A Computer Algebra Primer and Homework Exercises for use in an Intermediate Macroeconomics Course—A Student/Teacher Collaboration

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Introduction

This paper details some of the results of a student/teacher collaboration to develop computer algebra materials for use by students in an intermediate macroeconomics course. Many years ago the teacher required students solve macro models using a pencil and paper. For the most part these models were linear IS—LM models. One problem that arose with the pencil and paper solution is that introducing slight modifications to the model could require substantial solution time on the part of the students. This tended to reduce the complexity and sophistication of the models considered.

The teacher later converted these models to Excel. While Excel did reduce the computational burden on the students, the teacher was not satisfied with this solution. Excel cannot produce symbolic results. The teacher also thought that entering the models into Excel was rather awkward and did not resemble the equations that might appear in textbooks. Finally the operation of Excel's Solver often created difficulties. For these reasons the teacher decided to try using a computer algebra system (CAS) during the spring semester of 2004.

Computer algebra systems have certain advantages. They can produce symbolic solutions as well as numerical solutions. Symbolic solutions are often presented in textbooks giving students some confidence that the solutions really are valid. The teacher also found that using a CAS was far more efficient in a lecture than solving models on a blackboard.

The teacher wrote a primer discussing the features of the CAS that the students would need to complete projects given in the class as well as a series of classroom exercises. For the most part the exercises only required that the students copy the program, change parameters or exogenous variables and discuss the results of the changes. The teacher discussed the value of the exercises with a number of students after the semester ended and found that they thought the exercises were too easy. The teacher succeeded in getting one student to enroll in an independent study class to revise the primer and to reconstruct the exercise so they would be more meaningful to the students.

The choice of a CAS is discussed next, followed by a discussion of the prevalence of CASs in various disciplines, followed by the primer, then the exercises and lastly, some example lecture notes created using a CAS. Please note that all of these items are works in progress.

The prevalence of CASs in various disciplines

CASs are being used with increasing frequency in graduate, undergraduate and even secondary classrooms (Mahoney 2002). Their adoption in the business disciplines and economics seems to lag that of many other fields, however. This seems odd given an increasing emphasis of the use of mathematics in business and economics.

Mathematica is the most widely used CAS and the company maintains a large repository of information about how Mathematica is being used in industry, research, the government and education. The Mathmatica website lists 4038 published items involving its use (an item can be an article, book, proceedings, working paper or just about any sort or written output). Most of these items concern mathematics, engineering and the sciences. Only 93 of the items pertain to business or economics. Most of the business and economics items are about finance and economics and almost all of these are written for use by graduate students.

The site lists 374 books devoted to Mathematica. Only eleven of these involve business (finance really) or economics. This number should really be reduced to ten business and economics books because one was misplaced. This list could be reduced nine(Stojanovic; Varian 1993; Varian 1996; Huang and Crooke 1997; Shaw 1998; Lewis 2000; Cool 2001; Shone 2002; Stinespring 2002) because the only information available for one item was the book title. Only one of the remaining volumes was written with the undergraduate in mind and it was devoted to microeconomics(Stinespring 2002).

The teacher has only located two articles in economics journals about using a CAS in the classroom. One of these concerned graduate education (Belsley 2000)**.** The other article discussed using a CAS in intermediate microeconomics. (Boyd 1998). It seems that teachers of economics are not utilizing CASs as fully as teachers in other disciplines. We would like to help popularize using these tools in economics classrooms.

The choice of a CAS

Four criteria were used to determine which CAS to use. One criterion was that the cost to the student was to be modest, zero if possible. A second criterion was that the user should be able to enter equations in a manner similar to writing them on a piece of paper. A graphical user interface was considered desirable. The last qualifier is that the CAS should be able to produce reasonable graphics. Two CASs met the cost criterion, Maxima and MuPAD.

Maxima is in the public domain and can be obtained from

http://maxima.sourceforge.net/

Maxima is descended from Macsyma, at one time a commercially vended CAS developed by faculty at MIT. Later this became a Department of Energy product. The department donated this product to the University of Texan and the University later placed it in the public domain. Maxima is available in both Windows and Linux versions.

The other product considered was MuPAD. The vendors of this product make light version of the product that is available free for classroom use. This means students can use the product on their home machines and that the product can be installed on institution labs. MuPAD Light is available in both Windows and Linux versions. A commercial version of MuPAD for the McIntosh is also available. The product can be downloaded from

http://research.mupad.de/

The students were allowed to use either product. The students generally preferred to use MuPAD.

Maxima is maintained by a group of volunteers while MuPAD is a commercially vended product. Probably for the reason the documentation is rather better for MuPAD than Maxima. Both products have on line documentation but third party books are becoming available for MuPAD. In either case, the documentation is often terse and not written with students in mind which necessitated the development of primers and exercises for both products.

Maxima was initially developed for mainframe computers while MuPAD was created with a graphical user interface in mind. Perhaps for that reason, it is easier to create graphics within MuPAD than within Maxima although quite good graphics can be created using Maxima and linking the results to Gnuplot.

There is one drawback common to both products that might lead one to the adoption of a commercial or student version of Maple, Mathematica or MuPAD. Neither MuPAD nor Maxima permits commingling of comments and commands. This limitation would make either product undesirable for large projects. We found that the best procedure for the students to follow in completing a project was to

- Maintain a file of commands for any given project using a text editor.
- Enter a command in the file and then copy and paste it into the CAS.
- Execute the command.
- Copy and paste the result to a word processing package used for the final report.
- Repeat these steps until all commands for the project have been entered. Text can be entered into the document in the word processor at the time commands are entered or this the commentary can be delayed until all the commands have been entered.
- Save the text file of commands frequently.

Entering commands directly into the CAS proved frustrating for all but the most trivial efforts. While both packages permit recovery from most errors, it is easy often difficult to recall previous steps. Frequently the best thing to do is start over, hopefully with an initial set of commands that are known to be good. This initial set of good commands will be available in the text editor's file if the procedure listed above is followed.

Belsley, D. A. (2000). "Mathematica as an Environment for Doing Economics and Econometrics." Computational Economics **14**(1): 69-87.

Boyd, D. W. (1998). "On the Use of Symbolic Computation in Undergraduate Microeconomics Instruction." Journal of Economic Education **29**(3): 227-246.

Cool, T. (2001). Voting Theory for Democracy. Moray, Scotland, Gopher Publishers.

Huang, C. J. and P. S. Crooke (1997). Mathematics and Mathematica for Economists. Oxford, Blackwell Publishers, Ltd.

Lewis, A. L. (2000). Option Valuation under Stochastic Volatility with Mathematica Code, Finance Press.

Mahoney, J. F. (2002). "Computer algebra Systems in Our Schools: Some Axioms and Some Examples." Mathematics Teacher: 598-605.

Shaw, W. T. (1998). Modelling Financial Derivatives With Mathematica. Cambridge, Massachusetts, Cambridge University Press.

Shone, R. (2002). Economic Dynamics: Phase Diagrams and their Economic Application.

Stinespring, J. R. (2002). Mathematica for Microeconomics: Learning by Example. San Diego, California, Academic Press.

Stojanovic, S. Computational Financial Mathematics Using Mathematica: Optimal Trading in Stocks and Options, Birkhauser.

Varian, H. R., Ed. (1993). Economic and Financial Modeling with Mathematica. New York, TELOS/Springer-Verlag.

Varian, H. R., Ed. (1996). Computational Economics and Finance: Modeling and Analysis with Mathematica. New York, TELOS/Springer-Verlag.

MuPAD Primer

Luke Olsen and Max Jerrell

Introduction

This document is intended to provide the necessary hints and tools to allow undergraduate economics students with a familiarity with college algebra to use MuPAD to construct economic models and solve economic problems. If you want to take a closer look at any of these commands or to try to find a command that is not listed, use the MuPAD help browser (keyboard shortcut: F2).

MUPAD is a computer algebra system (CAS). It operates on symbols as well as number. X is certainly a symbol. Is 2 a symbol or a number? In MuPAD, it could be both.

This document is organized along the following topics:

- **Examples of elementary math operations**
- **MuPAD specific techniques**
- **Assigning values to variables**
- **Defining functions in MuPAD**
- **The Solve Command**
- **The Assume Command**
- **Derivatives and Differentiation**
- **Plotting graphs with MuPAD**
- **Removing Braces (curly brackets) and The op Command**

The sections should provide an adequate guide for creating and working with the economic models. Also available for reference are examples of complete projects. A few of the sections might not be necessary for the models assigned, but provide material for anyone interested in further exploring MuPAD's capabilities.

Examples of Elementary Math Operations in MuPAD

The following are examples of running elementary math operations. They are similar to what you would enter in a graphing calculator.

Addition

•2+2

4

•x+x

$2 x$

•x+y

$x + y$

Subtraction

•3-2

1

•x-x

0

•x-y

$x - y$

Multiplication

•2*4

8

•x*x

x^2

•x*y

$x y$

Division

- 8/2

$$4$$

- x/x

$$1$$

- x/y

$$\frac{x}{y}$$

Exponents

- x^2

$$x^2$$

- (x+1)^2

$$(x + 1)^2$$

- (x/y)^2

$$\frac{x^2}{y^2}$$

- (x^2)^3

$$x^6$$

Square Root

- sqrt(9)

$$3$$

Absolute Value

- abs(-99)

99

More Operations

This software is capable of more basic math operations, including trigonometric functions, ln() and log(), inequalities <, >=, >, <=, as well as transcendental numbers like E (2.7182818...) and PI (3.141592654...).

Derivatives and Differentiation

The derivative or differential of a function is the rate of change of that function. Graphically, the derivative is the slope of the function at the point at which you are measuring it.

An economic example that can be understood best as a derivative is the marginal propensity to consume (MPC). MPC is the rate at which consumption changes as income changes. Symbolically, MPC = $\Delta C/\Delta Y$, where ΔC is change in consumption and ΔY is change in income.

In MuPAD, there are two commands that you can use to find the derivative, which we will discuss in the next section.

MuPAD Specific Techniques

Here are a couple of techniques to improve the time using MuPAD.

Multiple commands

You can enter several commands on a line if they are separated by a semicolon.

- `2+2; 2/3 ; 3*(2/3);`

4

2/3

2

Suppress output

You can suppress output by using a colon instead of a semicolon.

- `2+2: 2/3 : 3*(2/3):`

Fractions and Decimals

MUPAD treats 2 and 3 as symbols. To make MuPAD compute the result as a number use decimal notation. Compare:

- `2/3;`

2/3

and

- `2.0/3.0;`

0.6666666667

Assigning Values to Variables

It is often useful to name a variable. Symbols can be assigned values using the **:=** operator.

- `x:=2;y:=3;x+y;x/y;`

2

3

5

2/3

As we mentioned above, if you want the result displayed in numerical form, use decimal notation:

- `x:=2.0;y:=3.0;x+y;x/y;`

2.0

3.0

5.0

0.6666666667

Defining Functions In MuPAD

Consider the function,

$$f(x) = x^2.$$

If we want to know the value of f(x = 2), our function becomes

$$f(2) = 2^2 = 4.$$

Functions have the same meaning in MuPAD. Here is how the above function can be created in MuPAD:

- `f:=x->x^2;`

```
x -> x^2
```

then you can plug in values of x to find values of the function:

- `f(2);`

```
4
```

- `f(3);`

```
9
```

- `f(y);`

```
 2
y
```

One of the chief reasons for using functions is to avoid typing the same information repeatedly. Consider:

- `f:=x->ln(x)*exp(x) + 100*sin(x)`

```
x -> ln(x)*exp(x) + 100*sin(x)
```

- `f(4.0)`

```
0.008857987733
```

- `f(10.0);`

50663.40968

It is not necessary to type out the entire expression each time a value of f(x) is needed.

