# PRICING AMERICAN OPTIONS UNDER STOCHASTIC VOLATILITY

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Abstract. This paper presents an extension of McKean's (1965) incomplete Fourier transform method to solve the two-factor partial differential equation for the price and early exercise surface of an American call option, in the case where the volatility of the underlying evolves randomly. The Heston (1993) square-root process is used for the volatility dynamics. The price is given by an integral equation dependent upon the early exercise surface, using a free boundary approximation that is linear in volatility. By evaluating the pricing equation along the free surface boundary, we provide a corresponding integral equation for the early exercise region. An algorithm is proposed for solving the integral equation system, based upon numerical integration techniques for Volterra integral equations. The method is implemented, and the resulting prices are compared with the constant volatility model. The computational efficiency of the numerical integration scheme is also considered.

**Keywords:** American options, stochastic volatility, Volterra integral equations, free boundary problem.

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## 1. Introduction

In this paper we seek to generalise the constant volatility analysis of American option pricing to the stochastic volatility case. We focus on Heston's (1993) square root model, and use the free surface series expansion of Tzavalis & Wang (2003) to produce analytic approximations for the early exercise surface and price of an American call option. We derive these results using the incomplete Fourier transform method of McKean (1965), accompanied by a simplification presented by Jamshidian (1992). In this way we demonstrate how Heston's results for the European call, there derived using characteristic functions, may also be found by means of Fourier transforms. We then proceed to numerically solve the resulting integral equation system for the free surface using an iterative method involving numerical integration, based on the techniques considered by Kallast & Kivinukk (2003) and Chiarella & Ziogas (2005). We consider the efficiency of such a solution method, and demonstrate the impact that stochastic volatility has on American options relative to those priced under the Black-Scholes model.

Stochastic volatility models for pricing derivative securities have been developed as an extension to the original, constant volatility model of Black & Scholes (1973). Many studies of market prices for option contracts have consistently found that implied volatilities vary with respect to both maturity and the moneyness of the option. Directly modelling the volatility of the underlying using an additional stochastic process provides one means by which this feature can be incorporated into the Black-Scholes pricing framework.

There are a number of methods one can use to model volatility stochastically. Hull & White (1987) model the variance using a geometric Brownian motion, as well as an Ornstein-Uhlenbeck process with mean-reversion related to the volatility. In the general case, mean-reversion is considered to be an essential feature of observed volatility, and thus all plausible models are of the Ornstein-Uhlenbeck type. Wiggins (1987) models the logarithm of the volatility with mean-reversion, whereas Scott (1987), Johnson & Shanno (1987), Heston (1993) and Stein & Stein (1991) model the variance using a square root process. Zhu (2000) also considers a double square root process, which is an extension of the basic square root process in which both the drift and diffusion coefficients involve the volatility. In this paper we focus on Heston's square root model, under which Heston (1993) provides an analytic expression for European option prices.

Pricing American options under stochastic volatility is a much more complicated task. In the constant volatility case, it is well known that the price of an American call option can be decomposed into the sum of a corresponding European call and an early exercise premium term. Kim (1990), Jacka (1991) and Carr, Jarrow & Myneni (1992) demonstrate this result using a variety of different approaches. The American call price takes the form of an integral equation involving the unknown early exercise boundary. By evaluating this equation at the free boundary, a corresponding integral equation for the early exercise condition is produced.

Since the cash flows arising from early exercise are typically independent of the volatility, generalising American option pricing theory from the constant volatility case to stochastic volatility is relatively straightforward. Lewis (2000) indicates that the free boundary becomes a two-dimensional free surface, in which the early exercise value of the underlying is a function of time to maturity and the volatility level. Touzi (1999) proves a number of fundamental properties for the free boundary and option price under stochastic volatility for the American put example, focusing on how the surface changes with respect to the volatility.

