fully confident about the fundamental value). Fig. B.3 in Appendix B plots the results for the following set of parameters:

$$(FH): \qquad \alpha = 0.5, \gamma = 0.8, \mu = 5, \delta = 0.85, \qquad m = -0.9, -0.5, 0, 0.9.$$

In this case, small values of m are close to Hopf boundary and large values of m are close to the flip boundary. As we can see from the AC patterns in Fig. B.3 in Appendix B that, as m increases, AC patterns change from strong under-reaction to under- and over-reaction, and to over-reaction, and then to strong over-reaction.

In all cases, the ACs decay and become insignificant after first few lags (the first 5 lags for under/over-reaction and the first 10 lags for strong reaction). Briefly, activity of the fundamentalists (either high fraction or high speed of price adjustment) are responsible for over-reaction AC patterns and extrapolation from the trend followers are responsible for the under-reaction AC patterns. In addition, a strong under-reaction AC patterns of SDS is in general associated with Hopf bifurcation of the DDS, a strong over-reaction AC pattern is associated with flip bifurcation, and under and over-reaction AC patterns are associated with both types of bifurcation (depending on their dominance). This statistical analysis leads us to some insights into how the AC structure of the SDS are affected by different types of bifurcation of the underlying DDS.

5. Conclusion

The model proposed in this paper introduces a market fractions model with heterogeneous traders in a simple asset-pricing and wealth dynamics framework. It also contributes to the literature by incorporating a realistic trading period, which eliminates the untenable risk-free rate assumption. The relationship between deterministic forces and stochastic elements by focusing on various aspects of financial market behaviour, including market dominance, under and over-reaction, profitability and survivability, and statistical properties, including autocorrelation structure, of the stochastic model is examined. Statistical analysis based on Monte Carlo simulations shows that the long-run behaviour and convergence of the market prices, long (short)-run profitability of the fundamental (trend following) trading strategy, survivability of chartists can be characterized by the dynamics of the underlying deterministic system. In particular, we show that various under and over-reaction autocorrelation patterns of returns can be characterized by the bifurcation nature of the deterministic system. Such analysis helps us to understand potential sources of generating realistic time series properties.

As one of the stylized facts, long-range dependence in volatility (i.e., hyperbolic decline of its autocorrelation function) has been focused in recent literature and we refer to Lux (2004) for an extensive survey on empirical evidence, models and mechanisms in financial power laws. Based on our understanding from this paper, it is interesting to know that if our model has a potential to generate realistic long-rang dependence in volatility. In fact, this issue is addressed in our separate paper He and Li (2004). It shows the model does have mechanism to generate realistic long-memory feature. The analysis is based on Monte Carlo simulations and estimates of GARCH and FIGARCH effects.

As we have seen that it is interesting and important to see how the deterministic dynamics and noise interact each other. Theoretical understanding on the connections between certain time series properties of the stochastic system and its underlying deterministic dynamics is important but difficult, and statistical analysis based on various econometric tools seems necessary. It is worth emphasizing that all these interesting qualitative and quantitative features arise from our simple market fraction model with fixed market fraction. The herding mechanism developed in Lux and Marchesi (1999) and the adaptive switching mechanism in Brock and Hommes framework (Brock and Hommes (1997, 1998)) are very important mechanisms in understanding the behaviour of real financial market. It would be interesting to extend our analysis from the current model to a changing fraction model, in which part of the market fractions are governed by herding mechanism and part follows some evolutionary adaptive processes. Taking together the herding and switching mechanisms and the findings in this paper, we hope we can better understand and characterize a large part of the stylized facts of financial data.