The Solve Command

We will often use MuPAD to solve systems of equations:

- `solve({x+y=3,x-y=1},{x,y})`

```
{[x = 2, y = 1]}
```

This situation specifies the solve() command to solve the system of equations:

$$x + y = 3$$
$$x - y = 1$$

for both x and y and it returned the solution x = 2, y = 1. We could also have it solve for only one of those variables. MuPAD can also solve more complex systems of equations with no trouble. Consider a larger system of equations:

```
•eq1:=x+4*y+z+4*w=1;
•eq2:=-3*x+2*y+2*z+3*w=2;
•eq3:=3*x-2*y+3*z+2*w=3;
•eq4:=4*x+y-4*z-w=1;
•solve({eq1,eq2,eq3,eq4},{x,y,z,w});
```

$$4w + x + 4y + z = 1$$

$$3w - 3x + 2y + 2z = 2$$

$$2w + 3x - 2y + 3z = 3$$

$$x - w + y - 4z = 1$$

{[w = 35/12, x = -5/12, y = -25/12, z = -23/12]}

Solving this system by hand is not rocket science, but would be time consuming.

The assume command

Consider the following example:

```
• solve(x^2=1,x);
```

```
{-1, 1}
```

which give two results, namely that x could be minus or plus one. It might be the case that we only want to consider positive values so we could restrict the values x can take on by using the assume statement.

```
• assume(x>0);
```

```
> 0
```

```
• solve(x^2=1,x);
```

```
{1}
```

In economics we know that many things should only have positive values such as income, prices, nominal interest rates, consumption, hours worked, etc. If we provide MuPAD with this information it can greatly reduce the number of solutions that MuPAD produces for some problems. MuPAD is quite sophisticated and can produce seemingly weird results that are actually correct. Consider:

```
• solve(x^4=8,x);
```

$$\{8^{1/4}, -8^{1/4}, -I\,8^{1/4}, I\,8^{1/4}\}$$

which gives four solutions, two of which are imaginary--indicated with the I in the solution--and two of which are real. If we want to restrict MuPAD to have only real solutions for x we can instruct it using the following assume command.

```
• assume(x, Type::Real);
```

```
Type::Real
```

```
• solve(x^4=8,x);
```

$$\{8^{1/4}, -8^{1/4}\}$$

and the imaginary solutions have gone away. We can further restrict x to only have positive values using

```
• assume(x>0, _and);
```

```
> 0
```

where the _and part combines Type::Real and x>0, otherwise Type::Real would be replaced with >0.

```
• solve(x^4=8,x);
```

$$\{8^{1/4}\}$$

which gives the only positive, real answer. We will use assume frequently.

Derivatives and Differentiation

As we mention above in the Examples of elementary math operations section, the derivative of a function measures the rate of change of a function. When we take the derivative of the function at a particular point, we measure the rate of change of a function at that point. Again, this rate of change at a particular point is described graphically as the slope of the function at that point.

The functions in MuPAD are the diff(f,x) and D(f), where f is the function you want to differentiate and x is the variable by which you are differentiating. Here is an example from MuPAD:

```
demand:=q->-2*q+6
```

```
q -> 6 - 2*q
```

- diff(demand(q),q)

-2

- D(demand)

-2

We defined our demand function to be P = -2q + 6. Notice that the slope of this function is -2. When we use the diff() command, we specify the independent variable by which we differentiate. In this case, the independent variable is q, quantity demanded. When we use the D() command, we do not specify the independent variable to differentiate.

When we come across a function with more than one independent variable, we must be careful. For more information on these commands, refer to the MuPAD help manual.

Plotting Graphs with MuPAD

Functions make it easy to plot items in MUPAD. To obtain a plot of a function, use plotfunc2d(f,x=a..b) command; where f is the name of the function to plot, x is the independent variable, a is the lower value of x, and b is the upper value of x. Consider the following example:

- `function1:=q->0.3*q+1; function2:=q->-0.3*q+36;`

```
q -> 0.3*q + 1

q -> 36 - 0.3*q
```

Now, we ask MuPAD to graph function1, from q = [1, 10].

- `plotfunc2d(function1(q),q=0..10);`

MuPAD then produces this graph in another window:

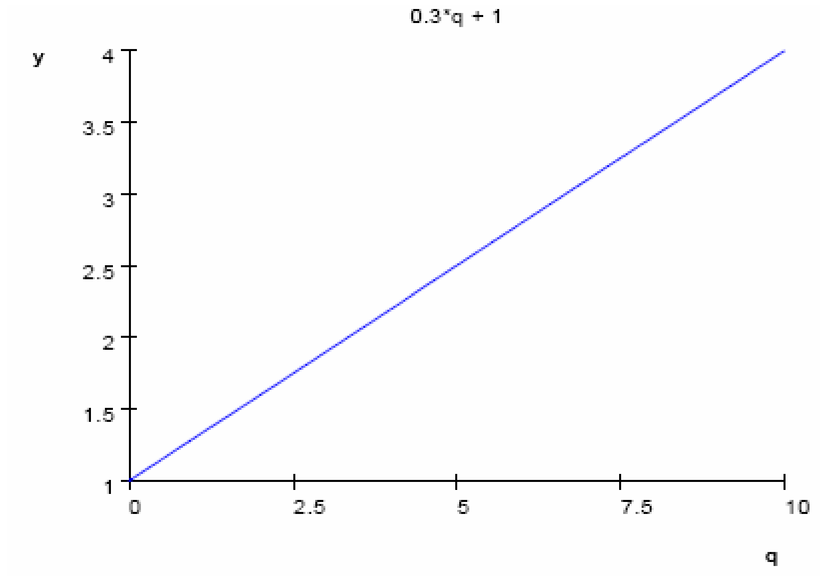


Figure 1

We can also use MuPAD to graph multiple functions at once:

- `plotfunc2d(function1(q),function2(q),q=0..100);`

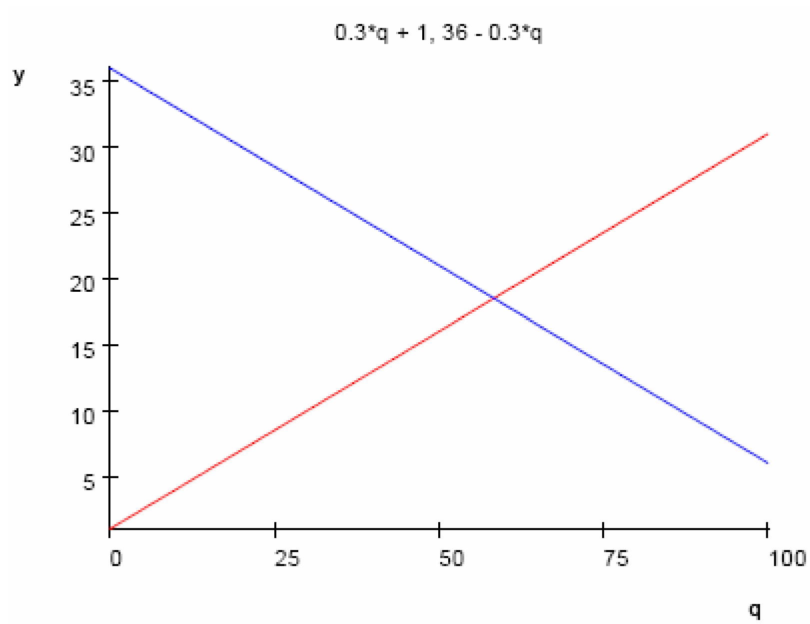


Figure 2

Notice here that we extended the graph along the horizontal axis in order to visually spot the point at which the functions cross.

MuPAD can graph more exciting functions as well:

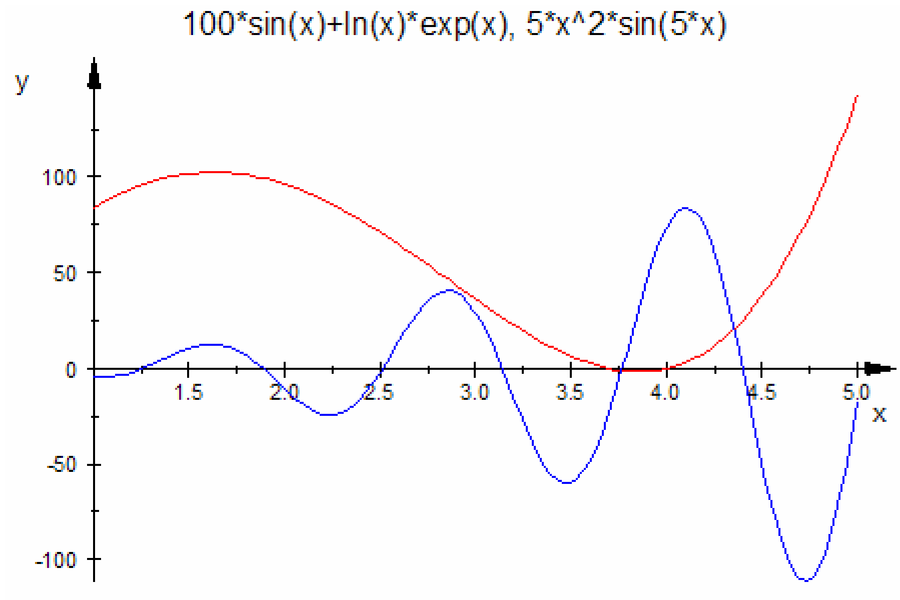


Figure 3

Removing Braces (curly brackets) and The op Command

Consider solving the following equation:

- `assume(a>0);`

```
> 0
```

- `eqn1:=x/a=1;`

```
x
- = 1
a
```

- `x:=solve(eqn1,x);`

```
{a}
```

- `x;`

```
{a}
```

Note that the solution is included in a set of braces,, or curly brackets { }. Many times equations have multiple solutions as in the following example.

- `solve(z^2=a,z);`

```
  1/2      1/2
{a    , - a    }
```

which has two solutions. What is included in the braces is a set of solutions. The problem with this is that we often can't operate on a set of solutions in the same fashion as we might operate on ordinary numbers.

For example:

- `x;`

```
{a}
```

- `solve(y=x,y);`

```
Error: Both sides of equation must be arithmetical expressions
[solve]
```

The problem is that x is not an arithmetical expression but a set which could contain several expressions. We can extract a member out of the set using the *op* command, however, and use it as a regular expression

- `assume(a>0);`

```
> 0
```

- `eqn1:=x/a=1;`

```
x
- = 1
a
```

- `x:=solve(eqn1,x);`

```
{a}
```

- `x;`

```
{a}
```

- `x:=op(x);`

```
a
```

Note that the braces have disappeared. Using this member of the set we can now solve the expression

- `solve(y=x,y);`

```
{a}
```

Note the solution is another set but that is what solve produces - sets of solutions.

The assign Command

This section shows how the MuPAD assign statement is used to select specific solutions out of a set of solutions.

First define some equations:

- `eqn1:=x^2+y^2=a ;`
- `eqn2:=x^2-y^2=a;`
- `a:=3;`

$$x^2 + y^2 = a$$

$$x^2 - y^2 = a$$

$$3$$

Now solve the system of equations and place the results in identifier solution:

- `solution:=solve({eqn1,eqn2},{x,y});`

$$\{[x = 3^{1/2}, y = 0], [x = - 3^{1/2}, y = 0]\}$$

The system of equations has two solutions. The following statement assigns the values of the first solution to identifiers x and y.

- `assign(op(solution,1)):x,y;`

$$3^{1/2}, 0$$

Display x and y to show that they do have the values of the first solution

- `x;`

$$3^{1/2}$$

- `y;`

The equations now how a numerical result, for example

- `eqn1;`

$$3 = 3$$

This does provide a check on the solution, to wit x^2+y^2 is equal to 3, but we may want to restore the original situation.