It becomes impossible, however, to derive analytic integral equations for the option price and free boundary under stochastic volatility without the use of asymptotic expansions. One such example is presented by Fouque, Papanicolaou & Sircar (2000), where they use a series expansion based around the square-root of the rate of mean reversion for the volatility. This method provides an adjustment to the Black-Scholes solution, taking advantage of the fact that the mean reversion feature dominates the volatility dynamics over a sufficient time period. The downside to this approach is that the expansion is poor close to expiry and near the early exercise surface, since the mean reversion will not dominate the volatility dynamics in these cases.

Since analytic expressions cannot be found without the use of assumptions or asymptotics, the most common solution methods in the literature involve a discrete grid or lattice. These are typically used to estimate the distribution of the underlying asset, or to numerically solve the partial differential equation for the option price. Using a GARCH model for the stochastic volatility price process, Ritchken & Trevor (1999) and Cakici & Topyan (2000) develop lattice methods for pricing European and American options, although such methods tend to provide poor free surface estimates.

Ikonen & Toivanen (2004) solve the second order PDE for the American put option under stochastic volatility using several implicit finite difference schemes for the space variables. They apply an operator splitting technique for the time component, and the method is easier to compute than fully implicit schemes. Multigrid methods are a related technique which involve solving a problem using a finite difference scheme on successively coarser grids to better control the error, and then interpolating the results back up to the original fine grid. The advantage is that less computation is required to solve the problem for a high level of accuracy, and this is of particular interest for multidimensional problems. Clarke & Parrott (1999), Oosterlee (2003) and Reisinger & Wittum (2004) all provide applications of these methods to option pricing under stochastic volatility. Furthermoe, Zvan, Forsyth & Vetzal (1998) solve the problem using a hybrid method, involving finite elements for the diffusion terms and finite volume for the convection terms.

There is currently very little work in the existing literature that deals with trying to find analytic integral equations for American options under stochastic volatility. One rare example is presented by Tzavalis & Wang (2003), in which they approximate the logarithm of the early exercise boundary using a Taylor series expansion around the long-run volatility. They cite the empirical findings of Broadie, Detemple, Ghysels & Torrès (2000) as justification for this assumption, and this leads them to a generalisation of Kim's (1990) expression for the early exercise premium, with an analytical form that generalises Heston's (1993) results for the European call option. To solve the resulting integral equation system, Tzavalis & Wang (2003) approximate the free boundary using Chebyshev polynomials, and are able to produce a fast approximation that leads to accurate option prices. In this paper we propose an iterative numerical integration scheme for estimating the free boundary, based on solution techniques typically applied to Volterra equations.

The remainder of this paper is structured as follows. Section 2 outlines the pricing problem for an American call option under Heston's square root process. Section 3 uses the Fourier transform method to provide a general solution to the problem, and shows that an analytic representation is not possible without making some assumption about the nature of the free boundary. Using the series expansion of Tzavalis & Wang (2003), we proceed to find an analytic integral equation for the option price in Section 4. A linked integral equation system for the early exercise boundary is provided in Section 5, along with a discussion on how this system may be solved using iterative techniques. Section 6 presents some numerical examples based on numerical integration techniques, along with an analysis of the impact that stochastic volatility has on the price and free boundary. Concluding remarks are presented in Section 7. Most of the lengthy mathematical derivations are given in appendices.

## 2. Problem Statement - the Heston Model

Let $C_A(S, v, \tau)$ be the price of an American call option written on $S$ with time to expiry $\tau$ and strike price $K$. We assume that the dynamics of $S$ are given by the stochastic differential equation

$$dS = \mu S dt + \sqrt{v} S dZ_1, \tag{1}$$

where $\mu$ is the instantaneous return per unit time, $v$ is the instantaneous squared volatility per unit time, and $Z_1$ is a standard Wiener process. We allow $v$ to also evolve stochastically, using the square root process of Heston (1993). The dynamics for $v$ are

$$dv = \kappa[\theta - v]dt + \sigma\sqrt{v} dZ_2, \tag{2}$$

where $\theta$ is the long-run mean for $v$, $\kappa$ is the rate of mean reversion, $\sigma$ is the instantaneous volatility of $v$ per unit time, and $Z_2$ is a standard Wiener process correlated with $Z_1$

