Teaching to do Economics with the Computer*

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Abstract

This paper presents the course "Doing Economics with the Computer" we taught since 1999 at the University of Bern, Switzerland. "Doing Economics with the Computer" is a course we designed to introduce sophomores playfully and painlessly into computational economics. Computational methods are usually used in economics to analyze complex problems, which are impossible (or very difficult) to solve analytically. However, our course only looks at economic models, which can (easily) be solved analytically. This approach has two advantages: First, relying on economic theory students have met in their first year, we can introduce numerical methods at an early stage. This stimulates students to use computational methods later in their academic career when they encounter difficult problems. Second, the confrontation with the analytical analysis shows convincingly both power and limits of numerical methods. Our course introduces students to three types of software: spreadsheet and simple optimizer (Excel with Solver), numerical computation (Matlab) and symbolic computation (Maple).

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Introduction

Computational methods are usually used in economics to analyze complex problems, which are impossible (or very difficult) to solve analytically. However, our course only looks at economic models, which can (easily) be solved analytically. This approach has two advantages: First, relying on economic theory students have met in their first year, we can introduce numerical methods at an early stage. This stimulates students to use computational methods later in their academic career when they encounter difficult problems. Second, the confrontation with the analytical analysis shows convincingly both power and limits of numerical methods.

This paper presents the course "Doing Economics with the Computer" we taught since 1999 at the University of Bern, Switzerland. "Doing Economics with the Computer" is a course we designed to introduce sophomores playfully and painlessly into computational economics. Our course introduces students to three types of software: spreadsheet and simple optimizer (Excel with Solver), numerical computation (Matlab) and symbolic computation (Maple). We stopped using statistical software as this is already very well covered in various econometric classes. The course consists of 10 sessions, we taught weakly each in a 3-hour lecture to 16 students. In the *1st part* of each session we present the economic problem, sometimes its analytical solution and introduce the software used. The *2nd part*, in the computer lab, starts the numerical implementation with step-by-step guidance. In this part, students work on exercises with clearly defined questions and precise guidance for their implementation. The *3rd part* is a workshop where students work in groups on exercises with still very clear defined questions but no help on their implementation. This part teaches students how to practically handle numerical questions in a well-defined framework. The *4th part* of a session is a graded take home assignment where students are asked to answer general economic questions. This part teaches students how to translate general economic questions into a numerical task and back into an economically meaningful answer. A short debriefing in the following week is *part 5* and completes each session.

The remainder of this paper contains the handouts that accompany the course.

Overview of the course

Session	Topic	Method	Software
1	Simple IS-LM Model	Function of Variables	Excel
2	Estimation of Consumption Function	Linear Regression	Excel
3	Introduction into Matlab 1: Basic Commands	Linear Algebra	Matlab
4	Two Country Mundell-Fleming Model	Linear Equation System	Matlab
5	The Cake-Eating Problem (Heuristic)	Dynamic Optimization	Solver Excel
6	Hog Cycle	Difference Equation	Excel
7	Introduction into Matlab 2: Scripts and Functions	Programming	Matlab
8	Mechanical Trading Rules	Programming	Matlab
9	Introduction into Maple	Analysis	Maple
10	Consumer Behaviour	Calculus	Maple

Session 1: Simple IS-LM Model

1 The Topic

Today's topic is the numerical implementation of a simple version of the IS-LM model. The IS-LM model is a business cycle model based on the assumption that there are barriers to the instantaneous adjustment of nominal prices.[1] Typically the price level is fixed and lies above the general-market-clearing value. The excessive price level leads to an excess supply of goods. This situation causes changes in the aggregate demand for goods at a given level of prices to affect the amount that firms produce. As a result, purely monetary disturbances can affect real variables like employment and output.

The set-up of the typical IS-LM begins by directly specifying relationships among aggregate variables, i.e. between aggregated production, consumption, investment, government spending, money supply, etc. In a simple version of the IS-LM model these relationships are static.

Such a simple model of aggregate demand can be used to analyse many issues such as the reaction of the economy to changes in policy variables or to changes in the specification of the interaction between endogenous variables. A very general description of the interaction between the different variables allows us to get *qualitative* results of these reactions, e.g. investment falls with increased government spending. However, often we are interested in

[1] There are other possibilities of nominal rigidities. For a more comprehensive discussion of the IS-LM model see e.g. Mankiw (1997), Romer (1996), or Barro (1997).

quantitative results, i.e. we want to know not only the direction but the extent of these reactions. To do so we have to make specific assumptions regarding the functional form of the interaction between the variables in the model. We consider here a particularly simple textbook version which involves contemporaneous variables, only; therefore, we call it a *static* version (compare Mankiw, 1997, pp. 294).[2]

1.1 An Algebraic Version of the IS-LM Model (Mankiw 1997, pp. 294)

exogenous variables:

- G government spending
- T tax on income
- M money supply
- P price level (fixed in the short-run)

endogenous variables:

- Y production
- C consumption
- I investments
- R interest rate

parameters:

a, b, c, d, e, f

assumption 1: output equals aggregate demand

$$Y = Y^d \equiv C + I + G \tag{1}$$

[2] A more sophisticated, dynamic version of the IS-LM model would also include lagged variables (as proxies for expected future income, e.g.) and a price adjustment equation (a so-called Phillips curve).

assumption 2: Keynesian consumption function

$$C = a + b(Y - T) \tag{2}$$

assumption 3: investment function

$$I = c - dR \tag{3}$$

assumption 4: monetary equilibrium

$$M = M^d \equiv P\left(eY - fR\right) \tag{4}$$

Assumption (1) to (3) lead to the IS-curve, the interest rate – savings (investment) relationship

$$Y = \frac{a+c}{1-b} + \frac{1}{1-b}G + \frac{-b}{1-b}T + \frac{-d}{1-b}R. \tag{5}$$

The rearranged assumption (4) is the LM-curve, the liquidity – money relationship

$$R = \frac{e}{f}Y - \frac{1}{f}M/P. \tag{6}$$

The equilibrium production in this simple model is already a rather complicated function:

$$Y^* = \frac{z(a+c)}{1-b} + \frac{z}{1-b}G + \frac{-zb}{1-b}T + \frac{d}{(1-b)[f + de/(1-b)]}M/P, \tag{7}$$

where $z = f/[f + de/(1-b)]$. The equilibrium values for the other variables can be found using the LM curve and assumption (1) to (3).

1.2 Calibration: Parameters for Switzerland

We assume[3] for Switzerland in the year 1998 (in billion[4] Swiss Francs):

a = 55, b = 0.63, c = 47, d = 1500, e = 0.6, f = 2700.

(For the policy variables and the fixed price level compare the Excel sheet below.)

2 The Method

The numerical implementation of the algebraic version of the simple IS-LM model does not need a lot of mathematics. The model is already solved analytically without the help of a computer. The equilibrium solution are functions for the endogenous variables *Y, R, I, C* with respect to the exogenous variables *G, T, M, P* and the (for the time being assumed) known parameters *a, b, c, d, e, d*. We can know calculate the equilibrium values for any given set of exogenous variables. Changing the value of one or more exogenous variables and recalculating the equilibrium allows us to do *comparative statics*.

[3] You will see in session 2 how to get the estimates for the consumption function. The model is calibrated for the year 1998: you may check the model's solution with Swiss economic data. These data can be found at http://www.statistik.admin.ch/stat_ch/ber00/imf.htm.

[4] A billion is in German called "eine Milliarde".

3 The Software

We will implement the above model in *Microsoft Excel*. We assume you to have basic knowledge of Excel. In particular, you should know how to assign a function of other cells to a cell, the difference between relative (e.g. B2) and absolute (B2) references and how to make graphs from data.

A feature in Excel that is very useful when dealing with complicated functions is the assignment of *names* to cells. This allows us to refer to the parameters and the variables with the same abbreviations as in the model, e.g. G, instead of referring to the cell, e.g. C8.

How to assign a name to a cell in Excel (German version):

(1) mark the cell you want to assign a name, e.g. B2

(2) click the menu item Einfügen/Namen/Festlegen ...

(3) fill the name, e.g. Y, in the field "Namen in der Arbeitsmappe" and press OK

(4) the new name will appear on the left side of the "Bearbeitungsleiste"

How to use a name in a formula:

(1) just use the new name in the formula instead of the cell reference, e.g.

= a+b*(Y-T) instead of = F16+I16*(F27-F9)

Note: Excel does not distinguish between capital and small letter. You cannot assign 'C' to one cell and 'c' to another.

4 References

Barro, Robert J. (1997), Macroeconomics, 5th edition, The MIT Press

Mankiw, N. Gregory (1997), Macroeconomics, 3rd edition, New York: Worth.

Romer, David (1996), Advanced Macroeconomics, McGraw-Hill

5 Today's Task

Exercise 1: numerical implementation of the model

We will produce an Excel worksheet that contains the values of exogenous variables, parameters and the endogenous variables in equilibrium. The appendix shows how such a worksheet may look like.

a) Make a sketch of what you want to put where on your Excel sheet.
b) Open a new worksheet in Excel.
c) Choose for each exogenous variable a cell in which you put its value. Assign names from the model to these cells. Describe their contents in another cell next to it.
d) Do the same for the model parameters.
e) Choose cells for the equilibrium values of the endogenous variables.
f) Insert equation 7 in the cell for the equilibrium production.
g) Insert the formulas for the further equilibrium values in their cell. You can refer to the already calculated equilibrium values.
h) Put in reasonable values for the exogenous variables and parameters.
i) Make the sheet look appealing by setting titles, number formats, additional descriptions, colours, etc.

Exercise 2: graphs

Let us display the Keynesian Cross (Mankiw 1997, pp. 250) and the IS-LM diagram (Mankiw 1997, pp. 269) in the worksheet from exercise 1.

a) Find a place for a table with 5 columns and 11 rows. The first column contains given values for the production Y from 0 to 500. The second column contains the corresponding consumption C from equation 2. The third column is the investment at the current equilibrium value of the

interest rate (equation 3). The fourth column is total expenditure $Y^d = C + I + G$. The last column stands for the 45°-line and repeats the Y values.

b) Make a scatterplot with Y on the horizontal and Y^d on the vertical axis. Also include a 45°-Line. This is the *Keynesian cross* at the current equilibrium interest rate.

c) Find a place for another table with 5 columns and 11 rows. Again, the first column contains given values for the production Y from 0 to 500 and the second column contains the corresponding consumption C from equation 2. The investment in the third column is now such that the Keynesian cross given Y is in equilibrium, i.e. equation 1 is satisfied. The fourth column is the interest rate in the IS-curve. It corresponds to the above investment according to equation 3. The last column is the interest rate in the LM curve that goes with the production in the first column (equation 6).

d) Make a scatterplot with Y on the horizontal and the IS and LM interest rate on the vertical axis. *This is the IS-LM diagram*. The crossing of the two curves yields the equilibrium interest rate and production.

Exercise 3: understand and use the numerical model

a) Suppose, the government cuts nondistorting (lump sum) taxes without changing government spending; the resulting budget deficit is financed by borrowing to the public (we call this a *deficit-financed tax cut*). If taxes are cut by 1 billion (without changing government spending), by how much do GDP, consumption, investment, and interest rates change?[5]

[5] From previous courses we know that in a dynamic general equilibrium model with flexible prices under certain conditions the Ricardian equivalence holds. That is, the public understands, that the government cannot cut taxes (leaving government expenditures

b) Go back to the initial setting. Suppose, the government increases government spending by 1 billion; the resulting budget deficit is covered with additional taxes. By how much do GDP, consumption, investment, and interest rates change?

c) Go back to the initial setting. Suppose, the government increases government spending by 1 billion; the resulting budget deficit is financed by borrowing to the public. Explain step by step how the increase in government spending influences production. Look how the curves move in the graphs. Does the change depend on the level of government spending? Does it depend on other exogenous variables? Derive the reaction of government purchases on equilibrium production analytically. Compare it with the numerical result. Is this the so-called Multiplier (Mankiw 1997, p. 253)?

d) Go back to the initial setting. Increase the money supply by one unit (= 1 billion) and study the implications. Explain step by step how the monetary transmission mechanism works. (Consult any good macro introductory textbook, if necessary.)

e) Go back to the initial setting. Suppose, the government increases government spending by 1 billion; the resulting budget deficit is financed by borrowing to the central bank. By how much do GDP, consumption,

unaffected) without raising taxes in the future to pay off principal and interest on the debt. Under those conditions, people would save the entire tax cut, buy the bonds issued by the government, and use the interest on the bonds to pay the higher future taxes; the deficit-financed tax cut would have no real effects. Not so in the IS-LM model! Here, a deficit financed tax cut makes people feel wealthier (possible reasons are consumers' shortsightedness or borrowing constraints -compare any good macro introductory textbook).