- `delete x,y;`
- `eqn1;`

$$x^2 + y^2 = 3$$

Change the problem

- `a:=4;`

```
4
```

- `eqn1;`

$$x^2 + y^2 = 4$$

Solve the new problem but this time select the second solution:

- `solution:=solve({eqn1,eqn2},{x,y});`
- `assign(op(solution,2)):x,y;`

```
{[x = -2, y = 0], [x = 2, y = 0]}

2, 0
```

Display x and y to see if it worked

- `x;y;`

```
2

0
```

Student Exercises

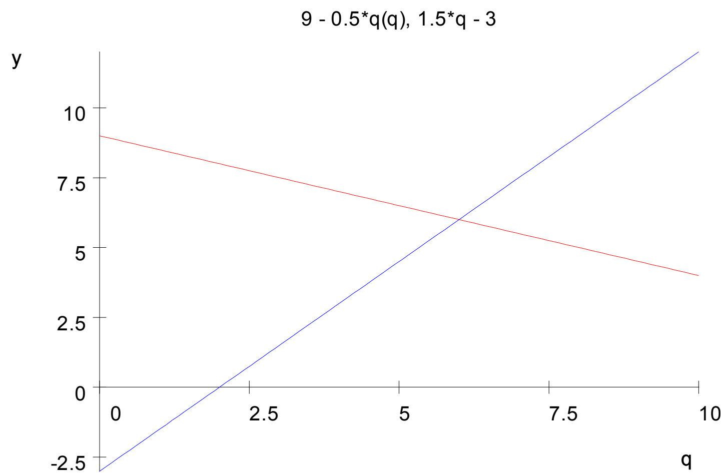
Project 1

This project is designed to get you familiar with the syntax of Mupad. Refer to the primer for any specific problems.

Demand line equation(d1):
p = -.5q + 9,

Supply line equation(s1):
p = 1.5q – 3,
where p is price and q is quantity demanded.

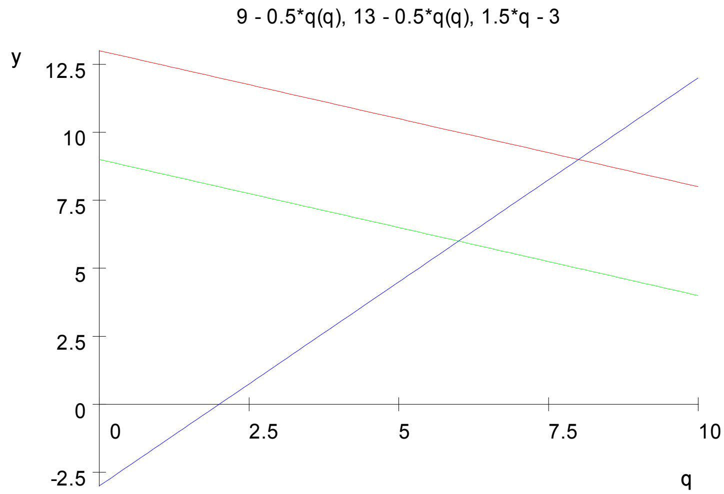
a graph of these equations looks like this in Mupad:



Try to reproduce this graph. Then, let's see what happens when the demand line shifts upward and to the right.

Now add a new demand equation(d2):
p = -.5q + 13,

The graph should look something like this:

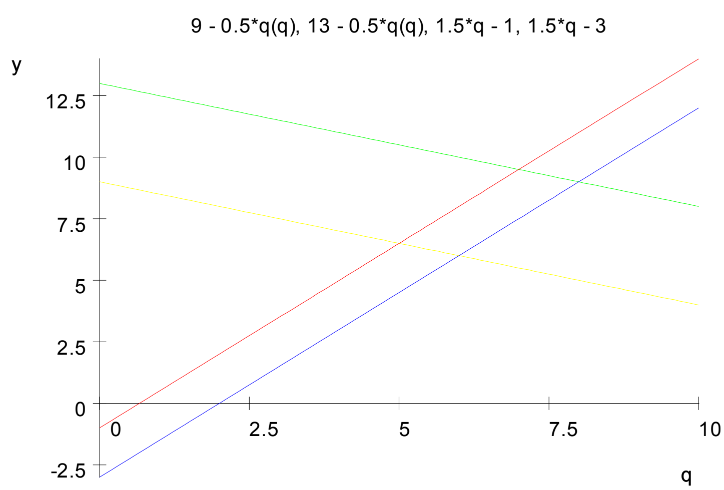


Don't worry if the colors do not match, this is simply a matter of the order in which you enter the equations in the plotfunc2d().

Now let's initiate a shrink in supply represented by the supply equation line moving up and to the left.

Now add a new supply equation(s2):
$p = 1.5q - 1$,

The graph should look like this:



Now that we have these four equations in Mupad, find the price and quantity demanded for each of these situations in Mupad: (d1,s1), (d2,s1), (d1,s2), (d2,s2). Look at the solve section in the primer for help.

Economics 385 Project 2:
The distribution of income in a classical macroeconomic model

The classical economists believed that the level of employment was determined in the labor market. The level of employment then determined the level of output and this, in turn, determined national income. The government cannot affect the level of employment unless it can change supply or demand in the labor market.

Suppose that the government increased government spending and bought more output. Would this increase the demand for labor to produce this increased demand for output?

Consider another question: how would the government fund its increased spending? There are two sources of income for government: taxes and loans.

Suppose the government raises taxes to pay for its spending increase. In the Classical view, taxed households will buy less goods by an amount that will offset the increase in government spending exactly. So aggregate output will remain at the same level.

The other source of funds for the government is borrowing from household savings. Realistically, government has a number of means by which it may borrow. But for the Classical model as well as future models, the details of this process are simplified by the assumption that the bond market is the only method through which industry and government borrow. A single variable r, represents the interest rate of the bond market and is the cost of borrowing money. If the government borrowed money from the bond market to finance an increase in spending, interest rates would rise.

What effects would a higher interest rate have on households and businesses? Households would have a greater incentive to save since they would earn a greater profit on the interest. An assumption we make in this class is that businesses do not have any retained earnings. So if businesses want to purchase new plants and equipment they must borrow the funds from the household sector. If interest rates increase, firms will find borrowing less attractive and they will be less likely to do so. So a higher interest rate would cause households to save more (spend less) and businesses to purchase less output (invest less). These decreases in demand for output from two sectors of the economy again offsets the government spending increase exactly.

A simple Classical economic model:

The classical household savings function is given by equation 1:

$$S(r) = S_0 + S_r * r,$$

where $S_r > 0$ and S_0 is autonomous savings.

The classical investment (business spending) function is given by equation 2:

$$I(r) = I_r * r,$$
where $I_r < 0$ and I_o is autonomous investment spending.

As we mentioned above, both business (equation 2) and government borrow from the household sector, (equation 1).

The amount the government borrows is determined by how much it spends, G, and how much it taxes, T. If G > T, the government has a deficit (G -T) for which it must borrow.

So total borrowing is given by:

$$I(r) + (G - T).$$

Interest rates will be determined by an equilibrium in the bond market where

savings (bonds bought) = borrowing (bonds sold)
or
$$S(r) = I(r) + (G - T)$$
or
$$S_r * r + S_0 = I_r * r + I_o + (G - T).$$

Assignment:
Use a Computer algebra system to explore the two scenarios of government funding -- increasing taxes and borrowing – within the classical model.

Initial parameter values:
{ $S_0 = 0$, $S_r = 5$, $I_o = 100$, $I_r = -5$, G = 100, T = 100 }.

Raise taxes:
{ G = 100, T = 110 }.

Increase government spending through borrowing:
{ G = 110, T = 100 }.

Present a paper with a graph of the model for each scenario -- the initial situation, raised taxes, and government borrowing.

After each graph, offer a short description and comment on what has happened.

Use the solve command to find out where output is in each case and what effect these two scenarios have on the interest rate, r. And what effect the change in r has on savings, S(r), and business investment, I(r).

Write a concise conclusion discussing the general implications of the model. Questions to consider: What effect does an increase in government spending have on our model? What effect would a decrease in government spending have on our model? Are either effective in accomplishing anything? What would the Classical economist recommend to a government leader given an economic recession?

Project 3:
The Varying components in the Simple Keynesian Model

Simple Keynesian Model

The Keynesian consumption function:

$$C = C_0 + C_y (Y - T),$$

Where $C_y > 0$ and $(Y - T)$ is real disposable income.

The investment spending function:

$$I = I_0 + I_r * r$$

Equilibrium condition:

$$Y = C + I + G$$

The savings residual:

$$S = Y - C - T$$

Assignment

Initial variable values:

{ $C_0 = 100$, $C_y = 0.75$, $I_0 = 100$, $I_r = -1$, $r = 10$, $G = 200$, $T = 100$ }

Set up the model using a computer algebra system such as Maxima or MuPAD for the given parameter values. Define functions representing a demand equation, a supply equation, and equilibrium equation Graph each scenario.

What is the initial level of output Y?

1. Now simulate a recession by letting autonomous investment spending, (I_0) fall from 100 to 90. Now what is the level of output?
{ $I_0 = 90$, $G = 200$, $T = 100$ }

2. Next try to offset the recession through government spending. Increase the value of government spending from 200 to 210. Record the new value of Y.
{ $I_0 = 90$, $G = 210$, $T = 100$ }

3. Next try to solve the recession using a tax cut. Reset the value of government spending to G = 200 and lower taxes, T = 90. Record the new value of Y.
{ $I_0 = 90$, $G = 200$, $T = 90$ }

Elements to cover in the Project

1. After each graph, offer a short description of each graph, does the graph make sense? If so, why?

2. Specify what happened to equilibrium income when autonomous investment spending fell. Does this seem to mimic the behavior one might expect in a recession?

3. Was the increase in government spending adequate to eliminate the recession (that is to restore the economy back to its original level of income)?
Was the decrease in taxes adequate to eliminate the recession?

4. Fill in a table like the one below with the appropriate value of Y for the values for autonomous investment spending, government spending and taxes:

Y	I_0	G	T
	100	200	100
	90	200	100
	90	210	100
	90	200	90

Notes:
1. These software packages work better if you can give it some a prioi knowledge about the expected values of some parameters. For example we expect that the demand curve will have a negative slope and the supply curve a positive slope. In Mupad, this can be accomplished by using the assume() command.

Economics 385 Project 4:
The IS-LM Model

In project 3, you used the IS model. The IS curve depicts one relationship: given interest rate, r, you can solve for output, Y, and vice versa, given Y you can find r. In this project you will add the money market to the model by including the LM curve. Now, given two relationships you can solve for both r and Y.

A Simple IS model:

$$C = C_0 + C_y * (Y - T),$$
$$I = I_0 + I_r * r,$$
$$Y = C + I + G.$$

A Simple LM model:

$$M^d = M_0 + M_Y * Y + M_r * r$$
$$M^s = M^d$$

Assignment:
Use a CAS program to set up the IS-LM Model given the initial parameters.

Initial parameter values:
{ C_0 = 100, C_y = 0.75, I_0 = 100, I_r = -1, G = 200, T = 100, M^s = 4100, M^d_0 = 5000, $M^d{}_y$ = 0.01, $M^d{}_r$ = -100}

Raise government spending: {G = 210, M^s = 4100}.

Increase the Money Supply: {G = 200, M^s = 4500}.

Explore the scenarios of an increase in government spending and an increase in the money supply. Graph these scenarios and explain what each entails.

Fill out the following table:

G	Ms	Y_eq	r_eq
100	4100		
110	4100		
100	4500		

Using the results in the table, do you think this model is one where either monetary or fiscal policy would be useful in stabilizing an economy? Explain.

Economics 385 Project 5:
The IS--LM Model, The monetarist view

In project 4 you used the IS--LM model. For project 5, we will keep the models the same save for the magnitude of the interest elasticity of money demand, M_r. In project 4 we gave the coefficient on interest rates the value, $M_r = -100$, modelling the Keynesian view of the economy, where a change in interest rates greatly affected money demand. In this project, we will compare this to a value more representative of the monetarist view, making interest elasticity of money demand much smaller, $M_r = -0.1$. We will also use a much smaller value for the marginal propensity to consume $C_y = 0.4$, compared with $C_y = 0.75$ used in project 4. Finally, we will make the influence of interest rates larger here than in project 4, $I_r = -10$ compared with $I_r = -1$. These changes represent the monetarist trust in the rational behavior of managers overcoming the Keynesian "animal spirits", I_0.