such that

$$\mathbb{E}[dZ_1 dZ_2] = \rho dt.$$

Let $r$ be the risk-free rate of interest, and assume that $S$ pays a continuously compounded dividend yield at rate $q$, Furthermore, we shall assume that the market price of volatility risk is equal to $\lambda v$, as per Heston (1993). Using standard hedging arguments and an application of Ito's lemma, it can be shown that the $C$ satisfies the partial differential equation (PDE)

$$\begin{aligned}\frac{\partial C_A}{\partial \tau} = \frac{vS^2}{2}\frac{\partial^2 C_A}{\partial S^2} + \rho\sigma v S\frac{\partial^2 C_A}{\partial S \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 C_A}{\partial v^2} + (r-q)S\frac{\partial C_A}{\partial S} \\ + (\kappa[\theta - v] - v\lambda)\frac{\partial C_A}{\partial v} - rC_A, \end{aligned} \quad (3)$$

in the region $0 \leq \tau \leq T, \quad 0 < S \leq b(v,\tau)$, and $0 \leq v < \infty$, where $b(v,\tau)$ denotes the early exercise boundary at time $\tau$ and volatility level $v$. The initial and boundary conditions for (3) in the case of an American call option are

$$C_A(S, v, 0) = \max(S - K, 0), \quad (4)$$

$$C_A(b(v,\tau), v, \tau) = b(v,\tau) - K, \quad (5)$$

$$\lim_{S\to b(v,\tau)} \frac{\partial C_A}{\partial S} = 1, \quad (6)$$

$$\lim_{S\to b(v,\tau)} \frac{\partial C_A}{\partial v} = 0. \quad (7)$$

Condition (4) is the payoff for the call. The additional boundary conditions are provided by Fouque et al. (2000), and they generalise the American call problem in the case of stochastic volatility. Equation (5) is the value-matching condition, and equations (6)-(7) are the smooth-pasting conditions. These collectively ensure that $C$, $\partial C/\partial S$ and $\partial C/\partial v$ will be continuous, thus preventing arbitrage. Figure 1 demonstrates the payoff, price profile and early exercise boundary for the American call under consideration.

The first step towards finding a solution to the free boundary value problem (3)-(7) involves a change of variable. Let $S \equiv e^x$, with $U_A(x,v,\tau) \equiv e^{r\tau}C_A(S,v,\tau)$. It is

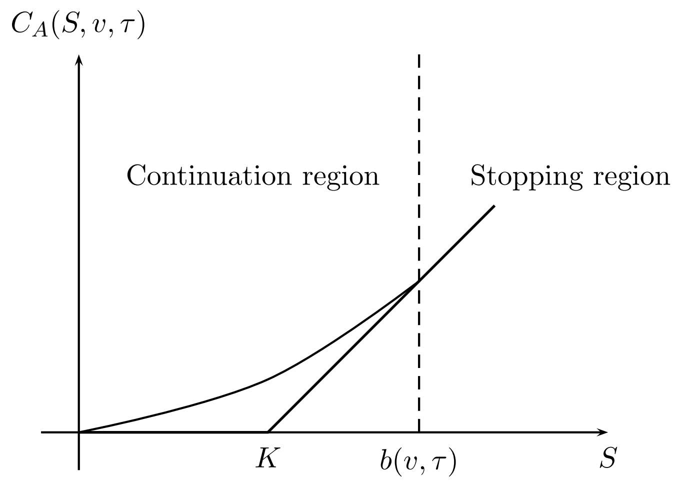


FIGURE 1. Continuation region for the American call option, for a given value of $v$.

