Wealth-Robust Intertemporal Incentive Contracts.[1]

by

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Abstract.

We study optimal incentive contracts in a continuous time principal-agent setting with hidden actions. The agent, whose effort controls the output, has a concave utility function which is non-separable in wealth and monetary cost of effort. The principal is risk neutral and optimally selects the effort to be induced and the contract design. Output follows a mean-reverting process with random coefficients. We characterize the class of *W-robust* compensation schemes that elicit a desired effort which is immune to the principal's mispecifications of the future wealth of the manager. We demonstrate the existence of a solution to the principal's problem, characterize the optimal effort policy, derive the optimal W-robust contract and show that our contract dominates randomized contracts.

KEYWORDS: Continuous time principal-agent problem, hidden actions, wealth robustness.

1 Introduction

In recent years, the study of non-market based contracts, notably the principal-agent paradigm, has been successful in explaining many economic phenomena that appeared inconsistent with the traditional efficient market-mediated contract theory. Despite its success, however, the principal-agent framework suffers from the limitation that even for moderately rich economic settings, it becomes complicated to solve for optimal contracts. For situations where contracts can be explicitly characterized, they appear formidable to implement in practice. Paradoxically, in real life multiperiod settings, incentive contracts are much simpler. In fact, they often take on linear forms. In addition, they are frequently specified in terms of a simple verifiable aggregate variable such as year-end profits, and not as functions of finer measures, even if those are also potentially verifiable. One could argue that it is costly to write optimal contracts that are complicated and so linear contracts, written on coarse aggregates, serve as approximations that are easy to implement. A more appealing approach would develop economic arguments suggesting that contracts written on aggregate measures are indeed optimal and robust. This is what we propose to do here within the framework of a general dynamic principal-agent setting with hidden actions.

One of the major difficulties associated with the practical implementation of a compensation scheme derived from theoretical considerations is the fact that many personal details may not, and in some instances cannot, be known with certainty by a principal offering a contract. This is particularly true for contracting in the private sector. Federal laws in the United States and many other countries as well, often augmented by state and local laws, prohibit the principal from seeking any information from the manager that is not connected with the service to be provided . Often included in the list of personal details are the private wealth situations of managers and the private wealth opportunities available to them. These details are particularly crucial for incentive contracting. Small errors by the principal in the assessing the agent's private wealth could obviate the objective of a finely tuned contractual relationship, which is to elicit a desired effort level from the manager.

Even without the legal restrictions, it may also be impractical to write contracts which depend on outcomes of other contractual obligations or a manager's financial portfolio, especially if the agent can enter into contractual obligations in the future. These problems can be particularly severe in a dynamic framework. Diffuse knowledge of a manager's trading patterns in financial markets is one example of a contingency which may lead to a breakdown of the contractual objectives. One typical example is that of a CEO (the manager) of an oil company who holds other companies stocks in the same industry. The presence of any correlation between traded securities and the reward scheme offered by shareholders of the oil company (the principal) could, in principle, be exploited by the CEO to hedge against undesirable outcomes, and not provide the effort sought by the principal. The shareholders could attempt to place additional restrictions on the trading activities of the CEO entering the contract, but it is inconceivable that an exhaustive list of contingencies and opportunities affecting the contractual outcome could ever be identified. Managers could always enter into private contracts with investment banks as well as other parties. Thus, the menu of possibilities far exceeds the set of traded assets available in financial markets and is only limited by the creativity of the parties. While, in a static setting it may be possible under certain circumstances, to offer a an array of contracts to separate managers in different initial wealth categories, it would be quite impossible to attempt this in a dynamic framework where the private wealth positions and opportunities are randomly and constantly changing over time.

Given all these difficulties associated with allowing managers to have private wealth opportunities, and given that the complexities increase rapidly as we go from static models to dynamic settings, it is not surprising that the prior principal-agent literature has mostly ignored these issues. We propose a simple way to circumvent these problems by restricting the space of contracts to those that are immune to the situations described. This leads to the requirement that the optimal effort elicited by the contractual relationship be robust to any perturbations of the terminal wealth of the manager that are correlated with the reward scheme. Contracts that satisfy these conditions are, what we call, *W-robust* (i.e. robust to wealth perturbations).

In this paper we characterize the class of W-robust contracts and derive the W-robust contract which is optimal for the principal in a general, dynamic principal-agent relationship with a risk neutral principal[2] and a risk and work averse agent. The agent's utility function is a concave function defined over wealth, i.e. compensation net of the monetary cost of effort. This is a major innovation since we do not require the usually assumed additive separability with respect to the reward and the cost of effort (see for example, Gjesdal (1982) regarding the complexities introduced by the absence of additive separability, including the possiblity that randomization of contracts may be optimal). In our model the agent's hidden actions influence the output of the firm. Output follows a mean reverting process with random coefficients, that is, we allow for a stochastic mean-reversion parameter and a stochastic volatility. The agent's effort influences the drift of the output process. Given mean-reversion, effort has both an immediate influence on the change in output as well as an impact on future output changes, as do shocks to output. Our only restriction is that the uncertainty affecting the coefficients of the model (such as interest rate, speed of mean reversion, volatility, etc...) be public information. Uncertainty affecting the local evolution of output, by contrast, is privately observed only by the manager. In this setting, any contract written on output and other relevant public information variables (in the coefficients of the model) can also be specified in terms of net output $\widetilde{x}$ (i.e., the difference between realized and the mean-reverting component of output) and the other public information. Net output $\widetilde{x}$ can then be interpreted as the managed account. This dynamic setting is inherently different from the static contracting framework since revisions over time in the effort exerted, as new information becomes available, affect the future values of output and the salary function.

Our first contribution is to characterize the class of W-robust contracts. We show that, in our setting, a contract is W-robust if and only if it consists of three parts, (i) a stochastic integral with respect to the change in the managed account $\int f d\widetilde{x}$, (ii) a

[2]Risk neutrality is for convenience only. Our results hold with minor modifications for risk averse principals.

random Riemann integral with respect to the level of the managed account $\int f\widetilde{x}dt$ and (iii) a random variable A. The main restriction in this specification is that the slope of the contract f (i.e. the integrand in the stochastic integral) and the random variable A are independent of the managed account $\widetilde{x}$. That, is the pair (A, f) depends only on public information and not on the (ex-ante) private information of the manager. This leaves us with a class of contracts which is still very large, in which the principal will select an optimal contract that is (i) incentive compatible and (ii) individual rational for the manager.

Our second contribution is at the level of the principal's problem. It is easily shown, given the general stochastic model of output, that many W-robust contracts (A, f) are incentive compatible and individual rational. Roughly speaking, these differ in the agent's compensation A provided for inducing participation. In effect, any random variable that depends on public information can be added onto the contract without distorting the incentives required to elicit the effort level desired by the principal. This contract will also satisfy the individual rationality constraint provided the agent is adequately compensated for the risk borne. Naturally, the principal will not be indifferent between all contracts satisfying these basic constraints. His/her choice set includes not only the design of the contract but also the selection of the effort level that is required of the agent. We examine and resolve this bivariate dynamic optimization problem. Our contribution in that respect is threefold: (i) we demonstrate the existence of a solution to the principal's problem, (ii) we characterize the optimal W-robust contract and the desired effort level and (iii) we show that randomized contracts do not improve welfare in the class of W-robust contracts. Establishing the existence of a solution to the principal's problem is a non-trivial issue and prior literature has often assumed that a solution exists. The reason the existence is hard to show is that the principal's choice problem is over an infinite dimensional space of controls which is not norm-compact. Hence, standard existence proofs do not apply. In spite of this technical difficulty, our paper resolves the existence question in the generality of our model with arbitrary agent's utility function and random coefficient of the output process. In short, the existence proof rests on the identification of a suitable weakly compact space of controls in which the maximizer is shown to lie. The characterization (ii) that we obtain is new and provides interesting economic insights about the structure of the optimal contract. Finally, result (iii) appears especially surprising in light of Gjesdal (1982) who showed that the principal is better off with randomized contracts when the agent's utility is not additively separable in wealth/reward and cost of effort. In our setting, randomized contracts are dominated even though the utility function is non-separable. This result reinforces the robustness of the class of contracts that we study. Given this space of contracts, the key factors underlying this result are the concavity of the principal's objective function and the convexity of the individual rationality constraint with respect to the principal's choice variables.

Our results provide new insights about the structure of the optimal contract in a general setting with arbitrary utility function for the agent. The optimal W-robust contract, that we obtain, is a linear, suitably weighted functional of the cumulative output and the change in output over the period of the contract.[3] The weights, in general, will be stochastic and path dependent. Moreover, in the absence of mean-reversion and when the coefficients of the model (interest rate, reinvestment rate, volatility, etc.) are constants the optimal weights in the contract become constant, even when the agent's utility function and the production function are concave. Under these conditions the principal requisitions a constant effort policy and the optimal contract becomes a linear function of the terminal value of the managed account. This is a striking result which extends Holmstrom and Milgrom (1987) and shows that simple, linear contracts are optimal, within our class, even for non-exponential utility functions.

The earliest paper that addressed the issue of dynamic contracts in continuous time settings is by Holmstrom and Milgrom (HM) (1987). They demonstrate the optimality of linear contracts based on the value of an aggregate counting measure of outcomes at the time of compensation in dynamic environments where the principal and the agent have negative exponential utility function and the agent can exert effort to influence a history independent technology. The absence of mean-reversion implies that the current effort and random shocks manifest themselves immediately in the outcome and have no influence on the change in output after that. Time independence also excludes discounting on the part of the principal and the agent. Further progress was achieved by Schattler and Sung (1993) who consider the principal-agent problem when (i) both parties have exponential utility and (ii) the output process is a Brownian semimartingale with history-dependent (progressively measurable) coefficients. In this setting they characterize the class of contracts which implement the principal's desired action and provide necessary conditions for optimality in the principal's problem. Moreover, they show that Holmstrom-Milgrom contracts rely on constant coefficients. They also give sufficient conditions when the technology has a Markovian evolution.[4] In proving these results they make extensive use of stochastic control and martingale methods.[5,6] Additional contributions to the continuous time contracting literature include Sung (1995) who allows for control over the drift and the volatility of a diffusion process with constant coefficients and Sung (1997) who deals with jump processes. Recent applications of Schattler and Sung include Ou-Yang (2000, 2001).

As noted before, our paper differs from this prior literature in that we consider utility functions for the agent that are non-separable in wealth and effort. Hence the contractual components designed to satisfy the agent's participation constraint cannot be chosen separately from the incentive part of the contract, which leads to non-trivial difficulties in proving the existence of an optimal solution to the principal's problem and raises the issue of randomized contracts discussed above. Our methodology also differs

[3] A functional $F(x)$ is said to be linear if $F(\lambda x) = \lambda F(x)$ for any $\lambda \in \mathbb{R}$.

[4] Their results extend to utility functions that are additively separable in reward and effort (see Schattler and Sung (1993), footnote 11).

[5] Govindaraj and Ramakrishnan (2001) extend HM to a setting where effort influences the mean path of earnings and earnings display mean-reversion. In obtaining an explicit optimal contract, they assume (i) exponential utility for the agent, (ii) that the agent's cost of effort is discounted at the interest rate and (iii) that earnings follows a linear stochastic equation with constant coefficients.

[6] The literature on intertemporal contracting is typically based on *exponential utility*.

from the prior literature. In our paper extensive use is made of perturbation methods as opposed to (i) variational inequalities, (ii) dynamic programming methods or (iii) convergence arguments. These provide crisp characterizations of the optimal controls in the principal's and the agent's problems which can be used to shed light on the structure of the optimal contract.

Our next section details the dynamic model and describes the principal-agent relation under study. W-robust contracts are characterized in section 3. Section 4 resolves the principal's problem. We first identify the class of incentive compatible (W-robust) contracts. Then we solve for the optimal W-robust contract: we prove the existence of a solution, characterize the optimal policy, address the issue of randomized contracts, and provide economic insights about the structure of the contract. Section 5 examines special cases and, in particular, shows the optimality of simple linear contracts in certain settings. Conclusions are formulated in section 6. All proofs are collected in the appendix.

2 The dynamic principal-agent problem

2.1 The output process

We postulate a general stochastic output process which encompasses most models that have appeared in the theoretical and empirical finance literature. Uncertainty in the evolution of output is captured by a bivariate Brownian motion process denoted by (B, Z). Let $\mathcal{F}^{B,Z}_{(\cdot)}$ denote the filtration associated with the pair (B, Z).

Let $x(t)$ be the output rate. Its dynamics is given by

$$dx(t) = [-\kappa(t)x(t) + g(e_t)]\,dt + \sigma(t)dB(t), \tag{1}$$

with initial value $x(0)$. In this expression the quantity e_t is the rate at which the agent provides effort, $g(e_t)$ is the productivity of effort, $\sigma(t)$ the volatility of output and $\kappa(t)$ the speed of mean reversion. The processes $\kappa(t)$ and $\sigma(t)$ can be stochastic, with the restriction that they be bounded and progressively measurable with respect to the filtration $\mathcal{F}^{Z}_{(\cdot)}$ generated by Z.[7] The volatility σ is bounded away from zero: there exists $\varepsilon > 0$ such that $\sigma > \varepsilon$. We assume that the production function satisfies the following conditions

Assumption 1: *The production function* $g(\cdot, t) : \mathbb{R}_+ \to \mathbb{R}$ *is*

(i) strictly increasing, concave and twice continuously differentiable, with $g(0, t) = 0$ *and* $\lim_{x\to 0} g'(x, t) = \infty$ *for all* $t \in [0, T]$.

