LONG MEMORY, HETEROGENEITY, AND TREND CHASING

XUE-ZHONG HE* AND YOUWEI LI**

*School of Finance and Economics

University of Technology, Sydney

PO Box 123 Broadway

NSW 2007, Australia

tony.he1@uts.edu.au

and

**School of Management and Economics

Queen's University Belfast

25 University Square

BT7 1NN, Belfast, UK

y.li@qub.ac.uk

ABSTRACT. Long-range dependence in volatility is one of the most prominent examples of applications in financial market research involving universal power laws. Its characterization has recently spurred attempts to provide theoretical explanations of the underlying mechanism. This paper contributes to this recent development by analyzing a simple market fraction asset pricing model with two types of traders—fundamentalists who trade on the price deviation from estimated fundamental value and trend followers who follow a trend which is updated through a geometric learning process. Our analysis shows that the heterogeneity, trend chasing through learning, and the interplay of noisy processes and a stable deterministic equilibrium can be the source of power-law distributed fluctuations. A statistical analysis based on Monte Carlo simulations is conducted to characterize the long memory. Realistic estimates of the power-law decay indices and the (FI)GARCH parameters are presented.

Date: Latest version: June 17, 2005.

Key words and phrases. Asset pricing, fundamentalists and trend followers, market fraction, stability, learning, long memory.

*Acknowledgements: Early part of this research was carried out during a sabbatical visit of He at the University of Amsterdam and Kiel University. We are indebted to Carl Chiarella, Bas Donkers, Cars Hommes, Thomas Lux and Bertrand Melenberg for many stimulating discussions. Financial support from ARC discovery funding is acknowledged.

1. Introduction

It is well known that (high-frequency) financial time series share some common features, the so called stylized facts,[1] including excess volatility (relative to the dividends and underlying cash flows), volatility clustering (high/low fluctuations are followed by high/low fluctuations), skewness, and excess kurtosis. Traditional economic and finance theory involving a representative agent and rational expectations has encountered great difficulties in explaining these facts. This has led to a rapidly increasing number of models incorporating heterogeneous agents and bounded rationality. These models characterize the dynamics of financial asset prices resulting from the interaction of heterogeneous agents having different attitudes to risk and having different expectations about the future evolution of prices.[2] In particular, Brock and Hommes (1997, 1998) proposed an *Adaptive Belief System* model of economic and financial markets. A key aspect of these models is that they exhibit feedback of the expectations—the agents' decisions are based upon predictions of future values of endogenous variables whose actual values are determined by equilibrium equations. The agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance. The resulting nonlinear dynamical system is, as Brock and Hommes (1998) and Hommes (2002) show, capable of generating the entire *zoo* of complex behaviour from local stability to high order cycles and chaos. They are also capable of explaining some of the stylized facts of financial markets. It is very interesting to find that adaption, evolution, heterogeneity, and even learning, can be incorporated into the Brock and Hommes type of framework. This framework can also give rise to many rich and complicated dynamics and might lead to an explanation and understanding of market behaviour.[3]

Among the stylized facts, volatility clustering and the long-range dependence (i.e., hyperbolic decline of its autocorrelation function) has been extensively studied since Ding, Engle and Granger's seminal paper in 1993. Recently, a number of universal power laws[4] have been

[1]See, e.g., Pagan (1996) for a comprehensive discussion of stylized facts characterizing financial time series.

[2]See, e.g., Arthur *et al.* (1997), Brock and Hommes (1997, 2002), Brock and LeBaron (1996), Bullard and Duffy (1999), Chen and Yeh (1997, 2002), Chiarella (1992), Dacorogna *et al.* (1995), Day and Huang (1990), De Long *et al.* (1990), Farmer and Joshi (2002), Frankel and Froot (1987), Iori (2002), LeBaron (2000, 2001, 2002), LeBaron et al (1999), Lux (1995, 1997, 1998) and Lux and Marchesi (1999)).

[3]See, e.g., Chiarella (1992), Chiarella *et al.* (2002), Chiarella and He (2001, 2002, 2003), Gaunersdorfer (2000), Hommes (2001, 2002) and De Grauwe and Grimaldi (2003) and Westerhoff (2003).

[4]They include cubic power distribution of large returns, hyperbolic decline of return autocorrelation function, temporal scaling of trading volume and multi-scaling of higher moments of returns.

found to apply in financial markets. This has spurred attempts at a theoretical explanation and the search for an understanding of the underlying mechanisms responsible for its presence.[5] This paper contributes to the development of this literature.

Various models have been developed to explain the power law behaviour. Among standard textbooks on theoretical and empirical finance, GARCH processes introduced in Engle (1982) model returns as a random process with a time-varying variance which shows autoregressive dependence. These models produce fat tails of the unconditional distribution and capture the short-run dynamics of volatility autocorrelations. However, the implied decay of the volatility autocorrelation is exponential rather than hyperbolic. In addition, the models do not provide an avenue towards an explanation of the empirical regularities.

As a consequence of rational bubble models, multiplicative stochastic processes (with multiplicative and additive stochastic components) have been used to explain the power law behaviour (see Kesten (1973) and Lux (2004)). The power-law exponent can be determined from the distribution of the multiplicative component, not the additive noise components. Unfortunately, as shown by Lux and Sornette (2002), the range of the exponent required for the rational bubble models is very different from the empirical findings. In addition, the rational bubble models share the conceptual problems of economic models with *fully* rational agents.

Herding models of financial markets have been developed to incorporate herding and contagion phenomena.[6] With a stripped down version of an extremely parsimonious herding model with fundamentalists (who trade on observed mispricing) and noise traders (who follow the mood of the market), Alfarano and Lux (2003) show that price changes are generated by either exogenous inflow of new information about fundamentals or endogenous changes in demand and supply via the herding mechanism. The model is able to produce relatively realistic time series for returns whose distributional and temporal characteristics are astonishingly close to the empirical findings. This is partly due to a bi-modal limiting distribution for the fraction of noise traders in the optimistic and pessimistic groups of individuals. It is also in part due to the stochastic nature of the process leading to recurrent switches from one majority to another and the increase in volatility that will last until lock-in reoccurs. It is very interesting to know

[5]We refer to Lux (2004) for a recent survey on empirical evidence, models and mechanisms of various financial power laws.
[6]See Kirman (1991, 1993), Lux (1995, 1997, 1998), Lux and Marchesi (1999), Chen *et al.* (2001), Aoki and Yoshikawa (2002), and Alfarano and Lux (2003).

that the corresponding dynamics of the underlying deterministic model displays back and forth movement through a Hopf bifurcation scenario (see Lux (1995)). This is related to the so-called on-off intermittency in physics. However, with the increase of the population size, the law of large numbers comes into effect and the intermittency and power-law statistics disappear.

As discussed earlier, the Brock and Hommes's framework and its various extensions are capable of explaining various market behaviours and important stylized facts. For example, a mechanism of switching between predictors and co-existing attractors is used in Gaunersdorfer and Hommes (2000) to characterize the volatility clustering. The highly nonlinear deterministic system may exhibit co-existence of different types of attractors and adding noise to the deterministic system may then trigger switches between low- and high-volatility phases. Their numerical simulation shows quite satisfactory statistics between the simulated and actual data. Compared to the herding mechanism, Brock and Hommes's framework allows an infinite population of speculators. However, like most of the analytical heterogeneous agent literature, the comparison with empirical records is mainly based upon visual inspection, or upon a few realizations of the model. A formal investigation of the differences between the time series properties of the heterogeneous agent models and the real world, including the estimation of power law indices, is still lacking.

Overall both the herding and switching models discussed above have shown their potential to explain the power-law behavior. To generate realistic time series, some kind of intermittent dynamics and self-amplification of fluctuations via herding or technical trading are necessary. As pointed out by Lux (2004), *one of the more important problems of these models is the relationship between system size, deterministic forces and stochastic elements.* Herding (and simulation) models suffer from a critical dependence of their *nice* results on the size of agent population, while switching models suffer from a critical dependence on the size of the noise.[7] In this paper, we study the market fraction (MF) model established in He and Li (2004). By examining the relationship between system size, deterministic forces, and stochastic elements, we find that the MF model provides a mechanism to address the power-law behavior and the results do not disappear by a law of large numbers. This is the main contribution of the paper. This

[7]For the switching model of Gaunersdorfer and Hommes (2000), the stochastic movement between the co-existing (locally stable) steady state and limit cycle of the deterministic system is indeed the mechanism in generating realistic time series. However, the noise level has to be adjusted in a way to counterbalance the deterministic core of their market dynamics. Very often, finding co-existence equilibria and the right noise level can be difficult.

mechanism shares the same spirit of the herding and switching mechanisms but in a different and much simpler way.

The MF model is a simple stochastic asset pricing model, involving two types of traders (fundamentalists and trend followers) under a market maker scenario. He and Li (2004) aims to explain various aspects of financial market behaviour and establish connections between the stochastic model and its underlying deterministic system. It shows that the long-run behaviour of asset prices, wealth accumulations of heterogeneous trading strategies, and the autocorrelation structure of the stochastic system can be characterized by the dynamics of the underlying deterministic system, the parameters driving traders' behaviour and the market fraction. In particular, a statistical analysis shows that convergence of market price to fundamental value, long- and short-run profitability of the two trading strategies, survivability of chartists and various under- and over-reaction autocorrelation patterns can be explained by the stability and bifurcations of the underlying deterministic system.

This paper builds on He and Li (2004) and reveals the potential of the MF model to explain some of the stylized facts of financial markets. Focusing on the long memory characteristics, essentially, we show that heterogeneity, trend chasing through learning, and the interplay of a stable deterministic equilibrium and stochastic noisy processes can be the source of power-law distributed fluctuations. This is further verified via a Monte Carlo simulation and a statistical analysis on the decay patterns of autocorrelation functions of returns, squared returns and absolute returns, and the estimates of (FI)GARCH $(1,1)$ parameters.

The remainder of the paper is organized as follows. Section 2 reviews the MF model established in He and Li (2004). Section 3 is devoted to a discussion on the potential generating mechanism of the power-law behavior. In Section 4 we estimate the autocorrelation of returns, squared returns and absolute returns and (FI)GARCH(1,1) parameters for the Standard & Poor 500 (hereafter S&P 500) stock market daily closing price index, which we use to represent the real world. The long memory properties of the market fraction model and comparison with the real world is analyzed in Section 5. Section 6 concludes.

2. THE MARKET FRACTION MODEL

The market fraction (MF) model is a standard discounted value asset pricing model with heterogeneous agents. It is closely related to the framework of Brock and Hommes (1997,

1998) and Chiarella and He (2002). We outline the model and refer the readers to He and Li (2004) for details.