straightforward to show that the PDE becomes

$$\frac{\partial U_A}{\partial \tau} = \frac{v}{2}\frac{\partial^2 U_A}{\partial x^2} + \rho\sigma v \frac{\partial^2 U_A}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 U_A}{\partial v^2} + \left(r - q - \frac{v}{2}\right)\frac{\partial U_A}{\partial x} + (\alpha - \beta v)\frac{\partial U_A}{\partial V}, \quad (8)$$

which must now be solved in the region $0 \leq \tau \leq T$, $-\infty < x < \ln b(v, \tau)$, and $0 \leq v < \infty$, where $\alpha \equiv \kappa\theta$ and $\beta \equiv \kappa + \lambda$. The initial and boundary conditions become

$$U_A(x, v, 0) = \max(e^x - K, 0), \quad (9)$$

$$U_A(\ln b(v, \tau), v, \tau) = (b(v, \tau) - K)e^{r\tau} \quad (10)$$

$$\lim_{x \to \ln b(v,\tau)} \frac{\partial U_A}{\partial x} = b(v, \tau)e^{r\tau}, \quad (11)$$

$$\lim_{x \to \ln b(v,\tau)} \frac{\partial U_A}{\partial v} = 0. \quad (12)$$

While it is possible to solve the free boundary value problem (8)-(12) using the incomplete Fourier transform method of McKean (1965), there is a more elegant approach available. Jamshidian (1992) shows that in the case of constant volatility, there exists an inhomogenous PDE in the unrestricted domain $-\infty < y < \infty$ that is equivalent to solving the Black-Scholes PDE for the American call price in a restricted domain $-\infty < y < b(v, \tau)$. We can derive a similar result for the stochastic volatility case using McKean's incomplete Fourier transform.

Define the Fourier transform of $U(x, v, \tau)$ with respect to $x$ as

$$\mathcal{F}\{U(x, v, \tau)\} = \int_{-\infty}^{\infty} U(x, v, \tau) e^{i\phi x} dx = \hat{U}(\phi, v, \tau), \tag{13}$$

with inversion

$$\mathcal{F}^{-1}\{\hat{U}(\phi, v, \tau)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}(\phi, v, \tau) e^{-i\phi x} dx = U(x, v, \tau). \tag{14}$$

Furthermore, we define the Heaviside step function, $H(x)$ as

$$H(x) = \begin{cases} 1, & x > 0, \\ \frac{1}{2}, & x = 0, \\ 0, & x < 0. \end{cases} \tag{15}$$

We can use these definitions to derive the Jamshidian (1992) formulation of the free boundary value problem.

**Proposition 2.1.** *The free boundary value problem (8)-(12) for $U(x, v, \tau)$ in the restricted domain $-\infty < x < b(v, \tau)$ is equivalent to the inhomogeneous PDE*

$$\begin{aligned} \frac{\partial U_A}{\partial \tau} &= \frac{v}{2} \frac{\partial^2 U_A}{\partial x^2} + \rho \sigma v \frac{\partial^2 U_A}{\partial x \partial v} + \frac{\sigma^2 v}{2} \frac{\partial^2 U_A}{\partial v^2} + \left(r - q - \frac{v}{2}\right) \frac{\partial U_A}{\partial x} \\ &\quad + (\alpha - \beta v) \frac{\partial U_A}{\partial v} + H(x - \ln b(v, \tau))\{e^{r\tau}(q e^x - rK)\}, \end{aligned} \tag{16}$$

*solved in the unrestricted domain $-\infty < x < \infty$, $0 < v < \infty$, $0 \leq \tau \leq T$, subject to the initial condition (9), where $H(x)$ is the Heaviside step function defined by (15). Note that the boundary conditions (10)-(12) still apply.*

**Proof:** Refer to Appendix 1.

□

It should be noted that the inhomogeneous term represents the cash flows received by a portfolio composed of a long position in one unit of $S$, and a loan of $K$ borrowed at the risk-free rate. This portfolio arises whenever the option holder exercises the American call early, and hence the cash flows only arise when $x > \ln b(v, \tau)$. Thus $qe^x - rK$

represents the net income from dividends earned and interest paid for the portfolio $S - K$.