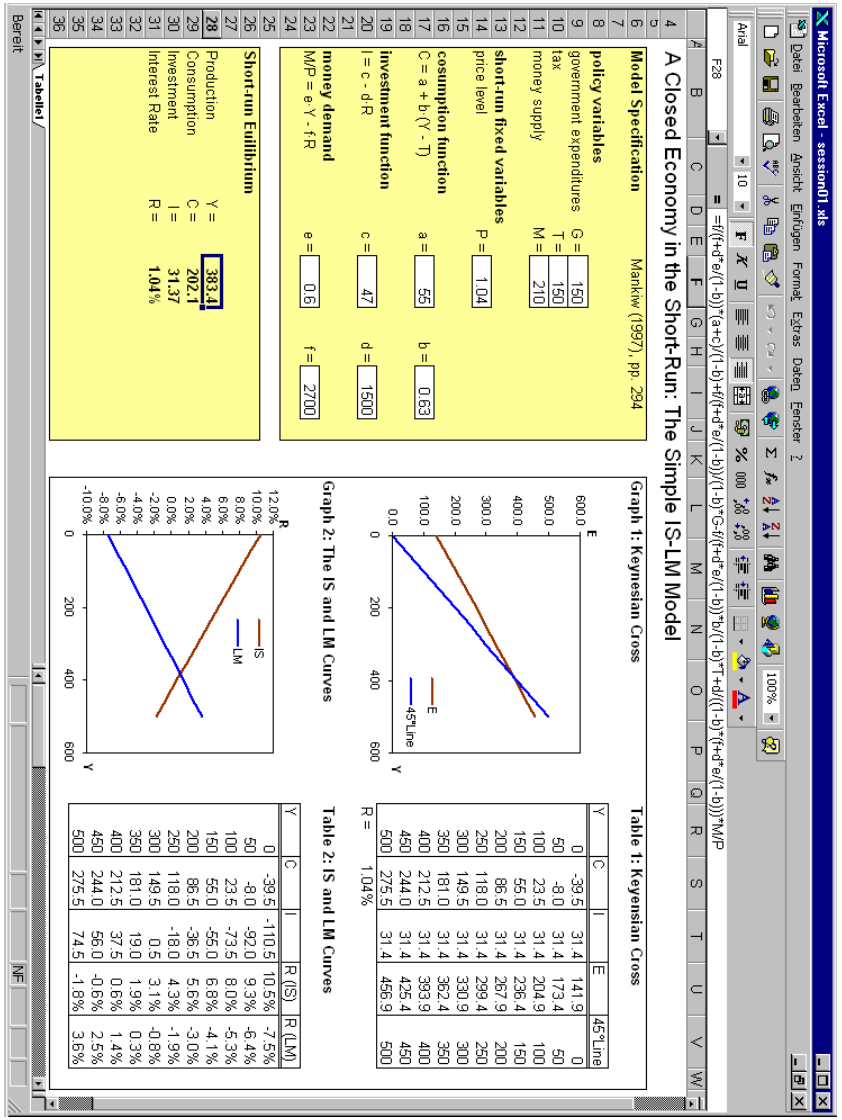
investment, and interest rates change? Compare the outcome to the debt financed increase of government consumption.

f) Which parameter describes the savings behaviour of the people.

g) Which parameter contains implicitly the size of the economy?

h) Try more.

Appendix: The Resulting Excel Worksheet



Microsoft Excel - session01.xls

F28 = f/(f+d*e/(1-b))*(a+c)/(1-b)+f/(f+d*e/(1-b))/(1-b)*G-f/(f+d*e/(1-b))*b/(1-b)*T+d/((1-b)*(f+d*e/(1-b)))*M/P

A Closed Economy in the Short-Run: The Simple IS-LM Model

Model Specification Mankiw (1997), pp. 294

policy variables

government expenditures	G =	150
tax	T =	150
money supply	M =	210

short-run fixed variables

price level	P =	1.04

cosumption function

C = a + b (Y - T)	a =	55	b =	0.63

investment function

I = c - d R	c =	47	d =	1500

money demand

M/P = e Y - f R	e =	0.6	f =	2700

Short-run Euilibrium

Production	Y =	383.4
Consumption	C =	202.1
Investment	I =	31.37
Interest Rate	R =	1.04%

Graph 1: Keynesian Cross

Table 1: Keynesian Cross

Y	C	I	E	45°Line
0	-39.5	31.4	141.9	0
50	-8.0	31.4	173.4	50
100	23.5	31.4	204.9	100
150	55.0	31.4	236.4	150
200	86.5	31.4	267.9	200
250	118.0	31.4	299.4	250
300	149.5	31.4	330.9	300
350	181.0	31.4	362.4	350
400	212.5	31.4	393.9	400
450	244.0	31.4	425.4	450
500	275.5	31.4	456.9	500
R =	1.04%			

Graph 2: The IS and LM Curves

Table 2: IS and LM Curves

Y	C	I	R (IS)	R (LM)
0	-39.5	-110.5	10.5%	-7.5%
50	-8.0	-92.0	9.3%	-6.4%
100	23.5	-73.5	8.0%	-5.3%
150	55.0	-55.0	6.8%	-4.1%
200	86.5	-36.5	5.6%	-3.0%
250	118.0	-18.0	4.3%	-1.9%
300	149.5	0.5	3.1%	-0.8%
350	181.0	19.0	1.9%	0.3%
400	212.5	37.5	0.6%	1.4%
450	244.0	56.0	-0.6%	2.5%
500	275.5	74.5	-1.8%	3.6%

Tabelle1

Bereit

Session 2: Estimation of the Consumption Function

1 The Topic

In the last session we encountered the IS-LM Model. This demand side macroeconomic model describes the short run behaviour of a closed economy. At the core of the model is the consumption function:

$$C = a + b(Y - T), \tag{1}$$

where a is called autonomous consumption expenditure and b marginal propensity to consume. The IS-LM model assumes consumption to depend only on disposable income and the relationship stable over time. Consequently, the interest rate does not have an important role. These are basically John Maynard Keynes conjectures. See Mankiw (1997, pp. 404, 2000, pp. 434) for a discussion of the appropriateness of these assumptions.

There are two parameters a and b in the above linear version of the consumption function. How do we know these parameters? If we look at a particular country, we can observe aggregate consumption C and aggregate disposable income (Y-T) in different time periods from national accounts. This allows us to study the relationship empirically.

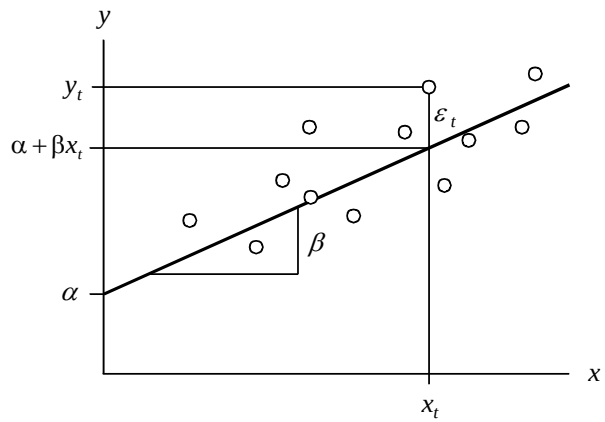
2 The Method

The statistical tool we use to find the parameters in the consumption function is *Linear Regression Analysis* (see for a detailed introduction Greene 1997, 2000, chapter 6).

In its simplest version the linear regression model deals with just two variables: the independent variable x and the dependent variable y. For all observations t (individuals, time periods, etc.) the dependent variable y is a function of x plus a random error:

$$y_t = \alpha + \beta x_t + \varepsilon_t . \tag{2}$$

The error term ε is on average zero, $E(x_t) = 0$, and has constant variance, $V(x_t) = \sigma^2$. The error terms are independent across observations, $\forall\, t \neq h$: $E(x_t, x_h) = 0$. Sometimes the error term is assumed normally distributed, $\varepsilon_t \sim N(0, \sigma^2)$.



Graph: observations and the regression line

The parameters α and β are chosen, such that the regression line $\alpha + \beta x_t$ lies as "close" as possible to the observed y_t. Usually the method of ordinary least squares (OLS) is used. OLS minimises the sum of the squared vertical distances between y_t and the regression line:

$$S(\alpha,\beta) = \sum_{t=1}^{T} \varepsilon_t^2 = \sum_{t=1}^{T} \left(y_t - \alpha - \beta x_t \right)^2 \to \min_{\alpha,\beta} \tag{3}$$

The solution to this minimisation problem is:

$$\hat{\beta} = \frac{s_{xy}}{s_{xx}} = \frac{\sum_{t=1}^{T} (y_t - \bar{y})(x_t - \bar{x})}{\sum_{t=1}^{T} (x_t - \bar{x})^2}, \tag{4}$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x},$$

where $\bar{x}$ and $\bar{y}$ are the mean of x and y. $\hat{\alpha}$ and $\hat{\beta}$ are called the OLS estimates of a and b. OLS estimates have nice properties if the above model is a good description of the reality: they are unbiased, efficient, consistent and (asymptotically) normally distributed (see Greene 1997, 2000, chapter 6.6 and 6.7).

After the estimation of the parameters $\hat{\alpha}$ and $\hat{\beta}$ you can predict the values of the dependent variable y by

$$\hat{y}_t = \hat{\alpha} + \hat{\beta} x_t. \tag{5}$$

The difference between the observed value and the predicted value is an estimate for the error term. It is called the residual e:

$$e_t \equiv \hat{\varepsilon}_t = y_t - \hat{y}_t = y_t - \hat{\alpha} - \hat{\beta} x_t. \tag{6}$$

The coefficient of determination is a measure of how well the regression line fits the data. In the case of one independent variable x it is defined as

$$R^2 = \frac{SSR}{SST} = \frac{\sum_{t=1}^{T} (\hat{y}_t - \bar{y})^2}{\sum_{t=1}^{T} (y_t - \bar{y})^2} = 1 - \frac{SSE}{SST} = 1 - \frac{\sum_{t=1}^{T} (e_t)^2}{\sum_{t=1}^{T} (y_t - \bar{y})^2}. \tag{7}$$

The coefficient of determination tells you what fraction of the variation of the dependent variable can be explained by the linear relationship.

The method can easily be generalised to a set of independent variables. The Model is then called multiple linear regression:

$$y_t = \beta_1 x_{t1} + \beta_2 x_{t2} + \cdots + \beta_K x_{tK} + \varepsilon_t \qquad t = 1,\ldots,T \tag{8}$$

The multiple regression is often expressed in matrix notation:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1K} \\ x_{21} & x_{22} & \cdots & x_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ x_{T1} & x_{T2} & \cdots & x_{TK} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_T \end{bmatrix}$$

$$\begin{array}{ccccccc} T \times 1 & & T \times K & & K \times 1 & & T \times 1 \\ \mathbf{y} & = & \mathbf{X} & & \boldsymbol{\beta} & + & \boldsymbol{\varepsilon} \end{array}$$

The OLS estimator becomes:

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{X} \right)^{-1} \mathbf{X}'\mathbf{y}. \tag{9}$$

3 The Software

Linear Regression is *the* basic tool in statistics. It is extensively used in all fields of natural and social sciences. Probably any statistic and econometric software package can do OLS estimation. We use Excel and its Regression Analysis Tool.

4 References

Greene, William H. (1997), *Econometric Analysis*, 3rd edition, Upper Saddle River: Prentice Hall.

Greene, William H. (2000), *Econometric Analysis*, 4th edition, Upper Saddle River: Prentice Hall.

Mankiw, N. Gregory (1997), *Macroeconomics*, 3rd edition, New York: Worth.

Mankiw, N. Gregory (2000), *Macroeconomics*, 4th edition, New York: Worth.

5 Today's Task

We will estimate the consumption function for Switzerland. Assuming an IS-LM type consumption income relationship as in equation 1, we estimate the unknown parameters *a* and *b* with OLS.

The raw data are stored in the Excel file consfctn.xls. This file contains annual data on nominal disposable income and nominal consumption in Switzerland from 1980 to 1998. The time series are:

YEAR	year of observation
CONS	private consumption of goods and services in Mio Swiss Francs[1]
INC	disposable gross income of private households in Mio Swiss Francs[2]
DEFL	GDP deflator, price base 1990[3]

Exercise 1: Explore the raw data

a) Open the file consfctn.xls in Excel.

b) What is the average consumption and disposable income of Swiss households? How big is the variation, measured as standard deviation, of these two variables?

c) Plot annual consumption and disposable income over time.

[1] Data source: BFS/seco,

[2] Data sources: 1990-1997: BFS; 1980-1989, 1998: KOF/ETH

[3] Data source: seco. Deflator Bruttoinlandprodukt zu Marktpreisen, 1990 = 100.

Exercise 2: Generate deflated time series

The consumption function describes a relationship between consumption and income in real terms. As our data set contains nominal values, we have to deflate the time series.

a) How do you generate an inflated series from the nominal series and the deflator?
b) Generate both a new time series *REALCONS* with the deflated private consumption and the real disposable income *REALINC*.
c) Make a scatterplot with real income on the horizontal and real consumption on the vertical axis.