Consider an economy with one risky asset and one risk free asset. It is assumed that the risk free asset is perfectly elastically supplied at gross return of $R = 1 + r/K$, where r stands for a constant risk-free rate per annual and K stands for the trading frequency measured in a year.[8] Let P_t be the price (ex dividend) per share of the risky asset at time t and $\{D_t\}$ be the stochastic dividend process of the risky asset. Then the wealth of a typical investor-h at $t+1$ is given by

$$W_{h,t+1} = RW_{h,t} + [P_{t+1} + D_{t+1} - RP_t]z_{h,t}, \tag{2.1}$$

where $W_{h,t}$ and $z_{h,t}$ are the wealth and the number of shares of the risky asset purchased of investor-h at t, respectively. Let $E_{h,t}$ and $V_{h,t}$ be the "beliefs" of type h traders about the conditional expectation and variance of quantities at $t+1$ based on their information set. Denote by R_{t+1} the excess capital gain on the risky asset at $t+1$, that is

$$R_{t+1} = P_{t+1} + D_{t+1} - R\,P_t. \tag{2.2}$$

Assume that traders have a constant absolute risk aversion (CARA) utility function with the risk aversion coefficient a_h for type h traders (that is $U_h(W) = -\exp(-a_h W)$) and their optimal demand on the risky asset $z_{h,t}$ are determined by maximizing their expected utility of wealth. Then

$$z_{h,t} = \frac{E_{h,t}(R_{t+1})}{a_h V_{h,t}(R_{t+1})}. \tag{2.3}$$

Given the heterogeneity and the nature of asymmetric information among traders, we consider two popular trading strategies corresponding to two types of boundedly rational traders—fundamentalists and trend followers. Assume the market fraction of the fundamentalists and trend followers is n_1 and n_2, respectively. Let $m = n_1 - n_2 \in [-1, 1]$.[9] Assume zero supply of outside shares. Then, using (2.3), the aggregate excess demand per investor $z_{e,t}$ is given by

$$z_{e,t} \equiv n_1 z_{1,t} + n_2 z_{2,t} = \frac{1+m}{2}\frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + \frac{1-m}{2}\frac{E_{2,t}[R_{t+1}]}{a_2 V_{2,t}[R_{t+1}]}. \tag{2.4}$$

[8]Typically, $K = 1, 12, 52$ and 250 for trading period of year, month, week and day, respectively. To calibrate the stylized facts observed from daily price movement in financial market, we select $K = 250$ in our following discussion.

[9]Obviously, $m = 1, -1$ corresponds to the case when all the traders are fundamentalists and trend followers, respectively.

To complete the model, we assume that the market is cleared by a market maker. The role of the market maker is to take a long (when $z_{e,t} < 0$) or short (when $z_{e,t} > 0$) position so as to clear the market. At the end of period t, after the market maker has carried out all transactions, he or she adjusts the price for the next period in the direction of the observed excess demand. Let μ be the speed of price adjustment of the market maker (this can also be interpreted as the market aggregate risk tolerance). To capture unexpected market news or speculators' excess demand, we introduce a noisy demand term $\tilde{\delta}_t$ which is an i.i.d. normally distributed random variable with $\tilde{\delta}_t \sim \mathcal{N}(0, \sigma_\delta^2)$. Based on those assumptions and (2.4), the market price is determined by

$$P_{t+1} = P_t + \frac{\mu}{2}\left[(1+m)\frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + (1-m)\frac{E_{2,t}[R_{t+1}]}{a_1 V_{2,t}[R_{t+1}]}\right] + \tilde{\delta}_t. \tag{2.5}$$

Now we turn to discuss the beliefs of fundamentalists and trend followers.

Fundamentalists—Denote by $F_t = \{P_t, P_{t-1}, \cdots; D_t, D_{t-1}, \cdots\}$ the common information set formed at time t. We assume that, apart from the common information set, the fundamentalists have *superior* information on the fundamental value, P_t^*, of the risky asset which is introduced as an exogenous news arrival process. The fundamentalists also realize the existence of non-fundamental traders, such as trend followers introduced in the following discussion. They believe that the stock price may be driven away from the fundamental value in the short-run, but it will eventually converge to the fundamental value in the long-run. More precisely, we assume that the relative return $(P_{t+1}^*/P_t^* - 1)$ of the fundamental value follows a normal distribution, and hence

$$P_{t+1}^* = P_t^*[1 + \sigma_\epsilon \tilde{\epsilon}_t], \qquad \tilde{\epsilon}_t \sim \mathcal{N}(0,1), \qquad \sigma_\epsilon \geq 0, \qquad P_o^* = \bar{P} > 0, \tag{2.6}$$

where $\tilde{\epsilon}_t$ is independent of the noisy demand process $\tilde{\delta}_t$. This specification ensures that neither fat tails nor volatility clustering are brought about by the exogenous news arrival process. Hence, emergence of any autocorrelation pattern of the return of the risky asset in our later discussion would be driven by the trading process itself, rather than news. We assume the conditional mean and variance of the fundamental traders follow

$$E_{1,t}(P_{t+1}) = P_t + \alpha(P_{t+1}^* - P_t), \qquad V_{1,t}(P_{t+1}) = \sigma_1^2, \tag{2.7}$$

where σ_1^2 stands for a constant variance on the price. Here parameter $\alpha \in [0,1]$ is the speed of price adjustment of the fundamentalist toward the fundamental value. It measures how fast the fundamentalists think the price converges to the fundamental value, and their confidence level in the fundamental value. In particular, for $\alpha = 1$, the fundamental traders are fully confident about the fundamental value and adjust their expected price at next period instantaneously to the fundamental value. For $\alpha = 0$, the fundamentalists become naive traders.

Trend followers—Unlike the fundamental traders, trend followers are technical traders who believe the future price change can be predicted from various patterns or trends generated from the historical price. The trend followers are assumed to extrapolate the latest observed price change over a long-run sample mean price and to adjust their variance estimate accordingly. More precisely, their conditional mean and variance are assumed to satisfy

$$E_{2,t}(P_{t+1}) = P_t + \gamma(P_t - u_t), \qquad V_{2,t}(P_{t+1}) = \sigma_1^2 + b_2 v_t, \tag{2.8}$$

where $\gamma, b_2 \geq 0$ are constants, and u_t and v_t are sample mean and variance, respectively, which may follow some learning processes. The parameter γ measures the extrapolation rate and high (low) values of γ correspond to strong (weak) extrapolation from the trend followers. The coefficient b_2 measures the influence of the sample variance on the conditional variance estimated by the trend followers who believe in more volatile price movements. Various learning schemes can be used to estimate the sample mean u_t and variance v_t. Here we assume that

$$u_t = \delta u_{t-1} + (1-\delta)P_t, \tag{2.9}$$

$$v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2, \tag{2.10}$$

where $\delta \in [0,1]$ is a constant. This process on the sample mean and variance is a limiting process of a *geometric decay process* when the memory lag length tends to infinity. Basically, a geometric decay probability process $(1-\delta)\{1, \delta, \delta^2, \cdots\}$ is associated to the history prices $\{P_t, P_{t-1}, P_{t-2}, \cdots\}$. The parameter δ measures the geometric decay rate.[10] The selection of this process is two fold. First, traders tend to put a high weight to the most recent prices and less weight to the more remote prices when they estimate the sample mean and variance. Secondly, we believe that this geometric decay process may contribute to certain autocorrelation patterns,

[10]For $\delta = 0$, the sample mean $u_t = P_t$, which is the latest observed price, while $\delta = 0.1, 0.5, 0.95$ and 0.999 gives a half life of 0.43 day, 1 day, 2.5 weeks and 2.7 years, respectively.

in particular the long memory feature observed in real financial markets. In addition, it has the mathematical advantage of tractability.

To simplify our calculation, we assume that the dividend process D_t follows $D_t \sim \mathcal{N}(\bar{D}, \sigma_D^2)$, the expected long-run fundamental value $\bar{P} = (R-1)\bar{D}$, and the unconditional variances of the price and dividend over the trading period are related by $\sigma_D^2 = q\sigma_1^2$.[11] Based on (2.7),

$$E_{1,t}(R_{t+1}) = \alpha(P^*_{t+1} - P_t) - (R-1)(P_t - \bar{P}), \qquad V_{1,t}(R_{t+1}) = (1+q)\sigma_1^2$$

and hence the optimal demand for the fundamentalist is given by

$$z_{1,t} = \frac{1}{a_1(1+q)\sigma_1^2}[\alpha(P^*_{t+1} - P_t) - (R-1)(P_t - \bar{P})]. \tag{2.11}$$

In particular, when $P^*_t = \bar{P}$,

$$z_{1,t} = \frac{(\alpha + R - 1)(\bar{P} - P_t)}{a_1(1+q)\sigma_1^2}. \tag{2.12}$$

Similarly, from (2.8),

$$E_{2,t}(R_{t+1}) = P_t + \gamma(P_t - u_t) + \bar{D} - R\,P_t = \gamma(P_t - u_t) - (R-1)(P_t - \bar{P}),$$

$$V_{2,t}(R_{t+1}) = \sigma_1^2(1 + q + b\,v_t),$$

where $b = b_2/\sigma_1^2$. Hence the optimal demand of the trend followers is given by

$$z_{2,t} = \frac{\gamma(P_t - u_t) - (R-1)(P_t - \bar{P})}{a_2\sigma_1^2(1 + q + b\,v_t)}. \tag{2.13}$$

[11] In this paper, we choose $\sigma_1^2 = (\bar{P}\sigma)^2/K$ and $q = r^2$. This can be justified as follows. Let $\sigma\bar{P}$ be the annual volatility of P_t and $\bar{D}_t = rP_t$ be the annual dividend. Then the annual variance of the dividend $\bar{\sigma}_D^2 = r^2(\bar{P}\sigma)^2$. Therefore $\sigma_D^2 = \bar{\sigma}_D^2/K = r^2(\bar{P}\sigma)^2/K = r^2\sigma_1^2$. For all numerical simulations in this paper, we choose $\bar{P} = \$100, r = 5\%$ p.a. $\sigma = 20\%$ p.a., $K = 250$. Correspondingly, $R = 1 + 0.05/250 = 1.0002, \sigma_1^2 = (100 \times 0.2)^2/250 = 8/5$ and $\sigma_D^2 = 1/250$.