## 3. General Solution using Fourier Transforms

Since the PDE (16) is an inhomogeneous variant of the PDE presented by Heston (1993) for the European call, we will consider a solution that includes Heston's result. Let $U(x, v, \tau)$ denote the Heston solution for the corresponding European call option, generalised to allow for a continuous dividend yield at rate $q$. We guess a solution to (16) of the form

$$U_A(x, v, \tau) = U(x, v, \tau) + V(x, v, \tau), \tag{17}$$

where we define

$$V(x, v, \tau) \equiv \int_0^{\tau} \int_0^{\infty} \int_{-\infty}^{\infty} H(u - \ln(b(w, \xi)))(qe^u - rK)e^{r\xi} F(x, v, \tau - \xi; u, w) du dw d\xi, \tag{18}$$

for the yet to be determined function $F(x, v, \tau - \xi; u, w)$.

Before proceeding, we provide some motivation for this "guess". Given that the payoff for the American call does not explicitly depend upon $v$, it is reasonable to assert that the American call price under both constant and stochastic volatility can be decomposed into a European component, here $U(x, v, \tau)$, plus an early exercise premium, in this case $V(x, v, \tau)$, as per the results of Kim (1990). $U$ is the complimentary function for the solution $U_A$, since it satisfies the homogeneous PDE (8) in the domain $-\infty < x < \infty$.

$V$ is the particular integral component of solution $U_A$. The form of $V$ is a generalization of the constant volatility case, in which the early exercise premium is given by the expected present value of $(qS - rK)$ in the stopping region, for all future times until expiry, $(\tau - \xi)$. The main adjustment for stochastic volatility addresses the fact that the free boundary depends upon $v$. Thus we must consider all possible values of $v$, and infer that $F$ will be the joint transition density for $x$ and $v$. Further motivation is provided by Tzavalis & Wang (2003) who analyse this problem using discounted expectations. Thus our objective is to determine the functions $U$ and $V$. The $U$ function has already been provided by Heston (1993). To find $V$ we will need to determine the functional form of $F$.

Substituting $U_A(x,v,\tau)$ from (17) into the PDE (16), we have

$$\begin{aligned}
&\frac{\partial U}{\partial \tau} + \int_0^\infty \int_{-\infty}^\infty H(u - \ln(b(w,\tau)))(qe^u - rK)e^{r\tau} F(x,v,0;u,w)dudw \\
&\qquad + \int_0^\tau \int_0^\infty \int_{-\infty}^\infty H(u - \ln(b(w,\xi)))(qe^u - K)e^{r\xi} F(x,v,\tau-\xi;u,w)dudwd\xi \\
&= \frac{v}{2}\frac{\partial^2 U}{\partial x^2} + \rho\sigma v \frac{\partial^2 U}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 U}{\partial v^2} + \left(r - q - \frac{v}{2}\right)\frac{\partial U}{\partial x} + (\alpha - \beta v)\frac{\partial U}{\partial v} \\
&\qquad + \int_0^\tau \int_0^\infty \int_{-\infty}^\infty H(u - \ln(b(w,\xi)))(qe^u - K)e^{r\xi} \\
&\qquad\quad \times \left\{ \frac{v}{2}\frac{\partial^2 F}{\partial x^2} + \rho\sigma v \frac{\partial^2 F}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 F}{\partial v^2} + \left(r - q - \frac{v}{2}\right)\frac{\partial F}{\partial x} + (\alpha - \beta v)\frac{\partial F}{\partial v} \right\} dudwd\xi \\
&\qquad\quad + H(x - \ln(b(v,\tau))e^{r\tau}(qe^x - rK). \qquad (19)
\end{aligned}$$