A Simple IS model:

$$C = C_0 + C_y * (Y - T),$$
$$I = I_0 + I_r * r,$$
$$Y = C + I + G.$$

A Simple LM model:

$$M^d = M_0 + M_Y * Y + M_r * r$$
$$M^s = M^d$$

Assignment:
Use a CAS program to set up the IS-LM Model given the initial parameters.

Use the following values for the parameters
{ $C_0 = 500$, $C_y = 0.4$, $I_0 = 200$, $I_r = -10$, $G = 100$, $T = 100$, $M^s = 850$, $M^d_0 = 740$, $M^d_y = 0.1$, $M^d_r = -0.1$}

Raise government spending: {$G = 110$, $M^s = 850$}.

Increase the Money Supply: {$G = 100$, $M^s = 860$}.

Explore the scenarios of an increase in government spending and an increase in the money supply. Graph these scenarios and explain what each entails.

Use a CAS to get the values to fill out the following table:

G	Ms	Y_eq	r_eq
100	850		
110	850		

100	860		

Using the results in the table do you think this model is one where monetary or fiscal policy would be useful in stabilizing an economy? Explain.

Macro Lecture Notes

The Basic Keynesian Model

The economy will be in equilibrium if

$$\text{household income} = Y = C(Y-T) + I(r) + G = \text{spending} = \text{GDP}$$

The relationship $Y = C + T + G$ is called an equilibrium condition. In this case if the income earned by households in producing output is just equal to the amount spent on that output the economy would be stable. If any of the spending components (*C,I,G*) then this equilibrium will be destroyed. Suppose household decide to spend more. In that case spending on output exceeds household income. In the Keynesian word inventories decrease, retailers will place orders for additional output to rebuild inventory, firms will have to hire additional workers to produce this output and household income will increase. This increase in household income will continue until a new equilibrium is established.

Symbolic Results

- `reset() ;`

- `assume(Co>0): assume(Cy>0): assume(Cy<1,_and): assume(T>0):`
- `assume(G>0): assume(Ir<0): assume(Io>0):`

Household consumption is given by

- `C:=Co+Cy*(Y-T);`

  $\mathrm{Co} - \mathrm{Cy} \cdot (T - Y)$

where *C* is total consumption, *Co* is autonomous consumption, *Cy* is the marginal propensity to consume, Y is household income, *T* is lump sum taxes paid by households and (*Y-T)* is household disposible income.

Investment (business) spending is

- `Inv:=Io ;`

  Io

where *Inv* is total investment spending and *Io* is autonomous investment spending. Keynes believed that managers were often controlled by "animal spirits" and would behave in unpredictable ways when in the grips of such spirits. Animal spirits here are represented by autonomous investment spending.

Equilibrium between spending an income occurs when

- `IS:=Y=C+Inv+G ;`

$$Y = Co + G + Io - Cy \cdot (T - Y)$$

Next solve the IS curve to get the relationship between

-
- `ans:=solve({IS},{Y});`

$$\left\{\left[Y = -\frac{Co + G + Io - Cy \cdot T}{Cy - 1}\right]\right\}$$

- `assign(op(ans)):Y ;`

$$-\frac{Co + G + Io - Cy \cdot T}{Cy - 1}$$

- `Y;`

$$-\frac{Co + G + Io - Cy \cdot T}{Cy - 1}$$

This symbolic result can be used to make predictions about how income will change if there is a change in the spending terms. Recall that (*Cy-1 < 0*). Increases in *Co, Io or G* cause income to increase. An increase in *T* causes income to decrease.

Numerical Results

It is possible to get numerical values for *Y* if values are given for the other terms in the equilibriums condition.

- `Co:=100 : Io := 100 : Cy := 0.9: T:=100: G:=100:`
- `Y;`

$$2000.0$$

and the equilibrium value of income is *Y=2000* with this set of values. Suppose now that managers fall under the spell of animal spirits and reduce business investment spending by 10 units.

- `Io:=90; Y;`

$$90$$

$$2000.0$$

The effect of a 10 unit decline in investment spending is a 100 unit decrease in household income. Changes in autonomous investment spending are a primary source of instability in the Keynesian model of the economy.

The classical distribution of income

This note demonstrates how the classical economists believed that income was distributed.

-
- `reset();`
- `assume(Sr>0): assume(Ir<0):`

Define savings to be a function of interest rates

- `S:=So+Sr*r;`

$$\mathrm{So} + \mathrm{Sr} \cdot r$$

Investment is also a function of interest rates

- `I_:=Io+Ir*r;`

$$\mathrm{Io} + \mathrm{Ir} \cdot r$$

Equilibrium is given by

- `Eq:=S=I_+G-T;`

$$\mathrm{So} + \mathrm{Sr} \cdot r = G + \mathrm{Io} - T + \mathrm{Ir} \cdot r$$

Solving this gives the equilibrium value for interest rates

- `ans:=solve(Eq,r);`

$$\left\{ -\frac{G + \mathrm{Io} - \mathrm{So} - T}{\mathrm{Ir} - \mathrm{Sr}} \right\}$$

- `ans:=op(ans);`

$$-\frac{G + \mathrm{Io} - \mathrm{So} - T}{\mathrm{Ir} - \mathrm{Sr}}$$

- `r:=ans;`

$$-\frac{G + \mathrm{Io} - \mathrm{So} - T}{\mathrm{Ir} - \mathrm{Sr}}$$

Next give values for the parameters and exogenous variables. Note that income is assumed to be fixed and determined by the supply and demand for labor.

- `Y:=2000: Sr:=10:  Ir:=-10: T:=100: G:=100: So:=0: Io:=100:`

The equilibrium values for interest rates, consumption and income are

- `r; C:=Y-S-T; I_; G;`

  5

  1850

  50

  100

Suppose that the government now decides to increase spending and finance the deficit by borrowing

- `G:=110;`

  110

Note that interest rates increase with the additional borrowing and this affects other spending components

- `r; C:=Y-S-T; I_; G;`

  $\frac{11}{2}$

  1845

  45

  110

Suppose that the government decides to finance the increase in spending with and increase in taxes

- `T:=110;`

  110

- `r; C:=Y-S-T; I_; G;`

  5

  1840

  50

  110

The IS Curve and Changes in Government Spending

The IS curve is made up of combinations of interest rates and household incomes (r,Y) that make household income equal to total spending in the economly.

$$\text{household income} = Y = C(Y-T) + I(r) + G = \text{spending} = \text{GDP}$$

where *Y* is household income, *T* is lump sum taxes, *r* indicates interest rates, *G* represents government spending, *C* representes household spending and *I* indicates business investment spending. A change in *r* causes a change in *I* that causes a change in *Y*. This change in *Y* causes a further change in *C* that leads to an additional change in *Y*, etc.

Symbolic Solution

```
• reset() ;
```

```
• assume(Co>0): assume(Cy>0): assume(Cy<1,_and): assume(T>0):
• assume(G>0): assume(Ir<0): assume(Io>0):
```

Household consumption is given by

```
• C:=Co+Cy*(Y-T);
```

$$\mathrm{Co} - \mathrm{Cy} \cdot (T - Y)$$

where *C* is total consumption, *Co* is autonomous consumption, *Cy* is the marginal propensity to consume, *Y* is household income, *T* is lump sum taxes paid by households and (*Y-T)* is household disposable income.

Investment (business) spending is

```
• Inv:=Io+Ir*r ;
```

$$\mathrm{Io} + \mathrm{Ir} \cdot r$$

where *Inv* is total investment spending, *Io* is autonomous investment, *r* is the interest rate and *Ir* is a parameter that determines how much investment spending will change if interest rates change. Equilibrium between spending an income occurs when

```
• IS:=Y=C+Inv+G ;
```

$$Y = \mathrm{Co} + G + \mathrm{Io} - \mathrm{Cy} \cdot (T - Y) + \mathrm{Ir} \cdot r$$

Next solve this for *Y*

```
ans:=solve({IS},{Y});
```

$$\left\{\left[Y = -\frac{\mathrm{Co} + G + \mathrm{Io} - \mathrm{Cy} \cdot T + \mathrm{Ir} \cdot r}{\mathrm{Cy} - 1}\right]\right\}$$

Numeric Solution

```
assign(op(ans)):Y ;
```

$$-\frac{\mathrm{Co} + G + \mathrm{Io} - \mathrm{Cy} \cdot T + \mathrm{Ir} \cdot r}{\mathrm{Cy} - 1}$$

Give numerical values to the parameters

```
Co:=100: Cy:=0.9 :  Ir:= -1 : G:=100: T:=100:
Io:=100: r:=10:
```

Store the results for *r* and *Y* in *r_is1* and *Y_is1* so they can be plotted out later on.

```
 r_is1:= r ; Y_is1:= Y;
```

$$10$$

$$2000.0$$

It can be useful to check to see if the equilibrium condition has been met. *Does C+I+G=Y*?

```
C ;Inv ; G; C+Inv+G ;
```

$$1810.0$$

$$90$$

$$100$$

$$2000.0$$

Because *Y=C+I+G* this is an equilibrium combination of (*r*,*Y*).

Suppose interest rates increase from *r=10* to *r=20*. This should cause investment to decrease. The reduction in business spending will cause a reduction in household income. Store the values of *r* and *Y* for plotting.

```
r:=20;  r_is2:=r:  Y_is2:= Y;
```

$$20$$

1900.0

Again check to see if this is an equilibrium

- `C; Inv; G; C+Inv+G;`

1720.0

80

100

1900.0

Again because *Y=C+I+G* this is an equilibrium combination of (*r*,*Y*). Next plot the equilibrium points--the plot is the IS curve.

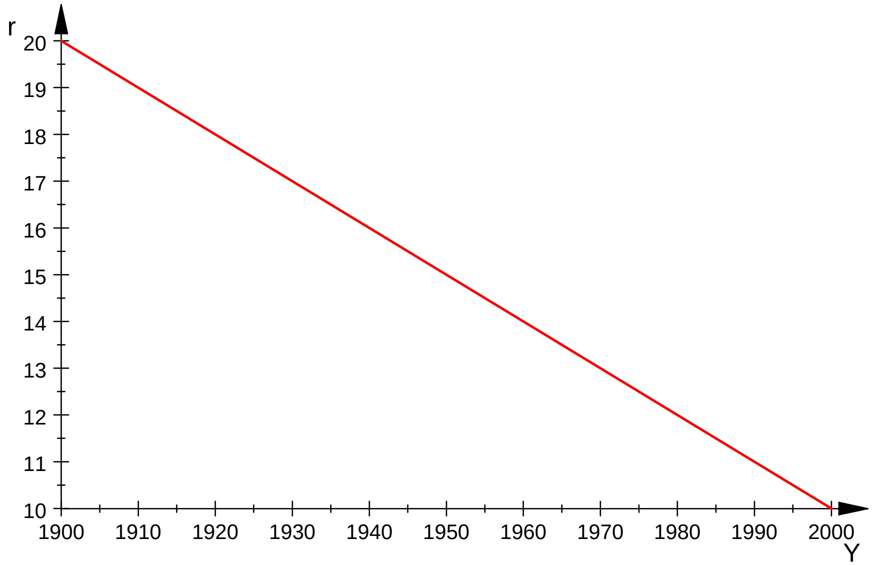
- `p1:=plot::Polygon2d([[Y_is1,r_is1], [Y_is2,r_is2]], Color=RGB::Red) ;`

plot::Polygon2d([[2000.0, 10], [1900.0, 20]])

- `CS1:=plot::CoordinateSystem2d(p1) ;`

plot::CoordinateSystem2d(...)