Subsisting (2.11) and (2.13) into (2.5), the price dynamics under a market maker is determined by the following 4-dimensional stochastic difference system

$$\begin{cases} P_{t+1} = P_t + \dfrac{\mu}{2}\Bigg[\dfrac{1+m}{a_1(1+q)\sigma_1^2}[\alpha(P^*_{t+1}-P_t)-(R-1)(P_t-\bar{P})] \\ \qquad\qquad + (1-m)\dfrac{\gamma(P_t-u_t)-(R-1)(P_t-\bar{P})}{a_2\sigma_1^2(1+q+b\,v_t)}\Bigg] + \tilde{\delta}_t, \\ u_t = \delta u_{t-1} + (1-\delta)P_t, \\ v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2, \\ P^*_{t+1} = P^*_t[1+\sigma_\epsilon\tilde{\epsilon}_t]. \end{cases} \tag{2.14}$$

The price dynamics and statistical properties of the stochastic model (2.14) have been studied in He and Li (2004) by using Monte Carlo simulation and statistical analysis. It is found that the long-run behaviour and convergence of the market prices, long (short)-run profitability of the fundamental (trend following) trading strategy, survivability of chartists, and various under and over-reaction autocorrelation patterns of returns can be characterized by the stability and bifurcations of the underlying deterministic system. The analysis provides some insights into the generating mechanism on various market behaviours (such as under/over-reactions), market dominance and stylized facts in high frequency financial markets. In the following discussion, we investigate the potential of the model to explain the long memory behavior by examining the autocorrelation pattern under different noise structures and by estimating the decay indices and (FI)GARCH parameters.

3. A Mechanism Analysis on Volatility Clustering and Long Memory

We now proceed with an analysis on the mechanism of volatility dynamics of the MF model. The analysis is conducted to explore possible sources of volatility fluctuations. In doing so, we provide some insights into the interplay between system size, deterministic forces and stochastic elements, in particular, the potential mechanism in generating realistic time series properties.

Aside from the parameters stated before, the parameters used for simulations are given by Table 3.1 with $m = 0$, $n_1 = n_2 = 0.5$. The volatility σ_ϵ of the fundamental price corresponds

TABLE 3.1. Parameter settings and initial values

α	γ	a_1	a_2	μ	m	δ	b	σ_ϵ	σ_δ	P_t	P_0^*
0.1	0.3	0.8	0.8	2	0	0.85	1	0.01265	1	100	100

to an annual volatility of 20% (hence $\sigma_\epsilon = (20/\sqrt{K})\%$ with $K = 250$) and the volatility of the noisy demand $\sigma_\delta = 1$, which is about 1% of the average fundamental price level $\bar{P} = \$100$.

Following from the stability and bifurcation analysis in He and Li (2004), the constant steady state fundamental price $\bar{P}$ of the corresponding deterministic system is asymptotically stable. To see how the price dynamics, in particular, the autocorrelation patterns of returns, are affected under different noisy processes, we consider four cases listed in Tabel 3.2. Case-00 corresponds

TABLE 3.2. Four Cases of the noisy effect

Cases	Case-00	Case-01	Case-10	Case-11
$(\sigma_\delta, \sigma_\epsilon)$	$(0, 0)$	$(0, 0.01265)$	$(1, 0)$	$(1, 0.01265)$

to the deterministic case. Case-01 (Case-10) corresponds to the case with noisy fundamental price (noisy excess demand) only and both noisy processes appear in Case-11.

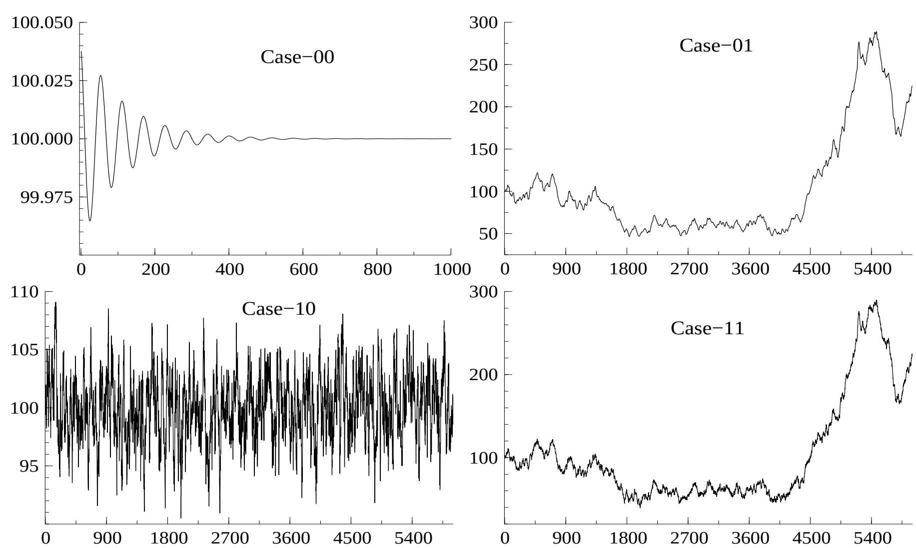


FIGURE 3.1. Time series of prices for four cases.

Fig.3.1 illustrates the price time series for the four cases. The corresponding time series and density distributions of the returns are given in Fig.3.2 for the three noisy cases. Fig.3.3 shows the ACs of returns, the absolute and squared returns. For comparison, a same set of noisy demand and fundamental processes is used for Case-11. Each simulation runs 6,000 time periods and the first 1,000 is dropped to wash out the initial effect for the estimates of density and ACs of returns to make the estimates robust.[12]

[12]Robust here means that the estimates of the density distributions and ACs are independent from the initial conditions. In fact, numerical simulations show that, for each of the three noisy cases, prices converge to an invariant distribution which characterizes the so-called stable random fixed point of the stochastic system in the random dynamics literature. Further discussion on the statistical analysis and test on the convergence of the market price to the fundamental value is given in He and Li (2004).

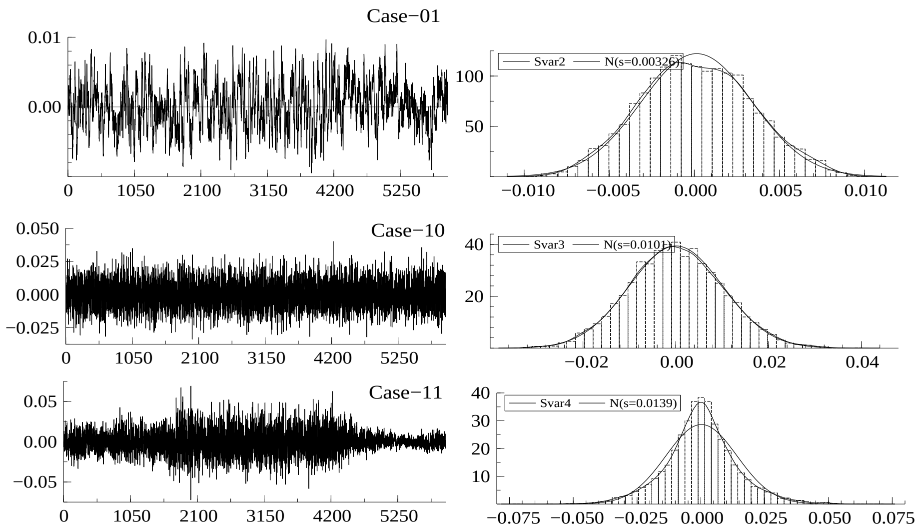


FIGURE 3.2. Time series and density distributions of the returns of Cases-01, 10 and 11.

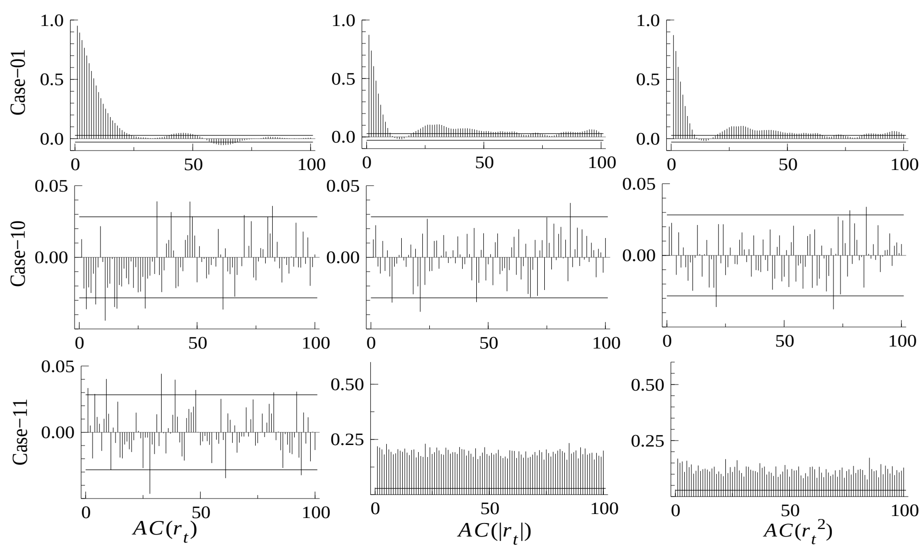


FIGURE 3.3. The ACs of returns (left column), the absolute returns (middle column), and the squared returns (right column) of Cases-01, 10 and 11.

Both Figs.3.2 and 3.3 show significantly different impacts of different noisy processes on the market return volatility. (i) For Case-01, the stochastic fundamental price process is the only noisy process.[13] In this case, the market price displays a *strong under-reaction*[14] AC pattern on returns, which is characterized by the significantly positive decaying ACs shown in the top left panel in Fig.3.3. This significant AC pattern is so strong and even carried forward to the

[13]For comparison, we include the fundamental price process used for our simulation in Fig.A.1 in Appendix A, which gives time series of price and return, return distribution density (compared with the normal distribution), and the ACs of returns, the absolute and squared returns.

[14]See He and Li (2004) for more detailed analysis on the generating mechanism for various under- and over-reaction AC patterns.

AC patterns for the absolute and squared returns. (ii) For Case-10, the noisy excess demand is the only noisy process. In this case, the market price displays no volatility clustering, which is characterized by insignificant AC patterns for return, the absolute and squared returns shown in the middle row in Fig.3.3. (iii) For Case-11, both the noisy excess demand and fundamental price processes appear. In this case, we observe the relatively high kurtosis in Fig.3.2 and insignificant ACs for returns, but significant ACs for both absolute and squared returns shown in the bottom panel in Fig.3.3.

The simple MF model appears to do the job of generating appropriate power laws for returns and volatility when both noisy processes are present. What is the reason for this outcome? Let us start our analysis with the dynamics of the underlying deterministic system. He and Li (2004) show that, for the deterministic system, a stable steady state, which is the constant fundamental value, can become unstable through either a flip or a Hopf bifurcation. Furthermore, the flip bifurcation is mainly due to the strong price adjustment of the fundamentalists towards the fundamental value, while the Hopf bifurcation is mainly due to the strong extrapolation of the trend followers towards the trend which itself follows a geometric decay learning process. For the chosen set of parameters, the Hopf bifurcation value, in terms of the extrapolation parameter γ of the trend followers, is given by $\bar{\gamma} \approx 0.32684$. In other words, the linearized deterministic system has a pair of complex eigenvalues $\lambda_{\pm}$ satisfying $|\lambda_{\pm}| < 1$ for $\gamma < \bar{\gamma}$ and $|\lambda_{\pm}| = 1$ for $\gamma = \bar{\gamma}$. In our case $\gamma = 0.3$, the solution is oscillating initially but converging to the steady state eventually, which is clearly demonstrated by the price series for Case-00 in Fig.3.1. Intuitively, the nature of the oscillating convergence to the steady state is due to the extrapolation and learning of the trend followers. The trend they are trying to learn follows a geometric probability process (with decay parameter δ) and this learning process is updated every time based upon historical price. As a result, the learning process smooths the price and leads to a lagged reaction to the market price. It is this *lagged learning* (on the fundamental value) that plays an important role for the dependent volatility.