(ii) For any given constant $e_t = e$ *the production process* $g(e, \cdot) \equiv \{g(\cdot, t) : t \in [0, T]\}$ *is an* $\mathcal{F}^{Z}_{(\cdot)}$*-progressively measurable stochastic process.*

[7] A process X is said to be progressively measurable with respect to the filtration generated by Z if X_t depends on time and on the trajectories of the Brownian motion Z up to time t, for all $t \in [0, T]$ (i.e. X_t is $\mathcal{F}^{Z}_{t} \times \mathcal{B}([0, t])$ measurable where $\mathcal{B}([0, t])$ is the Borel sigma–field on $[0, t]$).

These assumptions are standard. Condition (i), in particular, stipulates that there are decreasing returns to effort and that effort must be exerted to increase returns. Condition (ii) allows the production function to evolve randomly over time. This enables us to capture stochastic shocks to production which are unrelated to the effort of the manager.

The output model (1) formalizes the intuitive notion that changes in the output rate $dx(t)$ arise from effort exerted by the agent, $g(e_t)$, and from random shocks (innovations), $\sigma(t)dB(t)$. They are also influenced by a mean reverting component, $\kappa(t)x(t)$. In the absence of external uncertainty ($\sigma(t) = 0$) and effort ($e_t = 0$) the output rate would simply decay at the rate $\kappa(t)$. This captures the notion that net cash flows generated by the current projects of the firm depreciate and in the very long run will eventually dry up. In the presence of uncertainty, however, actual realizations $x(t)$ may deviate from the long term value due to random shocks affecting the production process. In the absence of structural changes in production or in market conditions, it is reasonable to expect that output does not drift too far away from the long term mean. This pull to the central tendency is captured by the stochastic coefficient $\kappa(t)$ which measures the speed of reversion to the mean. For large values of κ, the output rate exhibits faster reversion to the long term value. Similarly, large deviations from the long term value imply a larger magnitude of the reversion in the output rate.

The speed of mean-reversion κ affects the extent to which output shocks persist. If future effort after current time t is set equal to zero, then the expected future output is given by

$$E_t \left[x(t+\tau) \mid e_{t+v} = 0, \forall v \in [0, \tau] \right] = x(t) E_t \left[e^{-\int_t^\tau \kappa(v) dv} \right] \tag{2}$$

Observe that for $\kappa(v) = 0$ for all $v \geq t$, the output process is a martingale and all innovations will have permanent effects. The polar opposite is when κ approaches infinity. In this case, the effects of random shocks on the evolution of output is temporary or transient.

Another key aspect of the model is the information structure. We assume that the agent observes both Brownian motions: his/her information filtration is $\mathcal{F}_{(\cdot)}^{B,Z}$. In contrast the principal observes the output process x and the Brownian motion Z (but not B): his/her information is given by the filtration $\mathcal{F}_{(\cdot)}^{x,Z}$.[8] Since the principal does not observe the manager's effort either, there is asymmetric information between the two parties. It is the presence of this asymmetry that motivates the contractual problem to be studied. Note also that contracts based on output are feasible since x is in the information set of the principal $\mathcal{F}_t^{x,Z}$ at date t and this for all $t \in [0, T]$.

[8] As we have it here, observing Z is not informative about B. It would be a simple matter to extend the results to a situation where the two Brownian motions are correlated, i.e. where observing B tells the principal something about Z. In this case, the contracting is simply based on all the information shared by the parties; only the component of B that cannot be inferred from Z is left out of the contractible information set.

2.2 The agent's optimization problem

The manager's expected utility function is

$$U(\Psi, e) = Eu\left(\Psi - \int_0^T c(e_t, t)dt\right) \tag{3}$$

where Ψ is the compensation to be paid to the agent at terminal time T, and $c(e_t, t)$ is a stochastic monetary cost inflicted by effort. We will assume that

Assumption 2: *The utility function* $u(\cdot) : \mathbb{R} \to \mathbb{R}$ *is*

(i) strictly increasing, concave, and twice differentiable function, and

(ii) marginal utility has the limits $\lim_{x\downarrow-\infty} u'(x) = \infty$ *and* $\lim_{x\uparrow\infty} u'(x) = 0$.

(iii) Let $n(\xi)$ *be the standard normal density. The integrability condition* $\int_{-\infty}^{\infty} u(x + \gamma\xi)n(\xi)d\xi < \infty$ *holds, for all* $x \in \mathbb{R}, \gamma \in \mathbb{R}_+$.

Conditions (i)-(ii) are standard. Condition (iii) ensures that expected utility is finite when wealth is normally distributed. This is a weak condition which is automatically satisfied in standard examples such as exponential utility.

Assumption 3: *The monetary cost function* $c(\cdot, t) : \mathbb{R}_+ \to \mathbb{R}$ *is*

(i) strictly increasing, convex and twice continuously differentiable, with $c(0, t) = 0$ *and* $\lim_{x\to 0} c'(x, t) = 0$ *for all* $t \in [0, T]$.

(ii) For any given constant $e_t = e$ *the cost* $c(e, \cdot) \equiv \{c(\cdot, t) : t \in [0, T]\}$ *is an* $\mathcal{F}^Z_{(\cdot)}$*-progressively measurable stochastic process.*

(iii) The function $f(e, t) = c'(e, t)/g'(e, t)$ *is convex for all* $t \in [0, T]$.

Condition (i) in Assumption 3 is standard. Condition (ii) allows us to encompass a monetary cost of effort which changes stochastically over time as information changes. Our formulation is very general in that respect. It includes, for example, the specification $c(e, t) = \exp(-\int_0^t \beta_v dv)c(e)$ where the discount rate $\beta > 0$ applied to the cost of effort captures the notion that early effort is more costly than late effort. The stochastic nature of the discount rate β enables us to model situations in which the cost of early effort is state dependent. Finally, condition (iii) is a restriction on the ratio of the marginal cost to the marginal product of effort. Convexity of this ratio ensures that the objective function of the principal is concave with respect to his/her choice variables.

Given our information/uncertainty structure the compensation scheme will be some functional of the output process x. Moreover, we can write $\Psi \equiv \Psi(x(e))$. In fact, define the stochastic process

$$\widetilde{x}(t) \equiv x(t) + \int_0^t \kappa(v)x(v)dv = \int_0^t (g(e_v, v)dv + \sigma(v)dB_v)$$

for $t \in [0, T]$. Clearly, $\widetilde{x}$ is $\mathcal{F}^{x,Z}_{(\cdot)}$-adapted. Conversely, since

$$x(t) = x(0)e^{-\int_0^t \kappa(u)du} + \int_0^t e^{-\int_v^t \kappa(u)du} d\widetilde{x}(v)$$

we have that x is $\mathcal{F}^{\widetilde{x},Z}_{(\cdot)}$-adapted. Thus, the filtrations $\mathcal{F}^{x,Z}_{(\cdot)}$ and $\mathcal{F}^{\widetilde{x},Z}_{(\cdot)}$ coincide and any contract written in terms of (x, Z) can also be written in terms of $(\widetilde{x}, Z)$. Without loss of generality, we can then think of contracts as functionals of the form $\Psi \equiv \Psi(\widetilde{x}(e))$.

At date 0, the manager has two options: either taking the contract and working for the principal, or rejecting the contractual offer and idling. In the first instance, the agent seeks to maximize (3) by choosing a stochastic process for effort $e = \{e_t : t \in [0, T]\}$ taking the contractual structure $\Psi(\widetilde{x}(e))$ offered to him/her as given. In the second instance, welfare is provided by the reservation utility $\overline{u}$. Naturally the manager selects the best of these two options, and solves

$$\max\{\max_e U(\Psi(\widetilde{x}(e)), e), \overline{u}\}$$

where e is an $\mathcal{F}^{B,Z}_{(\cdot)}$-progressively measurable process and e controls the process (1).

2.3 The principal's problem

The risk neutral principal's problem is to select the effort level to be induced from the risk averse agent as well as the contract Ψ, so as to maximize the expected discounted payoff net of compensation. Suppose that the principal's payoff at time T, before managerial expenses, is an affine function of terminal output $x(T)$,

$$G_0 + G_1 x(T)$$

where G_0, G_1 are random variables which are strictly positive, bounded and $\mathcal{F}^Z_T$-measurable. Taking account of the compensation to the manager, leads to the principal's valuation of the firm at date 0,

$$V_0 = E\left[e^{-\int_0^T r(v)dv}\left(G_0 + G_1 x(T)\right)\right] - E\left[e^{-\int_0^T r(v)dv}\Psi(\widetilde{x}(\overline{e}))\right]$$

where $\overline{e}$ is the effort of the manager and $r \equiv \{r(v) : v \in [0, T]\}$ denotes the interest rate process in the market (locally riskless rate); we assume that r is a bounded $\mathcal{F}^Z_{(\cdot)}$-progressively measurable process.

Since the output process satisfies (1) over the course of the management period the following decomposition of the value of the firm holds,

Lemma 1: *The principal's valuation of the firm is* $V_0 = I_0 + x(0)I_1 + J(\overline{e}, \Psi)$ *where*

$$I_0 = E\left[e^{-\int_0^T r(v)dv} G_0\right], \qquad I_1 = E\left[e^{-\int_0^T (r(v)+\kappa(v))dv} G_1\right]$$

and
$$J(\overline{e},\Psi)=E\left[\int_0^T g(\overline{e}_s)D_s ds\right]-E\left[e^{-\int_0^T r(v)dv}\Psi(\widetilde{x}(\overline{e}))\right] \tag{4}$$
with $D_s \equiv e^{-\int_0^s r(v)dv}E_s\left[e^{-\int_s^T (r(u)+\kappa(u))du}G_1\right]$, *a strictly positive and bounded process.*

The value of the firm, for the principal, has two components, one which depends only on technological parameters, $I = I_0 + x(0)I_1$, and the other which is a function of the effort of the manager and of the contractual payment, $J(\overline{e},\Psi(\overline{e},\widetilde{x}))$. The first term of $J(\cdot,\cdot)$ is the present value of the benefits, to the principal, from managerial effort. The second term is the present value of the cost incurred to elicit the desired level of effort. Thus, $J(\overline{e},\Psi(\overline{e},\widetilde{x}))$ captures the net benefits, to the principal, from motivating managerial effort $\overline{e}$.

The owner of the firm seeks to maximize this net value subject to (i) inducing the manager to follow the effort policy $\overline{e}$, (ii) meeting the manager's individual rationality constraint, (iii) the information available, and (iv) the non-negativity constraint on desired effort level. Let
$$\mathcal{IC}=\left\{\Psi:\overline{e}\text{ solves }\max_e Eu\left(\Psi(\widetilde{x}(e))-\int_0^T c(e_t,t)dt\right)\right\}$$
denote the set of contracts that are incentive compatible (hence satisfy (i)) and let $\mathcal{A}$ denote the set of admissible pairs $(\overline{e},\Psi)$ which satisfy the constraints (i)-(iv) in the principal's optimization problem. This is the set of stochastic processes $\overline{e}$ and random functionals $\Psi(\widetilde{x}(\overline{e}))$ such that
$$\begin{cases}\Psi\in\mathcal{IC}\\ Eu\left(\Psi(\widetilde{x}(\overline{e}))-\int_0^T c(\overline{e}_t,t)dt\right)\geq\overline{u}\\ \Psi\text{ is }\mathcal{F}_T^{\widetilde{x},Z}\text{-measurable}\\ \overline{e}\text{ is }\mathcal{F}_{(\cdot)}^{\widetilde{x},Z}\text{-progressively measurable}\\ \overline{e}\geq 0.\end{cases} \tag{5}$$
The second condition in (5) is the IR constraint. Here we used the notation $\widetilde{x}(\overline{e})$ to emphasize the fact that the agent follows the action $\overline{e}$ desired by the principal, when $\Psi\in\mathcal{IC}$. The principal then solves the problem
$$\max_{\overline{e},\Psi} J(\overline{e},\Psi(\widetilde{x}(\overline{e}))) \quad \text{subject to} \quad (\overline{e},\Psi(\widetilde{x}(\overline{e})))\in\mathcal{A} \tag{6}$$
where $\mathcal{A}$ is defined in (5).

2.4 The space of contracts

Before turning to the resolution of the principal-agent problem described above, we specify the space of contracts. Clearly, the informational constraint $\Psi \in \mathcal{F}_T^{\widetilde{x},Z}$ must be satisfied. In addition, we need to ensure that the agent can identify the impact of a change in effort on compensation. In effect, this suggests a notion of differentiability for the functionals Ψ representing the reward scheme. In what follows, we restrict attention to contracts which induce interior effort levels. This is justified because, given our assumptions, starting from a contract which does not motivate interior effort, it is always possible to construct contracts that make both the principal and the agent better off at an interior point.

Formally, let $\widetilde{x}(\cdot)+\epsilon \int_0^{\cdot} h(v)dv$ denote a perturbation of $\widetilde{x}(\cdot)$, in the direction $\int_0^{\cdot} h(v)dv$ where h is an $\mathcal{F}_{(\cdot)}^{\widetilde{x},Z}$-progressively measurable process. Here $h(v)$ represents a perturbation in effort at time v. We consider the space of contracts $\mathcal{D}$ defined as follows,

Definition 1: *The space of contracts $\mathcal{D}$ is the class of $\mathcal{F}_T^{\widetilde{x},Z}$-measurable random variables $\Psi(\widetilde{x})$ such that,*

(i) there exists a kernel $\lambda^{\Psi} \in \mathcal{F}_T^{\widetilde{x},Z}$ such that the limit

$$\lim_{\epsilon \to 0} \frac{\Psi(\widetilde{x}(\cdot) + \epsilon \int_0^{\cdot} h(v)dv) - \Psi(\widetilde{x}(\cdot))}{\epsilon} = \int_0^T \lambda^{\Psi}(v,T)h(v)dv \tag{7}$$

exists (P-a.s.),

(ii) $E\left[\Psi(B)^2\right] < \infty$ where B is a Brownian motion,

(iii) there exists a random variables ω such that $E[\omega^2] < \infty$ and a function F such that $\lim_{\epsilon \downarrow 0} \frac{F(\epsilon)}{\varepsilon} < \infty$ and

$$\left| \Psi \left(B + \epsilon \int h(s)ds \right) - \Psi(B) \right| \leq \omega(B) F\left(\|h\|\right)$$

where$\|h\|$ denotes the sup norm, and

(iv) the agent's optimal effort e satisfies the integrability condition

$$E\left[\exp\left(\frac{1}{2}\int_0^T g(e_t)^2 dt\right)\right] < \infty.$$

(v) The process $\lambda^{\Psi}(v,T) > 0$, $l \otimes P - a.s.$[9]

Condition (i) is a notion of differentiability. In this representation $\lambda^{\Psi}(v,T)$ captures the impact of the perturbation $h(v)$ on the compensation collected at T (i.e. $\lambda^{\Psi}(v,T)$ is the sensitivity of the terminal reward with respect to a small change $h(v)$ in effort at v). The representation (7) holds, in particular, for functionals that are pathwise Frechet differentiable.