Exercise 3: Estimate the consumption function

We will know estimate α and β in the consumption function by regressing *REALCONS* on a constant and *REALINC*.

a) Use equation (4) to calculate the estimated slope β and intercept α. It may be convenient to calculate $y_t - \bar{y}$, $x_t - \bar{x}$, $(x_t - \bar{x}) \cdot (y_t - \bar{y})$ and $(x_t - \bar{x})^2$ for every time period first.
b) Calculate the predicted real consumption and the residual for any time period. Also calculate the sample variance of REALCONS, predicted REALCONS and the residuals.
c) Calculate the coefficient of determination.
d) Try to draw the regression line in the scatterplot from 2c.

Exercise 4: Use the statistical functions

a) Find out more about the Excel functions “Steigung”, “Achsenabschnitt“ and „Bestimmtheitsmass” in the Excel help.
b) Calculate the estimated parameters and the coefficient of determination using the above functions.

Exercise 5: Use the Analysis tools

a) Estimate the linear regression using the Regression tool in the Menu Extras/Analyse-Funktionen.
b) Examine the estimation output. Where is the intercept α and the slope parameter β? Is there a number that describes the precision of the estimates?
c) Give the 95%-confidence interval for the parameter β.
d) Test if β is significantly different from 1.

Exercise 6: Include the interest rate

Does the interest rate influence the consumption?

a) Search the real (after tax) Swiss interest rate for the years 1980 to 1998 in the world wide web.
b) Regress the consumption on income and the real interest rate.
c) Test if the interest rate has a significant influence on the consumption.

Exercises Session 3: Introduction into Matlab I

Today's Task

The best way to become familiar with Matlab is to read the guide "Introduction into Matlab" and directly try out what you learn there, step-by-step (this is actually what we will do during Session 3). After that you may want to solve the following problems, using your Matlab guide:

a) Go back to the Excel sheet *consfctn.xls* of session 2. There you find the time series *disposable income*, *private consumption*, and *GDP deflator*. Import the data into the Matlab workspace.

b) Generate a new time series *real private consumption*, `RCON`, with the deflated private consumption. Do the same for disposable income, `RINC`.

c) Plot a graph for both `RCON` and `RINC` versus their index.

d) Next, define the time axis `T=1980:1998`. Generate a new graph with `T` on the x-axis and `RCON` and `RINC` on the y-axis.

e) Plot `RCON` against `RINC`.

f) Now, calculate some basic descriptive statistics (like the mean of both series, the standard deviation of both series, the correlation between the two series) for `RCON` and `RINC`?

g) Now, you are going to estimate α and β in the consumption function

$$RCON = \alpha + \beta \cdot RINC$$

In matrix notation the linear regression model can be written as:

$$y = \begin{bmatrix} \mathbf{c} & \mathbf{x} \end{bmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \varepsilon$$

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

y denotes a vector where each element is an observation of the dependent variable. The dependent variable is `RCON`. **X** is matrix with the vector **c** containing just ones, and the data vector **x**, in our case `RINC`, where each element is an observation. $\boldsymbol{\beta}$ is the parameter vector, containing two elements, β_1 and β_2.

In a first step, generate a constant, **c**, as a ones-vector of the same length as `RCON` and `RINC`. Now, generate **y** and **X**.

h) Calculate the *ordinary least squares* estimator **b** by

$$\mathbf{b} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}\,.$$

i) Calculate the OLS-residuals**, e,** which are given by

$$\mathbf{e} = \left(\mathbf{y} - \mathbf{X}\mathbf{b}\right).$$

j) Insert the regression line in the plot of problem e).

k) Advanced[1]: Estimate the variance of the error term, σ^2, and the variance-covariance matrix of the estimate b, V**b**. Test some interesting hypothesis.

[1] If you like to test hypothesis about the estimated coefficients (or parameters) or form confidence intervals about the individual elements of **b**, consult Greene (1997). There you learn, among others, how to compute the *variance of the OLS-residuals*, how to compute the *variance of the estimated parameters* and how to implement tests.

Session 4: The Two Country Mundell-Fleming Model

1 The Topic

Today's model extends the IS-LM model to the case of an open economy. The most crucial assumptions are that there is perfect capital mobility and that investors do not expect the exchange rate to change. Assumptions concerning the exchange rate regime (floating or fixed) may differ. The original model, with an exogenous foreign country, was proposed by Robert A. Mundell[1] and J. Marcus Fleming in the early 60s. For an introduction to the Small-Country, Mundell-Fleming (MF) model see e.g. Mankiw (1997, pp. 299, 2000, pp. 312); for an introduction to the Large-Country or Two-Country MF model see the supplementary material handed out in class.

A simple version of the Two-Country MF model looks as follows. Suppose a world consisting of two countries. One country is the Home country, the other country is the Foreign country. Both economies can be described by means of a (slightly modified) static IS-LM model. Goods and bonds can be traded across countries. Goods and bonds sold by agents living in *H* must be paid in the *domestic currency*; goods and bonds sold by agents living in *F* must be paid in the *foreign currency*. There is a market where the currencies of the two countries can be exchanged; it is called *foreign exchange market*.

[1] He got the 1999 Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel.

Nominal goods prices in both countries are fixed. The imports of *H* are the exports of *F* and vice versa. Net exports of *H* decrease with its own output, but rise with *F*'s output. A rising real exchange rate (a deterioration of the terms of trade) has negative effects on net exports. Given perfect capital mobility and that investors do not expect the exchange rate to change, in a equilibrium situation the interest rate on both island must be equal. Otherwise an arbitrageur could make profits by borrowing money in the country with lower interest rate and invest the money on the other island.

1.1 An Algebraic Version of the Two Country Mundell-Fleming Model

This Section describes a linear version of the Two Country Mundell-Fleming model.

Exogenous variables:

G, G_w	government spending (home and foreign).
T, T_w	tax on income
M, M_w	money supply
P, P_w	price level (fixed)

Endogenous variables:

Y, Y_w	production (output)
C, C_w	consumption
I, I_w	investments
NX, NX_w	net exports (i.e., exports minus imports)
R, R_w	interest rate (real *and* nominal)
E	exchange rate (real *and* nominal) – as usual the home price for foreign currency (e.g. 1.50 CHF/EUR)

Parameters:

a, b, c, d, e, f, g, h, i, a_w, b_w, c_w, d_w, e_w, f_w

The assumptions of the model:

$$Y = C + I + G + NX \qquad Y_w = C_w + I_w + G_w + NX_w \qquad (1)\ (2)$$

$$C = a + b(Y - T) \qquad C_w = a_w + b_w(Y_w - T_w) \qquad (3)\ (4)$$

$$I = c - dR \qquad I_w = c_w - d_w R_w \qquad (5)\ (6)$$

$$M / P = eY - fR \qquad M_w / P_w = e_w Y_w - f_w R_w \qquad (7)\ (8)$$

$$NX = g - h(Y - Y_w) + iE \cdot P_w / P \qquad (9)$$

$$NX = -NX_w \qquad (10)$$

$$R = R_w \qquad (11)$$

1.2 Numerical Example

Start the analysis with two identical countries (see Hall 1991, exercise 4, chapter 19):

$a = a_w = 55$, $b = b_w = 0.63$, $c = c_w = 47$, $d = d_w = 1500$,

$e = e_w = 0.6$, $f = f_w = 2700$, $g = -10$, $h = 0.1$, $i = 10$.

$G = G_w = 150$, $M = M_w = 210$, $T = T_w = 150$ $P = P_w = 1$.

2 The Method

The above model as well as the IS-LM model in session 1 are linear models.[2] They are defined by a system of linear equations. We use *matrix algebra* to deal with these models.[3] Matrix algebra gives a compact notation, an exact criterion for testing the existence of a unique solution and an elegant way to solve the system.

Consider the following system of linear equations with n equations and m unknowns $x_1 \ldots x_m$:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m = b_n$$

Let us call x_m a *variable*, a_{nm} a *coefficient*, and b_n a *constant term*. There are $n \cdot m$ known coefficients, $a_1 \ldots a_{nm}$, and m known constant terms, $b_1 \ldots b_m$, of the column vector **b**.

[2] They are linear with respect to the unknown (endogenous) variables. However, they are not linear in prices. Therefore, the model becomes a non-linear model as soon as we skip the assumption that prices are fixed. This is e.g. the case in the *traditional IS-LM-Phillips curve* framework. IS-LM models that are non-linear in prices cannot be solved with matrix algebra. Programs such as GAMS use iterative algorithms to find the solution numerically.

[3] Matrix algebra is introduced in Chiang (1984). The department of econometrics provides a short introduction into matrix algebra.

The above linear equation system can be written in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{12}$$

$$\underset{\mathbf{A}}{(n \times m)} \quad \underset{\mathbf{x}}{(m \times 1)} = \underset{\mathbf{b}}{(n \times 1)}$$

where **A** is called the *coefficient matrix* and **x** is the vector of unknowns (or variables).

There is a *unique non-trivial solution* to a linear equation system if there are exactly as many unknown variables as equations and if there is no redundant information in the equations. These conditions can be expressed in mathematical terms: **A** has to be quadratic and nonsingular and $\mathbf{b} \neq 0$. (**A** is nonsingular if its determinant differs from zero, $|\mathbf{A}| \neq 0$.)

Premultiplying both sides of equation (12) with $\mathbf{A}^{-1}$ yields the unique solution of the equation system:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}\,. \tag{13}$$

If the numerical values of the parameters a_{ij} and b_i are known, the inverse of **A** can be calculated numerically. Computers can calculate the inverse of big matrices. Computing the inverse of a matrix, however, is a costly problem and is therefore avoidable whenever possible (Judd 1998, p. 84). A more efficient and numerically more accurate ways of solving the equation $\mathbf{Ax} = \mathbf{b}$

is for instance *Gaussian elimination* (Strang 1986, chapter 1).[4] Note that equation (13) can also be used to determine the analytical solution (well, it is already the analytical solution, but not as explicit as we like.)

3 The Software

We will use Matlab to solve linear equation systems numerically. We will use backslash or *matrix left division* in Matlab for solving linear equation systems, e.g.:

```
» X=A\B
```

If **A** is a square matrix, `A\B` is roughly the same as `inv(A)*B`, except it is computed in a different way. If **A** is an n-by-n matrix and **B** is a column vector with *n* components, or a matrix with several such columns, then **X** = **A\B** is the solution to the equation **AX** = **B** computed by Gaussian elimination. A warning message prints if **A** is badly scaled or nearly singular.

Notice: We will calculate the analytical solution with Maple in Session 9.

4 References

Chiang, Alpha C. (1984), *Fundamental Methods of Mathematical Economics*, Auckland: McGraw-Hill, chapter 4 and 5.

Hall, Robert E. and John B. Taylor (1991), *Macroeconomics*, 3rd edition, New York: Norton.

[4] Strang (1986, p. 56) shows how bad algorithms ruin the numerical accuracy of the solution. The Matlab *matrix left division* (\) uses Gaussian elimination with partial pivoting as described in Strang. Judd (1998, chapter 3) describes a variety of sophisticated numerical solving strategies in detail.

Mankiw, N. Gregory (1997), *Macroeconomics*, 3rd edition, New York: Worth.

Strang, G. (1986): *Linear Algeba and its Applications*, 3rd Edition, San Diego: Harcourt Brace Jovanovich.

5 Today's Task

Exercise 1: solve the static IS-LM model numerically

We start with the simple IS-LM Model from session 1.

a) Rewrite equation (1) to (4) from session 1, such that all endogenous variables are on the left hand side (lhs) and only exogenous variables are on the right hand side (rhs). Now, rewrite the 4 by 4 equation system as a matrix equation of the form $\mathbf{Ax} = \mathbf{b}$.

b) Let us open a new *Editor* window in Matlab. Let us proceed as follows. First, we define the parameters and, then, use the parameter names in the definition of the matrix **A** and the vector **b**. Let us use the parameters from session 1.

c) Test if the model has a non-trivial unique solution. Calculate the inverse of the matrix **A**. Calculate the solution vector $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Compare it with the results in Session 1.

d) Read the notes on the `inv` function in the Matlab html help.

e) Calculate the solution using *Gaussian Elimination*. Matlab performs Gaussian elimination with matrix left division (\).

f) Suppose, the government cuts nondistorting (lump sum) taxes without changing government spending; the resulting budget deficit is financed by borrowing to the public (i.e., a deficit-financed tax cut). If taxes are cut by 1 billion CHF (without changing government spending), by how much do GDP, consumption, investment, and the interest rate change?

