Reducing the curse of dimensionality in dynamic stochastic economic problems by decomposition methods

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Abstract

Despite the rapid growth in computing power and new developments in the literature on numerical dynamic programming, many economic problems are still quite challenging to solve. Economists are aware of the so-called *curse of dimensionality* and the limits placed on the ability to solve high-dimensional dynamic models. Many of the economic models subjected to the curse of dimensionality present some special structure that can be exploited in an efficient manner. This paper introduces a decomposition methodology, based on a mathematical programming framework, to compute the equilibrium path in dynamic models by breaking the problem into a set of smaller independent subproblems. We study the performance of the method solving a set of dynamic stochastic economic models. The numerical results reveal that the proposed methodology is efficient in terms of computing time and accuracy.

Keywords: Dynamic stochastic economic model, computation of equilibrium, mathematical programming, decomposition techniques.

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1 Introduction

In many situations, applied economists consider stochastic dynamic models for forecasting, testing economic theories, and designing economic policies. Dynamic programming has been extensively used in economics because this theoretical framework is flexible enough to represent the fluctuations of unemployment, prices, consumption, production and investment, among others. From a computational point of view, solving dynamic economic problems is still a quite challenging task, in spite of the noticeable increase in computing power, storage capacity and new approaches in the literature on computational economics. The search of major realism in the economic models have pushed economists to consider more and more complex dynamic stochastic specifications which generally challenge the best existing approaches. The solvability of many economic models suffers from the so-called *curse of dimensionality*: the computing time required to solve these models presents an exponential growth associated with the dimension of the models. Therefore, reducing the curse of dimensionality in economic growth and business cycle models gives the practitioners to enlarge substantially the class of questions that can be addressed with dynamic modelling.

Many different algorithms have been proposed during the past 15 years. There are two main approaches to approximate the solution of a dynamic economic problem: *the discrete approximation* and *the smooth approximation*. The first approach consists of discretizing the value of the policy function over a refined grid of points and then solving the Bellman operator for each point of the grid (see [24]). The second approach considers parametric approximations, based on Taylor series expansions, of the value of the policy function (see [15, 10, 18, 28], among others). In many problems, the Euler equation (which represents the first-order conditions for the problem corresponding to the Bellman equation) is used to approximate the value function. The log polynomials approximations (see e.g., [9]), projection methods as introduced in [14] and perturbation methods (see e.g., [16, 15]) are popular procedures for approximating the Euler equations. For a discussion and analysis of these approaches see [27]. More recently, in [17], Judd et al. review advances in computational methods for solving dynamic models and Borağan et al., in [3], study the performance and accuracy of different solution methods, making a clear recommendation of perturbation methods.

Despite the fact that most of these approaches show good performance in solving particular economic models, a computational methodology to deflate the curse of dimensionality is worthwhile and would be appreciated by practitioners. We aim to address this problem by splitting it into manageable pieces (subproblems) and by coordinating the solutions of these subproblems. To attain this goal, we have studied the special structure of economic dynamic stochastic models. Then, we have developed a general methodology based on the Lagrangian decomposition procedure to reduce the dimensionality problem associated with dynamic stochastic models. This methodology is based on a mathematical programming framework. It solves the original model by breaking the problem into a set of smaller independent problems. With the proposed methodology, we obtain two main computational advantages. First, the subproblems are, by definition, smaller than the original problem and therefore much faster to solve. Second, the subproblems could have special properties such as convexity and sparsity that enable the use of efficient algorithms to solve them.

Previous decomposition algorithms break into three groups: Danzting-Wolfe decomposition, Benders decomposition and augmented Lagrangian relaxation procedures. Both Danzting-Wolfe decomposition (see [7]) and Benders decomposition (see [2] and [11]) are efficient schemes to deal with convex optimization problems. Extension to nonconvex problems is attained by the augmented Lagrangian relaxation (see [5, 23, 25, 6]). These techniques are based on an estimate of the

Lagrange multipliers to decompose the problem into a set of subproblems. Then, their solutions are used to update the current estimate of the Lagrange multipliers. But augmented Lagrangian methods may converge slowly in practice (see [13, 4]).

Applications of decomposition methods to economics are originally due to Mansur and Whalley [21]. They apply Danzting and Wolfe's decomposition to compute the solution of a pure exchange general equilibrium model. A pure exchange general equilibrium problem represents a static economy where there are no sectors of production (for a detail description of the model see [8]). In contrast, the current paper considers an extension of Lagrangian decomposition methods for the computation of stochastic dynamic economic models. It must be noted that these models represent a more general and versatile tool in economics as they describe an economy with consumption and production sectors that evolves over an infinite number of time periods. Our approach first reduces the original problem to a finite-horizon problem and then solves decomposed subproblems obtained after fixing some of the decision variables and Lagrange multipliers.