[9] P is the probability measure and l is the Lebesgue measure.

Condition (ii) ensures that the contract has finite variance. It implies, in particular that the compensation has finite variance in the absence of managerial effort (i.e. if $e = 0$).

Condition (iii) is technical. It has the nature of a random Lipschitz condition. Condition (iv) is a Novikov condition which ensures that the Girsanov change of measure theorem applies. This condition is automatically satisfied if the production function $g(\cdot,\cdot)$ is bounded. The condition is only required to hold at the optimal effort level.

Lastly, condition (v) will imply that the agent's optimal effort is interior. Contracts which do not induce interior effort can be treated using similar techniques. However, these contracts are not relevant for the principal's problem in section 5.2, since they can be dominated by one which motivates interior effort (see footnote 11).

The class of contracts $\mathcal{D}$ is very general; it includes, for example, path-dependent contracts of the form $\max_{t\in[0,T]}(\widetilde{x}_t)$. Further examples are,

Example 1: *The class $\mathcal{D}$ includes all random variables of the form $\Psi(\widetilde{x}) = \Psi(\widetilde{x}_{t_1}, ..., \widetilde{x}_{t_n})$ where $\Psi : \mathbb{R}^n \to \mathbb{R}$ and $\Psi \in \mathcal{C}_b^1$ for some collection $\{t_i : i = 1, ..., n\}$ with $\frac{\partial \Psi}{\partial x_{t_i}} > 0$. In this instance,*

$$\lim_{\epsilon\to 0} \frac{\Psi(\widetilde{x}_{t_1}+\epsilon\int_0^{t_1} h(v)dv,...,\widetilde{x}_{tn}+\epsilon\int_0^{tn} h(v)dv)-\Psi(\widetilde{x}_{t_1},...,\widetilde{x}_{tn})}{\epsilon}$$

$$= \sum_{i=1}^n \frac{\partial \Psi}{\partial x_{t_i}} \int_0^{t_i} h(v)dv = \int_0^T \sum_{i=1}^n \frac{\partial \Psi}{\partial x_{t_i}} 1_{\{v\le t_i\}} h(v)dv.$$

In this case, a change in effort at time v affects compensation through all derivatives with respect to the values of the managed account at those times $t_i \geq v$. Contracts of the form $\Psi(\widetilde{x}_{t_1}, ..., \widetilde{x}_{t_n})$ include, in the limit, forcing contracts (Mirrlees (1974)).

Example 2: *Random Riemann-Stieljes integrals, stochastic integrals and combinations of the two are also included in $\mathcal{D}$. Thus, contracts of the form*

$$\Psi = \int_0^T \alpha(\widetilde{x}(\cdot), v)dv + \int_0^T \beta(\widetilde{x}(\cdot), v)d\widetilde{x}(v)$$

where $\alpha(\widetilde{x}(\cdot),\cdot)$ and $\beta(\widetilde{x}(\cdot),\cdot)$ are Frechet differentiable functionals, are admissible. In effect, this includes the class of contracts considered by Schattler and Sung (1993).

3 Robust contracts

We now formalize a notion of robustness to perturbations in the manager's environment. Let Y denote a random variable which is $\mathcal{F}_T^{B,Z}$-measurable. The random variable Y can be interpreted as a random cash flow collected by the manager at date T, due to his/her engagement in other activities which are not controllable by the principal.

Definition (W-robustness): *An incentive scheme $\Psi \in \mathcal{F}_T^{\widetilde{x},Z}$ is said to be W-robust if and only if any manager u, receiving compensation Ψ would provide the same optimal effort when terminal wealth equals $\Psi + Y$ where $Y \in \mathcal{F}_T^{B,Z}$.*

In essence W-robustness requires that the action induced by the contract be immunized against perturbations in the terminal wealth of the manager in $\mathcal{F}_T^{B,Z}$. This would ensure that the contract elicits the proper incentives even if the manager has access to other activities, outside the contractual relationship, which could produce correlated cash flows. The property seems especially relevant in economies with developed capital markets where managers typically have access to securities whose returns are correlated with the cash flows generated by the firm under management. Even in the absence of correlated traded securities financial technology usually provides the means to write contracts between managers and investment banks based on variables, such as B and Z, that are mutually observable (exotic derivatives are routinely quoted and purchased over-the-counter).

Our next theorem provides a complete characterization of W-robust contracts.

Theorem 1: *Suppose that the output process is given by (1) with stochastic speed of reversion and volatility $(\kappa(t), \sigma(t))$ and that assumption 1 holds. Suppose also that the risk-work averse agent has general utility function $u(\cdot)$ and cost of effort $c(\cdot, t)$ satisfying assumptions 2 and 3. A compensation contract in $\mathcal{D}$ is W-robust (i.e.$\Psi \in \mathcal{W}$) if and only if*

$$\Psi = A + \int_0^T f(v)\left(dx(v) + \kappa(v)x(v)dv\right) \tag{8}$$

where $A \in \mathcal{F}_T^Z$ and f is an $\mathcal{F}_{(\cdot)}^Z$-progressively measurable process, $f > 0$ $(l \otimes P - a.s.)$.

Theorem 1 gives a necessary and sufficient condition for a contract which induces interior effort to be W-robust.[10] It shows that $\mathcal{W}$ contracts can be identified with random variables consisting of a (stochastic) integral with respect to the change in the managed account $dx(v)$, augmented by a (Riemann-Stieljes) integral with respect to the level of the account $x(v)dv$ and a random variable depending on public information Z. The notable restrictions embedded in (8) are that the random variable A and the stochastic integrand f are independent of the managed account, i.e. that (A, f) depend solely on the public information available Z. The intuition for this structure is fairly straightforward. Any dependence of the slope f on the value of the account x would imply a correlation between the manager's private wealth and the impact of effort on the compensation. In turn, this would produce a dependence of the optimal effort on the private wealth of the manager, contradicting the definition of W-robustness.

Thus, W-robustness restricts the space of contracts to those satisfying (8). This leaves us with a very large class of compensation schemes, parametrized by $(A, f) \in \mathcal{F}_T^Z \times \mathcal{F}_{(\cdot)}^Z$, in which the principal will choose a maximum.

[10] If condition (v) in the definition of $\mathcal{D}$ is not included then a somewhat involved analysis shows that a W-robust contract must involve an incentive portion $\int_0^T f(v)dZ(v)$ where $f(v) \geq 0$ and $f(v) \in \mathcal{F}_{(\cdot)}^Z$. Other terms may depend on $\widetilde{x}$ but are easily shown to be dominated from the principal's perspective, using an analysis similar to that in section 5.

4 The optimal contract

In this section we derive the optimal W-robust contract that a risk neutral principal would offer a risk averse and work averse agent to motivate a given desired effort level. We first characterize the class of IC contracts (section 5.1). Then we solve the principal's problem (section 5.2): we successively (i) prove the existence of a solution to the principal's problem, (ii) characterize the optimal contract, (iii) resolve the issue of randomized contractual schemes and (iv) discuss the structure of the optimal contract.

4.1 The class of IC contracts

Suppose that the principal seeks to motivate stochastic effort $\{\overline{e}_t : t \in [0,T]\}$ with $\overline{e} > 0$ ($l \otimes P - a.s.$). Our first theorem characterizes the set of contracts in $\mathcal{W}$ that are incentive compatible.

Theorem 2: *Suppose that the output process is given by (1) with stochastic speed of reversion and volatility ($\kappa(t), \sigma(t)$) and that assumption 1 holds. Suppose also that the risk-work averse agent has general utility function $u(\cdot)$ and cost of effort $c(\cdot, t)$ satisfying assumptions 2 and 3. A W-robust compensation contract is $\mathcal{IC}$ if and only if*

$$\Psi(x(\overline{e})) = A(\overline{e}) + \int_0^T \frac{c'(\overline{e}_v, v)}{g'(\overline{e}_v, v)} \left(dx(v) + \kappa(v)x(v)dv\right) \tag{9}$$

$$A(\overline{e}) = -\int_0^T h(\overline{e}_v, v)dv + H \tag{10}$$

where $\overline{e}$ is $\mathcal{F}^Z_{(\cdot)}$-progressively measurable and $h(\overline{e}_v, v) \equiv f(\overline{e}_v, v)g(\overline{e}_v, v) - c(\overline{e}_v, v)$ with $f(\overline{e}_v, v) \equiv \frac{c'(\overline{e}_v,v)}{g'(\overline{e}_v,v)}$. Let $y \equiv \int_0^T f(\overline{e}_v, v)^2\sigma(v)^2 dv$ denote the $\mathcal{F}^Z_T$-conditional variance of the contract. The random variable H is any $\mathcal{F}^Z_T$-measurable random variable.

Theorem 2 provides a characterization of the set of W-robust contracts that are incentive compatible. In effect $\mathcal{IC}$ restricts the slope of the contract to be the ratio c'/g'. Evidently, many contracts (parametrized by A) will satisfy this constraint. The principal will choose over this general class to determine the contractual design that optimizes his/her welfare.

But first we elaborate on the structure of an $\mathcal{IC}$ contract. As revealed by the theorem such a contract has two components. To begin with consider the second component

$$\int_0^T \frac{c'(\overline{e}_v, v)}{g'(\overline{e}_v, v)} \left(dx(v) + \kappa(v)x(v)dv\right)$$

which represents the incentive part of contract. Both the change $dx(v)$ and the level $x(v)$ of residual output weighted by the mean reversion parameter $\kappa(v)$ figure in this incentive portion. As explained above, this structure is a consequence of W-robustness. The presence of the multiplier $\frac{c'(\overline{e}_v,v)}{g'(\overline{e}_v,v)}$, can also be intuitively explained. Marginal productivity

$g'(\overline{e}_v, v)$ appears in the denominator to counter any increase in compensation that comes due to the agent merely because of skill (higher endowment of productive capability) and not because of increased effort. It ensures that two agents with different productive capacities providing the same level of effort will be rewarded identically at the margin. The numerator $c'(\overline{e}_v, v)$ captures the fact that the agent has to be compensated for increasing marginal cost of effort. Finally, note that implementation of the optimal effort level $\overline{e}_v$ induced by the contract leads to an incentive compensation of

$$\int_0^T \frac{c'(\overline{e}_v, v)}{g'(\overline{e}_v, v)} \left(dx(v) + \kappa(v)x(v)dv\right) = \int_0^T \frac{c'(\overline{e}_v, v)}{g'(\overline{e}_v, v)} \left(g(\overline{e}_v, v)dv + \sigma(v)dB(v)\right) \tag{11}$$

at the payment date T. This incentive compensation consists of a reward for productivity as well as a reward induced by random realizations of abnormal output.

It is also interesting to note that, under our assumptions on preferences, the incentive part of the contract is impervious to the structure of the utility function: agents with different preferences will be motivated to take the desired action through the same incentive package. The sources of this perhaps surprising robustness of the incentive component of the contract are twofold. First, it should be noted that effort and productivity only affect the combined process $d\widetilde{x}(t) = dx(t) + \kappa(t)x(t)dt$, i.e. for a given productivity function the impact of effort is unrelated to the preferences of the agent. To the extent that W-robust contracts are written on this component the marginal local benefit of effort from the agent's point of view is the marginal impact on the combination $dx(t) + \kappa(t)x(t)dt$. Second, the agent values terminal wealth net of the cumulative cost of effort which is a summation of local costs. The linearity of this objective function implies that the marginal local benefit of effort balances the marginal local cost of effort at every point in time. Since this trade-off is independent of preference considerations the contractual weights that provide the incentives to select the effort desired by the principal are also independent of preferences. However, as we show later, preferences do matter for individual rationality.

The first component, A, of the contract will be selected by the principal so as to ensure individual rationality. Without loss of generality we can decompose A as

$$A(\overline{e}) = -\int_0^T h(\overline{e}_v, v)dv + H$$

where H is chosen by the principal. We provide further interpretation of this part of the contract after resolving the principal's problem.

4.2 The optimal contract

4.2.1 Existence

The characterization of W-robust $\mathcal{IC}$ contracts in Theorem 6 implies that

$$\begin{aligned}\Psi(x(\overline{e})) &= H + \int_0^T c(\overline{e}_v, v)dv + \int_0^T f(\overline{e}_v, v)\sigma(v)dB_v \\ &= H + \int_0^T c(\overline{e}_v, v)dv + \sqrt{y}\,\widetilde{\xi}\end{aligned}$$

where $\widetilde{\xi}$ is a random variable with standard normal distribution and

$$y \equiv \int_0^T f(\overline{e}_t, t)^2 \sigma(t)^2 dt$$

denotes the quadratic variation (variance) of the contract conditional on $\mathcal{F}_T^Z$.