APPENDIX A. PROOFS OF PROPOSITIONS

A.1. **Proof of Proposition 3.1.** For $P_t^* = \bar{P}$, the demand function for the fundamentalists becomes

$$z_{1,t} = \frac{(1-\alpha-R)(P_t-\bar{P})}{a_1(1+r^2)\sigma_1^2}.$$

Let $(P_t, u_t, v_t) = (P_0, u_0, v_0)$ be the steady state of the system. Then (P_0, u_0, v_0) satisfies

$$P_0 = P_0 + \frac{\mu}{2}\Big[(1+m)\frac{(1-\alpha-R)(P_0-\bar{P})}{a_1(1+r^2)\sigma_1^2} + (1-m)\frac{\gamma(P_0-u_0)-(R-1)(P_0-\bar{P})}{a_2\sigma_1^2(1+r^2+b\,v_0)}\Big], \tag{A.1}$$

$$u_0 = \delta u_0 + (1-\delta)P_0, \tag{A.2}$$

$$v_0 = \delta v_0 + \delta(1-\delta)(P_0-u_0)^2. \tag{A.3}$$

One can verify that $(P_0, u_0, v_0) = (\bar{P}, \bar{P}, 0)$ satisfies (A.1)-(A.3); that is the fundamental steady state is one of the steady state of the system (3.1). It follows from (A.2)-(A.3) and $\delta \in [0,1)$ that $P_0 = u_0, v_0 = 0$. This together with (A.1) implies that $P_0 = \bar{P}$. In fact, if $P_0 \neq \bar{P}$, then (A.1) implies that

$$\frac{1+m}{a_1}(1-\alpha-R) + \frac{1-m}{a_2}(1-R) = 0. \tag{A.4}$$

However, since $\alpha \in [0,1], R = 1+r/K > 1$ and $m \in [-1,1]$, equation (A.4) cannot be hold. Therefore the fundamental steady state is the unique steady state of the system.

A.2. **Proof of Proposition 3.2.** For $P_t^* = \bar{P}$ and $m = 1$, equation (3.1) becomes

$$P_{t+1} = P_t - \mu\frac{(R+\alpha-1)(P_t-\bar{P})}{a_1(1+r^2)\sigma_1^2}, \tag{A.5}$$

which can be rewritten as

$$P_{t+1} - \bar{P} = \lambda[P_t - \bar{P}], \tag{A.6}$$

where

$$\lambda \equiv 1 - \mu\frac{R+\alpha-1}{a_1(1+r^2)\sigma_1^2}.$$

Obviously, from (A.6), the fundamental price $\bar{P}$ is globally asymptotically attractive if and only if $|\lambda| < 1$, which in turn is equivalent to $0 < \mu < \mu_o$.

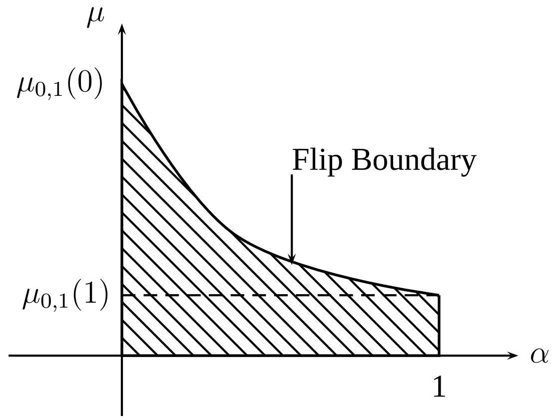


FIGURE A.1. Stability region and bifurcation boundary for $m = 1$.

A.3. **Proof of Propositions 3.3 and 3.4.** For $P_t^* = \bar{P}$, system (3.1) is reduced to the following 3-dimensional difference deterministic system

$$\begin{cases} P_{t+1} = F_1(P_t, u_t, v_t), \\ u_{t+1} = F_2(P_t, u_t, v_t), \\ v_{t+1} = F_3(P_t, u_t, v_t), \end{cases} \tag{A.7}$$

where

$$\begin{aligned} F_1(P, u, v) &= P + \frac{\mu}{2}\Big[(1+m)\frac{(1-\alpha-R)(P-\bar{P})}{a_1(1+r^2)\sigma_1^2} \\ &\qquad + (1-m)\frac{\gamma(P-u)-(R-1)(P-\bar{P})}{a_2\sigma_1^2(1+r^2+b\,v)}\Big], \\ F_2(P, u, v) &= \delta u + (1-\delta)F_1(P, u, v), \\ F_3(P, u, v) &= \delta v + \delta(1-\delta)(F_1-u)^2. \end{aligned}$$