We now turn to Case-01. When the fundamental price fluctuates stochastically, it leads to recurrent shifts of the fundamental values to different levels. When the shifting is so often to leave the trend followers not enough time to learn the true fundamental value, the lagged learning from the trend followers leads to a highly dependent volatility (measured by the absolute and squared returns) over the short-run and this is clearly demonstrated by the strong under-reaction AC pattern on returns in the top row in Fig.3.3. However, the lagged learning does not prevent trend followers from learning the constant fundamental values when the market price is perturbed by a small noisy excess demand with mean value zero, which is the Case-10. Consequently, the return distribution in the middle row in Fig.3.2 is close to normal and there is no significant AC patterns for return, the absolute return and squared return shown in the middle row in Fig.3.3.

When both the fundamental price and excess demand noisy processes are present, which is Case-11, the stochastic nature from the noisy excess demand and the weak extrapolation from the trend followers prevents the market price from forming any significant trend, leading to no significant AC pattern for returns. However, the volatility fluctuations due to the lagged learning from the trend followers are maintained. Because of the stochastic nature of the noisy excess demand, the strong AC patterns of the absolute and squared returns shown in Case-01 are washed out, but remain highly significant, which is demonstrated in the bottom row in Fig.3.3. Comparing the AC patterns from real financial data, such as the S&P 500 which we present in the following section, the volatility fluctuations characterized by our simulated data is very close to what we have observed in the actual data. It is worth emphasizing that neither one of the two noisy processes alone is responsible for this realistic feature.

Overall, we see that the interaction of speculators, trend chasing through learning, and the interplay of noise and a stable deterministic equilibrium can be a source of long-memory behaviour. Our analysis allows us to gain some insights into the origin of this realistic dynamic behavior. Basically, the system is characterized by a continuum of equilibria with a market price which fluctuates around and (on an average) equals the fundamental value (due to the fundamentalists), lagged learning and trend chasing (due to the trend followers), and balanced noise level (from the excess demand). Because neither group has an advantage in a situation where no arbitrage opportunities exist ($P = P^*$) and no deviations from the equilibrium price are expected (when there is no excess noisy demand), the system moves in an erratic manner along its continuum of equilibria. This mechanism shares the same spirt of Lux and Marchesi's (1999) herding model.

In principle, different types of dynamics could be the source of power-law distributed fluctuations as we have discussed in our introduction. Based on our analysis, it appears that the dynamics near the Hopf bifurcation boundary (or surface) plays an important role in this aspect. Of course, such Hopf behaviour can be generated by many mechanisms including herding (e.g. Lux (1995)) and adaptive switching (e.g. Brock and Hommes (1997)). Also the interplay of noise and dynamics of the deterministic system plays a crucial role. In particular, the size of the noisy process is a very subtle issue. For the herding mechanism in Lux and Marchesi (2001), a balanced disposition among noise traders is necessary. For the switching mechanism in Gaunersdorfer and Hommes (2000), the noisy component added to the excess demand is responsible for the switching between locally co-existing attractors, and hence the noisy level has to be large to obtain their realistic results. In our model, the distributed fluctuations due to the lagged learning and weak extrapolation from the trend followers needs to be balanced to the noisy level of the excess demand. At this stage, a theoretical analysis on the interplay of deterministic dynamics and noise seems difficult.

In the following discussion, we adopt statistical methods based on Monte Carlo simulation to estimate various models related to the long memory characterization. The estimates are for both the MF model and the S&P 500. We use the estimates for S&P 500 to represent the real world and then compare to those from the MF model.

4. Empirical Evidence and Long Memory Behaviour of S&P 500

As an empirical evidence and a benchmark for our comparison, this section provides a brief statistical analysis of the S&P 500 price index[15]. There are altogether 5306 observations from Oct 20, 1982 to Oct 27, 2003. Denote p_t as the price index for S&P 500 at time t ($t = 0, ..., 5305$) and log returns r_t are defined as $r_t = lnp_t - lnp_{t-1}$.

4.1. **Statistics and Autocorrelations of Returns.** Table 4.1 gives the summary statistics for r_t. We can see from Table 4.1 that the kurtosis for r_t, 44.76, is much higher than that of a normal distribution which is 3. The kurtosis and studentized range statistics (which is the range divided by the standard deviation) show the characteristics fat-tailed behavior compared with a normal distribution. The Jarque-Bera normality test statistic is far beyond the critical value which suggests that r_t is far from a normal distribution.

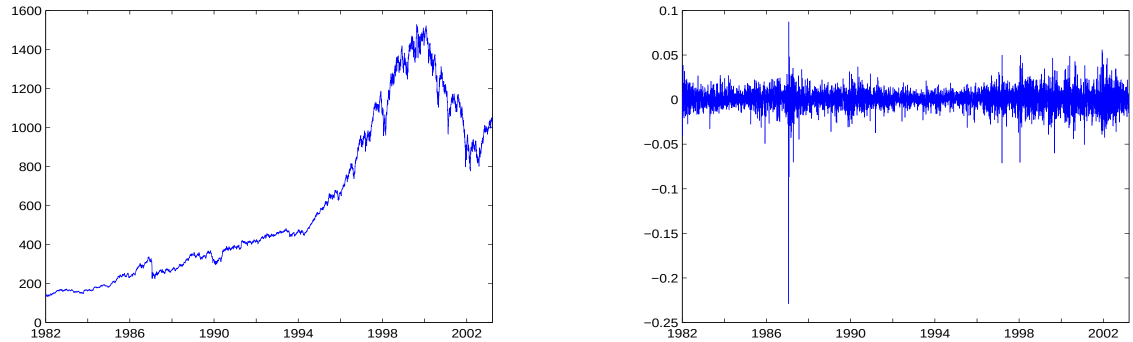
Figure 4.1 gives the plots of p_t, r_t. We can see that there is a clear trend for p_t but r_t is rather stable. The large absolute returns are more likely than small absolute returns to be followed

[15] We get the data from http://finance.yahoo.com.

TABLE 4.1. Summary statistics of r_t.

data	sample size	mean	std	skewness	kurtosis	min	max	studentized range	Jarque-Bera
r_t	5305	0.00037	0.0108	-1.933	44.76	-0.229	0.087	29.16	388510

by a large absolute return. The market volatility is changing over time which suggests that a suitable model for the data should have a time varying volatility structure as suggested by the ARCH model.



(a) (b)

FIGURE 4.1. Time series on prices (a) and log returns (b) of S&P 500 from Oct 20, 1982 to Oct 27, 2003.

A well known stylized fact of the stock return is that the returns themselves contain little serial correlation, but the squared returns r_t^2 and absolute return $|r_t|$ do have significantly positive serial correlation over long lags. For example, Ding, Granger, and Engle (1993) investigate autocorrelations of returns (and their transformations) of the daily S&P 500 index over the period 1928 to 1991 and find that the absolute returns and squared returns tend to have very slow decaying autocorrelations, and further, the sample autocorrelations for the absolute returns are greater than the sample autocorrelations for squared returns at every lag up to at least 100 lags. Table B.1 in Appendix B reports the autocorrelation coefficients for the returns, squared returns, and absolute returns and their corresponding confidence intervals, which are constructed by using the Newey-West corrected standard error. The autocorrelations are plotted in Figure 4.2, where the lines from the bottom to the top are the autocorrelation coefficients for the returns, squared returns, and absolute returns respectively. These results coincide with the findings in Ding, Granger and Engle (1993).

4.2. **Estimates of Power-Law Decay Index via ARFIMA.** Besides the visual inspection of autocorrelations of r_t, r_t^2 and $|r_t|$ for the S&P 500, one can also construct models to estimate the decay rate of the autocorrelations of r_t, r_t^2 and $|r_t|$. For instance, we consider the simple

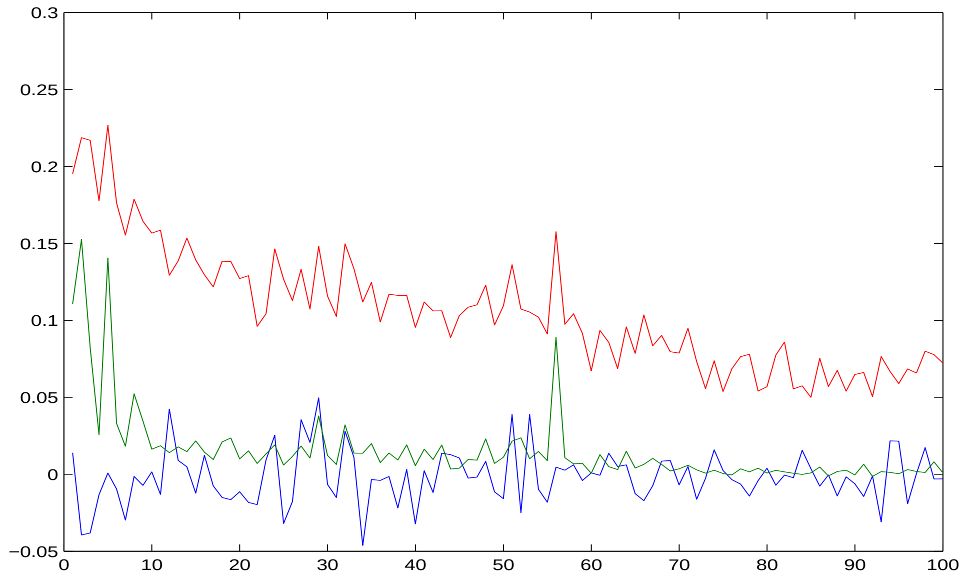


FIGURE 4.2. The autocorrelations of r_t, r_t^2 and $|r_t|$ for S&P 500.

ARFIMA$(0, d, 0)$ process (for example, see the review paper by Baillie(1996))

$$(1 - L)^d x_t = \varepsilon_t, \quad \varepsilon_t \sim NID(0, \sigma^2), \tag{4.1}$$

where L is the lag operator, and d is the order of integration. For $d = 0$, x_t is simply white noise and its autocorrelation function exhibits an exponential decay, whereas for $d = 1$, x_t is a random walk and hence has an autocorrelation function that remains at unity. For non-integer values of d, the autocorrelation function of x_t declines hyperbolically to zero. To be precise, the autocorrelations are given by

$$\rho_k = Ck^{2d-1},$$

where C is a constant, so the hyperbolic decay index $\mu \equiv 2d - 1$ depends upon d. For the daily return, absolute return, and squared return of the S&P 500, we estimate the ARFIMA$(0, d, 0)$ model; the estimates of parameter d are summarized in Table 4.2.