To validate the efficiency of the proposed methodology, we have solved several economic dynamic models. The numerical results are very encouraging, showing computational gains when applying to large-scale problems. The proposed approach has therefore the potential for application in many economics problems.

The paper proceeds as follows. In Section 2, for illustrative purposes, we consider a simple but important example, the traditional deterministic neoclassical growth model. We use this model to develop, in Section 3, the proposed decomposition methodology. In Section 4, we extend this methodology for dealing with uncertainty. In Section 5 we present and solve an international model which is typically hard to solve because of its high dimensionality. Finally, in Section 6, we discuss the results and provide conclusions.

2 The economic growth model

The neoclassical growth model, despite its age and recent developments in the growth literature, continues to be of great theoretical and empirical interest to study the long-term economic performance. In these models, agents have to decide (in each period) how to allocate their resources between consumption commodities, in order to provide instantaneous utility and capital commodities, and obtaining production for the next period. Therefore, these models are usually used for forecasting, testing economic theories, and designing economic policies that increase the growth rate in the long-term. For a formal discussion of these models, see e.g., [26].

The neoclassical growth model can be motivated as follows. Suppose a dynamic economy with infinitely lived agents, whose preferences are representable by the utility function $U(c) = \sum_{t=0}^{\infty} \beta^t u(c_t)$, where $\{c_t\}$ is the sequence of consumption at each period t, $u(c_t)$ is the utility function at each date and $\beta \in (0,1)$ is the discount factor. It is assumed that the agents aim at maximizing their present utility represented by $U(c)$. The physical capital is assumed to evolve according the law of motion $k_{t+1} = F(k_t) - c_t$, for $t = 0, 1, \ldots$, and where $F(k_t)$ denotes a production function, given an initial endowment $k_0 = \bar{k}_0 > 0$ of the capital stock.

Modern economic theory is based upon the concept of competitive equilibrium, which consists of an array of prices and allocations (the consumption and the physical capital decisions) equating aggregate supply and demand. Equilibrium is a basic descriptive and predictive tool for economists because it is expected that the forces acting on the economy will drive

to this array of allocations and prices. However, we can assume that both the consumption and the physical capital decision are made by a representative agent, the *social planner*. In welfare economics, a social planner is a decision-maker who attempts to achieve *Pareto optimality*, in which no one's outcome can be improved without worsening someone else's outcome. The social planner role can be thought of as if it were played by a government. Two important results in economics, called the *Two Fundamental Theorems of Welfare Economics*, link the concept of a Pareto-optimal allocation with that of a competitive equilibrium under certain conditions. See [22] and [26] for details. Since both welfare theorems hold in this economy, our problem reduces to solve the social's planner problem (i.e. to solve a Pareto-optimal allocation):

$$\begin{array}{ll} \max & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{subject to} & k_{t+1} - F(k_t) + c_t = 0, \quad t = 0, 1, \ldots, \end{array} \tag{1}$$

given an initial condition $k_0 = \bar{k}_0$. The social planner problem is usually easier to solve than the equilibrium problems.

3 The decomposition methodology

In this section, we describe a general decomposition methodology to compute economic equilibria in dynamic models. Decomposition is a classical solution approach for optimization problems based on the idea of partition. To simplify our exposition, we consider the standard deterministic neoclassical growth model (1), which encompasses most of discrete-time models proposed in the economic literature.

The first step of the proposed decomposition procedure follows the conventional domain truncation technique. Mainly, we consider a large but finite temporal horizon of problem (1), $0 < T < \infty$. Therefore, the truncated problem has the form:

$$\begin{array}{ll} \max & \sum_{t=0}^{T} \beta^t u(c_t) \\ \text{subject to} & k_{t+1} - F(k_t) + c_t = 0, \quad t = 0, \ldots, T, \\ & k_0 = \bar{k}_0. \end{array} \tag{2}$$