Denote

$$a = \frac{a_2}{a_1}, \qquad Q = 2a_2(1+r^2)\sigma_1^2.$$

At the fundamental steady state $(\bar{P}, \bar{P}, 0)$,

$$\begin{aligned} &\frac{\partial F_1}{\partial P} = A \equiv 1 + \frac{\mu}{Q}[(1+m)a(1-\alpha-R) + (1-m)(1+\gamma-R)], \\ &\frac{\partial F_1}{\partial u} = B \equiv -\frac{\mu\gamma(1-m)}{Q}, \qquad \frac{\partial F_1}{\partial v} = 0; \\ &\frac{\partial F_2}{\partial P} = (1-\delta)A, \qquad \frac{\partial F_2}{\partial u} = C \equiv \delta + (1-\delta)B, \qquad \frac{\partial F_2}{\partial v} = 0; \\ &\frac{\partial F_3}{\partial P} = \frac{\partial F_3}{\partial u} = \frac{\partial F_3}{\partial v} = 0. \end{aligned}$$

Then the Jacobina matrix of the system at the fundamental steady state J is given by

$$J = \begin{pmatrix} A & B & 0 \\ (1-\delta)A & C & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.8}$$

and hence the corresponding characteristic equation becomes

$$\lambda\Gamma(\lambda) = 0,$$

where

$$\Gamma(\lambda) = \lambda^2 - [A + \delta + (1-\delta)B]\lambda + \delta A.$$

It is well known that the fundamental steady state is stable if all three eigenvalues λ_i satisfy $|\lambda_i| < 1$ $(i = 1, 2, 3)$, where $\lambda_3 = 0$ and $\lambda_{1,2}$ solve the equation $\Gamma(\lambda) = 0$.

For $\delta = 0$, $\Gamma(\lambda) = \lambda[\lambda - (A+B)]$. The first result of Proposition 3.3 is then follows from $-1 < \lambda = A + B < 1$ and $\lambda = -1$ when $A + B = 1$.

For $\delta \in (0, 1)$, the fundamental steady state is stable if

(i). $\Gamma(1) > 0$;

(ii). $\Gamma(-1) > 0$;

(iii). $\delta A < 1$.

It can be verified that

(i). For $\alpha \in [0, 1]$, $\Gamma(1) > 0$ holds;

(ii). $\Gamma(-1) > 0$ is equivalent to

$$\text{either} \qquad \gamma \geq \gamma_2 \qquad \text{or} \qquad 0 < \gamma < \gamma_2 \qquad \text{and} \qquad 0 < \mu < \mu_1,$$

where

$$\gamma_2 = \frac{1+\delta}{2\delta}[(R-1) + a(R+\alpha-1)\frac{1+m}{1-m}],$$
$$\mu_1 = \frac{1+\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma_2 - \gamma}.$$

(iii). The condition $\delta A < 1$ is equivalent to

$$\text{either} \qquad \gamma \leq \gamma_1 \qquad \text{or} \qquad \gamma > \gamma_1 \qquad \text{and} \qquad 0 < \mu < \mu_2,$$