- `CS1::XAxisTitle:="Y" :`
- `CS1::YAxisTitle:="r" :`
- `plot(CS1);`



What effect will a change in government have on the IS curve? Let government spending increase from *G=100* to *G=110* and get one point on the new IS curve (*r_is3, Y_is3*) ;

- `G:=110 : r:=10 : Y : r_is3:=r : Y_is3 := Y :`

Change r and get a second point (*r_is4,Y_is4*)

- `r:=20 : r_is4:=r : Y_is4:= Y:`

Plot both IS curves

- `p2:=plot::Polygon2d([[Y_is3,r_is3], [Y_is4,r_is4]], Color=RGB::Green) ;`

  plot::Polygon2d([[2100.0, 10], [2000.0, 20]])

- `p3:=plot::Arrow2d([1950,15],[2050,15]) ;`

  plot::Arrow2d([1950, 15], [2050, 15])

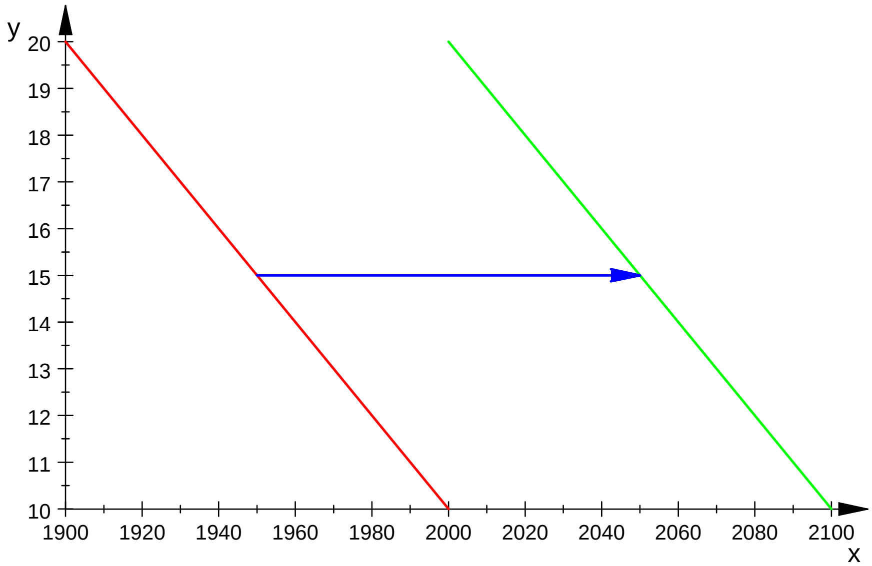
- `CS1::XAxisTitle:="Y" :`

- `CS1::YAxisTitle:="r" :`

- `CS2:=plot::CoordinateSystem2d(p1,p2,p3) ;`

  plot::CoordinateSystem2d(...)

- `plot(CS2) ;`



The effect of an increase in government spending is to shift the IS curve to the right.

The IS Curve and Changes in Autonomous Investment Spending

The IS curve is made up of combinations of interest rates and household incomes (r,Y) that make household income equal to total spending in the economy.

$$\text{household income} = Y = C(Y-T) + I(r) + G = \text{spending} = \text{GDP}$$

where *Y* is household income, *T* is lump sum taxes, *r* indicates interest rates, *G* represents government spending, *C* representes household spending and *I* indicates business investment spending. A change in *r* causes a change in *I* that causes a change in *Y*. This change in *Y* causes a further change in *C* that leads to an additional change in *Y*, etc.

Symbolic Solution

- `reset() ;`
- `assume(Co>0): assume(Cy>0): assume(Cy<1,_and): assume(T>0):`
- `assume(G>0): assume(Ir<0): assume(Io>0):`

Household consumption is given by

- `C:=Co+Cy*(Y-T);`

$$\text{Co} - \text{Cy} \cdot (T - Y)$$

where *C* is total consumption, *Co* is autonomous consumption, *Cy* is the marginal propensity to consume, *Y* is household income, *T* is lump sum taxes paid by households and (*Y-T)* is household disposible income. Investment (business) spending is

- `Inv:=Io+Ir*r ;`

$$\text{Io} + \text{Ir} \cdot r$$

where *Inv* is total investment spending, *Io* is autonomous investment, *r* is the interest rate and *Ir* is a parameter that determines how much investment spending will change if interest rates change. Equilibrium between spending an income occurs when

- `IS:=Y=C+Inv+G ;`

$$Y = \mathrm{Co} + G + \mathrm{Io} - \mathrm{Cy} \cdot (T - Y) + \mathrm{Ir} \cdot r$$

Next solve this for *Y*

- `ans:=solve({IS},{Y});`

$$\left\{\left[Y = -\frac{\mathrm{Co} + G + \mathrm{Io} - \mathrm{Cy} \cdot T + \mathrm{Ir} \cdot r}{\mathrm{Cy} - 1}\right]\right\}$$

Numeric Solution

- `assign(op(ans)):Y ;`

$$-\frac{\mathrm{Co} + G + \mathrm{Io} - \mathrm{Cy} \cdot T + \mathrm{Ir} \cdot r}{\mathrm{Cy} - 1}$$

Give numerical values to the parameters

- `Co:=100: Cy:=0.9 :  Ir:= -1 : G:=100: T:=100:`
- `Io:=100: r:=10:`

Store the results for *r* and *Y* in *r_is1* and *Y_is1* so they can be plotted out later on.

- ` r_is1:= r ; Y_is1:= Y;`

$$10$$

$$2000.0$$

It can be useful to check to see if the equilibrium condition has been met. Does *C+I+G=Y*?

- `C ;Inv ; G; C+Inv+G ;`

1810.0

90

100

2000.0

Because *Y=C+I+G* this is an equilibrium combination of (r,Y).

Suppose interest rates increase from *r=10* to *r=20*. This should cause investment to decrease. The reduction in business spending will cause a reduction in household income. Store the values of *r* and *Y* for plotting.

- `r:=20;  r_is2:=r:  Y_is2:= Y;`

20

1900.0

Again check to see if this is an equilibrium

- `C; Inv; G; C+Inv+G;`

1720.0

80

100

1900.0

Again because *Y=C+I+G* this is an equilibrium combination of (*r*,*Y*). Next plot the equilbrium points--the plot is the IS curve.

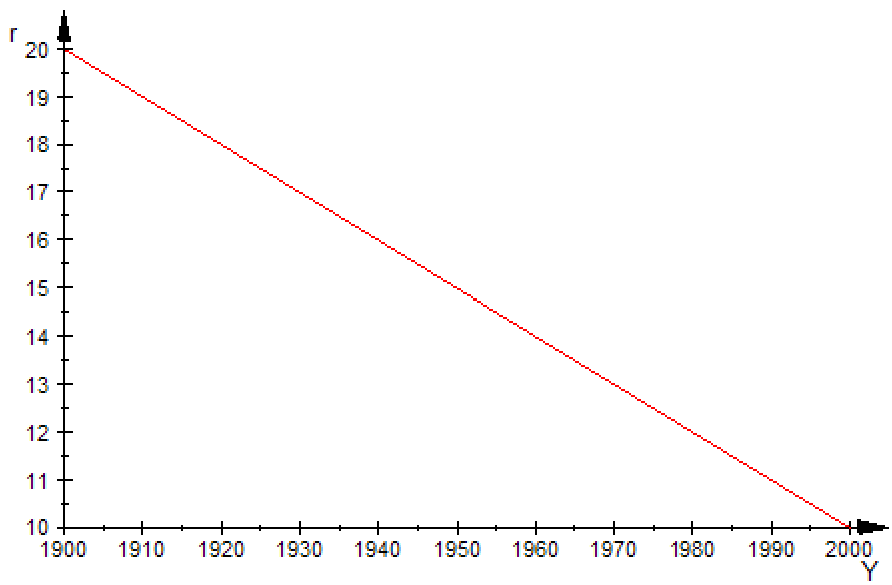
- `p1:=plot::Polygon2d([[Y_is1,r_is1], [Y_is2,r_is2]], Color=RGB::Red) ;`

plot::Polygon2d([[2000.0, 10], [1900.0, 20]])

- `CS1:=plot::CoordinateSystem2d(p1) ;`

plot::CoordinateSystem2d(...)

- `CS1::XAxisTitle:="Y" :`
- `CS1::YAxisTitle:="r" :`
- `plot(CS1);`



What effect will a change in autonomous investment spending have on the IS curve? Let autonomous investment spending decrease from Io=100 to Io=90 and get one point on the new IS curve (*r_is3, Y_is3*) ;

- `Io:=90 : r:=10 : Y : r_is3:=r : Y_is3 := Y :`

Change r and get a second point (*r_is4,Y_is4*)

- `r:=20 : r_is4:=r : Y_is4:= Y:`

Plot both IS curves

- ```
  p2:=plot::Polygon2d([[Y_is3,r_is3], [Y_is4,r_is4]], Color=RGB::Green) ;
  ```

plot::Polygon2d([[1900.0, 10], [1800.0, 20]])

- ```
  p3:=plot::Arrow2d([1950,15],[1850,15]) ;
  ```

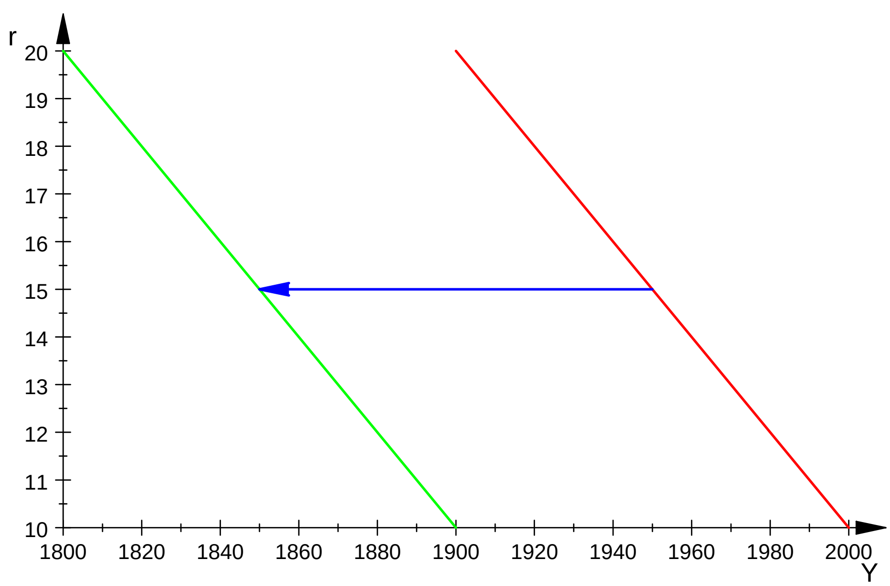
plot::Arrow2d([1950, 15], [1850, 15])

- ```
  CS2:=plot::CoordinateSystem2d(p1,p2,p3) ;
  ```

plot::CoordinateSystem2d(...)

- ```
  CS2::XAxisTitle:="Y" :
  ```
- ```
  CS2::YAxisTitle:="r" :
  ```

- ```
  plot(CS2) ;
  ```



The effect of an decrease in autonomous investment is to shift the IS curve to the left.