Before presenting the decomposition approach, we need to accommodate the transversality condition associated to problem (1), namely,

$$\lim_{T \to \infty} \lambda_T \, k_{T+1} = 0, \tag{3}$$

where λ_t denotes the Lagrange multiplier associated to the t-th constraint: $k_{t+1} - F(k_t) + c_t = 0$. To attain this goal, we propose to add a penalty to the capital stock at $T+1$, guaranteeing that an optimal solution for problem (2) is a solution of the following problem:

$$\begin{array}{ll} \max & \sum_{t=0}^{T} \beta^t u\left(c_t\right) + \epsilon_T \ln\left(k_{T+1}\right) \\ \text{subject to} & k_{t+1} - F(k_t) + c_t = 0, \quad t = 0, \ldots, T, \\ & k_0 = \bar{k}_0, \end{array} \tag{4}$$

where $\epsilon_T > 0$ and $\lim_{T \to \infty} \epsilon_T = 0$. However, even these truncated problems can be too large to be solved by standard algorithms.

For this reason, we propose to use the following decomposition approach which alleviates the high dimensionality by breaking problem (4) into a set of smaller independent subproblems. The separability of the subproblems is obtained by fixing the values of some variables.

At each iteration of the procedure, the following $T+1$ subproblems (SP) are solved:

SP0. For $t = 0$:

$$\begin{aligned} \max \quad & \beta^0 u(c_0) \\ \text{subject to} \quad & \bar{k}_1 - F(\bar{k}_0) + c_0 = 0, \\ & k_0 = \bar{k}_0, \end{aligned}$$

(with $\bar{k}_1$ fixed from previous iteration);

SPt. For $t = 1, \ldots, T-1$:

$$\begin{aligned} \max \quad & \beta^t u\left(c_t\right) - \bar{\lambda}_{t-1}(k_t - F(\bar{k}_{t-1}) + \bar{c}_{t-1}) \\ \text{subject to} \quad & \bar{k}_{t+1} - F(k_t) + c_t = 0, \end{aligned}$$

(with $\bar{\lambda}_{t-1}$, $\bar{k}_{t-1}$, $\bar{k}_{t+1}$ and $\bar{c}_{t-1}$ fixed from previous iteration);

SPT. For $t = T$:

$$\begin{aligned} \max \quad & \beta^T u\left(c_T\right) - \bar{\lambda}_{T-1}(k_T - F(\bar{k}_{T-1}) + \bar{c}_{T-1}) + \epsilon_T \ln\left(\bar{k}_{T+1}\right) \\ \text{subject to} \quad & \bar{k}_{T+1} - F(k_T) + c_T = 0, \end{aligned}$$

and $k_{T+1} = F(k_T) - c_T$ (with $\bar{\lambda}_{T-1}$, $\bar{k}_{T-1}$, $\bar{k}_{T+1}$ and $\bar{c}_{T-1}$ fixed from previous iteration).

The intuitive idea of this decomposition methodology is to consider a set of smaller independent problems whose first-order necessary conditions (at the optimal the solution) coincide with the corresponding conditions for problem (4). For each subproblem t, the decision variables are only the contemporary ones (for example, the subproblem SPt is solved in c_t and k_t). An economic interpretation of the decomposition draws on this partition of the decision variables into contemporary and non contemporary decisions taken among agents.

Once the solution for these subproblems have been computed, the multipliers and the fixed variables are updated to their last computed values. This procedure is repeated until the convergence criteria for the global finite problem (4) are satisfied. We have chosen the following stopping criterion (where the superscript ^{l} denotes the current iteration):

$$\frac{\left\| L\left(c^l, k^l\right) - L\left(c^{l-1}, k^{l-1}\right) \right\|}{1 + \left\| L\left(c^{l-1}, k^{l-1}\right) \right\|} \leq \epsilon, \tag{5}$$

where $L\left(c^l, k^l\right) = \sum_{t=0}^{T} \beta^t u\left(c_t^l\right) + \epsilon_T \ln\left(k_{T+1}^l\right)$ denotes the value of the objective function at iteration l.

The scheme of the decomposition algorithm is stated as follows.

Initialization: Set a truncation date T and a regularization parameter ϵ_T. Choose a starting point $\left(\bar{c}_0, \bar{c}_1, ..., \bar{c}_T, \bar{k}_1, ..., \bar{k}_{T+1}\right)$ and an initial set of multipliers $\left(\bar{\lambda}_0, \bar{\lambda}_1, ..., \bar{\lambda}_T\right)$. Set $k \leftarrow 0$.