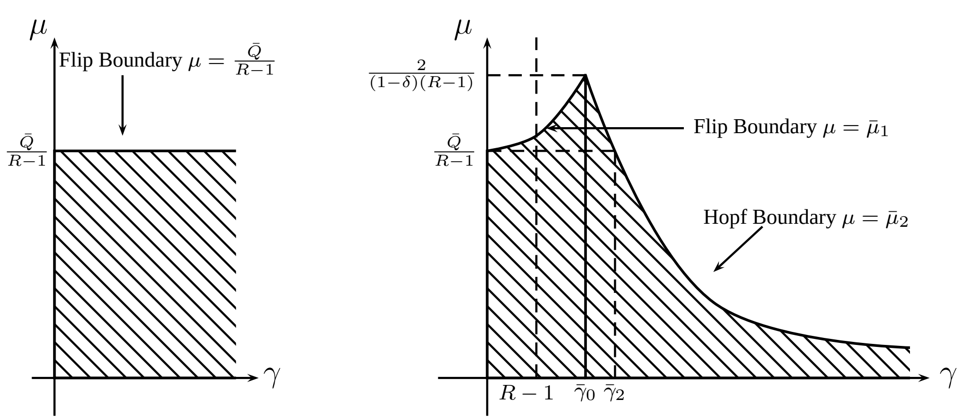
where

$$\gamma_1 = (R-1) + a(R+\alpha-1)\frac{1+m}{1-m},$$
$$\mu_2 = \frac{1-\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma-\gamma_1}.$$

Noting that, for $\delta \in (0,1)$, $\gamma_1 < \gamma_0 < \gamma_2$, where

$$\gamma_0 = \frac{(1+\delta)^2}{4\delta}\left[(R-1) + a(R+\alpha-1)\frac{1+m}{1-m}\right]$$

solves the equation $\mu_1 = \mu_2$. Also, μ_1 is an increasing function of γ for $\gamma < \gamma_2$ while μ_2 is a decreasing function of γ for $\gamma > \gamma_1$. Hence the two conditions for the stability are reduced to $0 < \mu < \mu_1$ for $0 \le \gamma \le \gamma_0$ and $0 \le \mu \le \mu_2$ for $\gamma > \gamma_0$. In addition, the two eigenvalues of $\Gamma(\lambda) = 0$ satisfy $\lambda_1 = -1$ and $\lambda_2 \in (-1,1)$ when $\mu = \mu_1$ and $\lambda_{1,2}$ are complex numbers satisfying $|\lambda_{1,2}| < 1$ when $\mu = \mu_2$. Therefore, a flip bifurcation occurs along the boundary $\mu = \mu_1$ for $0 < \gamma \le \gamma_0$ and a Hopf bifurcation occurs along the boundary $\mu = \mu_2$ for $\gamma \ge \gamma_0$.



(a) $\delta = 0$ (b) $\delta \in (0,1)$

FIGURE A.2. Stability region and bifurcation boundaries for the trend followers and market maker model with $\delta = 0$ (a) and $\delta \in (0,1)$ (b).

APPENDIX B. MONTE CARLO SIMULATIONS AND STATISTICAL RESULTS

Econometric Properties and Statistics of S&P 500 and AOI. In this appendix, we include time series plots on prices and returns for both S&P 500 and AOI (Australian Ordinary Index) from Aug. 10, 1993 to July 24, 2002 in Fig.B.1. The corresponding density distributions, autocorrelation coefficients (ACs) and statistics of the returns are also illustrated in Fig. B.1 and Table B.1.

FIGURE B.1. Time series on prices and returns and density distributions and autocorrelation coefficients (ACs) of the return for S&P 500 (a) and AOI (b) from Aug. 10, 1993 to July 24, 2002.