TABLE 4.2. Estimates of d for S&P 500

	d	Std.	P-value	95% CI
r	-0.0192	0.0112	0.086	[-0.0410, 0.0027]
r^2	0.1233	0.0102	0.000	[0.1033, 0.1433]
$\lvert r \rvert$	0.1762	0.0085	0.000	[0.1594, 0.1931]

We see that the results do provide evidence of long persistence for squared returns and absolute returns. It seems that the estimated d is not significant for the daily returns: We cannot reject the null hypothesis that d is zero. But it is significant for the absolute returns and squared returns, and the persistence in absolute returns is much stronger than that in squared returns. These results coincide with the well-established findings in the empirical finance literature.

4.3. **Volatility Clustering, Long Memory and (FI)GARCH Estimates.** Another striking feature of the return series observed from Figure 4.1 is *volatility clustering*. A lot of econometric models of changing conditional variance have been developed to test and measure the volatility clustering. The most widely used one is the family of ARCH (Autoregressive Conditionally Heteroskedastic) models introduced by Engle (1982) and its generalization, the GARCH model, introduced by Bollerslev (1986). Following their specification, for instance, if we model the returns as an AR(1) process, then a GARCH(p, q) model is defined by:

$$\begin{cases} r_t = a + b r_{t-1} + \varepsilon_t, & \varepsilon_t = \sigma_t z_t, \\ \sigma_t^2 = \alpha_0 + \alpha(L)\varepsilon_t^2 + \beta(L)\sigma_t^2, & z_t \sim N(0,1), \end{cases} \tag{4.2}$$

where L is the lag operator, $\alpha(L) = \sum_{i=1}^{q} \alpha_i L^i$ and $\beta(L) = \sum_{j=1}^{p} \beta_i L^j$. Defining $v_t = \varepsilon_t^2 - \sigma_t^2$, the process can be rewritten as an ARMA(m, p) process

$$[1 - \alpha(L) - \beta(L)]\varepsilon_t^2 = \alpha_0 + [1 - \beta(L)]v_t \tag{4.3}$$

with $m = \max\{p, q\}$. Table 4.3 reports the estimates of the GARCH $(1, 1)$ model, where the mean process follows an AR(1) structure. Based on the estimates, one can see that a small

TABLE 4.3. GARCH $(1, 1)$ Parameter Estimates for S&P 500

$a \times 10^3$	b	$\alpha_0 \times 10^5$	α_1	β_1
0.608	0.0359	0.113	0.0783	0.9145
(0.125)	(0.014)	(0.059)	(0.0304)	(0.0305)

Note: The numbers in parentheses are standard errors.

influence of the most recent innovation ($\alpha_1 < 0.1$) is accompanied by a strong persistence of the variance coefficient ($\beta_1 > 0.9$). It is also interesting to observe that the sum of the coefficients $\alpha_1 + \beta_1$ is close to one, i.e., the process is close to an integrated GARCH (IGARCH) process. A related literature (e.g. Pagan (1996)) shows that such parameter estimates are rather common when considering returns from high frequency daily financial data of both share and foreign exchange markets. However, GARCH implies that shocks to the conditional variance decay exponentially, and IGARCH implies that the shocks to the conditional variance persist indefinitely.

In response to the finding that most of the financial time series are long memory volatility process, Baillie, Bollerslev, and Mikkelson (1996) consider the Fractional Integrated GARCH (FIGARCH) process, where a shock to the conditional variance dies out at a slow hyperbolic rate of decay. Later on, Chung(1999) suggests a slightly different parameterization of the model:

$$\phi(L)(1-L)^d(\varepsilon_t^2-\sigma^2)=\alpha_0+[1-\beta(L)]v_t, \tag{4.4}$$

where $\phi(L)=1-\sum_{i=1}^{q}\phi_i L^i$, $\alpha_0=\phi(L)(1-L)^d\sigma^2$, and σ^2 is the unconditional variance of the corresponding GARCH model. Table 4.4 reports the estimates of the FIGARCH $(1,d,1)$ model, where the mean process follows an AR(1) model. The estimate for the fractional differencing parameter $\hat{d}$ is statistically very different from both zero and one. This is consistent with the well known findings that the shocks to the conditional variance dies out at a slow hyperbolic rate.

TABLE 4.4. FIGARCH $(1,d,1)$ Parameter Estimates for S&P 500

a	b	$\alpha_0 \times 10^4$	d	ϕ_1	β
-0.0258	0.0166	0.000017	0.3933	0.1012	0.7968
(0.00039)	(0.0083)	(0.1930)	(0.0091)	(0.0116)	(0.0035)

Note: The numbers in parentheses are standard errors.

5. Econometric Characterization of The Long Memory Properties of The MF Model

This section is devoted to an econometric analysis on the power-law behaviour and the volatility persistence of the MF model. Targeted for the results that we obtained in Section 4 for the S&P 500, various models are estimated using the MF model-generated data outlined in Section 3, and subsequently, these estimates are compared with those of the S&P 500 to see how close we are to the real world. The analysis and estimates are based on Monte Carlo simulations. For a chosen set of parameter and two noisy processes specified in Case-11 in Section 3, we ran 1,000 independent simulations over 6,306 time periods and discard the first 1,000 time periods to wash out the possible initial noise effect. For each run of the model we have 5,306 observations, which matches the sample size of S&P 500 that we used in previous section.

5.1. **Autocorrelations of Returns.** First, we look at the autocorrelation coefficients of returns, squared returns and absolute returns. It is interesting to see whether our simulation model can replicate the well known findings as described in Figure 4.2. By running 1,000 independent simulations, we estimate the autocorrelation coefficients and calculate Newey-West corrected standard errors of returns, squared returns and absolute returns for each run of the model, and then we take the average. The results for returns, squared returns and absolute returns are reported in Table B.2, B.3 and B.4 in Appendix B, respectively. We also plot the autocorrelation coefficients and their corresponding confidence interval in Figure 5.1.

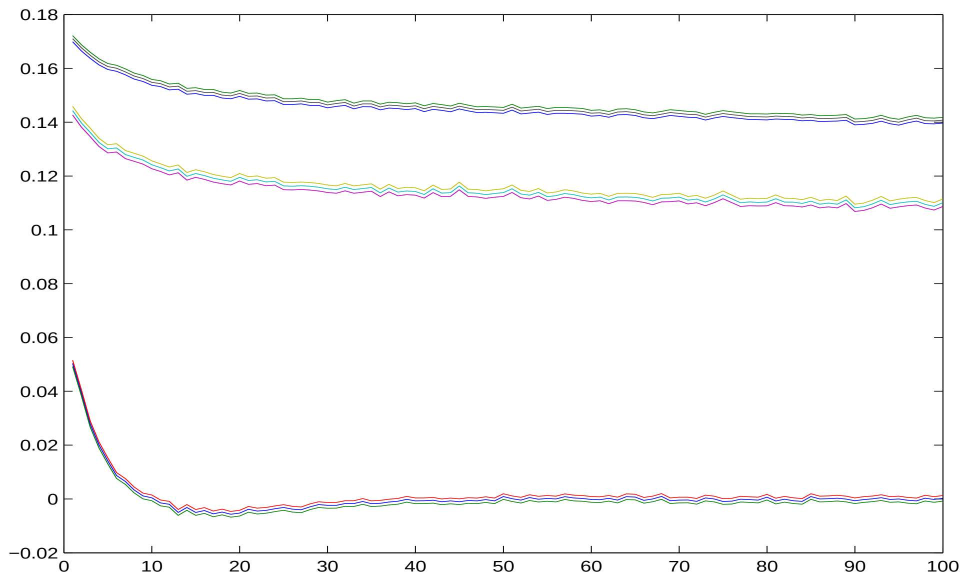


FIGURE 5.1. The autocorrelations of r_t, r_t^2 and $|r_t|$ for the MF model.

From Figure 5.1, we see that for the market fraction model, not only the sample correlations of r_t^2 and $|r_t|$ are all outside the 95% confidence interval of r_t but they also are all positive over long lags. Further, the sample autocorrelations for absolute returns are greater than the sample autocorrelations for squared returns at every lag up to at least 100 lags. Comparing to Figure 4.2 for the S&P 500, we see that the patterns of decay of the autocorrelation functions of return, squared return and absolute return are quite similar.

5.2. **Estimates of Power-Law Decay Index via ARFIMA.** We also look at the decay rate of the autocorrelations of returns, squared returns, and absolute returns that are estimated from the ARFIMA$(0, d, 0)$ model. The resulting estimates are reported in Table 5.1, where the second last column indicates the numbers that the corresponding estimates are significant at the 5% level over 1,000 independent simulations. Comparing with the estimated results of the S&P 500 in Table 4.2, we find that in both cases the estimate of d for returns is not significant.

There is a clear evidence of long memory for squared returns and absolute returns, and also the patterns of the estimates of d for the returns, squared returns, and absolute returns are the same.

TABLE 5.1. Estimates of d for the MF model

	d	Std.	P-value	95% CI	No. Sig
r	0.0341	0.0113	0.1684	[0.0334, 0.0348]	382
r^2	0.1381	0.0083	0.000	[0.1375,0.1386]	1000
$\|r\|$	0.1454	0.0081	0.000	[0.1449, 0.1459]	1000

5.3. **Volatility Clustering, Long Memory and (FI)GARCH Estimates.** We now check the ARCH/GARCH effects. We want to see whether the MF model is capable of capturing the feature of volatility clustering. We implement the test suggested by Engle (1982). The null hypothesis is that the residuals of a regression model are i.i.d. and the alternative hypothesis is that the errors are ARCH(q). Suppose the stock returns follow an AR(1) process with innovations ε_t. If the returns are homoskedastic, then the variance cannot be predicted and the variations in ε_t^2 will be purely random. However, if ARCH effects are present, large values of ε_t^2 will be predicted by large values of the past squared residuals. This idea leads to a TR^2 test statistic. In order to compute the test statistic, we first fit the returns series with an AR(1) model, and then regress the squared residuals ε_t^2 on a constant and $\varepsilon_{t-1}^2, ..., \varepsilon_{t-q}^2$. R^2 is then computed from this regression. Under the null hypothesis that there is no ARCH, the test statistic is asymptotically distributed as a chi-square distribution with q degrees of freedom. We implement the test for both the S&P 500 and the simulation model. The results are reported in Table 5.2. In both cases, the null hypothesis is strongly rejected. In terms of Engle's test, both the data from the S&P 500 and the MF model do have clear ARCH effects. So, we turn to look at the GARCH estimates, and the FIGARCH estimates which describe the volatility persistence.