We can then write the principal's objective function as $\widehat{J}(\overline{e}, H) \equiv J(\overline{e}, \Psi(x(\overline{e})))$ and the principal's problem (6) becomes

$$\max_{\overline{e},H} \widehat{J}(\overline{e}, H) \quad \text{subject to} \quad (\overline{e}, H) \in \mathcal{A}(\overline{e}, H) \tag{12}$$

where $\mathcal{A}(\overline{e}, H)$ is the set of $(\overline{e}, H)$ such that

$$\begin{cases} E\int_{-\infty}^{+\infty} u\left(H + \sqrt{y}\xi\right) n(\xi)d\xi \geq \overline{u} \\ H \text{ is } \mathcal{F}_T^Z - \text{measurable} \\ \overline{e} \text{ is } \mathcal{F}_{(\cdot)}^Z - \text{progressively measurable} \\ \overline{e} \geq 0. \end{cases} \tag{13}$$

The outside expectation in the first equation of (13) is taken with respect to Z. The inside integral is with respect to the normal density $n(\xi)$.

The main difficulty in proving the existence of an optimal policy $(\overline{e}^*, H^*)$ is that the set of admissible controls, $\mathcal{A}$, is not compact in the topology where the objective function is continuous. As a result an optimizer may not exist in the sense that the maximal value function may not be attainable. By the same token, standard existence theorems (such as Brouwer's theorem or Kakutani's theorem) which assume that the controls lie in a compact space, do not apply. In essence, these theorems rely on the ability to invert the first order conditions, which is not easily done in our setting. Yet, in spite of these difficulties we establish the following result (see appendix).

Theorem 3: *Problem (12)-(13) has a unique solution* $(\overline{e}^*, H^*)$.

To circumvent the failure of norm-compactness of the choice space we proceed as follows. We first prove that the maximizing choices are bounded and that the value function is finite. This enables us to restrict attention to a set of approximating sequences which is convex, norm-bounded and norm-closed (in the $L^2[0,T] \times L^2$-norm), hence weakly compact. Roughly speaking, these sequences have a weak limit point. Hence, sequences formed of convex combinations of elements of this sequence converge in norm, in fact strongly along some subsequence, to a limit point. Moreover, this limit point is bounded. Finally, using the concavity of the objective function we show that the value function evaluated at this limit point is maximal and that the limit point satisfies the IR constraint.

In order to provide intuition about the maximizing choices we next produce a characterization in the form of necessary and sufficient conditions for optimality.

4.2.2 Necessary and sufficient conditions for optimality

Let λ denote the Lagrange multiplier for the IR constraint and μ_t the stochastic Kuhn-Tucker multiplier for the non-negativity constraint on desired effort level. Necessary conditions for a maximum (see proof in the appendix) are given by

$$\begin{aligned} 0 &= D_v g'(\overline{e}_v, v) - E_v\left(e^{-\int_0^T r(v)dv}\right) c'(\overline{e}_v, v) + \mu_t \\ &\quad + \lambda E_v \left(\int_{-\infty}^{+\infty} u'\left(H + \sqrt{y}\xi\right) \frac{\xi}{\sqrt{y}} n(\xi) d\xi\right) f(\overline{e}_v, v) f'(\overline{e}_v, v) \sigma(v)^2 \end{aligned} \tag{14}$$

$$0 = -e^{-\int_0^T r(v)dv} + \lambda \int_{-\infty}^{+\infty} u'\left(H + \sqrt{y}\xi\right) n(\xi) d\xi \tag{15}$$

$$E\left[\int_{-\infty}^{+\infty} u\left(H + \sqrt{y}\xi\right) n(\xi) d\xi\right] - \overline{u} = 0, \quad \lambda > 0 \tag{16}$$

$$\mu_t \geq 0, \overline{e}_t \geq 0, \mu_t \overline{e}_t = 0. \tag{17}$$

where the first two conditions and the last one must hold for all $t \in [0,T]$.

These conditions are intuitive. Condition (14) states that the principal balances the productivity benefit of increased effort (the term involving $g'(\overline{e}_v, v)$) with the marginal cost of optimal contracting (the terms involving $c'(\overline{e}_v, v)$ and $f'(\overline{e}_v, v)$). Condition (15) expresses the optimality of the contract design: the principal equates the marginal cost of contracting, corresponding to the discount factor $-e^{-\int_0^T r(v)dv}$ to the marginal impact on the IR constraint. Condition (16) is the IR constraint. Finally condition (17) is a typical Kuhn-Tucker restriction which enforces the non-negativity of desired effort; in particular $\mu_t \overline{e}_t = 0$ represents the complementary slackness condition which must hold at all $t \in [0,T]$.

Condition (14) deserves additional discussion. The productivity benefit is reduced by the natural depreciation of output κ. In the absence of depreciation ($\kappa = 0$) the marginal product of effort exerted at date t results in incremental output at date T equal to $G_1(T)$. The present value of this incremental benefit is the process D_v defined in Lemma 1. If the speed of reversion is infinite ($\kappa = \infty$) there is no benefit to effort and $D = 0$. In intermediate cases the benefit of effort is curtailed by the speed of reversion which explains the discount factor involving κ in the expression for D. The marginal cost of effort to the principal is the additional compensation that flows to the manager. As shown in the earlier section this additional compensation arises both from the cost of effort to the manager and from the risk sharing component of the contract. Clearly risk aversion (embedded in the derivative u') only affects the risk sharing portion. Since $f'(\bar{e}_t, t)$ is positive (because the ratio of the marginal cost of effort over marginal productivity is an increasing function of effort), u'' is negative and the normal density is symmetric about 0 the risk sharing component increases the marginal contractual cost of effort. Therefore, as the riskiness of the contract increases, it increases the cost of the contract due to risk aversion. Naturally, the manager's cost of effort also increases at the margin, i.e., additional effort requires compensation.

Our next theorem establishes the sufficiency of (14)-(17).

Theorem 4: *(optimality) Suppose that $(\bar{e}, H)$ solves (14)-(17) and let $(\bar{e}', H')$ denote any alternative pair which is admissible, $(\bar{e}', H') \in \mathcal{A}$. Then $J(\bar{e}, H) \geq J(\bar{e}', H')$, i.e. $(\bar{e}, H)$ is optimal for the principal.*

Theorem 4 shows that pairs $(\bar{e}, H)$ which satisfy the conditions (14)-(17) dominate alternative admissible pairs $(\bar{e}', H')$. An essential ingredient behind the result is the assumption that f is convex, which ensures the convexity of the set of controls for the principal. The existence of a solution to (14)-(17) follows from Theorem 3.

Before providing further intuition about the structure of the optimal contract we elaborate on an issue raised by Gjesdal (1982) namely the possibility of improving welfare by using randomized contracts.

4.2.3 Randomized contracts

When preferences are not additively separable over wealth and cost of effort the principal's welfare may be improved by randomizing the compensation provided to the manager (Gjesdal (1982)). In short, this may occur when the benefits of the additional effort extracted from the manager offset the direct and indirect costs placed on the parties in the contractual relationship.

Randomized contracts are formalized as follows. Introduce a new probability space $(\Omega, \mathcal{G}, Q)$ where the sigma-algebra $\mathcal{G}$ is independent of $\mathcal{F}^{Z,B}$ and Q is a probability measure on $(\Omega, \mathcal{G})$. A *randomized* contract $(H^{\varepsilon}, \bar{e}^{\varepsilon}) \equiv (H + \varepsilon_1 g_1, \bar{e} + \varepsilon_2 g_2)$ is a perturbation of $(H, \bar{e})$ in the direction (g_1, g_2) where g_1, g_2 are $\mathcal{G}$-measurable and $\varepsilon_1, \varepsilon_2 \geq 0$. Since the probability space is arbitrary any random perturbation of the contract is included in our specification. Notice that this notion of perturbation can be implemented using wealth-robust contracts.

Our main result in this section identifies the optimal contract when randomization is permitted.

Proposition 1: *Let $(H^*, \bar{e}^*)$ denote the solution to the first order conditions (14)-(17) and consider the class of randomized contract $(H^{*\varepsilon}, \bar{e}^{*\varepsilon})$, $\varepsilon_1, \varepsilon_2 \geq 0$. The contract $\varepsilon_1 = \varepsilon_2 = 0$ is optimal (i.e. the non-randomized contract $(H^*, \bar{e}^*)$ is optimal).*

The intuition for the suboptimality of randomized compensation schemes is straightforward. A perturbation in a (g_1, g_2)-direction is welfare improving if and only if the principal stands to gain while meeting the manager's reservation utility and inducing the proper incentives. Evidently, this occurs if and only if the principal's expected payoff, evaluated at the agent's optimal effort choice, attains an IR-constrained maximum at $\varepsilon \equiv (\varepsilon_1, \varepsilon_2) \neq (0, 0)$. Simple analysis, however, shows that both the principal's objective function and the agent's expected utility are concave in ε. Moreover, the first order conditions with respect to ε are null at $\varepsilon = 0$. Combining these two properties establishes the optimality of the non-randomized contract $\varepsilon = (0, 0)$.

Gjesdal shows that randomization dominates if the Lagrangian associated with the principal's IR-constrained optimization problem is convex with respect to the perturbation parameter ε when evaluated at the point $\varepsilon = (0, 0)$. This convexity property can be restated in terms of a condition involving the risk aversions of the principal and the manager (Gjesdal (1982), proposition 3, eq. (19)). In our setting, Gjesdal's condition fails. More specifically, the convexity of the ratio of the marginal cost to the marginal productivity (assumption 3(iii)) ensures the concavity of the manager's expected utility function and, hence, the concavity of the principal's Lagrangian associated with the constrained problem. De facto, this violates Gjesdal's condition (19) and precludes any benefit from randomizing.

4.2.4 The structure of the optimal contract

Combining Theorems 1 and 2 and the first order conditions (14)-(17) enables us to further characterize the structure of the optimal contract and the associated optimal effort policy.

Corollary 1: *The optimal contract solves (14)-(17) and takes the form*

$$\Psi(\bar{e}, x) = H(y, \lambda) - \int_0^T h(\bar{e}_v, v)dv + \int_0^T \frac{c'(\bar{e}_v, v)}{g'(\bar{e}_v, v)} \left(dx(v) + \kappa(v)x(v)dv\right) \qquad (18)$$

where $H(y, \lambda)$ is the $\mathcal{F}_T^Z-$measurable random variable which uniquely solves the nonlinear equation

$$\int_{-\infty}^{+\infty} u'\left(H + \sqrt{y}\xi\right) n(\xi)d\xi = e^{-\int_0^T r(v)dv} \lambda^{-1} \qquad (19)$$

and λ is a constant which uniquely solves

$$E\left[\int_{-\infty}^{+\infty} u\left(H(y, \lambda) + \sqrt{y}\xi\right) n(\xi)d\xi\right] = \bar{u}.$$

The optimal effort level exists and solves the nonlinear equation

$$\begin{aligned} 0 &= D_v g'(\bar{e}_v, v) - E_v\left(e^{-\int_0^T r(v)dv}\right) c'(\bar{e}_v, v) \\ &\quad + \lambda E_v\left(\int_{-\infty}^{+\infty} u'\left(H(y, \lambda) + \sqrt{y}\xi\right) \frac{\xi}{\sqrt{y}} n(\xi)d\xi\right) f(\bar{e}_v, v)f'(\bar{e}_v, v)\sigma(v)^2. \qquad (20) \end{aligned}$$

Optimal effort is strictly positive at all times, $e_t > 0$ for all $t \in [0, T]$.

The incentive part of the contract was already discussed in section 4.1. We now elaborate on the component motivated by the need to induce the agent to accept the contract (the IR constraint) and work for the principal, i.e. $H(y, \lambda) - \int_0^T h(\bar{e}_v, v)dv$. The second term in this decomposition has two parts (recall that $h(\bar{e}_v, v) \equiv f(\bar{e}_v, v)g(\bar{e}_v, v) - c(\bar{e}_v, v)$). The first of these

$$-\int_0^T f(\bar{e}_v, v)g(\bar{e}_v, v)dv$$

is a reduction in compensation which precisely offsets the identical term of opposite sign in the incentive compensation received under the optimal effort policy (see (11)). This shows that, in the end, the agent is *not* additionally compensated for his production. However, he/she is compensated for the disutility for work required and the risk imposed. Therefore, the agent's reward stems from the second part

$$\int_0^T c(\bar{e}_v, v)dv$$

which provides compensation for disutility of work and from the component, H, which provides adequate compensation to induce work (i.e. it satisfies the IR constraint).

In section 4.1 we discussed the property that preferences (i.e. the utility function) do not impact the incentive part of the contract. In contrast preferences determine the first term of the optimal contract (10), more specifically its component H. There are two distinct reasons for preference dependence. The first is the fact that the contract must motivate work (rather than allow the agent to idle) at inception. Since different utility functions give rise to different reservation values for work the result follows: the random variable H is chosen to satisfy the IR-constraint (equation (32) in the appendix) and will therefore depend on the agent's characteristics. The second is the optimizing behavior of the principal. As explained above the class of IC-IR contracts is parametrized by random variables H which motivate work. The principal optimizes over this class to arrive at the selection of a specific contractual design.

5 Examples

5.1 Exponential utility (CARA)

When the agent exhibit constant absolute risk aversion ($u(x) = -\frac{1}{R}\exp(-Rx)$; $R \geq 0$) the optimal contract in corollary 1 simplifies to[11]

$$\Psi(\bar{e}, x) = H(y, \lambda) + \int_0^T c(\bar{e}_v, v)dv + \int_0^T \frac{c'(\bar{e}_v, v)}{g'(\bar{e}_v, v)} (dx(v) + \kappa(v)x(v)dv - g(\bar{e}_v, v)dv)$$

[11] The limit of the binomial model in Govindaraj and Ramakrishnan (2001) is a subcase of this model.

$$H(y,\lambda) = \frac{1}{2}R\left(\int_0^T f(\overline{e}_t,t)^2\sigma(t)^2dt\right) - \frac{1}{R}\log\left(\frac{e^{-\int_0^T r(v)dv}}{E\left(e^{-\int_0^T r(v)dv}\right)}\right)$$

where $\overline{e}_v$ is the unique solution of the equation

$$\frac{D_v}{E_v\left(e^{-\int_0^T r(v)dv}\right)}g'(\overline{e}_v,v) = c'(\overline{e}_v,v) + Rf(\overline{e}_v,v)f'(\overline{e}_v,v)\sigma(v)^2.$$

Simple calculations yield the following comparative static results

Proposition 2: *The optimal effort level $\overline{e}$ desired by the principal is*

(i) a decreasing function of the manager's risk aversion R and of the riskiness of the output process σ

(ii) an increasing function of the ratio $D_v/E_v \exp(-\int_0^T r(v)dv)$. Thus, an increasing function of the reinvestment rate q and a decreasing function of the speed of mean-reversion κ.