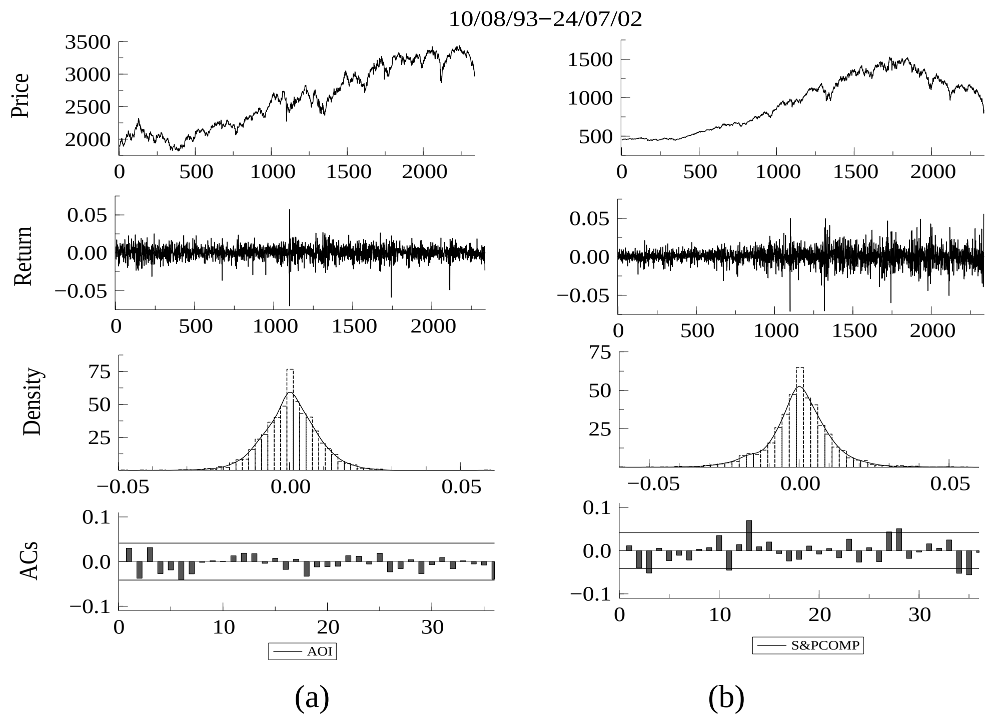


TABLE B.1. Statistics of returns series of for S&P500 and AOI from Aug. 10, 1993 to July 24, 2002.

Index	Mean	Median	Max.	Min.	Std. Dev.	Skew.	Kurt.	Jarque-Bera
S&P500	0.000194	0.0000433	0.057361	-0.070024	0.0083	-0.504638	8.215453	2746.706
AOI	0.000269	0.000106	0.055732	-0.071127	0.010613	-0.23127	7.263339	1789.96

FIGURE B.2. Average Monte Carlo time series of market prices, returns and absolute wealth share of the fundamentalists with $\alpha = 0, \gamma = 1, \mu = 0.4, \delta = 0.85, w_{1,0} = 0.5$, and $m = -0.5$ (top row), 0 (second row), 0.5 (third row), 1(4-th row), -1(the last row).