TABLE 5.2. Engle's test statistics for the presence of ARCH/GARCH effects

	Lag 1	Lag 2	Lag 5	Lag 100
S&P	72.88	174.26	270.62	342.00
MF	140.79 (987)	228.20 (993)	372.65 (998)	821.32 (999)

Note: The numbers in parentheses are the numbers that the test statistics are significant at 5% level over 1000 independent simulations.

We report the estimates of the GARCH and FIGARCH model in Table 5.3 and Table 5.4, respectively. The specifications of the models are the same as that we estimated for the S&P 500. Again, all these estimates are obtained from the estimation for each run of the simulation model and then averaged over independent simulations. The results from the GARCH model are astonishingly similar to what one usually extracts from real life data: a small influence of the most recent innovation ($\alpha_1 < 0.1$) is accompanied by strong persistence of the variance coefficient ($\beta_1 > 0.9$) and the sum of the coefficients $\alpha_1 + \beta_1 = 0.9928$ is close to one. For the estimates of FIGARCH$(1, d, 1)$, we see that the estimate of d is significantly different from zero and one.

TABLE 5.3. GARCH $(1, 1)$ Parameter Estimates of the MF Model

a	b	$\alpha_0 \times 10^4$	α_1	β
0.000074	0.0725	0.0078	0.0260	0.9738
(0.00023)	(0.0139)	(0.0035)	(0.0032)	(0.0033)
47	771	177	1000	1000

Note: The numbers in parentheses are the standard errors, and the number in the last row are the numbers that the test statistics are significant at 5% level over 1000 independent simulations.

TABLE 5.4. FIGARCH $(1, d, 1)$ Parameter Estimates of the MF Model

a	b	$\alpha_0 \times 10^4$	d	ϕ_1	β
0.0137	0.0769	0.3620	0.3797	0.3439	0.7933
(0.0010)	(0.0195)	(0.6112)	(0.0386)	(0.0281)	(0.0295)
412	726	356	876	831	985

Note: The numbers in parentheses are the standard errors, and the number in the last row are the numbers that the test statistics are significant at 5% level over 1000 independent simulations.

Overall, we find that the MF model do provide a mechanism for the long-range dependence in volatility. Now we turn to assess the differences between the MF model and the real world quantitatively.

5.4. **Comparing the MF model with the Real World.** We use the S&P 500 to represent the real world. Then, we compare the MF model with the real world in terms of the autocorrelation of returns, squared returns and absolute returns, power-law decay index, and (FI)GARCH(1,1) parameters, respectively.

In Figure 4.2 and 5.1, we plot the autocorrelation coefficients of returns, squared returns and absolute returns for the S&P 500 and the MF model respectively. For the purpose of comparison, we combine them together and plot the autocorrelation coefficients and their corresponding confidence interval in Figure 5.2.

For returns, we see from Figure 5.2 (b) that the confidence intervals of the simulation model lies inside the confidence intervals of the S&P 500. However, Figure 5.2 (c) and (d) indicate that

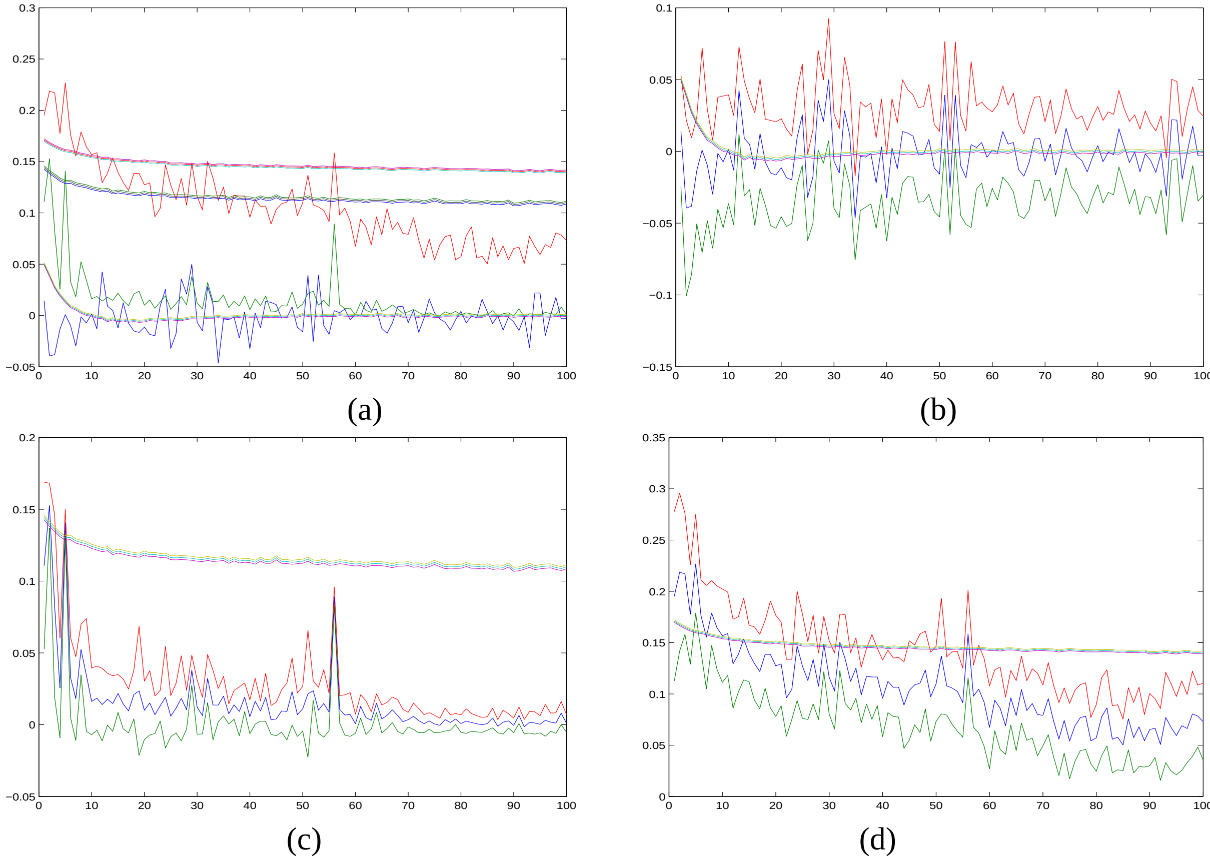


FIGURE 5.2. Autocorrelations of returns, squared returns and absolute returns of the S&P 500 and the MF model (with confidence intervals) (a). Autocorrelations and their confidence intervals of returns (b), squared returns (c), and absolute returns (d).

the speed of decay of the squared return and absolute return from the MF model are different from what we see from the S&P 500, especially for large lags.

For the decay index d for returns, squared returns or the absolute returns, we want to know whether the parameter d of the S&P 500 is the same as that of the MF model, in other words, we want to test $H_0 : d_{S\&P} = d_{MF}$. If we think that $\hat{d}_{S\&P}$ ($\hat{d}_{MF}$) is a good approximation of the true one, then we can check whether $\hat{d}_{MF}$ ($\hat{d}_{S\&P}$) lies in the confidence interval of $\hat{d}_{S\&P}$ ($\hat{d}_{MF}$) or not. Because both of the $d_{S\&P}$ and d_{MF} are estimated, the null hypothesis can be tested by the Wald test by assuming that both the number of simulations and the number of time periods for each simulation go to infinity. In the construction of Wald test

$$W = (\hat{d}_{S\&P} - \hat{d}_{MF})\hat{\Sigma}^{-1}(\hat{d}_{S\&P} - \hat{d}_{MF}),$$

$\hat{\Sigma}$ is simply the sum of sample variance of $\hat{d}_{S\&P}$ and $\hat{d}_{MF}$, because the outcomes of the MF model is statistically independent of the real world. We also notice that the sample variance of

$\hat{d}_{MF}$ is much smaller than that of $\hat{d}_{S\&P}$, this is because d_{MF} is estimated from the simulated data by running the MF independently many times. For a more general discussion on comparing the simulation models and comparing a simulation model with the real world, see Li *et al.* (2004). The resulting test statistics for the returns, the squared returns and the absolute returns are 22.624, 2.1040, and 13.1181 respectively. Noting that the critical value of the Wald test at 5% significant level is 3.84, we find that the null hypothesis for returns and absolute returns are rejected, but it is not rejected in case of the squared return. So, the differences between the estimated d of the S&P 500 and the MF model for returns and absolute returns are statistically significant, but the difference is not significant for the squared returns.

For (FI)GARCH parameters, first, we want to detect the differences between the GARCH estimates in Table 4.3 and 5.3 for the S&P 500 and the MF model respectively. Formally, for the parameter $\theta = (a, b, \alpha_0, \alpha_1, \beta)$, this is to test $H_0 : \theta_{S\&P} = \theta_{MF}$. This hypothesis can be tested again by the Wald test, which can be constructed similarly to that for parameter d. The resulting Wald statistic is 33.8971, which suggests that the null hypothesis is strongly rejected and hence the GARCH$(1, 1)$ estimates of the MF model and that of the S&P 500 are significantly different. Similarly, for the FIGARCH$(1, 1)$ estimates, we can also detect the difference between the estimates of $\vartheta = (a, b, \alpha_0, d, \alpha_1, \beta)$ of the MF model and that of the S&P 500. The null hypothesis becomes $H_0 : \vartheta_{S\&P} = \vartheta_{MF}$. The resulting Wald statistics is 1914, which is far beyond the critical value at any conventional significant level. So the estimates of FIGARCH$(1, d, 1)$ model of MF model are significantly different from that of the S&P 500.

The above analysis indicates that the simple market fraction model is able to replicate the long memory properties of the actual stock market qualitatively. However, the formal statistical tests find that the decay rate and (FI)GARCH estimates from the MF model are difficult to match that of the S&P 500 exactly. This is probably due to the simplicity of the MF model. The long memory mechanism of the MF model is different from either herding (for instance, the mechanism developed in Lux and Marchesi (1999)) or switching mechanisms (for instance, the adaptive switching mechanism in Brock and Hommes (1997, 1998)) in terms of modeling, but it shares the same spirit in a much simpler way. We should notice that it is this simplicity that makes it possible to identify potential sources and mechanisms to generate certain characteristics and this is one of the contributions of this paper.

6. CONCLUSION

Motivated by recent interest in the power law behaviour of high frequency financial market time series and the explanatory power of heterogeneous-agent asset-pricing models, this paper investigates the long memory properties of a simple market fraction model involving two types of traders (fundamentalists and trend followers). Extending earlier work on long-run asset price behaviour, profitability, survivability, various under- and over-reaction AC patterns, and their connections to the underlying deterministic dynamics, we are interested in the characterization of the power law volatility behaviour of the MF model and its comparison with the real world. We found that the heterogeneity, trend chasing through learning, and interplay of noise and stable deterministic equilibria can explain power-law distributed fluctuations.