Keynesian verses Monetarist models

```
• reset() ;

•
• assume(Co>0): assume(Cy>0): assume(Cy<1,_and): assume(T>0):
• assume(G>0): assume(Ir<0): assume(Io>0):
• assume(My>0) : assume(Mr<0) : assume(Ms>0) : assume(Mo>0) :
```

```
• C:=Y->Co+Cy*(Y-T);
```

$$Y \to \mathrm{Co} + \mathrm{Cy} \cdot (Y - T)$$

```
• Inv:=r->Io+Ir*r ;
```

$$r \to \mathrm{Io} + \mathrm{Ir} \cdot r$$

```
• IS:=Y=C(Y)+Inv(r)+G ;
```

$$Y = \mathrm{Co} + G + \mathrm{Io} - \mathrm{Cy} \cdot (T - Y) + \mathrm{Ir} \cdot r$$

$$(0, \infty)$$

```
• Md:=(r,Y)->Mo+My*Y+Mr*r ;
```

$$(r, Y) \to \mathrm{Mo} + \mathrm{My} \cdot Y + \mathrm{Mr} \cdot r$$

```
• LM:=Ms=Md(r,Y) ;
```

$$\mathrm{Ms} = \mathrm{Mo} + \mathrm{My} \cdot Y + \mathrm{Mr} \cdot r$$

```
• ans:=solve({IS,LM},{r,Y}):
• ans:=op(ans,[1,2]): assign(op(ans)):r,Y :
• r;
```

$$-\frac{\mathrm{Mo} - \mathrm{Ms} + \mathrm{Co} \cdot \mathrm{My} - \mathrm{Cy} \cdot \mathrm{Mo} + \mathrm{Cy} \cdot \mathrm{Ms} + G \cdot \mathrm{My} + \mathrm{Io} \cdot \mathrm{My} - \mathrm{Cy} \cdot \mathrm{My} \cdot T}{\mathrm{Mr} - \mathrm{Cy} \cdot \mathrm{Mr} + \mathrm{Ir} \cdot \mathrm{My}}$$

```
• Y;
```

$$\frac{\mathrm{Co} \cdot \mathrm{Mr} + G \cdot \mathrm{Mr} + \mathrm{Io} \cdot \mathrm{Mr} - \mathrm{Ir} \cdot \mathrm{Mo} + \mathrm{Ir} \cdot \mathrm{Ms} - \mathrm{Cy} \cdot \mathrm{Mr} \cdot T}{\mathrm{Mr} - \mathrm{Cy} \cdot \mathrm{Mr} + \mathrm{Ir} \cdot \mathrm{My}}$$

Keynesian vs Monetarists views

The Keynesian View

Now to compare Keynesian and monetarist models. These models will differ in the following respects: (a) the marginal propensity to consume will be larger for the the Keynesian model; (b) the rational part of investment, Ir, will be smaller for the Keynesian model; and (c) the speculative demand part of the money demand function, Mr, will be greater in the Keynesain case.f The values chosen for the coefficients in the Keynesian case are

- `Co:=100: Cy:=0.9 :  Ir:= -0.1 : G:=100: T:=100:`

  `Io:=100:`

- `My:=0.1: Mr:=-100: Ms:=4900: Mo:=5000:`
-
-
- `r; Y;`

  3.096903097

  2096.903097

Check to see if this really is an equlibirium point. Is Y = C(Y) +I(r) + G?

- `Y ; C(Y)+Inv(r)+G ;`

  2096.903097

  2096.903097

Is Ms=Md?

- `Ms; Md(r,Y) ;`

  4900

  4900.0

So this combination of interest rates and income is an equilibrium point in both the product and money markets. Suppose government spending changes from G=100 to G=110. Will this affect the equilibrium?

- `G:=110: r; Y ;`

  3.196803197

  2196.803197

Some things to note. Income increased by almost 100 units for the 10 unit increase in government spending. Interest rates only changed slightly, by something less than one tenth of a percentage point. These results are consistent with those of the full Keynesian mutliplier (where interest rates do not change).

Next consider the effects of an increase in money supply from Ms=4900 to Ms=4910 in the Keynesian model.

- `Ms:=4910: r; Y;`

  3.096903097

  2196.903097

The 10 unit increase in the money supply had little effect on interest rates or income in this case. Compare these results with those of the monetarist model that follows.

The monetarist view

The following set of coefficient values are representative of the monetarists' view. The marginal propensity to consume is smaller (0.5) than that used in the Keynesian model (0.9). The response of business managers to a change in interest rates is greater (-10) than that used in the Keynesian model (-0.1). This means that managers are viewed as being profit maximizers rather than being driven by animal spirits. Finally the speculative demand is smaller (-0.1) for the monetarist model than was used in the Keynesian model (-100). In this case the transactions demand for money is much greater than in the Keynesian case.

- `Co:=500: Cy:=0.5 :  Ir:= -10 : G:=100: T:=100:`

- `Io:=500:`

- `My:=0.1: Mr:=-0.1: Ms:=5200: Mo:=5000:`

The results for this set of coefficients are

- `r; Y;`

  4.761904762

  2004.761905

Check to see if these are equilibrium values

- `Y; C(Y)+Inv(r)+G;`

  2004.761905

  2004.761905

- `Ms; Md(r,Y);`

  5200

  5200.0

Let government spending increase by ten units, the same increase used in the Keynesian model

- `G:=110: r; Y;`

  5.714285714

  2005.714286

Check to see if these are equilibrium values

- `C(Y)+Inv(r)+G; Md(r,Y);`

  2005.714286

  5200.0

These results can be compared with the Keynesian results

$\Delta G = +10$	Δr	ΔY
Keynes	+0.10	+99.0
Monetarist	+0.95	+0.95

Increase money supply and compare these results with the Keynesian results.

- `Ms:=5210: r; Y;`

  0.9523809524

  2100.952381

•

• $\Delta M^s = +10$	Δr	ΔY
Keynes	0.10	0.10
Monetarist	-4.76	95.24

Finally check to see if these are equilibrium values

• `C(Y)+Inv(r)+G; Md(r,Y);`

2100.952381

5210.0

Micro Lecture Notes

How to find a maximum or a minimum

Economics is a study of optimization. Individuals maximize utility. Firms minimize costs. Some optimization methods will be covered here. Optimization problems can be either minimization problems or maximization problems. These problems are often call *extreme value* problems where an extreme value can refer to either a maximum or a minimum. Somewhat crudely, x^* is an extreme value (a maximum) of $f(x)$ if the value of the function at x^* is larger than at any other point near x^*. Likewise x^* will be a minimum of $f(x)$ if the value of the function at x^* is smaller than at any other point near x^*. It is also customary to call extreme values *stationary points*.

Using plots

The advent of the computer has made it feasible to use plots as a method of finding stationary points in some cases. Plots do not require any particular mathematical sophistication. Prior to the computer, however, computing the values for the plot and putting the values on the plot would likely be quite laborious.

Example: Find a stationary point of $f(x)=10x-2x^2$

- f:=x->10*x-2*x^2 ;

  $x \rightarrow 10 \cdot x - 2 \cdot x^2$

- `p1:=plot::Function2d(f(x), x=1..5) :`

- `plot(p1):`

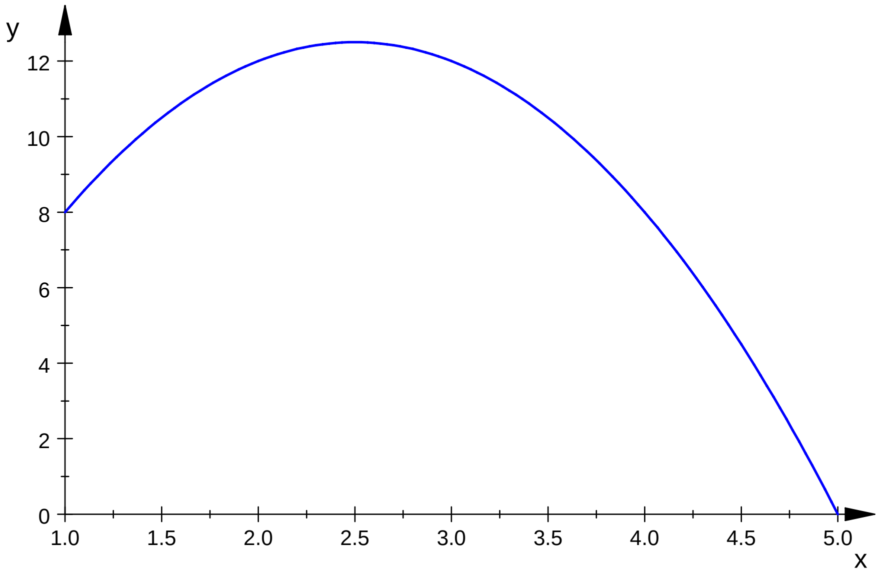


Figure 1. A plot of f(x)

A visual inspection of the plot indicates that the value of *x* that gives a maximum value of f(x) is approximately *x=2.5.* Next consider finding a stationary point of

```
• g:=x->-(10*x-2*x^2);
```

$$x \rightarrow -\left(10 \cdot x - 2 \cdot x^2\right)$$

```
• p1:=plot::Function2d(g(x), x=1..5) :
• plot(p1):
```

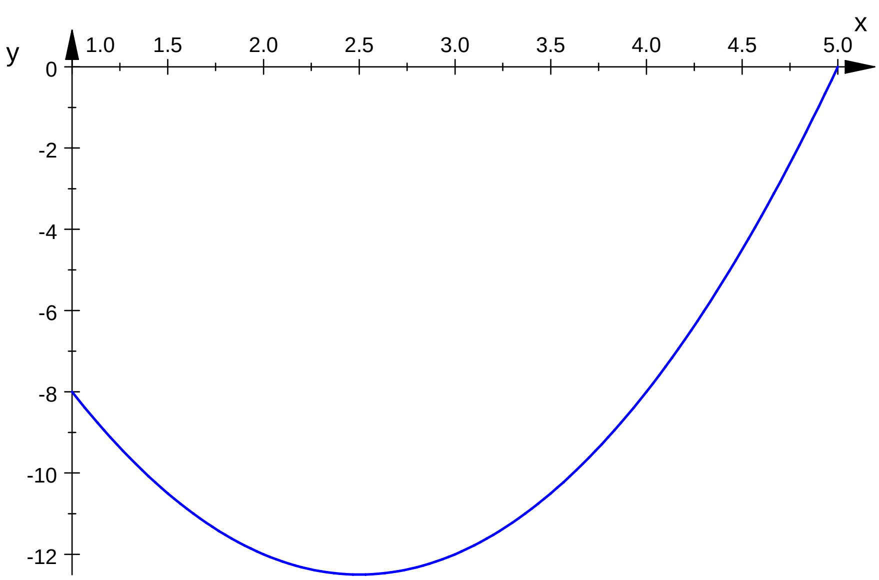


Figure 2. A plot of g(x)

In this case the stationary point is one that produces a minimum rather than a maximum. Again the value of the stationary point is approximately *x=2.5*. These two plots suggest the reason for the terminology *stationary* point. Pretend that the function is a roller coaster and that a ball is on the roller coaster at the highest or lowest point. The ball will stay at rest unless it is disturbed. If the ball is slightly displaced from the stationary point, it will now tend to remain in motion departing from the maximum or oscillating about the minimum.

Plotting can be used to locate stationary points in simple cases. Plots are not very useful if the problem involves more than two independent variables. They can also require significant trial and error. They are not useful if symbolic rather than numerical solutions are needed. They can only produce approximate answers and the approximation may not be particulalry accurate. More sophisticated method can be used to produce more accurate answers.

The D operator

MuPAD has a very useful operator called the D operator. An operator is something that performs an operation on a mathematical expression. Mathematicians call the D operator a derivative. Economists call the D operator a marginal change. The D operator computes how much f(x) changes for a given change in x. Suppose that the marginal change in f(x) is positive. That means that f(x) will be increasing as x increases. If the marginal change in f(x) is negative, then f(x) will be decreasing as as x increases. At a stationary point the function neither increases nor decreases. In other words the marginal change in f(x) will be zero at a stationary point.

This gives a method of locating a stationary point of a function f(x).