Exercise 2: The two country IS-LM Model under a floating exchange rate

Consider the two country model defined in section 1.1. Assume a *floating* exchange rate regime.

g) Which variables are exogenous and which endogenous? How many unknown variables are there? And how many equations? Is the system identified?

h) Use the above parameters and solve the model numerically in Matlab. Follow the steps in Exercise 1. Look at your solution for the variable *NX*: How do you interpret the result?

i) How does a 20 billion increase in government spending of the country affect its output, net exports and both countries' consumption. Compare this to the outcome in a closed economy. This answer was worth a Nobel price.

j) How does a 20 billion increase in the money supply of one country affect net exports and consumption in the two countries.

k) Use the model to study the effects of other policy changes and shifts in parameters.

l) Solve the two-country Mundell-Fleming Model under the assumption that the Home country unilaterally pegs its currency to the Foreign country's currency. Assume for instance that in the benchmark solution Home's endogenous money supply equals the level of the Foreign money supply (210); the implied value for E then is 1. Redo problem i).

Session 5: The Cake-Eating Problem

1 The Topic

Once upon a time there was a little girl who got a cake. The girl decided to eat the cake all alone. But she was not sure when she wanted to eat the cake. First, she thought of eating the whole cake right away. But then, nothing would be left for tomorrow and the day after tomorrow. Well, on the one hand, eating cake today is better than eating eat it tomorrow. On the other hand, eating too much at the same time might not be the best. She imagined that the first mouthful of cake is a real treat, the second is great, the third is also nice. But the more you eat, the less you enjoy it. In the end you're almost indifferent, she thought. So, she decided to eat only a bit of the cake everyday. Then, she could eat everyday another first mouthful of cake. The girl knew that the cake would be spoiled if she kept it more than nine days. Therefore, she would eat the cake in the first ten days. Yet, how much should she eat everyday? She thought of eating everyday a piece of the same size. But if eating cake today is better than waiting for tomorrow, how can it possibly be the best to do the same today as tomorrow? If I ate just a little bit less tomorrow and a little bit more today I would be better off, she concluded. — And she would eat everyday a bit less than the previous day and the cake would last ten days long and nothing would be left in the end.

1.1 The Formal Structure of the Story

Let's solve the girl's problem formally. Assume that the preferences for cake are in any period described by the following utility function:

$$U(c_t) = ln c_t \tag{1}$$

where, c_t is the amount of cake consumed in period t. This utility function yields decreasing marginal utility. Future consumption is discounted with the time-preference factor $\beta < 1$. The present value in period 0 of the whole consumption path $(c_0, c_1, ..., c_T)$ is, therefore,

$$V(c_0, c_1, ..., c_T) = \sum_{t=0}^{T} \beta^t U(c_t), \tag{2}$$

where T is the last day with consumption. In the above story T is 9 as "today" is 0. These preferences are called intertemporally additive. The person tries to maximise (2) by choosing its consumption in period t = 0, ..., T. The cake size in period t is the previous size less the previous consumption:

$$k_t = k_{t-1} - c_{t-1}. \tag{3}$$

Consumption in any period must not exceed the cake size in that period. Therefore, the cake size must be nonnegative in any period:

$$k_t \geq 0 \text{ for } t = 1 \ldots T + 1. \tag{4}$$

1.2 The Relevance of the Story

The cake-eating problem yields the basic mathematical structure of the optimal growth models in modern macroeconomics. The famous Ramsey

Model (see e.g. Romer 1996, 39) can in principle be reduced to the above cake-eating problem. A society has to choose between consumption or investments. The more a society invests, the less it can consume instantaneously but the more it can produce, and hence consume, in the future. Models of optimal growth determine the optimal level of investment in any period.

1.3 Advanced: The Analytical Solution

The Cake-Eating problem can be solved analytically. The optimal consumption path satisfies the so-called *Euler equation*:

$$U'(c_t) = \beta U'(c_{t+1}) \tag{5}$$

The Euler equation has an intuitive interpretation: at a utility maximum, the consumer cannot gain from feasible shifts of consumption between periods. A one-unit reduction in period t consumption lowers U_t by U_t'. This unit saved can be shifted to period $t + 1$ where it raises utility by U'_{t+1}. Discounted to period t this is $\beta \cdot U'_{t+1}$. In the optimum, these two quantities must be equal. Using utility function (1), the Euler-equation (5) is

$$c_{t+1} = \beta c_t . \tag{6}$$

In the optimum, nothing is left in the end:

$$k_{T+1} = 0 . \tag{7}$$

Recursive insertion of (3) and (6) in (7) yields the optimal initial period consumption:

$$c_0 = \frac{1-\beta}{1-\beta^{T+1}} k_0 . \tag{8}$$

Together with (6) the optimal consumption path is exactly described.

2 The Method

The general method to solve this kind of problems is called *Dynamic Optimisation*. In dynamic optimisation problems a path of variables is choosen in order to maximise a criterion function subject to a set of intertemporal constraints. This is very advanced mathematics and its foundations are hidden from most economists (the authors self-evidently included). Several textbooks show how dynamic optimisation is applied in economics. See e.g. Kamien and Schwartz (1995) or Stokey and Lucas (1989). We will leave this to theorists.

If the problem is small and nice it may be solved numerically. This is an example of numerical constrained optimisation. Using an already programmed algorithm, this is a handy way to solve the problem. (You will learn how to program such an algorithm in session 10.) However, numerical solutions have several disadvantages: they may not be feasible, the algorithm may not find a solution, the algorithm may yield inaccurate results, the algorithm may find a local rather than a global maximum or the algorithm may be very time consuming.

3 The Software

Excel contains a powerful solution algorithms for solving both linear and nonlinear programming problems. Read the separate handout if you are not already familiar with the solver in Excel.

4 References

Kamien, Morton I. and Nancy L. Schwartz (1995), *Dynamic Optimization : The Calculus of Variations and Optimal Control in Economics and Management*, Amsterdam: North-Holland.

Romer, David (1996), *Advanced Macroeconomics*, New York: McGraw-Hill.

Stokey, Nancy L. and Robert E. Lucas (1989), *Recursive Methods in Economic Dynamics*, Cambridge: Harvard University Press.

5 Today's Task

Exercise 1: A Spreadsheet of the Basic Problem

We will implement the above Cake-Eating problem in an Excel spreadsheet. The appendix shows how such a spreadsheet might look like.

a) Make a sketch of what you want to put where on your Excel sheet.

b) Choose a cell in which you can put the value of the time-preference factor β and the initial size of the cake k_0. Describe the contents of the cell in another cell next to it. You may assign names to the cells with the value.

c) Find a place for a table with 11 columns and 4 rows. The first row contains the period numbers from 0 to 10. The second row contains the corresponding period consumption c_t. The third row is the size of the cake

at the beginning of the period t. The fourth and the last row contain the period utility and the discounted period utility respectively.

d) Fill the consumption path with an arbitrary series of number, for example ones. Why is there no consumption in period 10?

e) Assign the initial cake size to period 0. Use equation (3) to compute the cake size in all further periods including period 10.

f) Use the utility function (1) to calculate the period utility.

g) Get the discounted utility from each period applying the inner part of equation (2).

h) Assign the sum of the discounted period utilities to a cell.

i) Which cell should be maximal in the optimal solution of the cake-eating problem?

j) Which cells can be changed in order to maximise the discounted utility.

k) Which cells are concerned with the constraints in (4).

l) Solve the Cake-eating problem using the Solver: Extras/Sover... .

m) Try different starting values. The Solver uses the current values in the cells changed as starting values. Fill, for example, the consumption path with zeros. Explain.

n) Advanced: Add a table with the analytical solution from section 1.3 and compare the result with the numerical optimisation. Try different convergence criterion in the Solver options to obtain more precise results.

Exercise 2: Understand and use the Model

a) How should the consumption path change as the discount factor rises. Check it in your spreadsheet.

b) How changes the solution with different initial cake sizes.

c) What are two fundamental assumptions that drive the resulting

consumption path. Do you think they are reasonable. Do you agree with the outcome.

Exercise 3: The Cake Gets Worse

So far, we assumed that the quality of the cake is constant until day 9 and then suddenly the cake is rotten. What happens if the cake gets worse during the first nine days. We capture this by assuming that the cake's quality depreciates at a constant rate δ. The 'quality adjusted size' of the cake follows therefore:

$$k_t = k_{t-1} - c_{t-1} - \delta(k_{t-1} - c_{t-1}) = (1-\delta)\cdot(k_{t-1} - c_{t-1}). \qquad (9)$$

Again, the cake is rotten after ten days, whatever its size at day 9 was.

d) Change your Excel spreadsheet to include this new feature.

e) Try different depreciation rates and study the reaction of the consumption path.

Exercise 4: A More General Utility Function

Consider the following constant relative risk aversion (CRRA) or constant elasticity of intertemporal substitution utility function:[1]

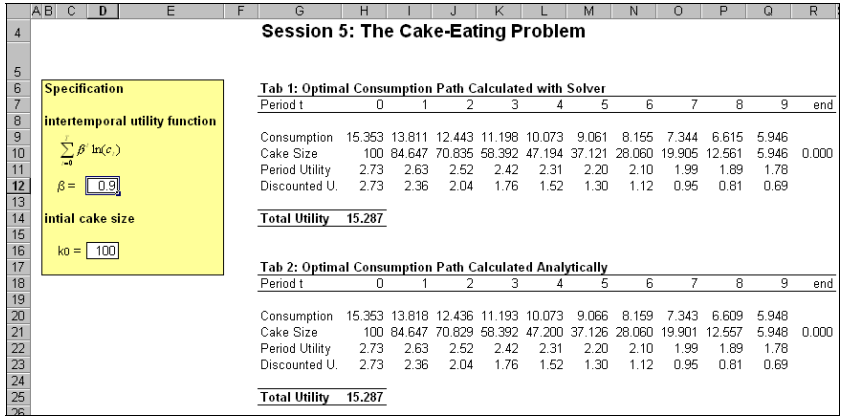
$$u(c) = \frac{c^{1-\theta}}{1-\theta}$$

f) Implement this utility function in your Spreadsheet.

g) Try different parameter values for β, δ, θ.

[1] The initial utility function ln(c) is the limit case of this new utility function as θ goes to 1, when the constant –1/(1-θ) is added. Remember from microeconomic theory that adding a constant does not change the preference structure.

6 Appendix: The Resulting Excel Sheet and the Solver Parameters



Session 5: The Cake-Eating Problem

Specification

intertemporal utility function

$\sum_{t=0}^{T} \beta^t \ln(c_t)$

$\beta =$ 0.9

intial cake size

k0 = 100

Tab 1: Optimal Consumption Path Calculated with Solver

Period t	0	1	2	3	4	5	6	7	8	9	end
Consumption	15.353	13.811	12.443	11.198	10.073	9.061	8.155	7.344	6.615	5.946	
Cake Size	100	84.647	70.835	58.392	47.194	37.121	28.060	19.905	12.561	5.946	0.000
Period Utility	2.73	2.63	2.52	2.42	2.31	2.20	2.10	1.99	1.89	1.78	
Discounted U.	2.73	2.36	2.04	1.76	1.52	1.30	1.12	0.95	0.81	0.69	

Total Utility 15.287

Tab 2: Optimal Consumption Path Calculated Analytically

Period t	0	1	2	3	4	5	6	7	8	9	end
Consumption	15.353	13.818	12.436	11.193	10.073	9.066	8.159	7.343	6.609	5.948	
Cake Size	100	84.647	70.829	58.392	47.200	37.126	28.060	19.901	12.557	5.948	0.000
Period Utility	2.73	2.63	2.52	2.42	2.31	2.20	2.10	1.99	1.89	1.78	
Discounted U.	2.73	2.36	2.04	1.76	1.52	1.30	1.12	0.95	0.81	0.69	

Total Utility 15.287

Session 6: Hog Cycle Model

1 The Topic

The Hog Cycle or Cobweb Model is a standard microeconomic supply and demand model for a single good with a dynamic extension.[1] It is called Hog[2] Cycle Model, because the supply of pork in the US and its price shows the features of this model.

Since only one commodity is being considered, it is necessary to include only three variables in the model: the quantity demanded of the commodity, QD_t, the quantity supplied of the commodity, QS_t, and its price, P_t, The quantity is measured, say, in tons per week, and the price in dollars. We make certain assumptions:

First, we must specify an equilibrium condition. The standard assumption is that equilibrium obtains in the market if and only if excess demand is zero ($QD_t - QS_t = 0$). Second we must describe how QD_t and QS_t itself are determined. We assume that QD_t is a decreasing linear function of P_t. QS_t is postulated to be an increasing linear function of the expected value of P_t, based on information available in $t-1$, with the proviso that no quantity is supplied unless the price exceeds a particular positive level. That is, we assume that the supplier's output decision must be made one period in

[1] As we shall see below, the static theory of price and quantity determination in an isolated market is modified by introducing a time delay in the supply function.

[2] Dt. Hausschwein.

advance of the actual sale – such as in agricultural production, where planting must precede by an appreciable length of time the harvesting and sale of the output. Third, we assume that farmers form their expectations in a naïve fashion: they base their decision in period $t-1$ for their supply in t, QS_t, simply on the observed price in period $t-1$, P_{t-1}.