It is interesting and important to see how the deterministic dynamics and noise interact with each other, and further, to understand the connections between the nonlinear dynamics of the underlying deterministic system and certain time series properties of the corresponding stochastic system. The theoretical analysis is important but difficult given the current state of knowledge. The statistical analysis with powerful econometric tools seems necessary. Based upon Monte Carlo simulations, statistical analysis, including estimates of (FI)GARCH parameters and related tests, shows that the MF model is able to explain some of the characteristics that are well established in the empirical finance literature. There is a clear evidence of long memory and GARCH effects. However, the exact decay rates of autocorrelation functions of returns, squared returns and absolute returns, and (FI)GARCH$(1,1)$ parameters are difficult to match with those of the S&P 500. It is worth emphasizing that all these interesting qualitative and quantitative features arise from a simple model with fixed market fractions. Further investigation and extension of the simple model seem necessary.

It may be interesting to extend the model to a changing fraction model, in which part of the market fractions are governed by herding and another part follows some adaptive switching process. One way to start might be to estimate the model first, and then implement misspecification tests. Econometric methods, such as efficient methods of moments could be used. Allowing for herding and switching mechanisms and these econometric estimation approaches, we may gain a better characterization and understanding of financial markets.

REFERENCES

Alfarano, S. and Lux, T. (2003), *Long memory in economics and econometrics*, Springer, Berlin, chapter A simple model of herd behavior with realistic time series properties. in press.

Aoki, M. and Yoshikawa, H. (2002), 'Demand saturation/creation and economic growth', *Journal of Economic Behavior and Organization* **48**, 127–154.

Arthur, W., Holland, J., LeBaron, B., Palmer, R. and Tayler, P. (1997), 'Asset pricing under endogeneous expectations in an artifical stock market', *Economic Notes* **26**(2), 297–330.

Baillie, R. T. (1996), 'Long memory processes and fractional integration in econometrics', *Journal of Econometrics* **73**, 5–59.

Baillie, R. T., Bollerslev, T. and Mikkelsen, H. (1996), 'Fractionally integrated generalized autoregressive conditional heteroskedasticity', *Journal of Econometrics* **74**, 3–30.

Bollerslev, T. (1986), 'Generalized autoregressive conditional heteroskedasticity', *Journal of Econometrics* **31**, 307–327.

Brock, W. and Hommes, C. (1997), 'A rational route to randomness', *Econometrica* **65**, 1059–1095.

Brock, W. and Hommes, C. (1998), 'Heterogeneous beliefs and routes to chaos in a simple asset pricing model', *Journal of Economic Dynamics and Control* **22**, 1235–1274.

Brock, W. and LeBaron, B. (1996), 'A structural model for stock return volatility and trading volume', *Review of Economics and Statistics* **78**, 94–110.

Bullard, J. and Duffy, J. (1999), 'Using Genetic Algorithms to Model the Evolution of Heterogeneous Beliefs', *Computational Economics* **13**, 41–60.

Chen, S.-H., Lux, T. and Marchesi, M. (2001), 'Testing for non-linear structure in an artificial financial market', *Journal of Economic Behavior and Organization* **46**, 327–342.

Chen, S.-H. and Yeh, C.-H. (1997), 'Toward a computable approach to the efficient market hypothesis: an application of genetic programming', *Journal of Economic Dynamics and Control* **21**, 1043–1063.

Chen, S.-H. and Yeh, C.-H. (2002), 'On the emergent properties of artificial stock markets: the efficient market hypothesis and the rational expectations hypothesis', *Journal of Economic Behavior and Organization* **49**, 217–239.

Chiarella, C. (1992), 'The dynamics of speculative behaviour', *Annals of Operations Research* **37**, 101–123.

Chiarella, C., Dieci, R. and Gardini, L. (2002), 'Speculative behaviour and complex asset price dynamics', *Journal of Economic Behavior and Organization* **49**, 173–197.

Chiarella, C. and He, X. (2001), 'Asset pricing and wealth dynamics under heterogeneous expectations', *Quantitative Finance* **1**, 509–526.

Chiarella, C. and He, X. (2002), 'Heterogeneous beliefs, risk and learning in a simple asset pricing model', *Computational Economics* **19**, 95–132.

Chiarella, C. and He, X. (2003), ‘Heterogeneous beliefs, risk and learning in a simple asset pricing model with a market maker’, *Macroeconomic Dynamics* **7**, 503–536.

Chung, C.-F. (1999), Estimating the fractionally integrated garch model, discussion paper, National Taiwan University.

Dacorogna, M., Muller, U., Jost, C., Pictet, O., Olsen, R. and Ward, J. (1995), ‘Heterogeneous real-time trading strategies in the foreign exchange market’, *European Journal of Finance* **1**, 383–403.

Day, R. and Huang, W. (1990), ‘Bulls, bears and market sheep’, *Journal of Economic Behavior and Organization* **14**, 299–329.

DeGrauwe, P. and Grimaldi, M. (2003), Exchange rate puzzles: a tale of switching attractor, manuscript, University of Leuven.

DeLong, J., Shleifer, A., Summers, L. and Waldmann, R. (1990), ‘Noise trader risk in financial markets’, *Journal of Political Economy* **98**, 703–738.

Ding, Z., Granger, C. and Engle, R. (1993), ‘A long memory preperty of stock market returns and a new model’, *Journal of Empirical Finance* **1**, 83–106.

Engle, R. (1982), ‘Autoregressive conditional heteroscedasticity with estimates of the variance of uk inflation’, *Econometrica* **50**, 987–1008.

Farmer, J. and Joshi, S. (2002), ‘The price dynamics of common trading strategies’, *Journal of Economic Behavior and Organization* **49**, 149–171.

Frankel, F. and Froot, K. (1987), ‘Using survey data to test propositions regarding exchange rate expectations’, *American Economic Review* **77**, 133–153.

Gaunersdorfer, A. (2000), ‘Endogenous fluctuations in a simple asset pricing model with heterogeneous agents’, *Journal of Economic Dynamics and Control* **24**, 799–831.

Gaunersdorfer, A. and Hommes, C. (2000), A nonlinear structural model for volatility clustering, Technical Report 00-02, CeNDF, University of Amsterdam. Working Paper.

He, X. and Li, Y. (2004), Heterogeneity, profitability and autocorrelations, manuscript, University of Technology, Sydney.

Hommes, C. (2001), ‘Financial markets as nonlinear adaptive evolutionary systems’, *Quantitative Finance* **1**, 149–167.

Hommes, C. (2002), ‘Modeling the stylized facts in finance through simple nonlinear adaptive systems’, *Proceedings of National Academy of Science of the United States of America* **99**, 7221–7228.

Iori, G. (2002), ‘A microsimulation of traders activity in the stock market: the role of heterogeneity, agents’ interactions and trade frictions’, *Journal of Economic Behavior and Organization* **49**, 269–285.

Kesten, H. (1973), ‘Random difference equations and renewal theory for products of random matrices’, *Acta Mathematica* **131**, 207–248.

Kirman, A. (1991), *Money and Financial Markets*, Blackwell, Cambridge, chapter Epidemics of opinion and speculative bubbles in financial markets, pp. 354–368.

Kirman, A. (1993), ‘Ants, rationality, and recruitment’, *Quarterly Journal of Economics* **108**, 137–156.

LeBaron, B. (2000), ‘Agent based computational finance: suggested readings and early research’, *Journal of Economic Dynamics and Control* **24**, 679–702.

LeBaron, B. (2001), ‘A builder’s guide to agent-based financial markets’, *Quantitative Finance* **1**(2), 254–261.

LeBaron, B. (2002), Calibrating an agent-based financial market to macroeconomic time series, Technical report, Brandeis University, Waltham, MA.

LeBaron, B., Arthur, W. and Palmer, R. (1999), ‘Time series properties of an artifical stock market’, *Journal of Economic Dynamics and Control* **23**, 1487–1516.

Lewwllen, J. and Shanken, J. (2002), ‘Learning, asset-pricing tests, and market efficiency’, *Journal of Finance* **57**(3), 1113–1145.

Li, Y., Donkers, B. and Melenberg, B. (2004), The econometric analysis of ms models, Manuscript, Tilburg University.

Lux, T. (1995), ‘Herd behaviour, bubbles and crashes’, *Economic Journal* **105**, 881–896.

Lux, T. (1997), ‘Time variation of second moments from a noise trader/infection model’, *Journal of Economic Dynamics and Control* **22**, 1–38.

Lux, T. (1998), ‘The socio-economic dynamics of speculative markets: Interacting agents, chaos, and the fat tails of return distributions’, *Journal of Economic Behavior and Organization* **33**, 143–165.

Lux, T. (2004), Financial power laws: Empirical evidence, models, and mechanisms, *in* ‘WEHIA2004’, Kyota.

Lux, T. and Marchesi, M. (1999), ‘Scaling and criticality in a stochastic multi-agent model of a financial markets’, *Nature* **397**(11), 498–500.

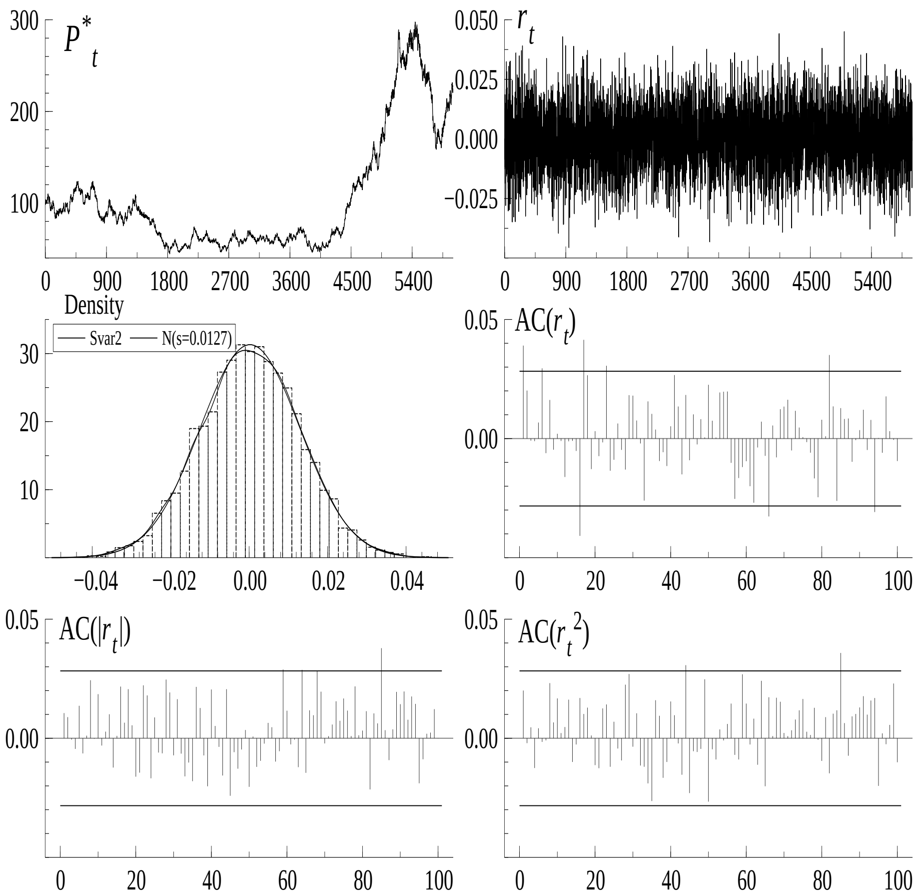
Lux, T. and Marchesi, M. (2001), ‘Volatility clustering in financial markets: A micro-simulation of interacting agents’, *Journal of Theoretical and Applied Finance* **3**, 675–702.