At this point we have placed no conditions on the function $F$, but we note that if we set

$$F(x,v,0;u,w) = \delta(u-x)\delta(w-v), \qquad (20)$$

where $\delta(x)$ is the Dirac-delta function, then the last term on the right-hand side of (19) will cancel with the first integral term on the left hand side. Equation (19) then becomes

$$\begin{aligned}
&\frac{\partial U}{\partial \tau} - \frac{v}{2}\frac{\partial^2 U}{\partial x^2} - \rho\sigma v \frac{\partial^2 U}{\partial x \partial v} - \frac{\sigma^2 v}{2}\frac{\partial^2 U}{\partial v^2} - \left(r - q - \frac{v}{2}\right)\frac{\partial U}{\partial x} - (\alpha - \beta v)\frac{\partial U}{\partial v} \\
&\quad = -\int_0^\tau \int_0^\infty \int_{-\infty}^\infty H(u - \ln(b(w,\xi)))(qe^u - K)e^{r\xi} \qquad (21) \\
&\qquad \times \left\{ \frac{\partial F}{\partial \tau} - \frac{v}{2}\frac{\partial^2 F}{\partial x^2} - \rho\sigma v \frac{\partial^2 F}{\partial x \partial v} - \frac{\sigma^2 v}{2}\frac{\partial F}{\partial^2 v^2} - \left(r - q - \frac{v}{2}\right)\frac{\partial F}{\partial v} - (\alpha - \beta v)\frac{\partial F}{\partial v} \right\} dudwd\xi
\end{aligned}$$

For equation (21) to hold for general functions $U$ and $F$, it must follow that

$$\frac{\partial U}{\partial \tau} = \frac{v}{2}\frac{\partial^2 U}{\partial x^2} + \rho\sigma v \frac{\partial^2 U}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 U}{\partial v^2} + \left(r - q - \frac{v}{2}\right)\frac{\partial U}{\partial x} + (\alpha - \beta v)\frac{\partial U}{\partial v}, \qquad (22)$$

and

$$\frac{\partial F}{\partial \tau} = \frac{v}{2}\frac{\partial^2 F}{\partial x^2} + \rho\sigma v \frac{\partial^2 F}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 F}{\partial v^2} + \left(r - q - \frac{v}{2}\right)\frac{\partial F}{\partial x} + (\alpha - \beta v)\frac{\partial F}{\partial v}. \qquad (23)$$

The PDE (22) is solved subject to the initial condition (9). As previously stated, $U(x, v, \tau)$ is given by generalising the results of Heston (1993) to include dividends, and can be found using characteristic functions or Fourier transforms. We derive $U$ using Fourier transforms in Appendix 5. Equation (23) must be solved subject to the initial condition (20).

It is possible to use the Fourier transform approach to reduce the dimensionality of (23). If we denote the Fourier transform of $F$ with respect to $x$ as

$$\hat{F}(\phi, v, \tau - \xi; u, w) = \int_{-\infty}^{\infty} e^{i\phi x} F(x, v, \tau - \xi; u, w) dx,$$

then the Fourier transform of (23) is

$$\frac{\partial \hat{F}}{\partial \tau} = \frac{\sigma^2 v}{2} \frac{\partial^2 \hat{F}}{\partial v^2} + (\alpha - (\beta + \rho\sigma i\phi)v) \frac{\partial \hat{F}}{\partial v} - \left( \frac{v}{2}\phi^2 + i \left( r - q - \frac{v}{2} \right) \phi \right) \hat{F}, \tag{24}$$

which must be solved subject to

$$\hat{F}(\phi, v, 0; u, w) = e^{i\phi w} \delta(w - v). \tag{25}$$

At this point we can proceed no further using this approach, as equation (24) can only be solved analytically for certain types of initial conditions. The initial condition (25) does not lead to an analytic solution for $\hat{F}$ because it involves the function $\delta(w - v)$. Thus we cannot find $\hat{F}$ analytically[1] for a general free boundary $b(v, \tau)$.