1. Apply the D operator to f(x). Call the result Df.
2. Find the value of x that makes Df=0.

Now let D "operate" on f(x)

- Df:=D(f) ;

$$x \to 10 - 4 \cdot x$$

- ans:=solve(Df(x)=0,x) ;

$$\left\{\frac{5}{2}\right\}$$

- stationary_x:=op(ans) ;

$$\frac{5}{2}$$

```
• p1:=plot::Function2d(f(x), x=1..5, Color=RGB::Red) :
• p2:=plot::Function2d(Df(x), x=1..5, Color=RGB::Blue) :
• p3:=plot::Line2d([stationary_x,0],[stationary_x,f(stationary_x)],
  Color=RGB::Green) :
• plot(p1,p2,p3, AxesTitles=["x","f(x)"]);
```

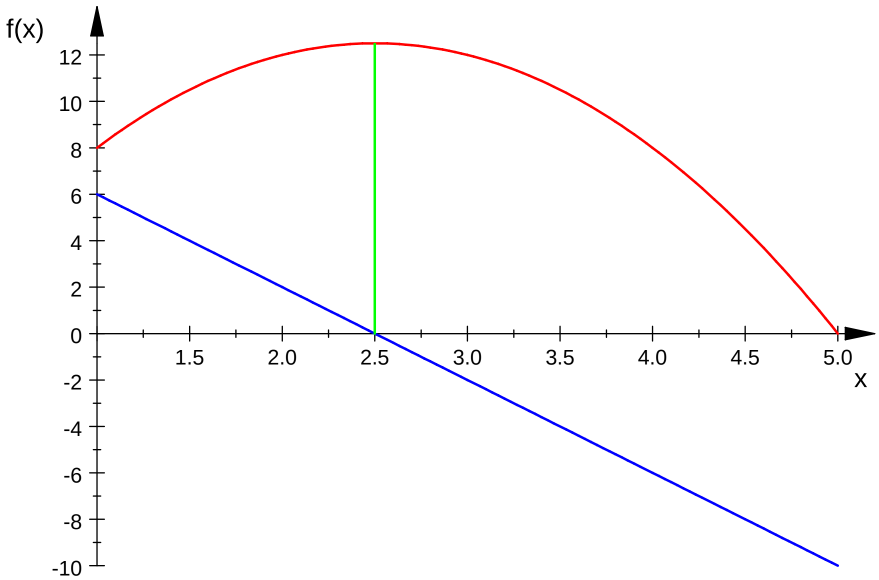


Figure 3. Locating the stationary point of f(x)

```
• Dg:=D(g) ;
```

$x \rightarrow 4 \cdot x - 10$

```
• ans:=solve(Dg(x)=0,x) ;
```

$\left\{\frac{5}{2}\right\}$

```
• stationary_x:=op(ans) ;
```

$\frac{5}{2}$

```
• p1:=plot::Function2d(g(x), x=1..5, Color=RGB::Red) :
• p2:=plot::Function2d(Dg(x), x=1..5, Color=RGB::Blue) :
• p3:=plot::Line2d([stationary_x,0],[stationary_x,g(stationary_x)],
  Color=RGB::Green) :
• plot(p1,p2,p3, AxesTitles=["x","g(x)"]);
```

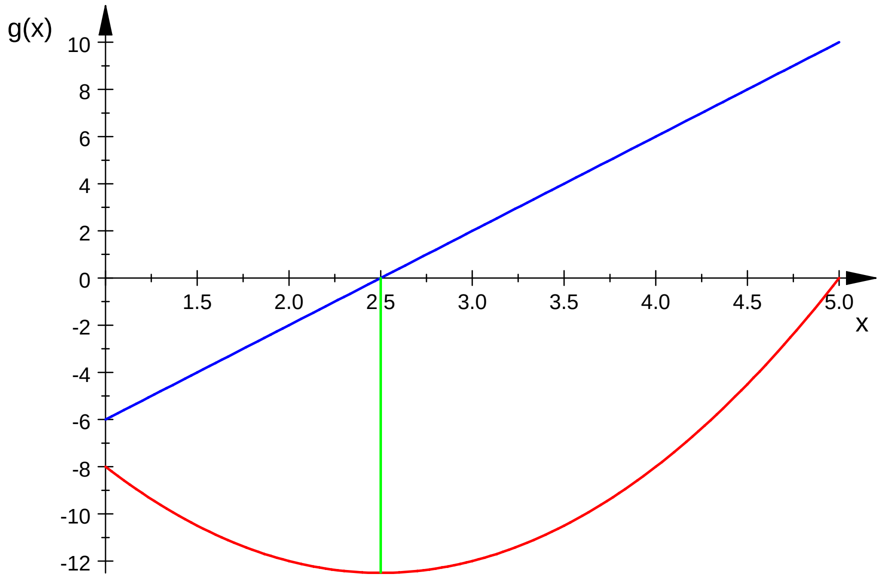


Figure 4. Locating a stationary point of g(x)

Actually this scheme locates stationary values of a function (they could be maxima or minima). You can also use the D operator to distinguish between maxima and minima. The following rule can be used to determine whether a stationary point x^* is a maximum or a minimum

1. Apply the D operator twice
2. Evaluate the resulting expression at x^*
3. If the value of the resulting expression is
   - negative then the stationary point is a maximum
   - positive then the stationary point is a minimum

Apply the D operator twice to f(x)

- delete ans ;

- ans:=x->D(D(f));

$$x \to f''$$

- ans(stationary_x);

$$-4$$

The negative result indicates that the stationary point *x=2.5* produces a maximum value of f(x).

Next apply the method to g(x) .

- `delete ans;`

- `ans:=x->D(D(g)) ;`

  $x \to g''$

- `ans(stationary_x);`

  4

The positive result shows that the stationary point x=2.5 produces a minimum value of g(x).

Summary

To locate a stationary point of a function f(x)

1. Apply the D operator to f(x). Call the result Df.
2. Find the value of x that makes Df=0.

To determine whether the stationary point is a maximum or a minimum

1. Apply the D operator twice
2. Evaluate the resulting expression at x^*
3. If the value of the resulting expression is
   - negative then the stationary point is a maximum
   - positive then the stationary point is a minimum

Profit maximization for a competitive firm

Note: You should read "How to find a maximum or a minimum" first.

A firm in a perfectly competitive industry takes price as a given and decides how much output to produce at that price in order to maximize profit. The firm will maximize profits when it produces a quantity of output that yields equality between marginal revenue and marginal cost.

```
• reset():
• assume(P>0): assume(c>0) :
```

First define total revenue (TR) and total cost (TC) functions. The total revenue function is

```
• TR:=Q->P*Q;
```

$$Q \to P \cdot Q$$

where *P* is the price received per unit of output and *Q* is the total number of units of output produced. The total cost function is

```
• TC:=Q->c*Q^2;
```

$$Q \to c \cdot Q^2$$

where *c* is the cost per unit of output.

Next compute marginal revenue (MR) and marginal cost (MC)

```
• MR:=D(TR) ;
```

$$Q \to P$$

```
• MC:=D(TC);
```

$$Q \to 2 \cdot Q \cdot c$$

A necessary condition for profit maximization is that marginal revenue should be equal to marginal cost. The value of output that satisfies this condition is found by setting marginal revenue equal to marginal cost and solving for Q

```
• optimal_Q:=solve(MR(Q)=MC(Q),Q);
```

$$\left\{\frac{P}{2 \cdot c}\right\}$$

where optimal_Q is the profit maximizing level of output. This is a symbolic result--it is in terms of the symbols *P* and *c* rather than a numerical value. Symbolic results are often more meaninful than numerical results. This symbolic result shows that an increase in the price level *P* will lead to an increase in the level of output--the optimal level of Q. An increase in unit cost however willlead to a decrease in the optimal level of output.

A numerical solution can be found by giving values to *P* and *c*

```
• P:=100; c:=10;
```

100

10

```
• optimal_Q:=op(optimal_Q) ;
```

5

So the profit maximizing level of output is *Q=5.* Plotting MR and MC also show that this is the level of output where they are equal

```
• f1:=plot::Function2d(P, x=0..10, Color = RGB::Red) :
• f2:=plot::Function2d(MC(x), x=0..10, Color = RGB::Blue) :
• f3:=plot::Line2d([optimal_Q,0],[optimal_Q,100], Color =
  RGB::Green) :
• plot(f1,f2, f3, AxesTitles=["Q","MC,MR"]);
```

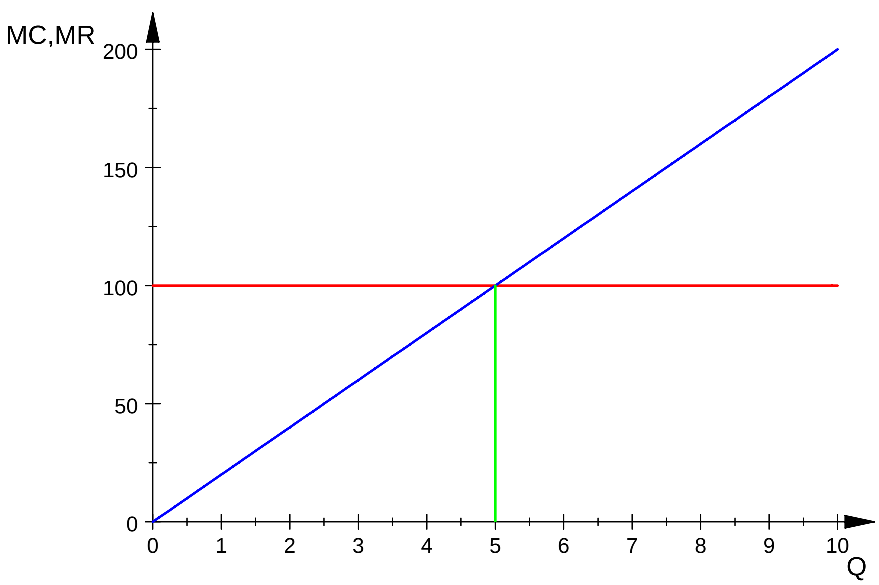


Figure 1. Margial revenue equals marginal cost at 5 units of output.

When only one choice variable is present it is often possible to locate an optimum value for that variable graphically.

•

```
f1:=plot::Function2d(TR(x), x=1..10, Color=RGB::Red) :

f2:=plot::Function2d(TC(x), x=1..10, Color=RGB::Red) :
```

• `f3:=plot::Line2d([optimal_Q,TC(optimal_Q)],[optimal_Q,TR(optimal_Q)], Color = RGB::Green) :`

• `plot(f1,f2,f3, AxesTitles=["Q","TC,TR"]) ;`

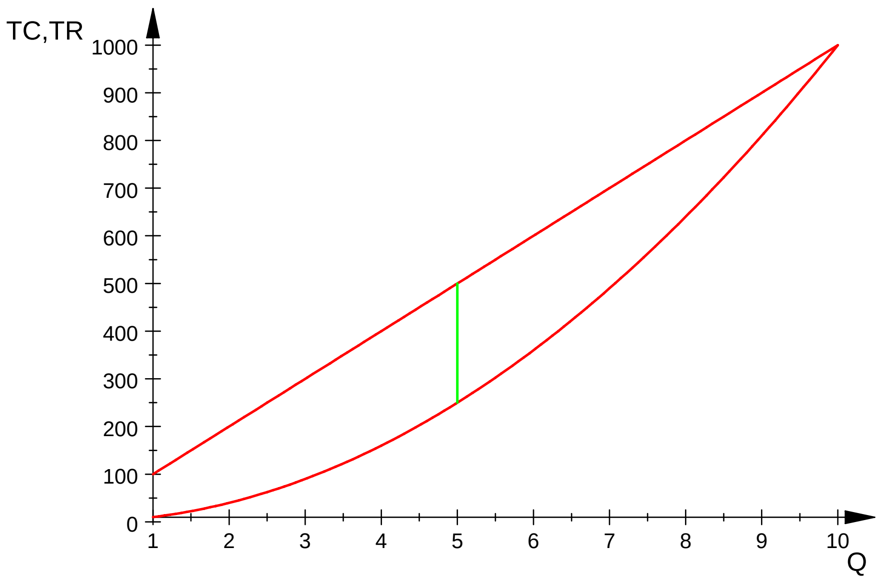


Figure 2. Locating the profit maximizing level of output graphically.

Next compute profit at optimal_Q and at points near optimal_Q. The profit at optimal_Q should be greater than that at nearby points if optimal_Q is the profit maximizing level of output. Defining a profit function will make the computations easier.

• `Profit:=Q->TR(Q)-TC(Q) ;`

$Q \to \mathrm{TR}(Q) - \mathrm{TC}(Q)$

• `Profit(optimal_Q);`

250

• `Profit(optimal_Q-0.001);`

249.99999

- `Profit(optimal_Q+0.001);`

  249.99999

It is also possible to locate the profit maximizing level of output graphically by plotting the profit function.