The Brownian model explored by Holmstrom and Milgrom (1987) corresponds to the parametrization $q = \kappa = 0, \sigma, r$ constants, $g(\overline{e},v) = \overline{e}, c(\overline{e},v) = c(\overline{e}), G_0 = 0$ and G_1 constant. Under these conditions the optimal effort is a constant $\overline{e}^*$ which uniquely solves

$$G_1 = c'(\overline{e}^*) + Rc'(\overline{e}^*)c''(\overline{e}^*)\sigma^2$$

and the optimal contract is linear in cumulative output

$$\Psi(\overline{e},x) = \frac{1}{2}Rc'(\overline{e}^*)^2\sigma^2T + c(\overline{e}^*)T + c'(\overline{e}^*)[x(T) - x(0) - \overline{e}^*T],$$

which corresponds to the solution derived by HM (see Theorem 7). Under these circumstances the optimal contract is path-independent and therefore easy to implement. Path-independence also holds when the drift of the output process is a function $g(\overline{e}_v,v) = g(\overline{e}_v)$ which does not depend on W-risk.

5.2 General utility function and constant coefficients

Suppose that we now place ourselves in the context of HM (i.e. $q = \kappa = 0, \sigma, r$ constants, $c'(\overline{e},v) = c'(\overline{e}), G_0 = 0$ and G_1 constant) except for the fact that the agent has a general concave utility function $u(\cdot)$ and we maintain the assumption of a concave technology $g(\overline{e})$. The optimal contract of Corollary 1 becomes

$$\Psi(\overline{e},x) = H(y,\lambda) + c(\overline{e}^*)T + \frac{c'(\overline{e}^*)}{g'(\overline{e}^*)}[x(T) - x(0) - g(\overline{e}^*)T] \tag{21}$$

where $H(y,\lambda)$ is the (non-random) function which uniquely solves the nonlinear equation

$$\int_{-\infty}^{+\infty} u'\left(H + \sqrt{y}\xi\right)n(\xi)d\xi = e^{-rT}\lambda^{-1} \tag{22}$$

and λ is a constant which uniquely solves

$$E\left[\int_{-\infty}^{+\infty} u\left(H(y,\lambda)+\sqrt{y}\xi\right)n(\xi)d\xi\right]=\overline{u}.$$

The optimal effort level exists and is a constant $\overline{e}^*$ which solves the nonlinear equation

$$G_1 g'(\overline{e}^*)=c'(\overline{e}^*)+\frac{\lambda}{e^{-rT}}\left(\int_{-\infty}^{+\infty} u''\left(H(y,\lambda)+\sqrt{y}\xi\right)n(\xi)d\xi\right)f(\overline{e}^*)f'(\overline{e}^*)\sigma^2. \tag{23}$$

where $y\equiv f(\overline{e}^*)^2\sigma^2 T$.

Even though we allowed for general utility function the optimal effort level is a constant. This is driven by the fact that the coefficients of the model are constants. Under these circumstances there is no reason for the principal to requisition an effort level that changes over time. The immediate consequence of this result is that the comparative static results of Proposition 2 are again valid. Moreover the optimal contract has the linear, path-independent, form (21) which is easy to implement in practice.

6 Concluding remarks

One of the main results from our analysis has been the identification of a class of contracts that are robust to small errors in the principal's assessment of the opportunities available to the manager and the actions pursued outside the contractual relationship. This leads to a class of W-robust incentive contracts that are linear functionals of the account under management. Contracts, within this class, that are optimal for the principal can be characterized explicitly, even in general environments when the managed account has stochastic coefficients and the utility of the manager is not separable over reward and effort. Moreover, the contractual structure obtained is reasonably simple and intuitively appealing. The most interesting aspect is perhaps the fact that somewhat similar contracts are observed in practice (Business Week (1993)). Indeed, it is common for a manager (agent) to be compensated not only on the level of cumulative aggregated output (typically a stock variable like output) that he/she influences, but also on the change in observed output (a flow variable that is typically the change in output).

>From an empirical perspective, there are clearly testable implications with respect to the relative weights placed on the levels and changes in output, and with respect to the slope coefficient of the incentive part of the contract. One prediction is that firms with low mean reversion in output (or κ small) will tend to put more weight on the change variable. This prediction extends a similar finding by Govindaraj and Ramakrishnan (2001) to more realistic incentive situations. Other potential implications relate to the slope which is affected by the agent's cost of effort and productivity. The nature and direction of these effects provides scope for empirical investigations.

7 Appendix: proofs

Proof of Theorem 1: Suppose that the optimal solution of the agent problem entails effort e^* and private wealth $\widetilde{Y}^*$ under the general contract $\Psi(\widetilde{x})$. We first show that there exists a wealth-robust contract which induces the same optimal effort level and leaves the agent equally well off.

Since $(e^*, \widetilde{Y}^*)$ is optimal for the agent it must be that

$$E\left[u'\left(\widetilde{Y}^* + \Psi(\widetilde{x}^*) - c(e^*)\right)(\Psi'(\widetilde{x}^*)g'(e^*) - c'(e^*))\right] = 0 \tag{24}$$

$$u'\left(\widetilde{Y}^* + \Psi(\widetilde{x}^*) - c(e^*)\right) = \lambda \tag{25}$$

where $\widetilde{x}^* = g(e^*) + \sigma\widetilde{B}$ and λ is a Lagrange multiplier for the budget constraint $E[\widetilde{Y}^*] = 0$. Simple arguments show that

$$\widetilde{Y}^* = I(\lambda) - \Psi(\widetilde{x}^*) + c(e^*)$$

where $I(\cdot)$ is the inverse marginal utility function and λ is the unique solution of the nonlinear equation

$$E\left[(I(\lambda) - \Psi(\widetilde{x}^*) + c(e^*))\right] = 0.$$

Combining (25) and (24) shows that

$$E\left[\Psi'(\widetilde{x}^*)\right] = \frac{c'(e^*)}{g'(e^*)}.$$

Now define

$$H(\widetilde{x}^*) = \Psi(\widetilde{x}^*) - \frac{c'(e^*)}{g'(e^*)}\widetilde{x}^*$$

and note that

$$\Psi(\widetilde{x}^*) = \frac{c'(e^*)}{g'(e^*)}\widetilde{x}^* + H(\widetilde{x}^*)$$

with

$$E[H'(\widetilde{x}^*)] = E[\Psi'(\widetilde{x}^*)] - \frac{c'(e^*)}{g'(e^*)} = 0.$$

Consider now the wealth-robust contract

$$\widetilde{\Psi}(\widetilde{x}) = \frac{c'(e^*)}{g'(e^*)}\widetilde{x} + E[H(\widetilde{x}^*)].$$

We show next that this contract will induce optimal effort e^* and leaves the agent equally well off as the general contract $\Psi(\widetilde{x})$. To prove this we must show that the agents first order conditions associated with $\widetilde{\Psi}(\widetilde{x})$ reduce to those associated with $\Psi(\widetilde{x})$.

Solving the agent's problem under $\widetilde{\Psi}(\widetilde{x})$ leads to the first order conditions

$$E\left[u'\left(\widetilde{Y} + \frac{c'(e^*)}{g'(e^*)}\widetilde{x} + E[H(\widetilde{x}^*)] - c(e)\right)\left(\frac{c'(e^*)}{g'(e^*)}g'(e) - c'(e)\right)\right] = 0$$

$$u'\left(\widetilde{Y}+\frac{c'(e^*)}{g'(e^*)}\widetilde{x}+E[H(\widetilde{x}^*)]-c(e)\right)=\widetilde{\lambda}.$$

Concavity of the objective function and convexity of the constraints implies that the second order conditions are satisfied. Simple manipulations show that e^* solves the first equation and is therefore the unique optimal effort. To verify that $\widetilde{Y}^*$ is optimal note that

$$\begin{aligned}\widetilde{Y} &= I(\widetilde{\lambda})-\frac{c'(e^*)}{g'(e^*)}\widetilde{x}^*-E[H(\widetilde{x}^*)]+c(e^*)\\ &= I(\widetilde{\lambda})-(\Psi(\widetilde{x}^*)-H(\widetilde{x}^*))-E[H(\widetilde{x}^*)]+c(e^*).\end{aligned}$$

Since the budget constraint mandates $E[\widetilde{Y}]=0$ we must have

$$\begin{aligned}0 &= E\left[I(\widetilde{\lambda})\right]-E\left[\Psi(\widetilde{x}^*)-H(\widetilde{x}^*)\right]-E[H(\widetilde{x}^*)]+E\left[c(e^*)\right]\\ &= E\left[I(\widetilde{\lambda})\right]-E\left[\Psi(\widetilde{x}^*)\right]+c(e^*)=E\left[\left(I(\widetilde{\lambda})-\Psi(\widetilde{x}^*)+c(e^*)\right)\right]\end{aligned}$$

which implies $\widetilde{\lambda}=\lambda$. We conclude that $\widetilde{Y}^*$ is also optimal for the agent under the wealth-robust contract. It follows now that the agent achieves the same welfare under the two contracts. Since the principal is risk neutral, and $E[\widetilde{\Psi}(\widetilde{x})]=E[\Psi(\widetilde{x})]$, it also follows that the principal is equally well off. ■

Proof of Lemma 1: Given the payoff to the principal and the compensation package, the value of the firm to the risk neutral principal at date 0 is given by,

$$V_0=E\left[e^{-\int_0^T r(v)dv}\left(G_0+G_1x(T)\right)\right]-E\left[e^{-\int_0^T r(v)dv}\Psi\right]$$

where G_0, G_1 are $\mathcal{F}_T^Z$-measurable random variables and

$$x(s)=x(0)e^{-\int_0^s \kappa(u)du}+\int_0^s e^{-\int_v^s \kappa(u)du}\left(g(e_v,v)dv+\sigma(v)dB(v)\right).$$

The second component of the firm value simplifies as follows

$$\begin{aligned}&E\left[e^{-\int_0^T r(v)dv}G_1x(T)\right]\\ =\ &E\left[e^{-\int_0^T r(v)dv}G_1\left(x(0)e^{-\int_0^T \kappa(u)du}+\int_0^T e^{-\int_v^T \kappa(u)du}\left(g(e_v,v)dv+\sigma(v)dB(v)\right)\right)\right]\\ =\ &x(0)E\left[e^{-\int_0^T r(v)dv}e^{-\int_0^T \kappa(u)du}G_1\right]+E\left[e^{-\int_0^T r(v)dv}\left(\int_0^T e^{-\int_v^T \kappa(u)du}g(e_v,v)dv\right)G_1\right]\end{aligned}$$

and, therefore,

$$\begin{aligned}V_0 &= E\left[e^{-\int_0^T r(v)dv}G_0\right]+x(0)E\left[e^{-\int_0^T (r(v)+\kappa(v))dv}G_1\right]\\ &\quad +E\left[e^{-\int_0^T r(v)dv}\int_0^T g(e_s,s)e^{-\int_s^T \kappa(u)du}G_1ds\right]-E\left[e^{-\int_0^T r(v)dv}\Psi\right].\end{aligned}$$

Defining the process D as in the lemma and substituting in the second line gives the decomposition announced. ■

Proof of Theorem 2: Consider a contract $\Psi(\widetilde{x}) \in \mathcal{D}$. Since we are interested in contracts which are W-robust, we can limit our attention to agents whose wealth Y is such that appropriate limits can be taken to get the first order conditions.

Given the space of contracts $\mathcal{D}$, the first order conditions for an interior optimum at e can be stated as

$$E\left[u'(T)\left(\int_0^T \left(\lambda^\Psi(v,T)g'(e_v,v) - c'(e_v,v)\right)(e_v - \overline{e}_v)\,dv\right)\right] = 0 \tag{26}$$

where $u'(T) \equiv u'\left(Y + \Psi - \int_0^T c(e_u,u)du\right)$ denotes marginal utility of terminal net compensation and $\lambda^\Psi(v,T) \in \mathcal{F}_T^{\widetilde{x},Z}$ for all $v \in [0,T]$. By definition a contract is W-robust if the agent's optimal effort choice is impervious to perturbations of terminal wealth in $\mathcal{F}_T^{B,Z}$. Thus, (26) must hold for all $u'(T) \in \mathcal{F}_T^{B,Z}$ and for all $\overline{e} \geq 0$ such that $\overline{e} \in \mathcal{F}_T^{\widetilde{x},Z}$. As a result we must have $\lambda^\Psi(v,T) \in \mathcal{F}_v^{\widetilde{x},Z}$ for all $v \in [0,T]$. That is, $\lambda^\Psi(v,T) = \lambda^\Psi(v)$ where $\lambda^\Psi(v) \in \mathcal{F}_v^{\widetilde{x},Z}$, for all $v \in [0,T]$ (i.e. there is no dependence on events posterior to v). Moreover, optimal choice of effort $e \in \mathcal{F}_{(\cdot)}^{\widetilde{x},Z}$.