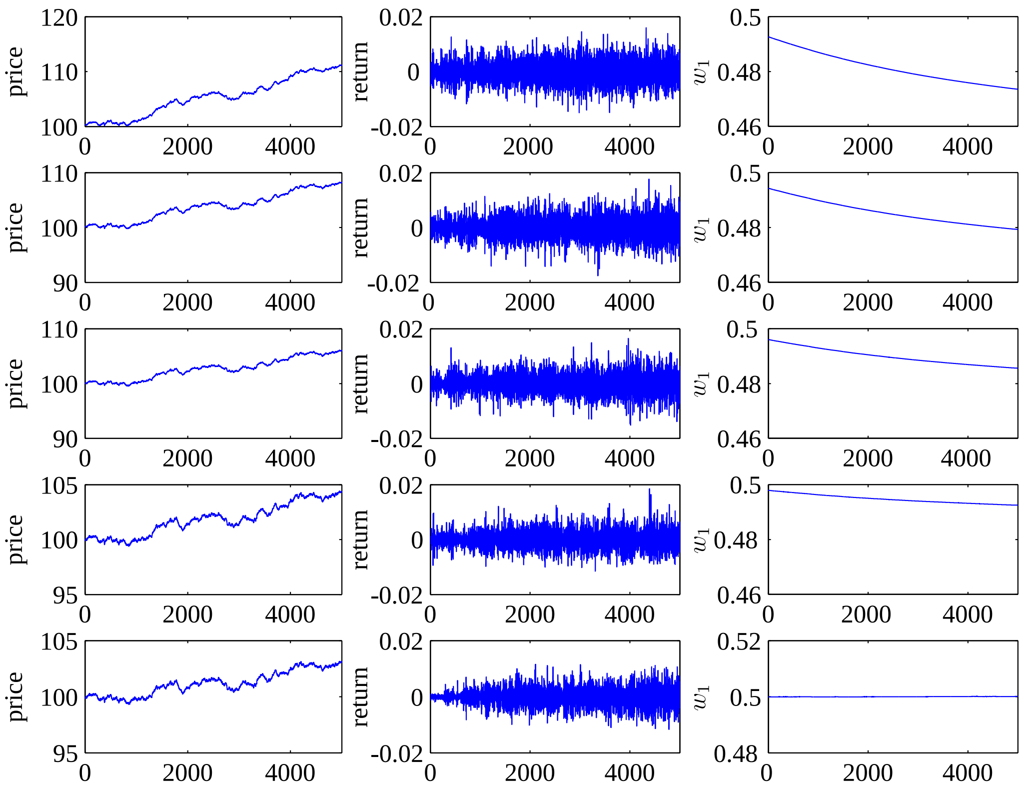


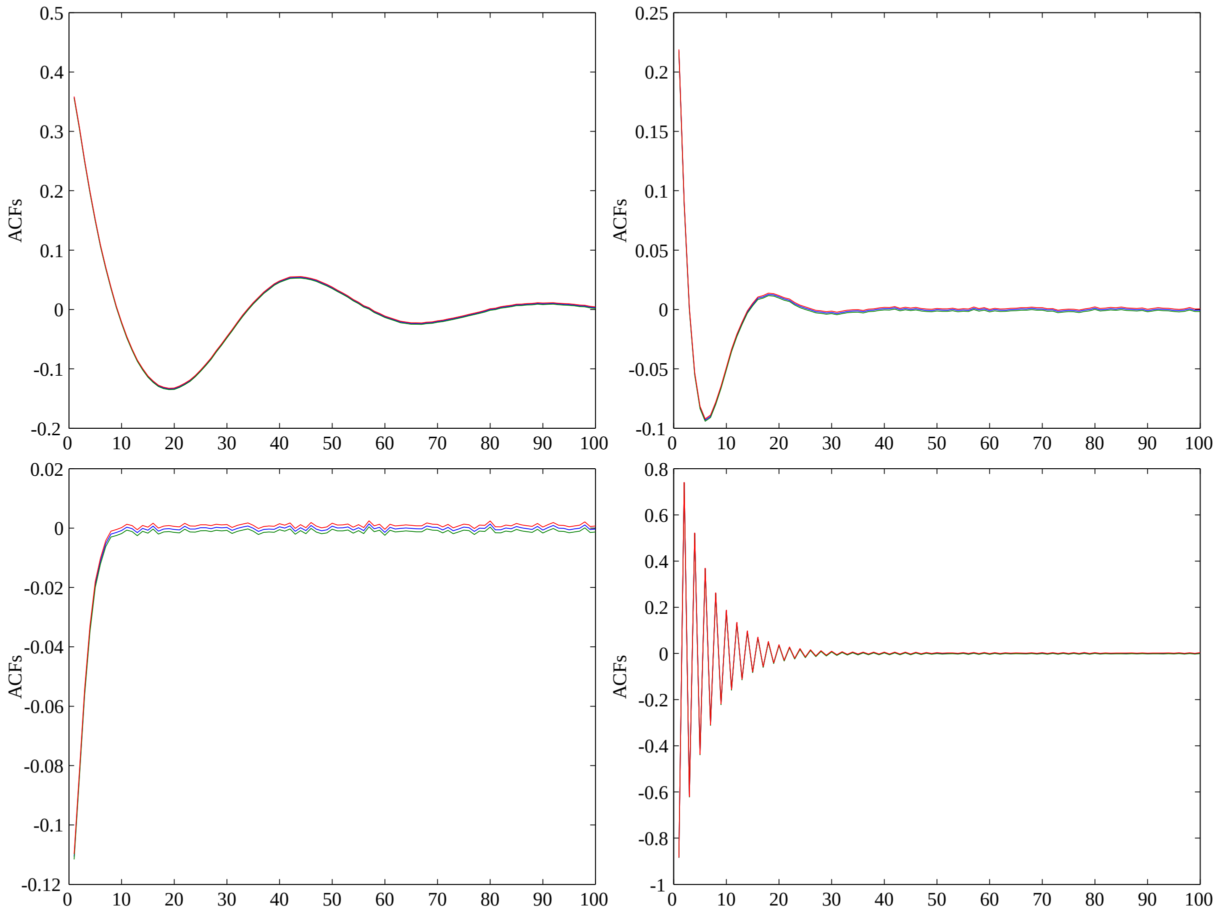
TABLE B.2. Autocorrelations of r_t for the flip-set parameter $(F1)$.