Translated into mathematical statements, the model can be written as

$$\left.\begin{aligned} QD_t &= QS_t \\ QD_t &= a - bP_t \qquad (a,b>0) \\ QS_t &= -c + d\,E_{t-1}(P_t) \quad (c,d>0) \\ E_{t-1}(P_t) &= P_{t-1} \end{aligned}\right| \qquad (1.1)$$

Four exogenous parameters, a, b, c, and d, appear in the two linear functions, and all of them are specified to be positive.

Equation (1.1) can be simplified as follow. We know that the solution values of the three endogenous variables, QD_t, QS_t, and P_t are those values that satisfy the three equations in (1.1) simultaneously. Hence, by substituting the last three equations into the first the model can be reduced to a single *linear first-order difference equation*

$$bP_t + dP_{t-1} = a + c \qquad (1.2)$$

or, equivalently,

$$P_t = \frac{-d}{b}P_{t-1} + \frac{(a+c)}{b} \qquad (1.3)$$

In the next section we shall ask ourselves how a dynamic system like (1.3) evolves over time and what the effects on P_t of changes in the parameter values a, b, c and d, are.

2 The Method

The purpose of this section is to provide you with the basics of a mathematical tool called *difference equations* which is widely used in econometrics and macroeconomics.[3]

2.1 Linear First-Order Difference Equations

We say a variable y_t follows a *linear first-order difference equation* if its dynamics can be described by

$$y_t = \phi y_{t-1} + w_t \,, \tag{2.1}$$

where ϕ is a parameter and w_t is an exogenous function of time. In other words, equation (2.1) relates the variable y_t to its previous value and a so called driving force. This is a first-order difference equation because only the first lag of the variable (y_{t-1}) appears in the equation. It is linear because no y term (of any period) is raised to power or is multiplied by a y term of another period. If the term w_t is nonzero the above difference equation is called *nonhomogenous*.

2.2 Time Path of *y*

How does a dynamic system of that kind evolve over time? For each date t we have an equation relating the value of y for that date to its previous value and the current value of w:

[3] A related topic are *differential equations*. Both difference and differential equations describe the evolution of a variable(s) over time, t, the former in discrete time (t = 1, 2, 3 ...) and the latter in continuous time.

Date	Equation	
0	$y_0 = \phi y_{-1} + w_0$	(2.2)
1	$y_1 = \phi y_0 + w_1$	(2.3)
2	$y_2 = \phi y_1 + w_2$	(2.4)
:	:	(2.5)
T	$y_t = \phi y_{t-1} + w_t$	(2.6)

Suppose we know the starting value of y for date t = -1 and the value of w for dates t = 0, 1, 2, … Hence, it is possible to calculate the value of y for any date. For example, if we know the value of y for t = -1 and the value of w for t = 0, we can calculate the value of y for t = 0 directly from (2.2). Given this value of y_0 and the value of w_1, we can calculate the value for y_1 (2.3), and so on.

2.3 Solution

A solution to a difference equation expresses the value y_t as a mathematical function of the (known) elements w_t and some given initial condition of y_0. Let's see how to solve (2.1).

Given the value of y_0 and the value of w_1 we can calculate:

$$y_1 = \phi y_0 + w_1$$
$$y_1 = \phi\left(\phi y_{-1} + w_0\right) + w_1$$
$$y_1 = \phi^2 y_{-1} + \phi w_0 + w_1$$

Given this value of y_1 and the value of w for $t = 2$, we can calculate the value of y for $t = 2$ (2.4):

$$y_2 = \phi y_1 + w_2$$
$$y_2 = \phi\left(\phi^2 y_{-1} + \phi w_0 + w_1\right) + w_2$$
$$y_2 = \phi^3 y_{-1} + \phi^2 w_0 + \phi w_1 + w_2$$

Continuing recursively in this fashion, the value that y takes on at date t can be described as a function of its initial value y_{-1} and the history of w between date 0 and date t:

$$y_t = \phi^{t+1} y_{-1} + \phi^t w_0 + \phi^{t-1} w_1 + \phi^{t-2} w_2 + \ldots + \phi w_{t-1} + w_t$$
$$y_t = \phi^{t+1} y_{-1} + \sum_{j=0}^{t} \phi^j w_{t-j} \qquad (2.7)$$

Of course (2.7) can alternatively be written as

$$y_{t+j} = \phi^{j+1} y_{t-1} + \phi^j w_0 + \phi^{j-1} w_1 + \phi^{j-2} w_2 + \ldots + \phi w_{t+j-1} + w_{t+j} \qquad (2.8)$$

This procedure is known as solving the difference equation (2.1) by *recursive substitution*. There are alternative methods. Those who are interested in that issue should consult one of the textbooks mentioned in References.

2.4 Dynamic Stability

Let's look at the solution in (2.7). Suppose $w_t = w$ for all t and y_{-1} is an arbitrary constant. Given sufficient time for the adjustment process, does our

dynamic system tend to converge to a particular equilibrium value, $\bar{y}$, or does it not?[4] (2.7) becomes to

$$\lim_{t\to\infty} y_t = \phi^{t+1} y_{-1} + w\left[\phi^t + \phi^{t-1} + \phi^{t-2} + \ldots + \phi + 1\right]$$

Evidently, the answer crucially depends on the parameter ϕ. *Dynamic stability* is achieved if and only if $|\phi| < 1$. In this case (2.7) can be written as

$$\lim_{t\to\infty} y_t = \frac{1}{1-\phi} w = \bar{y}$$

In general, there are six cases of dynamics: the stable monotone case, the stable oscillatory case, the unstable monotone case, the unstable oscillatory case, and two borderline cases. Borderline case 1 lies between the stable and the unstable monotone case. In this case, the output variable y is the sum of the historical inputs w plus the initial value of y, y_{-1}. Correspondingly boundary case 2 lies between the stable and the unstable oscillatory case.

2.5 Effects on *y* of changes in *w*

The fact that (2.7) expresses y_t as a linear function of the initial value y_{-1} and the historical values of w makes it very easy to calculate the effect of w_0 on y_t. If w_0 were to increase by one unit with y_{-1} and w_1, w_2, …, w_t taken as

---

[4] Formally, the linear difference equation $y_t = \phi y_{t-1} + w_t$ is called *stable* if, when $w_t = w$ for all t and y_{-1} is an arbitrary constant, then

$$\lim_{t\to\infty} y_t = \bar{y}$$

where $\bar{y}$ does not depend on y_{-1}.

unaffected (we call this a *transitory* change in w), the effect on y_t would be given by

$$\frac{\partial y_t}{\partial w_0} = \phi^t \tag{2.9}$$

or, alternatively,

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j \ . \tag{2.10}$$

Thus the so called *impulse-response function* in (2.10) depends only on j, the length of time separating the disturbance to the input (w_t) and the observed value of the output (y_{t+j}).

Again different values of ϕ in (2.1) can produce a variety of dynamic responses of y to w. If $0 < \phi < 1$, the impulse-response function decays geometrically toward zero the farther into the future one goes. If $-1 < \phi < 0$, the impulse-response function will alternate in signs. In this case an increase in w_t will cause y_t to be higher, y_{t+1} to be lower, y_{t+2} to be higher, and so on. The absolute value of the effect of a given change in w_t eventually dies out. If $\phi > 1$, the impulse-response function increases exponentially over time. A given increase in w_t has a larger effect the farther into the future one goes. For $\phi < -1$, the system exhibits explosive oscillation. In the two borderline cases the initial effect remains in action for ever.

So far we were looking for the effect of a purely transitory change in w. However, sometimes we might be interested in the consequences of a *permanent change* in w. A permanent change in w means that w_t, w_{t+1}, …, and w_{t+j} would all increase by one unit. From formula (2.10), the effect on y_{t+j} of a permanent change in w beginning in period t is given by

$$\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \frac{\partial y_{t+j}}{\partial w_{t+2}} + \ldots + \frac{\partial y_{t+j}}{\partial w_{t+j}} = \phi^j + \phi^{j-1} + \phi^{j-2} + \ldots + \phi + 1.$$

When $|\phi| < 1$, the limit of this expression as j goes to infinity, is sometimes described as the "long run" effect of w on y:

$$\lim_{j\to\infty}\left[\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \frac{\partial y_{t+j}}{\partial w_{t+2}} + \ldots + \frac{\partial y_{t+j}}{\partial w_{t+j}}\right] = 1 + \phi + \phi^2 + \ldots = \frac{1}{1-\phi} \tag{2.11}$$

Finally we might wonder about the *cumulative consequences* for y of a one-time change in w. As in the former case we consider a transitory disturbance to w but wish to calculate the sum of the consequences for all future values of y. It's evident that this is exactly the same as the long-run effect of a permanent change in w, i.e. equation (2.11).

3 The Software

We work in Excel. You already know all the required tools. We will repeat some of the exercises with Matlab in the next session.

4 References

Chiang, Alpha C. (1984), *Fundamental Method of Mathematical Economics*, McGraw-Hill.

Hamilton, James D. (1994), *Time Series Analysis*, Princeton University Press.

Obstfeld, M. and K. Rogoff (1996), *Foundations of International Macroeconomics*, Cambridge: MIT Press. Supplement C to chapter 2.

5 Today's Task

Exercise 1: the evolution of a dynamic system and the impulse-response functions

a) Let's consider equation (2.1). Suppose that $\phi = 0.7$. w_t is given by a constant term, let's say 0. Open a new spreadsheet in *Excel*. In a first column generate time $t = 1, 2, 3, \ldots 20$. Supposing $y_{-1} = 1.5$ you can compute the evolution of y over time (the *time path* of y) in a separate column. Plot y against t (with time on the horizontal axis).

b) What are the effects of a change in ϕ on the time path of y (given that $w_t = 0$ for all t and y_{-1} remain unchanged)? When is the system *dynamically stable*?

c) Suppose that w_1 is increased by 1 with y_{-1} and $w_2, w_3, w_4, \ldots$ taken as unaffected (a *temporary* change in w). Compute and plot the new time path for y.

d) Compute and plot the immediate and future effect on y of a temporary change in w, beginning in period 1 and ending in period 20. Does the impulse-response function only depend on the length of time separating the disturbance and the observed value of the y? (Hint: The difference between the two time series computed in b) and c) gives you the immediate and future effects on y of a one-time change in w.)

e) Assume that w_t is *permanently* risen by 1, i.e. $w_t = 1$ for all t. Plot a time path for y.

f) What are the immediate and future effects on y of a permanent change in w, beginning in period 1 and ending in period 20?

g) What are the *cumulative consequences* of a temporary change in w? Compare with f) and explain.

h) Optional: Let's assume that w_t is a stochastic variable which is distributed uniformly between -0.5 and 0.5. Generate w and then y . Plot the time path of y. How does the impulse response function change? What happens if $\phi = 1$?

i) Again consider equation (2.1). Suppose this time w_t is given by $\alpha + \beta x_t$. For example $\alpha = 2$ and $\beta = 1$ leads to $y_t = 0.7y_{t-1} + 2 + x_t$. Repeat a) to g).

Exercise 2: the cobweb model

a) Let's go back to equation (1.3). Given $a = 3.0$, $b = 0.51$, $c = 0.50$, $d = 0.60$, and initial price $P_0 = 3$. How does P_t evolve over time?

b) Assume different values for d and b. Show that there are three possible patterns. (Recall that $d, b > 0$.)

In order to illustrate the essence of the Hog Cycle Model imagine a conventional *Marshallian cross*. Given an initial price P_0, we can read off on the supply curve that the quantity supplied in the next period, period 1, will be Q_1 as a function of P_0 (according to the naïve expectation formation described above). In order to clear the market, the quantity demanded in period 1 must also be Q_1, which is possible if and only if price is set at the level of P_1. Now, via the supply curve, the price P_1 lead to Q_2 as the quantity supplied in period 2, and to clear the market in the latter period, price must be set at the level of P_2 according to the demand curve. Repeating this reasoning we can trace out the prices and quantities in subsequent periods in the diagram, thereby spinning a *cobweb* around the demand and supply curves.

c) Try to plot a graph which shows the demand and supply curves, and the cobweb. Redo b) and watch the cobweb.

Session 7: Introduction into Matlab II

1 The Topic

You already know a lot about Matlab. Among others you know how to build and manipulate matrices, how to plot graphs and how to import data. Today we would like to acquaint you with some additional, very handy features. In session 4 you made the experience that it can become very unsatisfying to input individual commands in the command window. At that time we mentioned that Matlab provides possibilities of processing frequently needed sequences of instructions more comfortably. Apart from the so-called *script M-files* there is yet another type of text files, called *function M-files*. In addition to this, like any other computer programming language, Matlab offers features that allow you to *control the flow* of command execution. Thus, *script M-files*, *function M-files*, and *controlling the flow* are the topics of today's session.

2 The Method

We are not going to introduce any new methods but take up some familiar ones like linear fist-order difference equations and simple linear regression. Besides 'Introduction into Matlab' you might want to have the handouts to session 1, 2, 3, and 6 at your disposal.

3 Today's Task

3.1 Script M-files

Example 1 : Open a new M-File, write

```
disp('hello world');
```

save the file as example1.m and run it from the Matlab prompt by typing example1.