Thus, for any perturbation $\widetilde{x}(\cdot) + \epsilon \int_0^{\cdot} h(v)dv$ of $\widetilde{x}(\cdot)$, where h is an $\mathcal{F}_{(\cdot)}^{\widetilde{x},Z}$-progressively measurable process, the gradient has the representation

$$\nabla^h \Psi = \int_0^T \lambda^\Psi(v)h(v)dv$$

where $\lambda^\Psi(\cdot)$ is $\mathcal{F}_v^{\widetilde{x},Z}$-progressively measurable. Since $\nabla^h \widetilde{x}(v) = \int_0^v h(s)ds$ we have $d\nabla^h \widetilde{x}(v) = h(v)dv$ and, hence,

$$\nabla^h \Psi = \int_0^T \lambda^\Psi(v)d\nabla^h \widetilde{x}(v) \tag{27}$$

for $\lambda^\Psi(v) \in \mathcal{F}_v^{\widetilde{x},Z}$. In particular note that $\lambda^\Psi(v)$ may involve a dependence on $\{\widetilde{x}(s) : s \leq v\}$.

We now introduce the stochastic process

$$B^*(s) = B(s) + \int_0^s \frac{g(e_v)}{\sigma(v)}dv = \int_0^s \frac{1}{\sigma(v)}d\widetilde{x}(v).$$

Notice that B^* is a Brownian motion under P^* defined by

$$\frac{dP^*}{dP} = \exp\left(-\frac{1}{2}\int_0^T \left(\frac{g(e_v)}{\sigma(v)}\right)^2 dv - \int_0^T \frac{g(e_v)}{\sigma(v)}dB(v)\right) \equiv \eta^*.$$

Additionally, since all our contracts are functionals written on $(\widetilde{x}, Z)$ and since the filtrations $\mathcal{F}_{(\cdot)}^{\widetilde{x},Z} = \mathcal{F}_{(\cdot)}^{B^*,Z}$ they can also be written on (B^*, Z). Since $\Psi \in L^2(P^*)$, by assumption (ii) in the definition of $\mathcal{D}$, the Martingale Representation Theorem applies to give

$$\Psi = E^* [\Psi] + \int_0^T \phi^1(v) dZ(v) + \int_0^T \phi^2(v) dB^*(v). \tag{28}$$

Consider perturbing $\widetilde{x}$ in the direction $\int_0^{\cdot} h(v)dv$, such that

$$\widetilde{x}^{\epsilon}(t) \equiv \widetilde{x}(t) - \epsilon \int_0^t h(v)dv = \int_0^t g(e_v)dv + \int_0^t \sigma(v)dB^{\epsilon}(v)$$

where $B^{\epsilon}(v) = B^*(t) - \epsilon \int_0^t \frac{h(v)}{\sigma(v)} dv$ is a $(P^{\epsilon}, \mathcal{F}_{(\cdot)}^{B^*,Z})$-Brownian motion with

$$\frac{dP^{\epsilon}}{dP^*} = \exp\left(-\frac{\epsilon^2}{2} \int_0^T \left(\frac{h(v)}{\sigma(v)} \right)^2 dv + \epsilon \int_0^t \frac{h(v)}{\sigma(v)} dB^*(v) \right) \equiv \eta^{\epsilon}.$$

Then

$$E^* [\Psi(\widetilde{x})] = E^{\epsilon} [\Psi(\widetilde{x}^{\epsilon})] = E^* \left[\eta^{\epsilon} \Psi \left(\widetilde{x} - \epsilon \int_0^{\cdot} h(v)dv \right) \right]$$

and, by simple algebra,

$$\begin{aligned} & \frac{1}{\epsilon} E^* \left[\Psi(\widetilde{x}) - \Psi\left(\widetilde{x} - \epsilon \int_0^{\cdot} h(v)dv \right) \right] \\ & + E^* \left[\frac{\eta^{\epsilon} - 1}{\epsilon} \left(\Psi(\widetilde{x}) - \Psi\left(\widetilde{x} - \epsilon \int_0^{\cdot} h(v)dv \right) \right) \right] \\ = \; & E^* \left[\frac{\eta^{\epsilon} - 1}{\epsilon} \Psi(\widetilde{x}) \right]. \end{aligned}$$

Letting $\epsilon \to 0$ and using assumption (iii) of the definition of the space of contracts $\mathcal{D}$ gives

$$E^* \left[\bigtriangledown^h \Psi(\widetilde{x}) \right] = E^* \left[\Psi(\widetilde{x}) \int_0^T \frac{h(v)}{\sigma(v)} dB^*(v) \right].$$

Substituting $\bigtriangledown^h \Psi(\widetilde{x}) = \int_0^T \lambda^{\Psi}(v) h(v) dv$ with $\lambda^{\Psi}(v) \in \mathcal{F}_v^{\widetilde{x},Z}$ on the left hand side of this equality, and using the representation (28) on the right hand side to compute the cross variation gives

$$E^* \left[\int_0^T \lambda^{\Psi}(v) h(v) dv \right] = E^* \left[\Psi(\widetilde{x}) \int_0^T \frac{h(v)}{\sigma(v)} dB^*(v) \right] = E^* \left[\int_0^T \phi^2(v) \frac{h(v)}{\sigma(v)} dv \right]$$

so that

$$\phi^2(v) = \lambda^{\Psi}(v) \sigma(v).$$

This relation implies that $\phi^1(v)$ is independent of $\widetilde{x}$, i,e., $\phi^1(v) \in \mathcal{F}_v^Z$ (otherwise $\lambda^{\Psi}(v)$ would also depend on the gradient of $\phi^1(v)$). Moreover, since, $\lambda^{\Psi}(v)$ is the gradient of Ψ (by definition) and λ^{Ψ} is adapted, we must also have $\lambda^{\Psi}(v) \in \mathcal{F}_v^Z$. Thus, the contract must have the form

$$\begin{aligned}\Psi &= E^*[\Psi] + \int_0^T \phi^1(v)dZ(v) + \int_0^T \lambda^\Psi(v)\sigma(v)dB^*(v) \\ &= E^*[\Psi] + \int_0^T \phi^1(v)dZ(v) + \int_0^T \lambda^\Psi(v)d\widetilde{x}(v)\end{aligned}$$

with

$$E^*[\Psi] = k, \qquad \phi^1(v) \in \mathcal{F}_v^Z, \qquad \text{and} \qquad \lambda^\Psi(v) \in \mathcal{F}_v^Z$$

for some constant k. We conclude that wealth-robust contracts are linear functionals of $\widetilde{x}$.

Finally, notice that with this form of contract, second order conditions are satisfied (since u, f and $-c$ are concave). Solutions of the first order conditions above are therefore global maxima for the agent's problem. ■

Proof of Theorem 3: Consider the W-robust contract

$$\Psi = A + \int_0^T f(t)\left(dx(t) + \kappa(t)x(t)dt\right)$$

where A is a random variable measurable with respect to public information $\mathcal{F}_T^Z$ but independent of x and f a stochastic process in $\mathcal{F}_{(\cdot)}^Z$ but independent of x. Substituting the equation for output (1) gives

$$\Psi = A + \int_0^T f(t)\left(g(e_t,t)dt + \sigma(t)dB(t)\right).$$

Given this contract, the agent solves the optimization problem

$$\max_e Eu\left(A + \int_0^T \left(f(t)g(e_t,t) - c(e_t,t)\right)dt + \int_0^T f(t)\sigma(t)dB(t)\right). \tag{29}$$

A stochastic process for effort $e(t)$ is optimal only if it cannot be dominated by any feasible perturbation. Consider an ε-perturbation in the feasible direction $e'(t)$ defined by $e(t)+\varepsilon(e(t)-e'(t))$, where $\varepsilon \geq 0$ and $e'(t)$ is an arbitrary nonnegative and progressively measurable effort policy. The first order condition evaluated at $\varepsilon = 0$ is (see for example Luenberger (1969))

$$F(e) \equiv E\left[u'(T)\int_0^T \left(f(t)g'(e_t,t) - c'(e_t,t)\right)(e(t)-e'(t))dt\right] = 0; \quad \forall\, e' \neq e,$$

where $u'(T)$ is defined as the marginal utility at date T (i.e. evaluated at $X_T = \Psi - \int_0^T c(e_t,t)dt$). The expression $F(e)$ on the left hand side of this equality can be written as

$$\begin{aligned}F(e) &\equiv E\left[\int_0^T u'(T)\left(f(t)g'(e_t,t) - c'(e_t,t)\right)(e(t)-e'(t))dt\right] \\ &= E\left[\int_0^T E_t[u'(T)]\left(f(t)g'(e_t,t) - c'(e_t,t)\right)(e(t)-e'(t))dt\right]\end{aligned}$$

where we used the law of iterated expectations and the fact that

$$(f(t)g'(e_t,t) - c'(e_t,t))\,(e(t) - e'(t))$$

is known at date t and therefore factors out of the conditional expectation. Since this equation must hold for all progressively measurable processes $e'(t)$ we must have

$$E_t[u'(T)]\,(f(t)g'(e_t,t) - c'(e_t,t)) = 0;\ \text{ for all } t \in [0,T] \text{ and all } A \in \mathcal{F}_t$$

(otherwise it is easy to find feasible effort policies that improve on $e(t)$), i.e this equality must hold at all times and in all states of nature that materialize at time t. Since marginal utility is strictly positive we conclude that

$$f(t)g'(e_t,t) - c'(e_t,t) = 0. \tag{30}$$

From (30) it follows that the policy $e_t = \overline{e}_t$ is optimal only if the slope of the contract equals $f(t) = f(\overline{e}_t,t) \equiv \frac{c'(\overline{e}_t,t)}{g'(\overline{e}_t,t)}$. Moreover since the function $f(e,t) = \frac{c'(e,t)}{g'(e,t)}$ is strictly increasing in e the slope $f(\overline{e}_t,t)$ implements a unique policy $\overline{e}_t$. Finally, Lemma A1 below shows that condition (30) is sufficient for a maximum. This shows that the effort policy $e_t = \overline{e}_t$ is the unique maximizer when $f(t) = f(\overline{e}_t,t)$.

Under this optimal policy, $e_t = \overline{e}_t$, the net compensation at time T to the agent is given by X_T, where

$$X_T = \Psi - \int_0^T c(e_t,t)dt = A + \int_0^T (h(\overline{e}_t,t)dt + f(\overline{e}_t,t)\sigma(t)dB(t))$$

with $h(\overline{e}_t,t) = f(\overline{e}_t,t)g(\overline{e}_t,t) - c(\overline{e}_t,t)$. Furthermore, since A is a function of the principal's information we can, without any loss of generality, perform the change of variable

$$A = -\int_0^T h(\overline{e}_t,t)dt + H$$

where H is an $\mathcal{F}_T^Z$-measurable random variable. ■

We now prove Lemma A1 which was used in the proof of Theorem 6 above.

Lemma A1: *Suppose that the policy e satisfies the necessary condition (30) for all $t \in [0,T]$. Then there is no other effort policy $\widehat{e}$ such than $\widehat{e}$ dominates e for the agent (i.e. $U(\widehat{e}) > U(e)$).*

Proof of Lemma A1: Let e denote the solution of (30) and let $\widehat{e}$ denote any other arbitrary effort policy. To simplify notation let

$$X_T(e) \equiv \Psi_T(e) - \int_0^T c(e_t,t)dt$$

and note that

$$\begin{aligned} X_T(e) - X_T(\widehat{e}) &= \Psi_T(e) - \int_0^T c(e_t,t)dt - \left(\Psi_T(\widehat{e}) - \int_0^T c(\widehat{e}_t,t)dt\right) \\ &= \int_0^T \left(f(\overline{e}_t,t)g(e_t,t) - c(e_t,t) - (f(\overline{e}_t,t)g(\widehat{e}_t,t) - c(\widehat{e}_t,t))\right)dt \\ &\geq \int_0^T \left(f(\overline{e}_t,t)g'(e_t,t) - c'(e_t,t)\right)(e_t - \widehat{e}_t)dt \\ &= 0, \end{aligned}$$

where the inequality follows from the concavity of the function $f(\overline{e}_t,t)g(e,t) - c(e,t)$ with respect to e and the last line follows from the fact that e satisfies (30). Concavity of the utility function then implies

$$\begin{aligned} u\left(X_T(e)\right) &\geq u\left(X_T(\widehat{e})\right) + u'\left(X_T(e)\right)\left(X_T(e) - X_T(\widehat{e})\right) \\ &\geq u\left(X_T(\widehat{e})\right) + u'\left(X_T(e)\right)\int_0^T \left(f(\overline{e}_t,t)g'(e_t,t) - c'(e_t,t)\right)(e_t - \widehat{e}_t)dt \\ &= u\left(X_T(\widehat{e})\right). \end{aligned}$$

Taking the expectation on both sides of the last inequality shows $U(e) \geq U(\widehat{e})$, i.e. e dominates any other effort policy $\widehat{e}$. ■

Proof of Theorem 4: Let

$$J(e,H) = E\left[\int_0^T g(e_s,s)D_s ds\right] - E\left[e^{-\int_0^T r_v dv}\left(\int_0^T c(e_v,v)dv + H\right)\right]$$

We wish to show the existence of $(e^*,H^*) \in \mathcal{A}$ such that

$$\sup_{(e,H)\in\mathcal{A}} J(e,H) = J(e^*,H^*)$$

where

$$\mathcal{A} = \left\{(e,H) : E\left[u\left(H + \sqrt{y(e)}\,\widetilde{\xi}\right)\right] \geq \overline{u}, e \text{ is } \mathcal{F}^Z_{(\cdot)}\text{-progressively measurable}, H \in \mathcal{F}^Z_T\right\}$$

$$y(e) = \int_0^T f(e_v,v)^2\sigma_v^2 dv$$

and $\widetilde{\xi}$ is standard normal independent of Z. If $\mathcal{A}$ is empty there is a trivial solution; so we assume that $\mathcal{A}$ is non-empty.