Repeat:

1. Solve subproblems **SPt** in c_t and k_t for all $t = 0, 1, ..., T$. Denote by $\left(\widetilde{c}_0, \widetilde{c}_1, ..., \widetilde{c}_T, \widetilde{k}_1, ..., \widetilde{k}_{T+1}\right)$ the solution of these subproblems for all $t = 0, 1, ..., T$ and $\left(\widetilde{\lambda}_0, \widetilde{\lambda}_1, ..., \widetilde{\lambda}_T\right)$ the associated optimal multipliers.
2. Update new point and multipliers:

$$\left(\bar{c}_0, \bar{c}_1, ..., \bar{c}_T, \bar{k}_1, ..., \bar{k}_{T+1}\right) \leftarrow \left(\widetilde{c}_0, \widetilde{c}_1, ..., \widetilde{c}_T, \widetilde{k}_1, ..., \widetilde{k}_{T+1}\right)$$

$$\left(\bar{\lambda}_0, \bar{\lambda}_1, ..., \bar{\lambda}_T\right) \leftarrow \left(\widetilde{\lambda}_0, \widetilde{\lambda}_1, ..., \widetilde{\lambda}_T\right)$$

and set $k \leftarrow k + 1$.

Until convergence (condition (5) is satisfied):

The convergence properties of this algorithm can be addressed using arguments analogous to those considered in [6]. These properties do not require an optimal solution of subproblems **SP0**, **SPt** and **SPT**. It is enough to compute their solution up to a certain degree of accuracy (near the solution, it could be enough to perform a single iteration for each subproblem). Therefore, the proposed decomposition technique considerably increases the speed of computations.

Next a numerical experiment is introduced to illustrate the computational gains of the decomposition method. We have considered the neoclassical growth model presented in Section 2. In this model, we have chosen a Cobb-Douglas production function $F(k) = k^{\alpha}$ with capital share $\alpha = 0.33$, and a utility function $u(c) = c^{\rho}/\rho$ with $\rho = 0.4$. The discount factor is $\beta = 0.8$ and the regularization parameter has been fixed to $\epsilon_T = 10^{-4}$.

We have implemented the decomposition algorithm using MATLAB 6.5 on an Intel Centrino Pentium M 1.6 GHz with machine precision 10^{-16}. Each subproblem **SPt**, for $t = 0, ..., T$, has been solved using the MATLAB subroutine `fmincon` corresponding to the *Optimization toolbox*. This routine is suited for optimization problems with nonlinear objective function and constraints.

We have computed the solution of neoclassical growth model by two procedures: i) a direct algorithm (i.e., we use the subroutine `fmincon` to solve the finite problem (4)) and ii) the proposed decomposition algorithm solving the subproblems to optimality by the subroutine `fmincon` (which is worst-case scenario in terms of computational cost). The decomposition algorithm stops whenever $\epsilon = 10^{-8}$.

Table 1 reports a comparison of the running times (in seconds) until convergence obtained from both procedures.

We can see a better view of these running times (in logarithmic scale) in Figure 1.

From this figure, it can be shown a clear advantage of the proposed decomposition algorithm over the direct approach. It is remarkable that the computing time until convergence (or curse of dimensionality) required to solve the original problem (2) is much better using the decomposition methodology than using a direct approach. Table 1 shows that the proposed methodology is an effective and useful tool for solving large-scale economic problems as it breaks down a high-dimensional problem into many low-dimensional ones, hence reducing the curse of dimensionality.

Table 1: Running times (in seconds) until convergence for different T's.

T	Direct	Decomposition	T	Direct	Decomposition
75	1.1	1.7	775	267.2	15.4
125	1.9	2.7	825	315.9	16.1
175	4.5	3.6	875	373.4	17.3
225	8.7	4.7	925	459.4	19.0
275	15.7	6.0	975	597.5	19.7
325	22.6	6.5	1000	642.2	20.4
375	34.8	7.4	1050	754.3	24.1
425	45.9	8.5	1100	1077.4	22.3
475	67.5	9.3	1150	1532.0	30.6
525	87.2	11.7	1200	1915.8	25.0
575	130.4	11.8	1250	1860.2	27.3
625	142.5	13.1	1300	2408.2	27.3
675	176.1	13.3	1350	3253.5	29.8
725	213.8	14.3			

Moreover, to make the decomposition algorithm comparable to standard approaches in computational economics literature, we consider the accuracy of the solution measured by the normalized Euler equation error over T periods:

$$EE_N = \max_{t=0,1,...,T} \frac{|E_t|}{u'(c_t)\ c_t} \tag{6}$$

where E_t represents, in some sense, the first-order necessary conditions for problem (1). The use of Euler equation error is a common and reliable feature of most traditional approaches (see [14]). In our example,

$$E_t = c_t^{\rho-1} - \beta\alpha c_{t+1}^{\rho-1} k_t^{\alpha-1}, \quad \text{for all } t = 0, 1, ..., T. \tag{7}$$