Lag	$m=-0.8$	$m=-0.5$	$m=-0.3$	$m=0$
1	0.2933 (0.0169) 993	-0.0256 (0.0149) 455	-0.3076 (0.0136) 1000	-0.8602 (0.0084) 1000
2	0.1664 (0.0162) 988	-0.0760 (0.0152) 935	-0.0278 (0.0169) 720	0.6939 (0.0161) 1000
3	0.0636 (0.0161) 883	-0.0782 (0.0157) 915	-0.0328 (0.0168) 456	-0.5899 (0.0205) 1000
4	-0.0112 (0.0164) 297	-0.0621 (0.0158) 826	-0.0102 (0.0168) 115	0.5123 (0.0233) 998
5	-0.0630 (0.0168) 868	-0.0420 (0.0158) 625	-0.0058 (0.0167) 79	-0.4528 (0.0250) 986
6	-0.0958 (0.0168) 949	-0.0262 (0.0158) 379	-0.0034 (0.0167) 70	0.4033 (0.0262) 978
7	-0.1116 (0.0169) 968	-0.0134 (0.0158) 163	-0.0014 (0.0167) 72	-0.3631 (0.0269) 969
8	-0.1148 (0.0169) 976	-0.0052 (0.0158) 57	-0.0006 (0.0166) 54	0.3282 (0.0274) 955
9	-0.1102 (0.0169) 966	-0.0015 (0.0159) 58	-0.0010 (0.0167) 53	-0.2981 (0.0278) 934
10	-0.0989 (0.0169) 953	0.0008 (0.0159) 63	-0.0009 (0.0167) 57	0.2712 (0.0280) 916
20	0.0248 (0.0179) 338	-0.0006 (0.0160) 51	-0.0001 (0.0167) 57	0.1188 (0.0278) 690
30	-0.0036 (0.0181) 96	0.0002 (0.0160) 51	0.0002 (0.0167) 54	0.0565 (0.0268) 463
40	-0.0020 (0.0180) 88	0.0005 (0.0160) 39	0.0007 (0.0167) 47	0.0291 (0.0262) 299
50	0.0015 (0.0180) 77	0.0006 (0.0160) 66	0.0009 (0.0167) 56	0.0150 (0.0259) 230
60	-0.0017 (0.0181) 99	-0.0014 (0.0161) 56	-0.0013 (0.0167) 54	0.0059 (0.0259) 218
70	0.0012 (0.0181) 84	0.0003 (0.0161) 54	0.0001 (0.0167) 50	0.0046 (0.0259) 197
80	0.0005 (0.0180) 74	0.0013 (0.0161) 76	0.0014 (0.0167) 64	0.0032 (0.0258) 181
90	-0.0006 (0.0181) 84	-0.0006 (0.0161) 64	-0.0007 (0.0167) 54	0.0016 (0.0259) 184
100	-0.0003 (0.0181) 69	-0.0005 (0.0162) 48	-0.0001 (0.0168) 52	0.0023 (0.0258) 192