- `f1:=plot::Function2d(Profit(x), x=0..10) :`

- `f2:=plot::Line2d([optimal_Q,0],[optimal_Q,Profit(optimal_Q)], Color = RGB::Green) :`

- `plot(f1, f2, AxesTitles=["Q","Profit"]) ;`

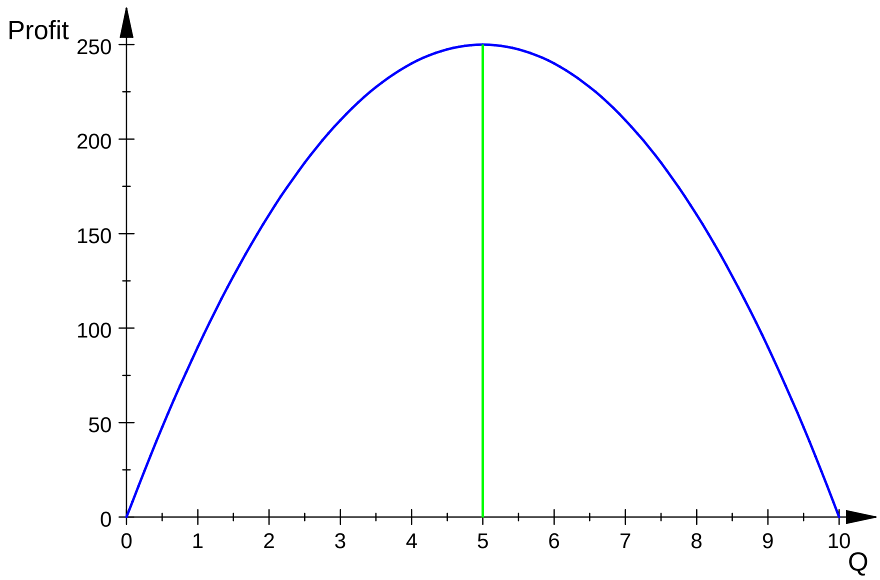


Figure 3. Another way of grahcially locating the profit maximizing level of output.

- Using the rules stated in "How to find a maximum or minimum" lead to a more elegant solution. First take the derivative (marginal change) of the profit function

- `Df:=D(Profit) ;`

  $Q \rightarrow 100 - 20 \cdot Q$

Next find the value of Q that makes Df equal to zero

- `ans:=solve(Df(Q)=0,Q) ;`

  $\{5\}$

This result is a stationary point. It could be a maximum or a minimum. The next operation is needed to determine which it is. Take the second derivative of the profit function. The second derivative of the profit function is negative indicating that this really is a maximum rather than a minimum.

- `D(D(Profit ));`

  -20

Profit maximization for a monopolist

- `reset() ; assume(c>0):  assume(P<0) :`

- `assume(Qo>0) ;`

  $(0, \infty)$

The monopolist has some pricing power. The monopolist can determine market price by increasing or decreasing the level of output. The resulting price is determined by the demand curve

- `P:=Q->Qo-Q;`

  $Q \to \mathrm{Qo} - Q$

Total revenue, as always, is given by the product of price and output. Note however that price is determined by output

- `TR:=Q->P(Q)*Q;`

  $Q \to P(Q) \cdot Q$

Total cost is given by

- `TC:=Q->c*Q^2 ;`

  $Q \to c \cdot Q^2$

where *c* is the per unit cost of output. As with the competitive firm, profits are maximized by locating that level of output that make marginal revenue equal to marginal cost.

- `MR:=D(TR);`

  $Q \to \mathrm{Qo} - 2 \cdot Q$

- `MC:=D(TC) ;`

  $Q \to 2 \cdot Q \cdot c$

- `ans:=solve(MR(Q)=MC(Q),Q) ;`

  $\left\{\frac{\mathrm{Qo}}{2 \cdot c + 2}\right\}$

The symbolic solution shows that the optimal level of output increases if Qo increases (the demand curve shifts right). The optimal level of output will decrease if per unit cost *c* increases.

A numerical solution can be optained if the parameters *Qo* and *c* are given values.

- `Qo:=1000: c:=10 :`

- `optimal_Q:=float( op(ans) ) ;`

  45.45454545

Next check to see if this value of output is one that makes marginal revenue equal to marginal cost

- `MC(optimal_Q) ; MR(optimal_Q) ;`

  909.0909091

  909.0909091

  909.0909091

  909.0909091

Next a graphical solution for the profit maximizing level of output can be sought.

```
• f1:=plot::Function2d(MC(x), x=10..100) :
• f2:=plot::Function2d(MR(x), x=10..100) :
• f3:=plot::Line2d([optimal_Q,0],[optimal_Q,MR(optimal_Q)],
  Color=RGB::Green) :
• plot(f1,f2,f3, AxesTitles=["Q", "MR,MC"]);
```

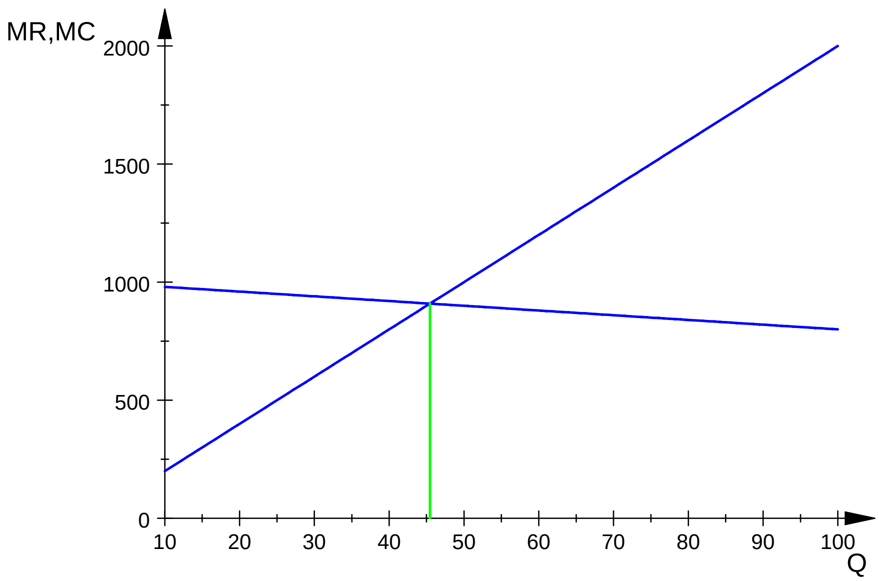


Figure 1. Graphically locating the profit maximizing level of output by plotting MR and MC

t is also prossible to locate the profit maximizing level of output grahpicall by plotting the total cost and total revenue functions directly.

- `f1:=plot::Function2d(TR(x), x=10..100, Color=RGB::Red) :`
- `f2:=plot::Function2d(TC(x), x=10..100, Color=RGB::Green) :`
- `f3:=plot::Line2d([optimal_Q,TR(optimal_Q)],[optimal_Q, TC(optimal_Q)], Color=RGB::Blue) :`
- `plot(f1,f2,f3, AxesTitles=["Q","TR,TC"] );`

I

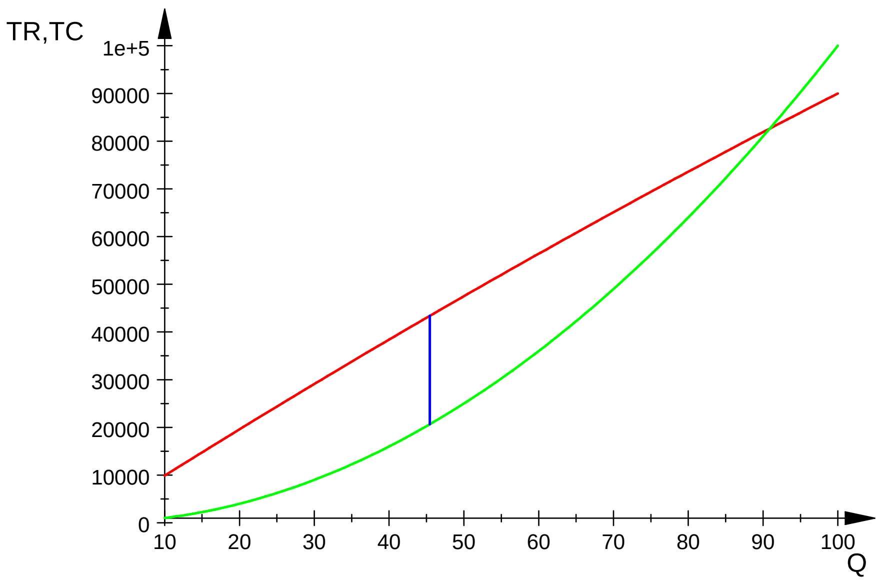


Figure 2. Graphically locating the profit maximizing level of output by plotting TR and TC

Finally the profit maximizing level of output can be locate graphically by plotting the profit function directly.

```
• Profit:=TR-TC ;
```

$$(Q \to P(Q) \cdot Q) - (Q \to c \cdot Q^2)$$

```
• f1:=plot::Function2d(Profit(x), x=10..100) :
• f2:=plot::Line2d([optimal_Q,0],[optimal_Q, Profit(optimal_Q)],
  Color=RGB::Green) :
• plot(f1,f2, AxesTitles=["Q","Profit"] );
•
```

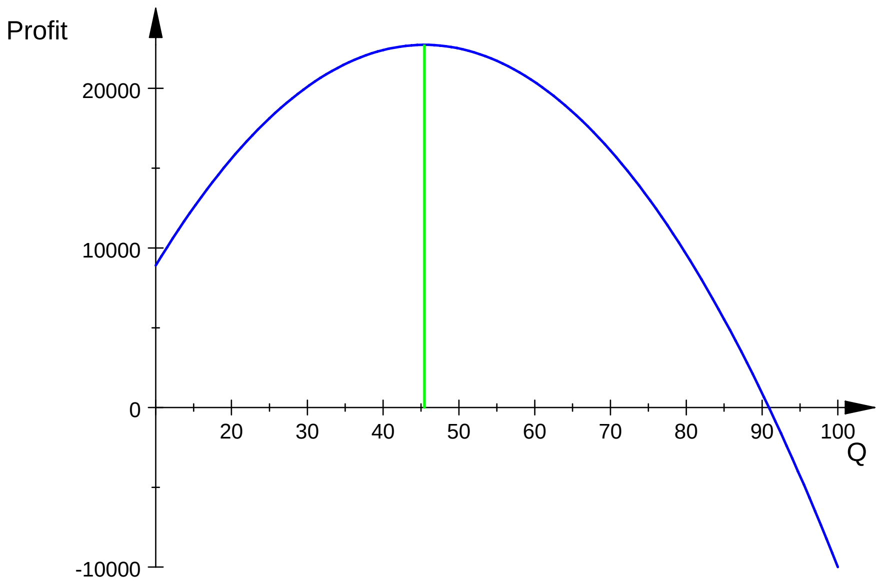


Figure 3. Locating the optimal level out output by plotting the profit function.

In any case, the graphical methods only locate the profit maximizing level of output approximately, apparently a value slightly bigger than Q=45. Using the MR=MC rule produced a more accurate result Q=45.45454545. The profit maximizing level of output can also be located by using the profit function directly

```
• delete optimal_Q :
```

Take the first derivative of the profit function

```
• Df:=D(Profit) ;
```

$$(Q \to 1000 - 2 \cdot Q) - (Q \to 20 \cdot Q)$$

•

Find the level of output that makes Df=0.

- `ans:=solve(Df(Q)=0,Q) ;`

  $\left\{\frac{500}{11}\right\}$

- `optimal_Q:=float( op(ans) );`

  45.45454545

-

The optimal level of output, optimal_Q, is a stationary point. It could be a maximum or a minimum. An inication of whether this is a maximum can be had by computing profit at the optimal level of output and at poitnts near the optimal level. If the optimal level of output is a maximum, then the value of the profit function should be larger here than at nearby points.

- `Profit(optimal_Q) ;`

  22727.27273

- `Profit(optimal_Q-0.001) ;`

  22727.27272

- `Profit(optimal_Q+0.001) ;`

  22727.27272

The value of profit is larger at the optimal level of Q than at the selected nearby points and this suggests that a maximum has be located. Taking the second derivative and evaluating it will determine whether optimal_Q is a point that generates a maximum level of profit or a minimum level of profit.

- `D( D(Profit)) ;`

  -22

Because the value of the second derivative is negative optimal_Q is a level of output that generates a maximum level of profit for the monopolist.