Judd and Guu, in [16], interpret this error as the relative optimization error incurred by the use of the approximated policy rule. Our algorithm achieves a good approximation of the solution of model (1) as the maximum Euler error is

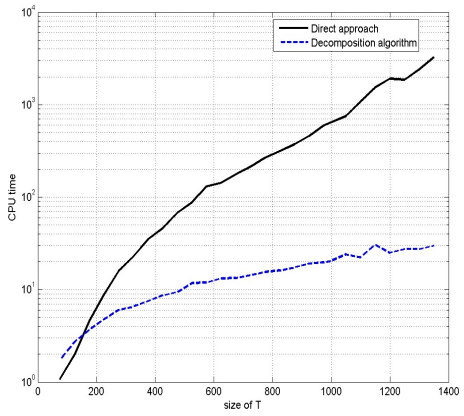


Figure 1: Running times (in logarithmic scale) until convergence for different T's.

8.62×10^{-7} for $T = 1350$. This means that the agent is making a cent mistake for each $\$10^6$ of dollars spent. This accuracy is comparable to that reported by Judd in [14], taking into account that we have set the termination tolerance for the subproblems as $\epsilon = 10^{-8}$.

Though the results are very encouraging, in terms of computing time and accuracy of the solution, however, this example is too simple to draw any firm conclusion. The next sections include models with uncertainty in some parameters and which are more realistic from a practical point of view.

4 Solving stochastic growth models

In macroeconomics, it is well-known that it is essential to incorporate uncertainty into the decision-making problems. The resulting stochastic models provide a tighter link between the specification of the economic theory and the empirical facts. In this section, we show how to extend the proposed decomposition methodology presented in Section 3 to solve stochastic dynamic economic problems.

We present the decomposition algorithm in the context of the following stochastic optimal growth problem:

$$\begin{array}{ll} \max & E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \\ \text{subject to} & k_{t+1} - F(k_t, \theta_t) + c_t = 0, \quad t = 0, \ldots, \infty, \\ & k_0 = \bar{k}_0, \end{array} \tag{8}$$

where θ_t denotes the stochastic shock to technology at time t.

After choosing T, the truncated problem is defined as follows:

$$\begin{array}{ll} \max & E\left[\sum_{t=0}^{T} \beta^t u(c_t) + \epsilon_T \ln\left(k_{T+1}\right)\right] \\ \text{subject to} & k_{t+1} - F(k_t, \theta_t) + c_t = 0, \quad t = 0, \ldots, T, \end{array} \tag{9}$$

where $k_0 = \bar{k}_0$, $\epsilon_T > 0$ and $\lim_{T\to\infty} \epsilon_T = 0$. Then, the decomposition procedure to solve (8) is similar to the deterministic case after representing uncertainties in a form suitable for computation.

As we are using a mathematical programming framework, the uncertainty presented in these problems must be represented in such a manner that its effect on present decision-making can properly be taken into account. A common representation of the uncertainty is to work with scenarios, which are particular models of how the future might unfold. Within this framework, simulations can be used to generate a batch of scenarios. Moreover, for the class of problems studied in this paper, one can act with perfect foresight, that is, the decisions can be made after the value of the uncertainty becomes known. Therefore, this kind of problems can be decomposed by scenarios and hence, we only need to solve an optimization problem for each scenario that is generated.

In model (9), uncertainty is represented by a scenario tree S. Each node of the tree is a history of exogenous aggregate shocks $s_t = \{s_0, s_1, ..., s_t\}$, where s_0 is the root of the tree given by some fixed event. The path to event s is a partial scenario with probability ω_s along the path (see e.g., [12] for further details on this terminology).

Hence, problem (9) can be written with the following deterministic equivalent formulation:

$$\begin{array}{ll} \max & \sum_{s\in S} \omega_s (\sum_{t=0}^{T} \beta^t u(c_{t,s}) + \epsilon_T \ln\left(k_{T+1,s}\right)) \\ \text{subject to} & k_{t+1,s} - F(k_{t,s}, \theta_{t,s}) + c_{t,s} = 0, \quad t = 0, \ldots, T, \ s \in S. \end{array} \tag{10}$$

As commented before, for each scenario $s \in S$, problem (10) can be broken into a set of independent problems of the form

$$\begin{array}{ll} \max & \sum_{t=0}^{T} \beta^t u(c_{t,s}) + \epsilon_T \ln(k_{T+1,s}) \\ \text{subject to} & k_{t+1,s} - F(k_{t,s}, \theta_{t,s}) + c_{t,s} = 0, \quad t = 0, \ldots, T, \end{array}$$

that can be solved similarly to problem (4). Therefore, we can decompose the stochastic dynamic problem by scenarios and by time, obtaining the solution path $\{c_{t,s}, k_{t,s}\}_t$ for each scenario $s \in S$.