Example 2 : Write and run the following script M-file:

```
a = [1 2];
b = [3 4];
c = a + b
```

Note the semicolons. Type the name of the variables a, b, c and D in the Matlab prompt and see what happens.

Example 3 : Add comments to the above script:

```
% Define some variables
a = [1 2];     % this is a vector
b = [3 4];     % this is another vector
% Calculate new variables
c = a + b      % add
```

Exercise 1: the simple OLS regression revisited

Recall the simple OLS regression introduced in session 2 and 3.

a) Import *real private consumption* and into the workspace and call it **y**. Do the same for *real disposable income* and call it **x**.
b) Write a m-file that calculates the ordinary least squares estimator **b** , which contains the constant and the slope coefficient. Note that the m-file can access the vectors **y** and **x** in the workspace.
c) Advanced: Extend your m-file to calculate the coefficients' standard errors and t-statistics for the Null Hypothesis that the coefficient is zero.

3.2 Controlling the flow – for loop

Example 4 :

```
for i = 1:10;
   disp(i);
end;
```

Example 5 :

```
x = 0
for i = 1:10;
   x(i)=2^i;
end;
x
```

Example 6 :

```
x = zeros(10,1)
for i = 1:10;
   x(i)=2^i;
end;
x
```

Example 7 :

```
A = zeros(5,5)
for i = 1:5;
   for j = 1:5;
      A(i,j) = i^j;
   end;
end;
A
```

Example 8 :

```
for i = 1:2:9;
   disp(i);
end;
```

Example 9 :

```
x = zeros(10,1)
for i = [2 4 6 8 10];
   x(i) = 2^i;
end;
x
```


Exercise 2: Geometric series

a) Write a m-file that calculates the (scalar) value of a geometric series $s(n) = 1 + q + q^2 + q^3 + ... + q^{n-1}$ for $q = 0.5$ and $n = 10$.

b) Calculate the values of the geometric series for $n = 1 \ldots 10$ and store the results in a vector **s**.

c) Store the values of the geometric series for $n = 1 \ldots 10$ and $q = 0.2$ in the first row of a matrix **S**. Store the values for $q = 0.5$ and $q = 0.8$ in row 2 and 3.

Exercise 3: Difference equations revisited

We go back to the simple linear difference in the handout of session 6: $y_t = \phi y_{t-1} + w_t$. Assume that $\phi = 0.7$, $y_1 = 1.5$ and $w_t = 0$ for all periods t.

a) Generate the time path of y for $t = 1, 2, \ldots, 20$ with the means of a for loop. Plot the time path of y. What happens if $\phi = 1$?

b) Consider a temporary change in w: $w_2 = 1$. How is the time path of y affected? Compute and plot the impulse response function.

c) Consider a permanent change in w: $w_t = 1$ for $t = 2, 3, \ldots$. How is the time path of y affected? Compute and plot the immediate and future effects.

d) Optional: Let's assume that w_t is a stochastic variable, a normally distributed random variable: $w_t \sim N(0,\sigma^2 = 0.1)$. Generate a sequence $\{w_t\}$ for $t = 2, \ldots, 20$. Compute and plot the time path of y. Compare this to the time path under a). How does the impulse response function change?

3.3 Controlling the flow – conditioning

Example 10 :

```
a = 2
b = 3
if a > b;
   disp('a is bigger than b');
end;
```

Example 11 :

```
a = 2
b = 3
if a > b;
   disp('a is bigger than b');
else;
   disp('a is smaller or equal than b');
end;
```

Example 12 :

```
a = 2
b = 3
if a > b;
   disp('a is bigger than b');
elseif a == b;
   disp('a is equal to b');
else;
   disp('a is smaller than b');
end;
```

Example 13 :

```
a = 2
b = 3
c = 3
if (a < b) & (b ~= c);
   disp('a is smaller than b and b is not equal to c');
end;
```


Example 14 :

```
x = 1
for i = 1:1000;
   x = x*2;
   if x > 100;
      break;
   end;
end;
x
```

Exercise 4: an (almost) infinite geometric series

Write a m-file that calculates the (scalar) value of a geometric series $s(n) = 1 + q + q^2 + q^3 + ... + q^{n-1}$ for $q = 0.5$ and $n \to \infty$ using a for loop. Stop the loop when the value of $s(n)$ equals approximately $s(n\text{-}1)$. Example 14 may give you an idea of how to do that.

Exercise 5: long term effect of permanent change in a difference equation

The long term effect to a permanent change of w in the difference equation $y_t = \phi y_{t-1} + w_t$ converges towards a certain value (see plot in exercise 3c). Write a m-file that calculates this value. Use a for loop and terminate the loop when the value of the effect in t equals approximately the effect in t-1. Example 14 may give you an idea of how to do that.

3.4 Controlling the flow – while loop

Example 15 :

```
i = 0
while i <= 9;
   i = i+1;
   disp(i);
end;
```

Example 16 :

```
x = 1
while x <= 100;
   x = x*2;
end;
x
```

Exercise 6: and yet another (almost) infinite geometric series

Do exercise 4 using a while loop. Example 16 may give you a hint of how to do that.

Exercise 7: and yet another (almost) long term effect

Do exercise 5 using a while loop. Example 16 may give you a hint of how to do that.

3.5 Function m-file

Example 17 : Save the following function m-file as parabola.m

```
function y = parabola(x)
a = 0;
b = 0;
c = 1;
y = a + b*x + c*x^2;
```

Switch to the Matlab prompt and apply your function by typing

```
parabola(4)
```

or

```
y = parabola(4)
```

or

```
z = parabola(5)
y
```

Note how the variables in the function affect the variables in the workspace.

Try the following command in the Matlab prompt

```
x = [1:5]
parabola(x)
```

Analyse what happened and generalize your function accordingly.

Example 18 : Save the following function m-file as stat.m

```
function [mx, sx] = stat(x)
% function [mx, sx] = stat(x)
% compute mean and standard deviation of vector x
mx = mean(x);
sx = std(x);
```

Switch to the Matlab prompt and apply your function by typing

```
z = randn(10,1)
stat(z)
```

or

```
[mz, sz] = stat(z)
```

or

```
help stat
```

Exercise 8: a regression function

a) Turn your script m-file from exercise 1 into a function. This function shall take a vector **y** with observations of the dependent variable and a vector **x** with observations of the independent variable as input and return the coefficient vector **b**. Call your function linreg.

b) Advanced: Let the function also return the standard errors and the t-statistics for the Null Hypothesis that the coefficient is zero.

Session 8: Mechanical Trading Rules

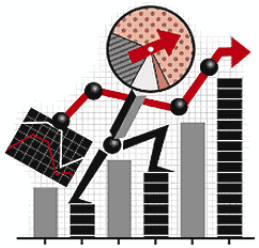
1 The Topic

Heads, shoulders and broadening bottoms

The Economist, Aug 17th 2000

Economists used to dismiss as mere snake-oil salesmen technical analysts who use simple patterns in past data to predict future share prices. Now, some think that stock charts can reveal valuable information

THE popularity of business television channels such as CNBC has done wonders for the careers of Wall Street's technical analysts, who claim to be able to predict future share prices by spotting trends in past prices. Their market charts, showing descriptively named patterns such as "head and shoulders"—a big peak surrounded by two smaller peaks—or "broadening bottoms"—a series of troughs, each lower than the preceding one—make ideal graphics for television producers in need of something pretty for viewers to watch while the glamorous reporters are busy off-camera, catching up on the latest market gossip. The simple trading advice conveyed by charts is, for CNBC's stock-tip-hungry viewers, manna from heaven.



But technical analysis is not merely for gullible CNBC-watchers. It has been around a long time, dating back a century to Charles Dow, the founder of Dow Jones, who invented the "Dow theory" for identifying trends in share prices. Charts are used by some of the world's most successful investors, such as Stanley Druckenmiller, who until recently managed George Soros's Quantum hedge fund, and Jeffrey Vinik, who used to run Fidelity's huge Magellan mutual fund and is now a star hedge-fund manager.

Nevertheless, economists who study financial markets have long regarded technical analysis as mumbo jumbo, bearing much the same relationship to rigorous economic "fundamental analysis" that astrology does to astronomy. Since the 1960s, economists have believed, more or less, in "efficient-market theory". In an efficient market, prices reflect all available information, and so scouring past prices for patterns can tell you nothing useful about whether in future prices will go up or down. Instead, prices will move unpredictably, in a "random walk".

In the past decade some economists have challenged efficient-market theory, by finding numerous examples of apparently predictable movements in share prices. But there is still a fierce debate about whether these movements are predictable enough for investors to make money trading on the basis of expected price changes. The evidence was described at length in "A Non-Random Walk Down Wall Street" (Princeton University Press, 1999), a book by Craig MacKinlay, of the Wharton School, and Andrew Lo, of the Massachusetts Institute of Technology.

Mr Lo and two new co-authors have now come to the defence of technical analysis.* Using American share prices during 1962-96, they investigated the predictive ability of five pairs of widely used technical patterns: "head and shoulders", "inverse head and shoulders", "broadening tops and bottoms", "triangle tops and bottoms", and "double tops and bottoms". The study focused on technical indicators that were especially controversial among economists. It did not test many other widely used indicators, such as the relationship between the current price of a share and its 200-day moving average.

The results showed that the various technical patterns mostly occurred far more frequently than they would have done if they were truly random events. The most common patterns were double tops and bottoms—two peaks (or two troughs) at similar prices to each other—followed by head and shoulders and inverted head and shoulders. In general, the charts contained useful information about future share prices. The study does not test whether this information was useful enough to allow investors to make sufficient profit trading on it to justify the extra risk. But Mr Lo thinks this is likely.

However, there was a significant difference between the results for different markets. For shares listed on the New York Stock Exchange and American Stock Exchange, only seven of the ten technical patterns had enough predictive power to be statistically significant. For Nasdaq shares, all ten of the patterns did.

Why the discrepancy? Individual investors account for a much larger slice of trading in Nasdaq shares than they do in NYSE and Amex shares, where institutional investors matter more. It may be that the predictive power of these technical patterns comes from the behaviour of individual investors. Certainly, that is Mr Lo's hunch: that updating the study for the past three years, when individual share-trading has grown to an unprecedented level, will show that the technical measures have worked even more strongly for Nasdaq stocks.

The study does not explore what causes the patterns: that is for future research. But one possibility is that they reflect changes in investor psychology. The appetite of investors for risk may change in predictable ways in response to particular changes in share prices. Nor is it clear, yet, whether the predictive powers of the technical patterns would be—as one might expect—eroded if investors traded to take advantage of them. Since investors have been using charts for 100 years or so and they still seem to work, the patterns may be so deeply ingrained that their predictive powers will persist come what may.

The traditional patterns used in technical analysis were, of necessity, fairly crude, determined by what was readily visible to the eye. As a result, technical analysis has always been more of an art than a science. Using computers, Mr Lo next hopes to model traditional patterns in a much more precise, systematic way—and perhaps to detect entirely new patterns that do not leap out so clearly from share-price charts. This is an intriguing prospect for investors. But if Mr Lo really wants his new, more scientific technical analysis to catch on, he may need to find a way to make it look good on the telly.

*"Foundations of technical analysis: computational algorithms, statistical inference, and empirical implementation", by Andrew Lo, Harry Mamaysky and Jiang Wang, *Journal of Finance*, August 2000.

1.1 Our Inquiry

In this session, we will study the performance of technical analysis in the context of foreign exchange trading. Consider the options open to a Swiss bank that temporarily has excess funds to be invested overnight. The bank could simply lend that money in the domestic overnight bank money market. The net return on each Swiss franc (CHF) would be the interbank overnight rate on Swiss deposits, i_t^{CHF}. Hence, one invested Swiss Franc will yield

$$1 + i_t^{CHF}. \tag{1}$$

The bank has yet another investment option. It could instead convert its money to a foreign currency, let's say the U.S. dollar (USD), lend its money in the overnight U.S. money market (at the U.S interbank overnight rate) and then convert it back to CHF tomorrow. The revenue of one Swiss Franc invested this way is then

$$\frac{1}{s_t}\left(1 + i_t^{USD}\right)s_{t+1}, \tag{2}$$

where i_t^{USD} is the date t U.S. interbank overnight rate and s_t is the date t nominal exchange rate CHF/USD.

Now, we propose you the following game. Suppose you are an investor who wants to make a lot of money in foreign exchange trading. You think that there are predictable patterns in the exchange rate data that allow you to buy US dollars just before an appreciation of the U.S. dollar and to sell them before a depreciation. Your claim is that following this strategy you can outperform the strategy implied by equation (1), that is, of investing a given amount of money in the Swiss overnight market. In other words, your job is to uncover trends in the exchange rate data and to forecast the future exchange rate behavior by extrapolating these past trends into the future and make the appropriate trading decisions.