## 4. Approximate Solution using Taylor Series

Since we are unable to find an analytic expression for $F(x, v, \tau - \xi; u, w)$ for a general early exercise boundary $b(v, \tau)$, we shall consider an approximation for the free boundary suggested by Tzavalis & Wang (2003). Given the empirical findings of Broadie et al. (2000), there is evidence that $\ln b(v, \tau)$ is well approximated by a function that is linear in $v$. If we expand $\ln b(v, \tau)$ in a Taylor series about $\theta$ we can approximate $b(v, \tau)$ according to

$$\ln b(v, \tau) \approx b_0(\tau) + v b_1(\tau), \tag{26}$$

[1]Of course one could solve the PDE (24) numerically for $\hat{F}$, but this offers no benefits over solving (8) numerically, given that the function we ultimately wish to find is $F$, not $\hat{F}$.

where $b_0$ and $b_1$ are functions of $\tau$. We now use Fourier transforms to demonstrate that this approximation will lead to an analytic solution for the early exercise premium $V(x,v,\tau)$.

Firstly, rewrite the early exercise premium (18) as

$$V(x,v,\tau) = \int_0^\tau \int_0^\infty \int_{-\infty}^\infty H(u - \ln b(w,\xi))(qe^u - rK)e^{r\xi} \times \left\{ \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\phi u} \hat{f}_2(x,v,\tau-\xi;\phi,w)d\phi \right\} dudwd\xi$$

where we define

$$\hat{f}_2(x,v,\tau-\xi;\phi,w) \equiv \int_{-\infty}^\infty e^{i\phi u} F(x,v,\tau-\xi;u,w)du. \tag{27}$$

Making the change of integration variable $y = u - \ln b(w,\xi)$ we have

$$V(x,v,\tau) = \int_0^\tau \int_0^\infty \int_{-\infty}^\infty H(y)(qe^{y+\ln b(w,\xi)} - rK)e^{r\xi} \times \left\{ \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\phi(y+\ln b(w,\xi))} \hat{f}_2(x,v,\tau-\xi;\phi,w)d\phi \right\} dydwd\xi.$$

Substituting (26) for $\ln b(v,\tau)$ we find that

$$V(x,v,\tau) = \int_0^\tau \int_0^\infty \int_0^\infty (qe^{y+b_0(\xi)+wb_1(\xi)} - rK)e^{r\xi} \times \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\phi(y+b_0(\xi)+wb_1(\xi))} \hat{f}_2(x,v,\tau-\xi;\phi,w)d\phi dydwd\xi. \tag{28}$$

We now express $V(x,v,\tau)$ in a form that will lead us towards an analytic solution.

**Proposition 4.1.** *The early exercise premium $V(x,v,\tau)$ can be written as*

$$V(x,v,\tau) = I_1(x,v,\tau) - I_2(x,v,\tau), \tag{29}$$

*where*

$$I_1(x,v,\tau) \equiv \int_0^\tau \int_{-\infty}^\infty \frac{qe^{r\xi}e^{-i\phi b_0(\xi)}}{2\pi} \left( \int_0^\infty e^{-i\phi y} dy \right) f_2(x,v,\tau-\xi;\phi - i, -b_1(\xi)\phi) d\phi d\xi, \tag{30}$$

*and*

$$I_2(x,v,\tau) \equiv \int_0^\tau \int_{-\infty}^{\infty} \frac{rKe^{r\xi}e^{-i\phi b_0(\xi)}}{2\pi} \left( \int_0^\infty e^{-i\phi y} dy \right) f_2(x,v,\tau-\xi;\phi,-b_1(\xi)\phi) d\phi d\xi, \tag{31}$$

*with*

$$f_2(x,v,\tau;\phi,\psi) \equiv \int_{-\infty}^{\infty} H(w)\hat{f}_2(x,v,\tau-\xi\phi,w)e^{i\psi w} dw. \tag{32}$$

*Note that $f_2$ is the Fourier transform of $\hat{f}_2$ with respect to $w$.*

**Proof:** Refer to Appendix 2.

□

Thus under the assumption (26) for the free boundary, we are now faced with the task of finding the functional form of $f_2$.