We first prove three auxiliary lemmas. The first of these demonstrates useful bounds on the value function and candidate optimizing random variables H,

Lemma A2: *The following bounds hold:*

(i) $\sup_{(e,H)\in\mathcal{A}} J(e,H) \leq V < \infty$

(ii) $0 \leq E[H] \leq cV$ *for some constant* c.

Proof of Lemma A2: Define

$$V = \sup_e J(e,0) = \sup_e \left(E\left[\int_0^T g(e_s,s) D_s ds\right] - E\left[e^{-\int_0^T r_v dv} \int_0^T c(e_v,v) D_v dv\right]\right).$$

Note that this optimization problem has a unique maximizer given by

$$\widehat{e}_t \equiv f^{-1}\left(\frac{D_t}{E_t \exp\left(-\int_0^T r_v dv\right)}\right)$$

(using the fact that the function f has a unique inverse), and therefore $V < \infty$. Furthermore, using the IR constraint, the concavity of the utility function and Jensen's inequality gives

$$u\left(E[H]\right) \geq E\left[u\left(H + \sqrt{y(e)}\,\widetilde{\xi}\right)\right] \geq \overline{u},$$

so that $E[H] \geq 0$. We conclude

$$\sup_{(e,H)\in\mathcal{A}} J(e,H) \leq V.$$

This establishes (i). Moreover, since $(0,0) \in \mathcal{A}$, any potential (e,H) must satisfy $J(e,H) \geq J(0,0) \geq 0$. Thus,

$$E\left[e^{-\int_0^T r_v dv} H\right] \leq J(e,0) \leq V.$$

Since $e^{-\int_0^T r_v dv} \geq k > 0$ (because r is bounded) the upper bound on $E[H]$ follows. This demonstrates (ii). ■

Our second auxiliary result shows that e is bounded by the unconstrained profit maximizing choice $\widehat{e}$,

Lemma A3: *Let* $(e,H) \in \mathcal{A}$ *and for all* $t \in [0,T]$ *define* $\overline{e}_t = (e_t \wedge \widehat{e}_t) \vee 0$. *Then* $J(\overline{e},H) \geq J(e,H)$ *and* $(\overline{e},H) \in \mathcal{A}$.

Proof of Lemma A3: Notice that $\overline{e}$ is uniformly bounded above and independent of H since $\widehat{e}$ is uniformly bounded above. Moreover the pair $(\overline{e},H)$ remains IR. Indeed, state by state in $\mathcal{F}_T^Z$,

$$\int_{-\infty}^{\infty} u\left(H + \sqrt{y(\overline{e})}\xi\right) n(\xi) d\xi \geq \int_{-\infty}^{\infty} u\left(H + \sqrt{y(e)}\xi\right) n(\xi) d\xi$$

since

$$\frac{\partial}{\partial y}\int_{-\infty}^{\infty} u\left(H + \sqrt{y}\xi\right) n(\xi) d\xi = \int_{-\infty}^{\infty} u'\left(H + \sqrt{y}\xi\right) \frac{\xi}{\sqrt{y}} n(\xi) d\xi < 0$$

(i.e. expected utility is decreasing in y). Integrating over Z gives

$$E\int_{-\infty}^{\infty} u\left(H+\sqrt{y(\overline{e})}\xi\right)n(\xi)d\xi \geq E\int_{-\infty}^{\infty} u\left(H+\sqrt{y(e)}\xi\right)n(\xi)d\xi \geq \overline{u}.$$

Also, $J(\overline{e},H) \geq J(e,H)$ follows directly. ■

Our third auxiliary result demonstrates that we can focus attention on random variables H which obey uniform bounds: given any uniformly bounded e paired with an arbitrary H, we can improve the principal's welfare by replacing H by some $\overline{H}$ which lies in a uniformly bounded set.

Lemma A4: *Let $(e,H)\in\mathcal{A}$. Then $J(\overline{e},\overline{H}) \geq J(e,H)$, for some $\overline{H}$ which obeys uniform bounds that are independent of $\overline{e}$, and $(\overline{e},\overline{H})\in\mathcal{A}$.*

Proof of Lemma A4: Consider the optimization problem (over H)

$$\max_H -E\left[e^{-\int_0^T r_v dv}H\right]$$

subject to

$$E\left[\int_{-\infty}^{+\infty} u\left(H+\sqrt{y(\overline{e})}\xi\right)n(\xi)d\xi\right] \geq \overline{u},$$

where the IR constraint is evaluated at $\overline{e}$. The solution is a constant (Lagrange multiplier) λ and a function $H(y(\overline{e}),\lambda)$ which solve the system of equations

$$e^{-\int_0^T r_v dv} = \lambda\int_{-\infty}^{+\infty} u'\left(H+\sqrt{y(\overline{e})}\xi\right)n(\xi)d\xi$$

$$E\int_{-\infty}^{+\infty} u\left(H+\sqrt{y(\overline{e})}\xi\right)n(\xi)d\xi = \overline{u}.$$

Note that a solution $\lambda, H(y(\overline{e}),\lambda)$ exists and is unique. Furthermore, $H(y(\overline{e}),\lambda)$ is an increasing function of λ which satisfies the limits

$$\lim_{\lambda\uparrow\infty} H(y(\overline{e}),\lambda) = \infty \text{ and } \lim_{\lambda\downarrow 0} H(y(\overline{e}),\lambda) = -\infty.$$

Let $\overline{\lambda}$ denote the unique solution of the IR constraint. The facts that $H(\cdot,\cdot)$ is continuous, $y(\overline{e})$ is bounded (i.e. $\overline{e}$ is bounded), combined with the results from Lemma A-2 (and the fact that the function $E\int_{-\infty}^{+\infty} u\left(H+\sqrt{y(\overline{e})}\xi\right)n(\xi)d\xi$ is an increasing function of H) yield uniform bounds on $\overline{\lambda}$. We conclude that $\overline{H}\equiv H(y(\overline{e}),\overline{\lambda})$ is uniformly bounded. From the optimality of $\overline{H}$ and lemma A3, we have $J(\overline{e},\overline{H}) \geq J(\overline{e},H) \geq J(e,H)$. ■

We now complete the proof of Theorem 4. Given the bound in lemma A4 we can restrict attention to maximizing sequences (see Ekeland and Turnbull (1983)) $(e^n,H^n)\in\mathcal{A}$ such that

$$J(e^n,H^n) \longrightarrow \sup_{(e,H)\in\mathcal{A}} J(e,H)$$

and

$$\left(E\left[\int_0^T (e_v^n)^2 dv\right]\right)^{1/2} + \left(E[(H^n)^2]\right)^{1/2} < K, \text{ for some constant } K \tag{31}$$

and

$$E\left[u\left(H^n + \sqrt{y(e^n)}\,\widetilde{\xi}\right)\right] = \overline{u}. \tag{32}$$

Moreover, without loss of generality, we can focus on increasing maximizing sequences (i.e. sequences $(e^n, H^n) \in \mathcal{A}$ such that $J(e^n, H^n)$ increases). With this $L^2[0,T] \times L^2$-norm we have a Hilbert space with inner product

$$\langle (e_1, H_1), (e_2, H_2) \rangle = E\left[\int_0^T e_{1v} e_{2v} dv\right] + E[H_1 H_2].$$

Let $\mathcal{B}^K$ denote the set of pairs (e, H) satisfying (31)-(32). The closure of the convex hull $\overline{c}_0(\mathcal{A} \cap \mathcal{B}^K)$ is norm bounded, convex and norm closed, hence, weakly compact by Alaoglu's theorem (see Conway (1990), theorem V.1.4 and V.3.1). From lemmas A2-A4 we can take (e^n, H^n) to satisfy uniform bounds. Thus,

$$(e^n, H^n) \xrightarrow{w} (e^*, H^*) \in \overline{c}_0(\mathcal{A} \cap \mathcal{B}^K)$$

(weak convergence) and there exist convex combinations

$$(e_n^\alpha, H_n^\alpha) = \sum_{i=n}^{N_n} \alpha_{n,i} (e^i, H^i)$$

where $\sum_{i=n}^{N_n} \alpha_{n,i} = 1$, such that (see Conway, exercise V.1.7)

$$(e_n^\alpha, H_n^\alpha) \xrightarrow{\|\|} (e^*, H^*),$$

(convergence in norm), hence $a.s.\ l \otimes P \times P$, along some subsequence (see Ekeland and Turnbull (1983, Theorem II.3). Moreover, (e^*, H^*) must in fact be bounded.

Concavity of the objective function implies

$$\sum_{i=n}^{N_n} \alpha_{n,i} J(e^i, H^i) \leq J\left(\sum_{i=n}^{N_n} \alpha_{n,i}(e^i, H^i)\right)$$

so that

$$\overline{\lim_n} \sum_{i=n}^{N_n} \alpha_{n,i} J(e^i, H^i) \leq \overline{\lim_n} J\left(\sum_{i=n}^{N_n} \alpha_{n,i}(e^i, H^i)\right) = \lim_{n\to\infty} J\left(\sum_{i=n}^{N_n} \alpha_{n,i}(e^i, H^i)\right) = J(e^*, H^*).$$

(The inequality, on the left hand side, follows from concavity and the next two equalities, on the right hand side, from dominated convergence). But

$$\sum_{i=n}^{N_n} \alpha_{n,i} J(e^i, H^i) \geq \inf_{k \geq n} J(e^k, H^k) = J(e^n, H^n),$$

where the equality follows since the sequence $J(e^k, H^k)$ is increasing, and therefore, taking the inf of the sup on each side,

$$\begin{aligned}\inf_{n_0} \sup_{n \geq n_0} \left(\sum_{i=n}^{N_n} \alpha_{n,i} J(e^i, H^i) \right) &\geq \inf_{n_0} \sup_{n \geq n_0} J(e^n, H^n) \\ &= \lim_{n \to \infty} J(e^n, H^n).\end{aligned}$$

We conclude that $J(e^*, H^*) \geq \lim_{n\to\infty} J(e^n, H^n)$ (which equals $\sup_{(e,H)\in\mathcal{A}} J(e, H)$).

To complete the proof we need to show that (e^*, H^*) is individual rational. But by concavity of u, convexity of f^2 and the fact that the function $\int_{-\infty}^{+\infty} u\left(H + \sqrt{y}\xi\right) n(\xi) d\xi$ is decreasing in y (see Lemma A.5 below), we obtain

$$\begin{aligned}\overline{u} &= \sum_{i=n}^{N_n} \alpha_{n,i} E\left[u\left(H^i + \sqrt{y(e^i)\widetilde{\xi}}\right)\right] \\ &\leq E\left[u\left(\sum_{i=n}^{N_n} \alpha_{n,i} H^i + \sqrt{y\left(\sum_{i=n}^{N_n} \alpha_{n,i} e^i\right)\widetilde{\xi}}\right)\right].\end{aligned}$$

By dominated convergence, using assumption 2(iii),

$$\lim_{n\to\infty} E\left[u\left(\sum_{i=n}^{N_n} \alpha_{n,i} H^i + \sqrt{y(\sum_{i=n}^{N_n} \alpha_{n,i} e^i)\widetilde{\xi}}\right)\right] = E\left[u\left(H^* + \sqrt{y(e^*)\widetilde{\xi}}\right)\right].$$

Thus,

$$\overline{u} \leq E\left[u\left(H^* + \sqrt{y(e^*)\widetilde{\xi}}\right)\right].$$

This completes the proof of the theorem. ■

Lemma A5: *The agent's expected utility function*

$$E\left[\int_{-\infty}^{+\infty} u\left(H^\varepsilon + \sqrt{y(\overline{e}^\varepsilon)}\xi\right) n(\xi) d\xi\right]$$

is decreasing in the conditional standard deviation $\sqrt{y}$*, and concave in* $\overline{e}$*.*

Proof of Lemma A5: Let $y(\overline{e}^\varepsilon) = y$, a constant. Taking the derivative with respect to $\sqrt{y}$ and using integration by parts gives

$$E\left[\int_{-\infty}^{+\infty} u'\left(H^\varepsilon + \sqrt{y}\xi\right)\xi n(\xi) d\xi\right] = \sqrt{y} E\left[\int_{-\infty}^{+\infty} u''\left(H^\varepsilon + \sqrt{y}\xi\right) n(\xi) d\xi\right].$$

Since $u(\cdot)$ is concave we conclude that expected utility is a decreasing function of $\sqrt{y}$.

Assumption 3(iii) on the convexity of $f(e,t) = c'(e,t)/g'(e,t)$ ensures that

$$f(\alpha e_1 + (1-\alpha)e_2, t) \leq \alpha f(e_1, t) + (1-\alpha) f(e_2, t).$$

With the notation $f_1 \equiv f(e_1,t)$ and $f_2 \equiv f(e_2,t)$, this convexity property and the triangle inequality for norms (see Luenberger (1969)) give

$$\begin{aligned}
\sqrt{y(\alpha e_1+(1-\alpha)e_2)} &\equiv \sqrt{\int_0^T f(\alpha e_{1s}+(1-\alpha)e_{2s},s)^2\sigma(s)^2ds} \\
&\leq \sqrt{\int_0^T \left(\alpha f(e_{1s},s)+(1-\alpha)f(e_{2s},s)\right)^2\sigma(s)^2ds} \\
&\leq \sqrt{\int_0^T \left(\alpha f(e_{1s},s)\right)^2\sigma(s)^2ds}+\sqrt{\int_0^T \left((1-\alpha)f(e_{2s},s)\right)^2\sigma(s)^2ds} \\
&= \alpha\sqrt{\int_0^T f(e_{1s},s)^2\sigma(s)^2ds}+(1-\alpha)\sqrt{\int_0^T f(e_{2s},s)^2\sigma(s)^2ds} \\
&= \alpha\sqrt{y(e_1)}+(1-\alpha)\sqrt{y(e_2)}.
\end{aligned}$$

In the second line above, we used the convexity of f; the third line follows from the triangle inequality and the fourth from the fact that α is a constant.