You may refuse to participate with the argument that if there were a way to make money with little risk from past prices, market participants would employ it until they bid away the money to be made. In this case we ask you to forget your knowledge about the efficient market hypothesis and to participate anyway – since it will be fun, as you will find out in a minute! [1]

2 The Method

2.1 Technical analysis

Professional exchange rate forecasters who believe that current and past price movements contain just the information needed to allow profitable trading, that asset prices move in trends, and who are convinced that asset traders will tend to react the same way when confronted by the same conditions are called technical analysts.

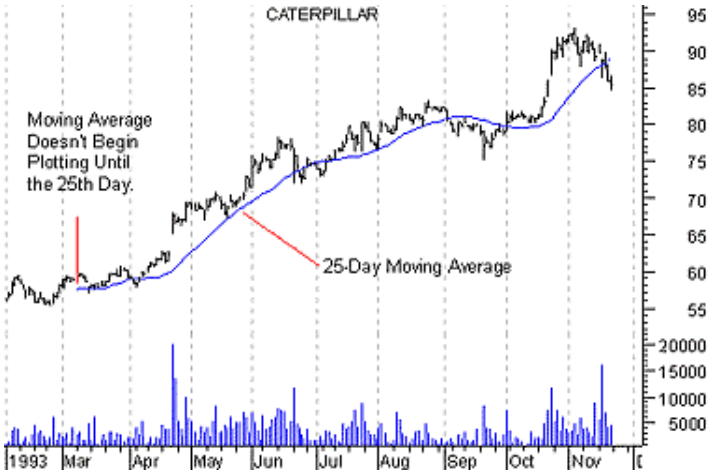
To distinguish trends from shorter-run fluctuations and to spot turning points, technical analysts employ two types of techniques. *Charting*, the older of the two methods, involves graphing the history of prices over some period – determined by the practitioner - to predict future patterns in the data from the existence of past patterns. Charting requires the analyst to use judgement

[1] You may use another argument: You may argue that economists believe that asset prices behave like random walks. But this means that one cannot get rich by observing past histories: future price changes are independent of past history. The current price predicts the future price.

and skill in finding and interpreting patterns. The second type of method, *mechanical trading rules*, imposes consistency and discipline on the technician by requiring him to use rules based on mathematical functions of present and past exchange rates. Since technical trading strategies requiring judgement and skill are impossible to quantify and therefore unsuitable for testing we focus on the second type of technical analysis.

Let's consider the following of mechanical trading rules: a) A *filter rule* suggests buying (selling) when the exchange rate rises (falls) x percent above its value a given period earlier. The analyst must make two choices to construct a filter rule. First, she must decide how much the exchange rate has to rise, i.e. what the size of the filter is. The size of the filter is chosen from past experience and lays generally between 0.5 percent and 3 percent. Second, the analyst must determine how far back the rule should go in finding a recent minimum. A typical window length is about five trading days.

b) A second variety of mechanical trading rules is the *moving average* class (see figure below). A moving average is the average closing price of the exchange rate over a given number of previous trading days. The length of the moving average window – the number of days in the moving averages – governs whether the moving average reflects long- or short-run trends: Any moving average will be smoother than the original exchange rate series, and long moving averages will be smoother than short moving averages. The window of a short moving average would typically lay between 1 and 5 days and the one of a long moving average between 10 and 50 days. A moving average trading rule prescribes a buy (sell) signal when a short moving average crosses a longer moving average from below (above).



c) A simple type of *oscillator rule* is given by the difference between two moving averages. They suggest buying (selling) the foreign currency when the difference takes a low (high) value. Oscillators are said to be useful in non-trending markets, i.e., when the exchange rate is not trending up or down strongly. Note that the oscillator rule as a difference between moving averages also generates buy/sell signals from a moving average rule when the index crosses zero. Therefore an oscillator rule is also useful for generating moving average rule signals.

2.2 Performance analysis

Technical analysis is a short-horizon trading method: positions last a few hours or days only. Switching between long and short positions, technical traders try to earn the best return on their money. Let's consider some typical ways in measuring the return and evaluate the performance of a given trading

rule. The ex post *return* from a short position is given by $\left(1+i_t^{CHF}\right)$, the ex post return from a long position is given by $s_{t+1}\left(1+i_t^{USD}\right)/s_t$.

The ex post *excess return* from a long position is given by

$$R_t^{USD} \equiv \frac{s_{t+1}}{s_t}\left(1+i_t^{USD}\right)-\left(1+i_t^{CHF}\right) \tag{2.3}$$

The ex post excess return to a short position is the negative of (2.3).

A first performance measure is given by the *overall return* to the trading rule, which is approximately the sum of daily returns or the wealth the trader ends up with (denominated in home currency).[2] The trader's wealth can be compared to the one generated by an appropriate benchmark strategy (e.g., lend the money in the overnight bank money market, day after day).

A second performance measure is given by the *mean return* over a given period of time. The trader's mean return can be compared to the return of the benchmark strategy (e.g., the mean of the overnight Swiss rate). Alternatively the *mean excess return* can be computed.

Comparing different investment strategies we would like to punish those who bear a higher risk. The *Sharpe ratio*, a third performance measure, normalises the average excess return of the portfolio over the riskfree rate (here the overnight Swiss rate) with the total risk of the investment strategy, approximated by the standard deviation of the return to a particular trading rule. Alternative methods would be the Jensen regression, the Treynor regression, and the Henriksson-Merton regression.

[2] Usually, transactions costs for each trade must be taken into account. Since we want to keep the problem as simple as possible we abstract from transaction costs at this point.

3 The Software

We will provide you the *Matlab code* which implements a simple moving average rule. You will analyze the script, run the script and interpret the output. Finally you will modify the code and compute the results to your own trading rule.

4 References

Caves, R.E., J.A. Frankel, and R.W. Jones (1999), World Trade and Payments – An Introduction, Addison Wesley

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5 Today's Task

Exercise 1: analyzing the code

The first step of implementing a job like the one at issue is to translate your problem in a consistent story:

(1) There is a trader. She starts with some starting capital (or initial wealth). In the morning of the first trading day she looks at the screen and learns the actual price of the CHF/USD exchange rate. She has to decide how to invest her capital overnight; in the Swiss overnight money market or in U.S. overnight money market. As a technician she bases her decision solely on the past and present CHF/USD exchange rate data. If she decides to invest in the Swiss overnight money market, her money on CHF account, `VCHF(t)`, remains untouched. If she decides to invest in the U.S. overnight money market, her money on the CHF account is transferred to a USD account, `VUSD(t)`, at the actual exchange rate. After having made her decision she goes home. During the night her money generates a return. If the trader's money is on the CHF account, the return equals the overnight Swiss rate, `iCHF(t)`. If the trader's money is on the USD account, the return equals the overnight U.S. rate, `iUSD(t)`. Thus, the trader's wealth of the subsequent period is either given by `VCHF(t+1)= VCHF(t)*(1+iCHF(t))` or by `VUSD(t+1)=VUSD(t)*(1+iUSD(t))`, depending on where the money has been invested. The following day the trader observes the most recent exchange rate. Based on past and present exchange data she decides whether to buy USD (or CHF, respectively) or not - and so forth, day after day.

(2) The trading rule implemented here is a simple moving average rule. The window of the short moving average is one day (i.e., the original exchange rate series) and the one of the long moving average is 10 days.

(3) Let's look at the variables (in terms of matrices) and parameters needed to implement our story. The relevant time series are the CHF/USD exchange rate, `sCHFUSD`, the overnight Swiss rate, `iCHF` and the overnight U.S. rate, `iUSD`. There is the time horizon (`T`); the starting capital (`capital`); the length of the long moving average window (`windL`); the value of the CHF account (`VCHF`); the value of the USD account (`VUSD`) and the moving average (`MaL`).

(4) Since our trading rule is a moving average rule we cannot start trading before day `windL+1`. Therefore we have to define `day1`, the first trading day. Our starting capital is CHF 100´000. Thus the `day1` value of Swiss account, `VCHF(day1)`, is 100´000.

(5) Every trading day is started by applying the trading rule. The long moving average, `MaL`, is given by the sum of the last ten closing prices of the exchange rate divided by `windL`. If the short moving average (given by `sCHFUSD(t)`) crosses the long moving average from below we interpret this as a buy signal. Since the two values will barely be the same we use relational operators. If the short moving average crosses the longer moving average from above we interpret this as a sell signal. If none of the above happens (i.e., no crossing) we don't trade. If we buy USD the value of the USD account is given by the value of the CHF account divided by the actual exchange rate, while the value of the CHF account becomes zero. If instead we buy CHF the value of the CHF account is given by the value of the USD account multiplied by the actual exchange rate, while the value of the USD account gets zero. If we don't trade nothing happens.

(6) The value of the t+1 CHF account `VCHR(t+1)` is given by `VCHFt*(1+iCHF(t))`; the value of the t+1 USD account `VUSD(t+1)` by `VUSD(t)*(1+iUSD(t))`. In case the day t value of one account is zero its return is zero as well.

(7) Since the trader wants to compare the performance of her trading rule with a benchmark case we have to calculate the evolution of the trader's wealth denominated in CHF, `Wtrade`. `Wtrade` is given by the addition of `VCHF` and `VUSD`, where each elements `VUSD(t)` is multiplied by `sCHFUSD(t)`, i.e. denominated in CHF. In the benchmark case, `Wbench`, the trader invests her capital on the overnight Swiss money market, day after day. Note that the original `iCHF` series is in terms of annualized interest rates. Hence, we have to divide `iCHF(t)` by 360.

(7) Next we calculate the return generated by the trading rule and the benchmark strategy. The return to `Wtrade`, `Rtrade`, is given by the percentage change in `Wtrade`. For convenience we define return in period t as

$$R_t = \frac{W_{t+1} - W_t}{W_t},$$

where W_t denotes period t wealth. The return to `Wbench`, `Rbench`, is simply given by `iCHF`. Since `iCHF` are annualized rates we have to multiply `Rtrade` by 360. The excess return, `Rxs`, is given by the difference of `Rtrade` and `Rbench`.

(8) Finally we calculate some performance measures. We look at the final values of both vectors, `Wtrade` and `Wbench`, respectively. Then we compute `mean(Rtrade)`, `mean(Rbench)`, and `mean(Rxs)`. At last we compute the Sharpe ratio.

The implementation of this story in Matlab is printed in the appendix of this handout. See if you understand what the program does.

Exercise 2: running the code

a) To run the script M-file *`session8.m`* the above mentioned three data series are required which are contained in the Excel file *irexrate.xls*. Load them to your workspace. Make sure that the interest rates are in the required form: annualised and expressed in percentages.
b) Plot the three series against time.
c) Copy the script M-file *`session8.m`* to your directory and run the script.
d) Plot the exchange rate and the long moving average against time. What's the number of transactions?
e) Plot `Wtrate` and `Wbench` against time. What's the number of transactions?
f) Generate a chart with the relevant performance measures.

Exercise 3: modifying the code 1: implement an alternative MA-rule

Implement a short moving average longer than one day. You may also want to modify the `windL`.

Exercise 4: modifying the code 2: introduce transaction costs

So far we have assumed that there are no transaction costs. As trading is not free let's define transaction costs as a fraction of the sale or purchase amount, e.g. 0.025 percent (or 2.5 basis points) of 100´000 USD. How do transaction costs affect the performance of your trading rule?

Appendix: the Matlab File of Exercise 1 and 2

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                                              %
% Import data; define variables and parameters                                 %
%                                                                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% dataimport
% load the three column vectors sCHFUSD, iCHF, and iUSD to your MATLAB
% workspace (note that they should be of the same dimension).