Pagan, A. (1996), ‘The econometrics of financial markets’, *Journal of Empirical Finance* **3**, 15–102.

Westerhoff, F. (2003), ‘Speculative markets and the effectiveness of price limits’, *Journal of Economic Dynamics and Control* **28**, 439–508.

APPENDIX A. TIME SERIES PROPERTIES OF THE FUNDAMENTAL PRICE PROCESS

FIGURE A.1. Time series of the fundamental price and return, the return distribution density and the corresponding ACs of returns, the absolute returns, and the squared returns.



APPENDIX B. TABLE OF AUTOCORRELATIONS

TABLE B.1. Autocorrelations of r_t, r_t^2 and $|r_t|$ for S&P 500.

Lag	r_t	r_t^2	$\|r_t\|$
1	0.0140 (0.0199)	0.1108 (0.0297)	0.1952 (0.0421)
	[-0.0250,0.0530]	[0.0526,0.1689]	[0.1127,0.2777]
2	-0.0395 (0.0312)	0.1525 (0.0080)	0.2187 (0.0392)
	[-0.1007,0.0217]	[0.1369,0.1682]	[0.1418,0.2957]
3	-0.0382 (0.0243)	0.0824 (0.0324)	0.2171 (0.0302)
	[-0.0858,0.0094]	[0.0190,0.1459]	[0.1580,0.2762]
4	-0.0133 (0.0190)	0.0257 (0.0177)	0.1776 (0.0248)
	[-0.0505,0.0239]	[-0.0090,0.0604]	[0.1290,0.2262]
5	0.0008 (0.0363)	0.1406 (0.0047)	0.2268 (0.0245)
	[-0.0703,0.0719]	[0.1313,0.1499]	[0.1787,0.2749]
6	-0.0096 (0.0198)	0.0330 (0.0142)	0.1764 (0.0177)
	[-0.0484,0.0292]	[0.0051,0.0609]	[0.1417,0.2110]
7	-0.0298 (0.0192)	0.0182 (0.0149)	0.1554 (0.0257)
	[-0.0674,0.0078]	[-0.0110,0.0474]	[0.1051,0.2058]
8	-0.0014 (0.0199)	0.0523 (0.0090)	0.1788 (0.0161)
	[-0.0404,0.0376]	[0.0346,0.0699]	[0.1472,0.2104]
9	-0.0073 (0.0234)	0.0346 (0.0200)	0.1645 (0.0208)
	[-0.0532,0.0386]	[-0.0046,0.0738]	[0.1238,0.2052]
10	0.0015 (0.0193)	0.0162 (0.0119)	0.1568 (0.0232)
	[-0.0363,0.0393]	[-0.0070,0.0395]	[0.1113,0.2024]
20	-0.0114 (0.0174)	0.0100 (0.0104)	0.1273 (0.0251)
	[-0.0455,0.0227]	[-0.0103,0.0304]	[0.0780,0.1766]
30	-0.0066 (0.0168)	0.0123 (0.0097)	0.1161 (0.0182)
	[-0.0395,0.0263]	[-0.0067,0.0312]	[0.0804,0.1519]
40	-0.0324 (0.0154)	0.0056 (0.0060)	0.0958 (0.0190)
	[-0.0626,-0.0022]	[-0.0062,0.0174]	[0.0584,0.1331]
50	-0.0159 (0.0152)	0.0111 (0.0079)	0.1098 (0.0187)
	[-0.0457,0.0139]	[-0.0045,0.0266]	[0.0732,0.1464]
60	0.0009 (0.0136)	0.0006 (0.0034)	0.0675 (0.0206)
	[-0.0258,0.0276]	[-0.0060,0.0073]	[0.0271,0.1079]
70	-0.0069 (0.0141)	0.0035 (0.0026)	0.0791 (0.0151)
	[-0.0345,0.0207]	[-0.0016,0.0086]	[0.0494,0.1088]
80	0.0040 (0.0139)	0.0008 (0.0031)	0.0572 (0.0166)
	[-0.0232,0.0312]	[-0.0053,0.0068]	[0.0248,0.0897]
90	-0.0062 (0.0132)	-0.0004 (0.0026)	0.0652 (0.0180)
	[-0.0321,0.0197]	[-0.0055,0.0047]	[0.0299,0.1005]
100	-0.0030 (0.0140)	0.0009 (0.0032)	0.0729 (0.0192)
	[-0.0304,0.0244]	[-0.0052,0.0071]	[0.0352,0.1105]

Note: The numbers in parentheses are Newey-West corrected standard errors, and 95% confidence intervals indicate by square brackets.

TABLE B.2. Autocorrelations of r_t for the MF model.

Lag	β	Min.	Max.	95% CIs
1	0.0504 (0.0186)	-0.2987	0.5411	[0.0492,0.0515]
2	0.0393 (0.0182)	-0.1556	0.5060	[0.0382,0.0404]
3	0.0279 (0.0182)	-0.1412	0.4546	[0.0267,0.0290]
4	0.0200 (0.0178)	-0.1303	0.4197	[0.0189,0.0211]
5	0.0141 (0.0178)	-0.1008	0.3721	[0.0130,0.0152]
6	0.0087 (0.0178)	-0.1125	0.3619	[0.0076,0.0098]
7	0.0055 (0.0177)	-0.1002	0.3325	[0.0054,0.0076]
8	0.0034 (0.0176)	-0.0799	0.2934	[0.0023,0.0044]
9	0.0011 (0.0175)	-0.1191	0.2668	[0.0000,0.0022]
10	0.0004 (0.0174)	-0.0900	0.2435	[-0.0007,0.0015]
20	-0.0053 (0.0172)	-0.2243	0.0937	[-0.0063,-0.0042]
30	-0.0024 (0.0171)	-0.0592	0.0566	[-0.0034,-0.0013]
40	-0.0007 (0.0170)	-0.0652	0.0572	[-0.0018,0.0004]
50	0.0009 (0.0170)	-0.0580	0.0793	[-0.0002,0.0019]
60	-0.0002 (0.0170)	-0.0646	0.0887	[-0.0013,0.0009]
70	-0.0004 (0.0170)	-0.0615	0.0689	[-0.0015,0.0006]
80	0.0006 (0.0171)	-0.0627	0.0802	[-0.0004,0.0017]
90	-0.0007 (0.0171)	-0.0761	0.0795	[-0.0017,0.0004]
100	0.0002 (0.0170)	-0.0763	0.0723	[-0.0008,0.0013]

TABLE B.3. Autocorrelations of r_t^2 for the MF model.

Lag	β	Min	Max	95% CIs
1	0.1443 (0.0256)	0.0135	0.4917	[0.1427,0.1459]
2	0.1397 (0.0256)	0.0049	0.5457	[0.1381,0.1413]
3	0.1362 (0.0254)	-0.0059	0.4338	[0.1346,0.1378]
4	0.1325 (0.0247)	0.0076	0.4431	[0.1309,0.1340]
5	0.1301 (0.0246)	-0.0051	0.3251	[0.1286,0.1316]
6	0.1304 (0.0249)	-0.0084	0.4107	[0.1289,0.1320]
7	0.1280 (0.0243)	-0.0002	0.4056	[0.1265,0.1295]
8	0.1270 (0.0240)	-0.0026	0.3644	[0.1255,0.1284]
9	0.1259 (0.0240)	-0.0035	0.3683	[0.1245,0.1274]
10	0.1242 (0.0234)	-0.0066	0.3219	[0.1227,0.1256]
20	0.1195 (0.0226)	0.0009	0.5453	[0.1181,0.1209]
30	0.1153 (0.0226)	-0.0056	0.4194	[0.1139,0.1167]
40	0.1143 (0.0222)	-0.0040	0.3041	[0.1129,0.1156]
50	0.1138 (0.0226)	-0.0039	1.2611	[0.1124,0.1152]
60	0.1119 (0.0221)	-0.0063	0.4257	[0.1105,0.1133]
70	0.1122 (0.0228)	-0.0144	0.7911	[0.1108,0.1136]
80	0.1103 (0.0222)	-0.0078	0.5088	[0.1089,0.1117]
90	0.1082 (0.0220)	-0.0038	0.3497	[0.1068,0.1095]
100	0.1101 (0.0224)	-0.0093	0.4121	[0.1087,0.1115]

TABLE B.4. Autocorrelations of $|r_t|$ for the MF model.

Lag	β	Min	Max	95% CIs
1	0.1710 (0.0185)	0.0111	0.5923	[0.1699,0.1722]
2	0.1676 (0.0186)	0.0074	0.5018	[0.1664,0.1688]
3	0.1649 (0.0186)	-0.0030	0.4928	[0.1637,0.1660]
4	0.1624 (0.0183)	0.0026	0.5154	[0.1613,0.1636]
5	0.1607 (0.0181)	-0.0029	0.4567	[0.1596,0.1618]
6	0.1600 (0.0181)	-0.0055	0.4892	[0.1589,0.1612]
7	0.1587 (0.0181)	-0.0035	0.4918	[0.1576,0.1598]
8	0.1572 (0.0180)	-0.0004	0.4684	[0.1560,0.1583]
9	0.1562 (0.0179)	-0.0024	0.4954	[0.1551,0.1573]
10	0.1548 (0.0177)	-0.0067	0.4642	[0.1537,0.1559]
20	0.1507 (0.0175)	0.0033	0.5045	[0.1496,0.1518]
30	0.1464 (0.0174)	-0.0018	0.4620	[0.1453,0.1475]
40	0.1461 (0.0174)	-0.0014	0.4826	[0.1451,0.1472]
50	0.1444 (0.0173)	-0.0174	0.4781	[0.1433,0.1455]
60	0.1433 (0.0174)	-0.0069	0.4716	[0.1423,0.1444]
70	0.1432 (0.0174)	-0.0085	0.4970	[0.1422,0.1443]
80	0.1419 (0.0174)	-0.0113	0.4974	[0.1409,0.1430]
90	0.1401 (0.0173)	-0.0043	0.4863	[0.1390,0.1412]
100	0.1407 (0.0174)	-0.0069	0.5069	[0.1397,0.1418]