The concavity of the agent's expected utility now follows from the fact that

$$E\left[\int_{-\infty}^{+\infty} u\left(H^\varepsilon+\sqrt{y(\overline{e}^\varepsilon)}\xi\right)n(\xi)d\xi\right]$$

is a decreasing function of $\sqrt{y(\overline{e}^\varepsilon)}$ and that the composition of a decreasing function and a convex function is concave. ■

Derivation of the first order conditions (14)-(16): The principal maximizes

$$J(\overline{e},H)=E\left[\int_0^T g(\overline{e}_v,v)D_vdv\right]-E\left[e^{-\int_0^T r(u)du}\left(\int_0^T c(\overline{e}_v,v)dv+H\right)\right]$$

subject to the individual rationality constraint

$$E\left[\int_{-\infty}^{+\infty} u\left(H+\sqrt{y}\xi\right)n(\xi)d\xi\right]-\overline{u}=0$$

where

$$y=\int_0^T f(\overline{e}_t,t)^2\sigma(t)^2dt.$$

Let λ denote the Lagrange multiplier for the IR constraint and let $\Delta_t^e \equiv \overline{e}'_t-\overline{e}_t$ and $\Delta^H \equiv H'-H$. Taking an $\varepsilon=(\varepsilon_1,\varepsilon_2)$-perturbation in the direction $(\overline{e}'_t,H'_t)$ where $\overline{e}'_t \neq \overline{e}_t$ and $H'_t \neq H_t$ yields the first order conditions

$$\begin{aligned}
0 &= E\left[\int_0^T D_vg'(\overline{e}_v,v)\Delta_v^edv\right]-E\left[e^{-\int_0^T r(u)du}\left(\int_0^T c'(\overline{e}_v,v)\Delta_v^edv\right)\right] \\
&\quad +\lambda E\left[\int_{-\infty}^{+\infty} u'\left(H+\sqrt{y}\xi\right)\frac{\xi}{\sqrt{y}}\left(\int_0^T f(\overline{e}_v,v)f'(\overline{e}_v,v)\sigma(v)^2\Delta_v^edv\right)n(\xi)d\xi\right]
\end{aligned}$$

$$0 = -E\left[e^{-\int_0^T r(u)du}\Delta^H\right] + \lambda E\left[\int_{-\infty}^{+\infty} u'\left(H + \sqrt{y}\xi\right)\Delta^H n(\xi)d\xi\right]$$

$$E\left[\int_{-\infty}^{+\infty} u\left(H + \sqrt{y}\xi\right)n(\xi)d\xi\right] - \overline{u} = 0$$

where $\lambda > 0$. Since this must hold for all progressively measurable processes $\overline{e}'_t \neq \overline{e}_t$, (i.e. for all progressively measurable Δ^e) and for all $\mathcal{F}_T^Z$-measurable random variables $H' \neq H$ (i.e. all measurable Δ^H) it must be that

$$\begin{aligned} 0 &= D_v g'(\overline{e}_v, v) - E_v\left(e^{-\int_0^T r(v)dv}\right)c'(\overline{e}_v, v) \\ &\quad + \lambda E_v\left(\int_{-\infty}^{+\infty} u'\left(H + \sqrt{y}\xi\right)\frac{\xi}{\sqrt{y}}n(\xi)d\xi\right) f(\overline{e}_v, v) f'(\overline{e}_v, v)\sigma(v)^2 \end{aligned}$$

$$0 = -e^{-\int_0^T r(v)dv} + \lambda\int_{-\infty}^{+\infty} u'\left(H + \sqrt{y}\xi\right)n(\xi)d\xi$$

$$E\left[\int_{-\infty}^{+\infty} u\left(H + \sqrt{y}\xi\right)n(\xi)d\xi\right] - \overline{u} = 0$$

where the first two equations must hold for all $v \in [0, T]$. ■

Proof of Theorem 5: Suppose that $(\overline{e}, H)$ solves (14)-(17) and let $(\overline{e}', H') \in \mathcal{A}$ denote an alternative admissible policy.

Concavity of the production function $g(\cdot)$ and convexity of the cost function $c(\cdot)$ (i.e. concavity of $-c(\cdot)$) imply that

$$g(\overline{e}_s, s) \geq g(\overline{e}'_s, s) + g'(\overline{e}_s, s)(\overline{e}_s - \overline{e}'_s) \quad \text{for all } s \in [0, T]$$

$$-c(\overline{e}_s, s) \geq -c(\overline{e}'_s, s) - c'(\overline{e}_s, s)(\overline{e}_s - \overline{e}'_s) \quad \text{for all } s \in [0, T]$$

or, multiplying the first inequality by D_s and the second one by $e^{-\int_0^T r(v)dv}$ gives

$$D_s g(\overline{e}_s, s) \geq D_s g(\overline{e}'_s, s) + D_s g'(\overline{e}_s, s)(\overline{e}_s - \overline{e}'_s) \quad \text{for all } s \in [0, T]$$

$$-e^{-\int_0^T r(v)dv}c(\overline{e}_s, s) \geq -e^{-\int_0^T r(v)dv}c(\overline{e}'_s, s) - e^{-\int_0^T r(v)dv}c'(\overline{e}_s, s)(\overline{e}_s - \overline{e}'_s) \quad \text{for all } s \in [0, T].$$

Let $\Delta_s^{\overline{e}} \equiv \overline{e}_s - \overline{e}'_s$ and $\Delta^H \equiv H - H'$. Integrating over the product measure $P \otimes l$ (where l is Lebesgue measure) gives,

$$
\begin{aligned}
J(\overline{e}, H) &= E\left[\int_0^T g(\overline{e}_s, s) D_s ds\right] - E\left[e^{-\int_0^T r(v)dv}\int_0^T c(\overline{e}_s, s)ds + e^{-\int_0^T r(v)dv} H\right] \\
&\geq E\left[\int_0^T g(\overline{e}'_s, s) D_s ds\right] - E\left[e^{-\int_0^T r(v)dv}\int_0^T c(\overline{e}'_s, s)ds + e^{-\int_0^T r(v)dv} H'\right] \\
&\quad + E\left[\int_0^T D_s g'(\overline{e}_s, s)\Delta_s^{\overline{e}} ds\right] \\
&\quad - E\left[e^{-\int_0^T r(v)dv}\int_0^T c'(\overline{e}_s, s)\Delta_s^{\overline{e}} ds + e^{-\int_0^T r(v)dv}\Delta^H\right] \\
&= J(\overline{e}', H') + E\left[\int_0^T \left(D_s g'(\overline{e}_s, s) - e^{-\int_0^T r(v)dv} c'(\overline{e}_s, s)\right)\Delta_s^{\overline{e}} ds - e^{-\int_0^T r(v)dv}\Delta^H\right] \\
&= J(\overline{e}', H') \\
&\quad + E\left[\int_0^T \left(D_s g'(\overline{e}_s, s) - E_s\left(e^{-\int_0^T r(v)dv}\right) c'(\overline{e}_s, s)\right)\Delta_s^{\overline{e}} ds - e^{-\int_0^T r(v)dv}\Delta^H\right].
\end{aligned}
$$

Using (14) enables us to write

$$
\begin{aligned}
J(\overline{e}, H) &\geq J(\overline{e}', H') \\
&\quad - E\int_0^T \left(\mu_s + \lambda E_s\left(\int_{-\infty}^{+\infty} u'\left(H + \sqrt{y}z\right)\frac{z}{\sqrt{y}} n(z)dz\right) f(\overline{e}_s, s) f'(\overline{e}_s, s)\sigma(s)^2\right)\Delta_s^{\overline{e}} ds \\
&\quad - E e^{-\int_0^T r(v)dv}\Delta^H. \qquad (33)
\end{aligned}
$$

Note that the function $u\left(H + \sqrt{y}\xi\right)$ is concave with respect to $y \equiv \int_0^T f(\overline{e}_s, s)^2\sigma(s)^2 ds$. The assumption that $f(\overline{e}, s)^2$ is convex with respect to $\overline{e}$ implies that y is a convex function of effort $\overline{e}$. Since $\int_{-\infty}^{+\infty} u\left(H + \sqrt{y}\xi\right) n(\xi)d\xi$ is also decreasing in y it follows that it is concave in the pair $(\overline{e}, H)$. We can then write

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} u\left(H + \sqrt{y}\xi\right) n(\xi)d\xi \\
\geq\; &\int_{-\infty}^{+\infty} u\left(H' + \sqrt{y'}\xi\right) n(\xi)d\xi + \int_{-\infty}^{+\infty} u'\left(H' + \sqrt{y'}\xi\right) n(\xi)d\xi\Delta^H \\
&+ \left(\int_{-\infty}^{+\infty} u'\left(H + \sqrt{y}\xi\right)\frac{\xi}{\sqrt{y}} n(\xi)d\xi\right)\left(\int_0^T f(\overline{e}_s, s) f'(\overline{e}_s, s)\sigma(s)^2\Delta_s^{\overline{e}} ds\right)
\end{aligned}
$$

where we used the $\mathcal{F}_T^Z$-measurability of H, H' in the second line and the $\mathcal{F}_{(\cdot)}^Z$-progressive measurability of $\overline{e}, \overline{e}'$ in the third line. Multiplying both sides of the inequality by λ and using the fact that the expression $\int_{-\infty}^{+\infty} u'\left(H + \sqrt{y}\xi\right)\frac{\xi}{\sqrt{y}} n(\xi)d\xi$ on the third line does not depend on the integrator s enables us to write

$$
\begin{aligned}
& \lambda \int_{-\infty}^{+\infty} u\left(H+\sqrt{y}\xi\right) n(\xi) d\xi \\
\geq\; & \lambda \int_{-\infty}^{+\infty} u\left(H'+\sqrt{y'}\xi\right) n(\xi) d\xi + \lambda \int_{-\infty}^{+\infty} u'\left(H+\sqrt{y}\xi\right) n(\xi) d\xi \Delta^H \\
& + \lambda \int_0^T \left(\int_{-\infty}^{+\infty} u'\left(H+\sqrt{y}\xi\right) \frac{\xi}{\sqrt{y}} n(\xi) d\xi \right) f(\overline{e}_s, s) f'(\overline{e}_s, s) \sigma(s)^2 \Delta_s^{\overline{e}} ds.
\end{aligned}
$$

Since condition (15) implies $\lambda \int_{-\infty}^{+\infty} u'\left(H+\sqrt{y}\xi\right) n(\xi) d\xi = e^{-\int_0^T r(v)dv}$ we then obtain

$$
\begin{aligned}
& \lambda \int_{-\infty}^{+\infty} u\left(H+\sqrt{y}\xi\right) n(\xi) d\xi \\
\geq\; & \lambda \int_{-\infty}^{+\infty} u\left(H'+\sqrt{y'}\xi\right) n(\xi) d\xi + e^{-\int_0^T r(v)dv} \Delta^H \\
& + \lambda \int_0^T \left(\int_{-\infty}^{+\infty} u'\left(H+\sqrt{y}\xi\right) \frac{\xi}{\sqrt{y}} n(\xi) d\xi \right) f(\overline{e}_s, s) f'(\overline{e}_s, s) \sigma(s)^2 \Delta_s^{\overline{e}} ds.
\end{aligned}
$$

Taking unconditional expectations on each side of this inequality, using the $\mathcal{F}_s^Z$-measurability of $\overline{e}_s, \overline{e}'_s$ and rearranging the terms then gives

$$
\begin{aligned}
& \lambda E \int_{-\infty}^{+\infty} u\left(H+\sqrt{y}\xi\right) n(\xi) d\xi \\
\geq\; & \lambda E \int_{-\infty}^{+\infty} u\left(H'+\sqrt{y'}\xi\right) n(\xi) d\xi + E e^{-\int_0^T r(v)dv} \Delta^H \\
& + \lambda E \int_0^T E_s \left(\int_{-\infty}^{+\infty} u'\left(H+\sqrt{y}\xi\right) \frac{\xi}{\sqrt{y}} n(\xi) d\xi \right) f(\overline{e}_s, s) f'(\overline{e}_s, s) \sigma(s)^2 \Delta_s^{\overline{e}} ds.
\end{aligned}
$$

Since H satisfies (16) and since H' satisfies the IR constraint (by admissibility) this becomes

$$
\begin{aligned}
\overline{u} \;\geq\; & \overline{u} + E e^{-\int_0^T r(v)dv} \Delta^H \\
& + \lambda E \int_0^T E_s \left(\int_{-\infty}^{+\infty} u'\left(H+\sqrt{y}\xi\right) \frac{\xi}{\sqrt{y}} n(\xi) d\xi \right) f(\overline{e}_s, s) f'(\overline{e}_s, s) \sigma(s)^2 \Delta_s^{\overline{e}} ds
\end{aligned}
$$

or,

$$
-\lambda E \int_0^T E_s \left(\int_{-\infty}^{+\infty} u'\left(H+\sqrt{y}\xi\right) \frac{\xi}{\sqrt{y}} n(\xi) d\xi \right) f(\overline{e}_s, s) f'(\overline{e}_s, s) \sigma(s)^2 \Delta_s^{\overline{e}} ds - E e^{-\int_0^T r(v)dv} \Delta^H \geq 0
$$

Substituting in (33) then produces

$$
J(\overline{e}, H) \geq J(\overline{e}', H') - E \int_0^T \mu_s \Delta_s^{\overline{e}} ds.
$$

The complementary slackness condition (17) stipulates that $\mu_s \overline{e}_s = 0$ for all $s \in [0, T]$. Admissibility of $\overline{e}'$ and, again, condition (17) imply $\mu_s \overline{e}'_s \geq 0$ for all $s \in [0, T]$. We conclude that

$$J(\overline{e}, H) \geq J(\overline{e}', H') + E \int_0^T \mu_s \overline{e}'_s ds \geq J(\overline{e}', H')$$

i.e. $(\overline{e}, H)$ is optimal for the principal. ■

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