% parameters
T = length(sCHFUSD);   % time horizon
capital = 100;         % starting capital, in 1'000 CHF
windL = 10;            % length of the long moving average window

% initialize variables
VCHF   = zeros(T,1);   % the value of the CHF account
VUSD   = zeros(T,1);   % the value of the USD account
MaL = zeros(T,1);      % moving average long

% first trading day
day1 = windL+1;         % first trading day
VCHF(day1) = capital;   % the day1 value of the CHF account

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                                              %
% Implement trading rule                                                       %
%                                                                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% day1-1 element of MaL (element 1 to day1-2 remain zero)
MaL(day1-1) = sum(sCHFUSD(1:day1-1))/windL;

% loop for trading day (day1 to T-1)
for t=day1:T-1

   % calculate moving average
   MaL(t) = sum(sCHFUSD(t-windL+1:t))/windL;

   % buy or sell decision
   if sCHFUSD(t-1) < MaL(t-1) & sCHFUSD(t) >= MaL(t)
      % buy USD, sell CHF
      VUSD(t) = VUSD(t) + VCHF(t)/sCHFUSD(t);
      VCHF(t) = 0;
   elseif sCHFUSD(t-1) > MaL(t-1) & sCHFUSD(t) <= MaL(t)
      % sell USD, buy CHF
      VCHF(t) = VCHF(t) + VUSD(t)*sCHFUSD(t);
      VUSD(t) = 0;
   else
      % no trade
      % nothing happens
   end; % if
```



```
   % add the interest rate to get next days value
   VCHF(t+1) = VCHF(t)*(1+iCHF(t)/(100*360));
   VUSD(t+1) = VUSD(t)*(1+iUSD(t)/(100*360));

end; % for t

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                                              %
% calculate wealth, return and performance measures                            %
%                                                                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% initialization of additional variables
Wtrade = zeros(T,1);    % trader's wealth
Wbench = zeros(T,1);    % wealth of benchmark strategy
Rtrade = zeros(T,1);    % return of trading rule
Rbench = zeros(T,1);    % return of benchmark strategy
Rxs    = zeros(T,1);    % excess return

% calculate the evolution of trader's wealth, denominated in CHF
Wtrade = VCHF + VUSD.*sCHFUSD;

% calculate the wealth of the benchmark strategy, denominated in CHF
Wbench(day1) = capital;
for t=day1:T-1
   Wbench(t+1) = Wbench(t)*(1+iCHF(t)/(100*360));
end; %for t

% calculate the annualised return generated by the trading rule
Rtrade(day1:T-1) = ((Wtrade(day1+1:T)-Wtrade(day1:T-1))./Wtrade(day1:T-
1))*(100*360);

% calculate the annualised return generated by the benchmark strategy
Rbench(day1:T-1) = iCHF(day1:T-1);

% calculate the excess return
Rxs = Rtrade-Rbench;

% performance analysis
disp('Wealth Trader');
Wtrade(T)
disp('Wealth Benchmark');
Wbench(T)
disp('Mean Return Trader');
mean(Rtrade(day1:T-1))
disp('Mean Return Benchmark');
mean(Rbench(day1:T-1))
disp('Mean Excess Return Trader');
mean(Rxs(day1:T-1))
disp('Sharpe Ratio Trader');
mean(Rxs(day1:T-1))./std(Rxs(day1:T-1))
```

Exercise Session 9: Introduction into Maple

1 Today's Task

This exercise set accompanies the script 'Introduction into Maple 6'.

Exercise 1: Basics

a) Calculate the following expressions

$$4+5;\quad 3\cdot 6;\quad 9/3;\quad 2^4;$$

$$e^0;\quad \ln(10);\quad \sqrt{50};\quad |-5|;\quad \sin(\pi);$$

$$\frac{1}{0.5^{(2/3)}+e^4}$$

b) Calculate 10/3. Show it as floating point number with 3 digits. Set the number of digits back to the default value if it is changed.

c) Define the function $y = 2 + 5x$. Display the contents of the variable *y*. Define the function $Y = 4x$. Display the variables *y* and *Y*. Define the function $z = exp(\ y\ )$. Display the variable *z*. Set *x* = 2. Display the variable *z* again. Show it as a floating point number. Set the cursor to the initial definition of y and press Enter. What happens? Set the cursor to the end of the worksheet. Unassign *x* and display the variable *z* once more.

d) Save the current worksheet. Close the current worksheet. Open a new worksheet. Display the variable *z*. What do you conclude? Clear the workspace and display *z* again. Open the worksheet and recalculate it.

Exercise 2: Manipulation of Expressions

a) Cross multiply $a(x + y)$.

b) Factorise the result in a).

c) Write $1/a + 1/b$ as one ratio.

Exercise 3: Summation

Calculate $1 + q + q^2 + q^3 + \ldots q^{n-1}$. Simplify the result.

Exercise 4: Differentiation

a) Differentiate $\ln(x)$ wrt *x*.

b) Differentiate the above result in a) again wrt *x*.

c) Calculate

$$\frac{\partial x^{1/3}y^{2/3}}{\partial x},\frac{\partial x^{1/3}y^{2/3}}{\partial y},\frac{\partial^2 x^{1/3}y^{2/3}}{\partial x^2},\frac{\partial^2 x^{1/3}y^{2/3}}{\partial y^2},\frac{\partial^2 x^{1/3}y^{2/3}}{\partial x\partial y},\frac{\partial^2 x^{1/3}y^{2/3}}{\partial y\partial x}.$$

Exercise 5: Integration

Calculate $\int xdx$, $\int_2^4 xdx$ and $\int_a^b xdx$.

Exercise 6: Limits

a) Find the limit of $(1/x) - 2$ as *x* goes to infinity.

b) Find the limit of x^x as *x* goes to 0.

c) $\lim_{x\to 3}\frac{x^2-9}{x-3}$.

Exercise 7: Solving Equations

a) Solve $5z = 3$.

b) Find the roots (Nullstellen) of $0.5z^2 + 3z - 5$.

c) Find the roots of $0.5z^2 + 3z + 5$.

d) Find the roots of $az^2 + bz + c$.

e) Solve $z^4 + 5z - 5 = 0$. Get the explicit solutions. Obtain the floating-point solutions.

Exercise 8: Graphs

a) Plot $y = x^2 + 3x$ in a interesting range of *x*.

b) Plot the functions $y = x^2 + 3x$ and $y = 2x^2 + 3x$ in one graph.

Exercise 9: Solve the simple IS-LM model analytically

Consider the IS-LM Model in session 1. Derive the analytical solution using Maple.

a) Assign names to equation (1) to (4). Remember the difference between the assignment sign (:=) and the equal sign (=). Note: the name "I" cannot be assigned in Maple, as it is used for $\sqrt{-1}$.

b) Solve the equation system. Compare your result with the solution (7) given in session 1's handout.

c) Assign the solution of the equation system. Compute analytically the effect of a unit increase of government expenditures, *G*, on output, *Y*.

d) Compute the above marginal effect for the parameter values given in session 1. Compare it with the results there).

Exercise 10: Solve the Mundell-Fleming analytically

Consider the two-country Mundell-Fleming model in session 4. Whereas the simple IS-LM model could easily be calculated without the help of a computer, the two country model defined in section 1.1 of session 4, is troublesome. Assume a floating exchange rate regime.

a) Which variables are exogenous and which endogenous. How many unknown variables are there? And how many equations? Is the system identified?

b) Define the equation system in Maple and solve it analytically. You know now what we mean by troublesome!

Exercise 11: An Application from Microeconomics

Consider the following constant relative risk aversion (CRRA) utility function:

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta}$$

a) Draw the utility function for some values of θ.

b) Calculate the marginal utility $u'(c)$. Can you simplify the result?

c) Draw the marginal utility for $\theta = 1/2$.

d) Calculate $u''(c)$.

e) Determine the coefficient of relative risk aversion:

$$\rho = -\frac{u''(c) \cdot c}{u'(c)}$$

You may have to simplify the result.

f) Find the limit of $u(c)$ as θ goes to 1.

Session 10: Consumer Behaviour

1 The Topic

This is a session on the classical theory of consumer behaviour. As this topic is treated in any textbook on microeconomic theory (see the reference list), we will not discuss the theory. Let us just present some definitions for the two goods case. Our notation follows Varian (1993,1992).

We start with a household, whose preferences can be described with a *utility function* that satisfies the standard properties

$$u(x_1, x_2),$$

where x_1 and x_2 is the consumption of good 1 and good 2 respectively. The *indifference curve* is the set of all x_1, x_2 combinations, which lead to the same utility.

The *marginal utility* with respect to (wrt) good *i* describes by how much a change in the consumption of a good *i* increases the household's utility:

$$MU_1 = \frac{\partial u(x_1, x_2)}{\partial x_1}, \quad MU_2 = \frac{\partial u(x_1, x_2)}{\partial x_2}.$$

The *marginal rate of substitution* is the rate at which a consumer is just willing to substitute a small amount of good 2 for a certain amount of good 1:

$$MRS = \frac{dx_2}{dx_1} = -\frac{MU_1}{MU_2}.$$

The consumer has income *m* and faces prices p_1 and p_2 for good 1 and 2. She maximises her utility given her *budget constraint*. The utility maximisation problem is:[1]

$$\max_{x_1, x_2} u(x_1, x_2) \quad \text{such that } p_1 x_1 + p_2 x_2 \le m \ .$$

This constrained optimisation problem can be solved through the use of Langrange's method. The *Langrangian* is:

$$L = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - m),$$

where λ is the *Lagrange multiplier*. The optimal consumption must satisfy the following first order conditions (FOC):[2]

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial \lambda} = 0.$$

The optimal consumption is a function of income and prices:

$$x_1(p_1, p_2, m), \quad x_2(p_1, p_2, m).$$

This function is called the *Marshallian demand*. The function that gives us the maximum utility at given prices and income is called *indirect utility function*:

$$v(p_1, p_2, m) = u\left[x_1(p_1, p_2, m), x_2(p_1, p_2, m)\right].$$

[1] We assume strong monotonicity of the consumer's preferences for the budget constraint to hold strictly.

[2] The FOCs are necessary but not sufficient conditions for an optimum. One has to check if the second order sufficient conditions are satisfied. See Chiang (1984, p. 381).

More advanced students should also refresh their understanding of Roy's identity, expenditure minimisation, Sheppard's lemma, and the relationships between utility maximisation and expenditure minimisation.

2 The Method

The main mathematical issue that is used in this session is *calculus*. We determine the derivative of a continuous function and use the Lagrange-multiplier method optimisation with equality constraints. This method does also demand the solution of a non-linear equation system. Chiang (1984) covers all these topics.

3 The Software

Maple is used for symbolical computation of the derivative and for solving the non-linear equation system.

4 References

Mas-Colell Andreu, Michael D. Whinston and Jerry R. Green (1995), *Microeconomic Theory*, Oxford: Oxford University Press, chapter 7.

Varian Hal R. (1992), *Microeconomic Analysis*, third ed., New York: Norton, chapter 7.

Varian Hal R. (1993), *Intermediate Microeconomics*, third ed., New York: Norton, chapter 4, 5 and its appendices.

On Calculus:

Chiang, Alpha C. (1984), *Fundamental Methods of Mathematical Economics*, Auckland: McGraw-Hill, chapter 4 - 12.

5 Today's Task

A household can consume two goods x_1 and x_2 and is characterised by the following Cobb-Douglas utility function

$$u(x_1,x_2)=\frac{1}{4}ln(x_1)+\frac{1}{2}ln(x_2).$$

The household has an income m and faces prices p_1 and p_2 for good 1 and good 2 respectively.

Exercise 1 to 3 can easily be calculated without the help of the computer. The results to exercise 1 and 2 are given in Varian (1993, p.93). Compare the Maple computations with your own calculations. Exercise 2i and 3 are addressed to people with a bit more advanced knowledge of microeconomic theory.

Exercise 1: the Cobb-Douglas utility function

a) Open a new worksheet in Maple.
b) Define the utility function u.
c) Draw a three-dimensional plot of the utility function.
d) Plot the indifference curves of the utility function.
e) Derive the marginal utility wrt (with respect to) good 1.
f) Calculate the marginal rate of substitution?
g) Do the same exercise for the utility function $u(x_1,x_2)=x_1^{1/4}x_2^{1/2}$. Compare the indifference curves and the marginal rate of substitution.
h) Use other Coefficients instead of ¼ and ½.

Exercise 2: utility maximisation

a) Define the Lagrangian L of the utility maximisation problem.
b) Derive the first derivative of L wrt x_1 , x_2 and λ.
c) Solve the equation system of the three first order conditions and name it.
d) Use the function **assign(...)** to attribute the above solution to x_1 and x_2, which makes them a function of prices and income. Show the contents of x_1 and x_2 to see the assignments Maple performed. The variables represent Marshallian demand. Note: these variables are now programming variables. You cannot recalculate the operations in exercise 2 before you **unassign** the variables.
e) Assume $m = 5$: assign this value to yet unassigned variable m. Plot the Marshallian demand for good 1. Don't assume anything on m anymore: unassign m.
f) Show the contents of u. Why did the function u change. What does the function depend on now? Define v = u; How is this function called?
g) Assume $p_1 = 1$ and $p_2 = 1$. Both x_1 and x_2 are now functions of the income m only. Plot both x_1 and x_2 as a function of the income m. What is this?
h) Still Assume $p_1 = 1$ and $p_2 = 1$. Draw a scatterplot with the implicitly defined pair $x_1(m)$, $x_2(m)$ that correspond to a certain income. What is the meaning of this curve and how is it called? Don't assume anything on prices anymore.
i) Advanced: check the identity of Roy.

Exercise 3: expenditure minimisation (advanced)

a) Solve the expenditure minimisation problem and derive Hicksian demand. These expressions may need simplifying.
b) Plot Hicksian demand.
c) Calculate and simplify the expenditure function.
d) Check Shepphard's lemma.
e) Check the two demand identities.
f) Check the two relationships between indirect utility function and expenditure function.

Exercise 4: More

a) Redo exercise 2. Derive the Marshallian demand for the general Cobb-Douglas utility function with coefficients c and d.
b) Study a CES (constant elasticity of substitution) utility function:

$$u(x_1, x_2) = \left[x_1^{\rho} + x_2^{\rho} \right]^{1/\rho}$$

and the behaviour of a household with this utility function.