TABLE B.3. Autocorrelations of r_t for the Hopf-set parameter $(H1)$.

Lag	$m=-0.95$	$m=-0.5$	$m=0$	$m=0.5$
1	0.0746 (0.0345)	0.1037 (0.0196)	0.0688 (0.0176)	0.0205 (0.0168)
	898	964	582	730
2	0.0825 (0.0326)	0.0802 (0.0189)	0.0429 (0.0174)	0.0064 (0.0169)
	811	868	469	687
3	0.0720 (0.0315)	0.0593 (0.0187)	0.0241 (0.0173)	-0.0020 (0.0170)
	788	672	434	618
4	0.0631 (0.0309)	0.0426 (0.0183)	0.0116 (0.0173)	-0.0059 (0.0171)
	756	493	422	529
5	0.0535 (0.0301)	0.0294 (0.0182)	0.0023 (0.0174)	-0.0079 (0.0171)
	721	380	436	418
6	0.0456 (0.0292)	0.0185 (0.0182)	-0.0050 (0.0173)	-0.0099 (0.0171)
	677	301	398	339
7	0.0388 (0.0288)	0.0107 (0.0180)	-0.0080 (0.0173)	-0.0085 (0.0170)
	587	272	366	244
8	0.0333 (0.0287)	0.0049 (0.0179)	-0.0095 (0.0171)	-0.0068 (0.0170)
	498	257	325	161
9	0.0309 (0.0278)	-0.0009 (0.0178)	-0.0111 (0.0173)	-0.0066 (0.0170)
	433	290	313	154
10	0.0250 (0.0268)	-0.0050 (0.0177)	-0.0116 (0.0172)	-0.0055 (0.0170)
	358	281	245	106
20	0.0021 (0.0230)	-0.0152 (0.0175)	-0.0048 (0.0171)	-0.0012 (0.0170)
	88	228	62	53
30	-0.0035 (0.0215)	-0.0058 (0.0174)	0.0002 (0.0171)	0.0003 (0.0170)
	78	76	53	58
40	-0.0066 (0.0201)	-0.0013 (0.0175)	-0.0003 (0.0172)	-0.0004 (0.0170)
	84	54	50	47
50	-0.0053 (0.0191)	0.0002 (0.0177)	0.0001 (0.0172)	0.0002 (0.0170)
	80	56	63	62
60	-0.0059 (0.0193)	-0.0005 (0.0175)	-0.0012 (0.0172)	-0.0013 (0.0171)
	85	53	60	54
70	-0.0045 (0.0190)	0.0008 (0.0175)	0.0006 (0.0172)	0.0006 (0.0171)
	72	61	59	56
80	-0.0034 (0.0186)	0.0008 (0.0175)	0.0009 (0.0172)	0.0010 (0.0170)
	73	61	61	58
90	-0.0046 (0.0185)	-0.0013 (0.0176)	-0.0008 (0.0172)	-0.0009 (0.0171)
	73	60	65	63
100	-0.0037 (0.0183)	-0.0001 (0.0178)	-0.0002 (0.0173)	-0.0003 (0.0171)
	56	55	50	43

FIGURE B.3. Monte Carlo simulation on the average ACs of return for $m = -0.9$ (top left), -0.5 (top right), 0 (bottom left), 0.9 (bottom right) for parameter set (FH).



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