With this approach, the expected values of $\{c_{t,s}, k_{t,s}\}_t$ will be consistent estimates of the expected value of the solution path, that is,

$$E[c_t^*] = \sum_{s \in S} \omega_s c_{t,s}, \qquad E[k_t^*] = \sum_{s \in S} \omega_s k_{t,s}, \text{ for all } t = 0, 1, ..., T,$$

(where * denotes optimal solution) and the optimal value function can be estimated by means of

$$E[V^*] = \sum_{t=0}^{T} \beta^t \sum_{s \in S} \omega_s u(c_{t,s}).$$

In general, scenario specifications of stochastic problems can be easily formulated but not solved. In applications we confine ourselves with some approximate distributions of random data comprising only finitely many scenarios. In other words, we assume S is a finite set, $S = \{1, ..., |S|\}$. Hence, we compute the approximate solution $\{c_{t,s}^*, k_{t,s}^*\}_t$ for each $s \in \{1, ..., |S|\}$ and report the main statistical properties of the approximation. For example, the mean and variance of the solution path can be computed as follows:

$$\begin{aligned} \widehat{E}[c_t^*] &= \frac{1}{|S|} \sum_{s=1}^{|S|} c_{t,s}^*, \qquad \widehat{E}[k_t^*] = \frac{1}{|S|} \sum_{s=1}^{|S|} k_{t,s}^*, \text{ for all } t = 0, 1, ..., T, \\ \widehat{V}[c_t^*] &= \frac{1}{|S|} \sum_{s=1}^{|S|} \left(c_{t,s}^* - \widehat{E}[c_t^*] \right)^2, \qquad \widehat{V}[k_t^*] = \frac{1}{|S|} \sum_{s=1}^{|S|} \left(k_{t,s}^* - \widehat{E}[k_t^*] \right)^2, \text{ for all } t = 0, 1, ..., T, \end{aligned}$$

Note that we do not require stationarity as we estimate the statistical properties of the solution process from independent simulations. The Law of Large Numbers and Central Limit Theorems can be applied when the standard finite moments assumptions hold.

Confidence intervals can be made applying standard statistical methods. For example, for large $|S|$, we can compute a confidence interval (with probability $1 - \alpha$) for the expected value of the consumption at time t as:

$$E[c_t^*] \in \widehat{E}[c_t^*] \pm z_{\alpha/2} \sqrt{\frac{\widehat{V}[c_t^*]}{|S|}} \tag{11}$$

where $z_{\alpha/2}$ is the appropriate percentile of the standard normal distribution. Analogously, the confidence intervals for other estimations can be established. Moreover, as in Section 3, we consider the accuracy of the solution measured by the normalized Euler equation error over T periods (with its corresponding confidence interval).

Next, the results obtaining by solving different cases of model (8) are presented. We have considered a Cobb-Douglas production function $F(k_t, \theta_t) = \theta_t k_t^\alpha$ with capital share $\alpha = 0.33$, and a utility function c^ρ/ρ with $\rho = 0.4$. The discount factor is $\beta = 0.99$ and the regularization parameter has been set to $\epsilon_T = 10^{-4}$. The shocks $\{\theta_t\}$ are assumed to take the values $\{0.9, 1, 1.1\}$, following a first-order Markov chain given by the following transition matrix:

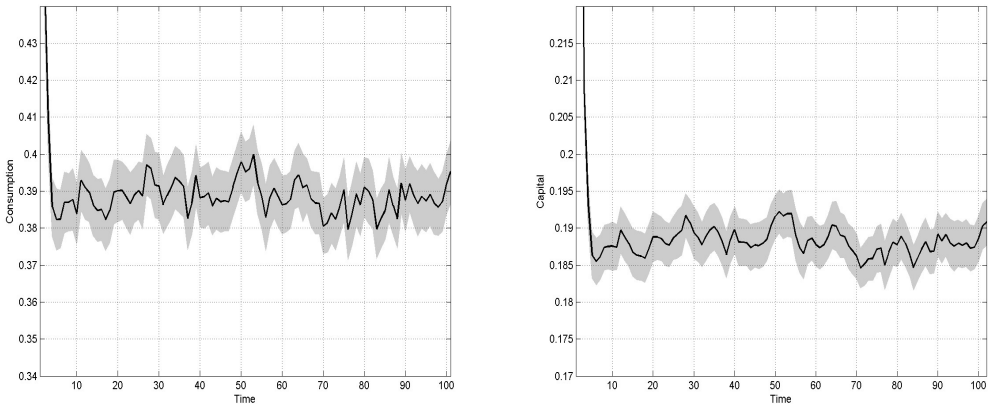
$$\pi = \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}.$$

Therefore, a sample of 3^{T+1} possible scenarios is obtained as θ_t can take three different values. Table 2 reports the main properties of the optimal value function for different simulation lengths and temporal horizons.

Table 2: Properties of the optimal value function.

	$T = 50$		$T = 100$	
	$\|S\| = 25$	$\|S\| = 50$	$\|S\| = 50$	$\|S\| = 100$
Estimation of $E[V^*]$	69.15	69.37	109.65	109.67
Standard error of $E[V^*]$	0.13	0.07	0.11	0.08
Confidence Interval (95%) of $E[V^*]$	[68.89, 69.41]	[69.23, 69.51]	[109.43, 109.87]	[109.51, 109.83]

In Figure 2, the optimal paths for $E(c_t)$ and $E(k_t)$, respectively, are shown with their corresponding confidence intervals (at 95% level and with $T = 100$ and $|S| = 100$).



(a) Consumption path (with confidence intervals)

(b) Capital path (with confidence intervals)

Figure 2: Paths for $T = 100$ and $|S| = 100$

In Figure 3, the optimal path for the normalized Euler equation errors is shown with their corresponding pointwise confidence intervals (at 95% level and with $T = 100$ and $|S| = 100$).

From Table 2 and figures 2-3 it can be deduced that the proposed decomposition methodology obtains sufficiently accurate results for practical purposes. For instance, the accuracy obtained for this type of models by Borağan et al., in [3], is around 10^{-8} (though these authors consider another normalization of the Euler equation). Note that, with the proposed methodology, we obtain an accuracy around of 10^{-7} (with a termination tolerance for the subproblems of $\epsilon = 10^{-8}$). Moreover, the maximum error obtained by Borağan et al. is around 10^{-4} while the maximum error obtained by us is around 10^{-6} (corresponding to the upper level of the confidence interval, as shown in Figure 3).

5 An international model with uncertainty

In this section, we consider a model with a finite number of heterogeneous agents. These models has been fruitfully

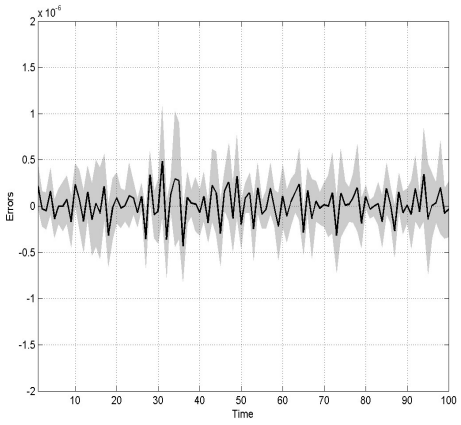


Figure 3: Euler's error path (with confidence intervals) for $T = 100$ and $|S| = 100$.

applied to study many questions in international economics, for example the co-movement of output, investment and consumption across countries and international capital flows between countries (for a review, see for instance [1]). Because of its high dimension, the computational solution of this problem is a quite challenging task.

The formulation of this model is as follows. Suppose a dynamic economy with N countries where each country is populated by a representative consumer, over a number of years, $t = 0, 1, \ldots$. A social planner maximizes a weighted sum of the expected lifetime utilities of the countries' representative consumers subject to the resource constraint. The resulting model has the following form:

$$\begin{aligned} \max \quad & E_0 \sum_{n=1}^{N} \tfrac{1}{N} \sum_{t=0}^{\infty} \beta^t u(c_t^n) \\ \text{subject to} \quad & k_{t+1}^n + \left(\tfrac{\varphi}{2} \left(k_{t+1}^n - k_t^n \right)^{\xi} \right) \Big/ k_t^n = i_t^n + (1-\delta)\, k_t^n, \quad \text{for all } t \text{ and } n, \\ & \sum_{n=1}^{N} c_t^n + \sum_{n=1}^{N} i_t^n = \sum_{n=1}^{N} a_t^n \left(k_t^n \right)^{\alpha}, \end{aligned}$$

where c_t^n, k_t^n, i_t^n denote consumption, capital and investment, respectively, at time t and in country n, β is the discount parameter, δ is the depreciation rate and φ is the adjustment cost parameter. In this case, the stochastic process a_t^n determining the technology shock is supposed to follow the law of motion:

$$\ln a_t^n = \eta \ln a_{t-1}^n + \sigma \left(\varepsilon_t - \varepsilon_t^n \right), \; n = 1, \ldots, N, \; t = 1, \ldots,$$

where ε_t and ε_t^n are i.i.d. random variables with a standard Normal distribution.

We have considered a utility function $u(c) = (c^{\rho} - 1)/\rho$ with $\rho = 0.4$. The parameters in the constraints are $\delta = 0.025$, $\alpha = 0.025$ and $\xi = 2$. The discount factor is $\beta = 0.99$ and the regularization parameter has been set to $\epsilon_T = 10^{-8}$. For the stochastic shock simulations we have selected $\eta = 0.95$ and $\sigma = 0.95$.

Table 3 reports the computational time (in seconds) that the proposed decomposition methodology needs to find an optimal solution for a temporal horizon $T = 100$ and $S = 100$ replications. Several cases were run to study the computational time as a function of the number of countries and the nonlinearity of the problem. The nonlinearity is measured by the parameter φ. When this parameter is greater than zero then the problem becomes strongly nonlinear. Moreover, Table 3 reports the confidence intervals of the optimal value function for the different number of countries and values of the parameter φ.

Table 3: Performance indices for $T = 100$ and $S = 100$ replications.

N	φ	CPU Time	Confidence Interval (95%) of $E[V^*]$
2	0	6.81e+003	$[32.88,\ 33.61]$
5	0	7.82e+003	$[28.65,\ 29.38]$
10	0	1.23e+004	$[25.51,\ 26.22]$
20	0	3.01e+004	$[21.67,\ 22.48]$
2	1	6.43e+003	$[30.53,\ 31.23]$
5	1	9.32e+003	$[27.52,\ 28.60]$
10	1	1.29e+004	$[24.14,\ 25.20]$
20	1	5.08e+004	$[22.90,\ 24.11]$
2	2	1.0672e+004	$[29.82,\ 30.80]$
5	2	1.4905e+004	$[26.20,\ 27.82]$
10	2	1.8201e+004	$[22.01,\ 23.44]$
20	2	4.2074e+004	$[18.25,\ 19.40]$

From Table 3 it can be deduced that the proposed decomposition methodology obtains sufficiently accurate results for practical purposes. Moreover, the required computed time to solve the problems is reasonable, taking into account the dimension of them. To the best of our knowledge, no results exist in the literature showing solutions of the presented model for $N > 10$.

Attempts to solve this problem using parametric approximations are unsatisfactory. Krueger et al., in [19], introduce the Smolyak's algorithm to compute solutions of large-scale dynamic economic models. To document the performance of the method they consider this model with $N < 4$ countries (and $\varphi \in [0, 2]$). Its computing time to converge to the solution is 4 hours and 15 minutes for $N = 4$ and $\varphi = 0$, and 12 hours and 31 minutes for $N = 4$ and $\varphi = 2$. Maliar and Maliar, in [20], describe a version of the simulation-based Parameterized Expectations Algorithm, introduced in [9], that is only successful in finding the solutions to models with a number of countries (N) less than 10. Their computing time to solve the problems vary from 5 minutes (for a model with $N = 2$ countries and $\varphi = 10$) to 5.5 hours (for a model with $N = 10$ countries and $\varphi = 10$).

6 Conclusions and further extensions

In this paper, we make a computational contribution for a rich *toolbox* of optimization models that economists use to analyze various facets of the economy. The solvability of these models suffers from the curse of dimensionality, which limits practitioners from the modelling standpoint. In this sense, we have introduced a novelty decomposition methodology for the computation of solutions of dynamic stochastic economics problems.

The proposed approach deflates the dimensionality of the models by breaking the problem into a set of smaller independent subproblems. We have shown the decomposition method works very well in practice, better than a direct method (when this is feasible).

We have solved several high-dimension problems. The numerical results have revealed the efficiency of the methodology in terms of computing time and accuracy, concluding that the proposed approach is promising for application in many economics problems with similar structure.

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