Recall from Section 3 that $F(x,v,\tau-\xi;u,w)$ satisfies the PDE (23) subject to the initial condition (20). If we take the two-dimensional Fourier transform of (23) with respect to $u$ and $w$, which according to equation (32) is

$$f_2(x,v,\tau-\xi;\phi,\psi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(w)e^{i\phi u}e^{i\psi w}F(x,v,\tau-\xi;u,w)dudw,$$

the PDE (23) becomes

$$\frac{\partial f_2}{\partial \tau} = \frac{v}{2}\frac{\partial^2 f_2}{\partial x^2} + \rho\sigma v \frac{\partial^2 f_2}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 f_2}{\partial v^2} + \left(r - q - \frac{v}{2}\right)\frac{\partial f_2}{\partial x} + (\alpha - \beta v)\frac{\partial f_2}{\partial v} \tag{33}$$

solved subject to the initial condition

$$f_2(x,v,0;\phi,\psi) = e^{i\phi x}e^{i\psi v}, \tag{34}$$

in the domain $-\infty < x < \infty$, $0 \le v < \infty$. This initial condition will lead to an analytic solution for $f_2$, as we now demonstrate.

**Proposition 4.2.** *The solution to the PDE (33) subject to the initial condition (34) is*

$$f_2(x,v,\tau-\xi;\phi,\psi) = \exp\{g_0(\phi,\psi,\tau-\xi) + g_1(\phi,\psi,\tau-\xi)x + g_2(\phi,\psi,\tau-\xi)v\}, \tag{35}$$

*where*

$$\begin{aligned} g_0(\phi,\psi,\tau-\xi) &= (r-q)i\phi(\tau-\xi) \\ &\quad + \frac{\alpha}{\sigma^2}\left\{(\beta-\rho\sigma i\phi+D_2)(\tau-\xi) - 2\ln\left[\frac{1-G_2(\psi)e^{D_2(\tau-\xi)}}{1-G_2(\psi)}\right]\right\}, \qquad (36) \\ g_1(\phi,\psi,\tau-\xi) &= i\phi, \qquad (37) \\ g_2(\phi,\psi,\tau-\xi) &= i\psi + \frac{\beta-\rho\sigma i\phi-\sigma^2 i\psi+D_2}{\sigma^2}\left[\frac{1-e^{D_2(\tau-\xi)}}{1-G_2(\psi)e^{D_2(\tau-\xi)}}\right], \qquad (38) \end{aligned}$$

*with*

$$D_2^2 \equiv (\rho\sigma i\phi-\beta)^2 + \sigma^2\phi(\phi+i), \qquad (39)$$

*and*

$$G_2(\psi) \equiv \frac{\beta-\rho\sigma i\phi-\sigma^2 i\psi+D_2}{\beta-\rho\sigma i\phi-\sigma^2 i\psi-D_2}. \qquad (40)$$

**Proof:** Refer to Appendix 3.

□

With a closed form expression established for $f_2$, we can now express the early exercise premium, $V(x,v,\tau)$, in terms of the inverse Fourier transform of $f_2$. In particular, we shall express $V$ in a form that generalises the American call early exercise premium term presented by Kim (1990) to the case where volatility changes randomly.

**Proposition 4.3.** *The early exercise premium, $V(x,v,\tau)$, is given by*

$$\begin{aligned} &V(x,v,\tau) \\ &\quad = e^x e^{r\tau}\int_0^\tau q e^{-q(\tau-\xi)}\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-b_0(\xi)i\phi}}{i\phi}f_1(x,v,\tau-\xi;\phi,-b_1(\xi)\phi)\right)d\phi\right)d\xi \\ &\qquad - \int_0^\tau rKe^{r\xi}\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-b_0(\xi)i\phi}}{i\phi}f_2(x,v,\tau-\xi;\phi,-b_1(\xi)\phi)\right)d\phi\right)d\xi, \quad (41) \end{aligned}$$

*where*

$$f_1(x,v,\tau-\xi;\phi,\psi) \equiv e^{-x}e^{-(r-q)(\tau-\xi)}f_2(x,v,\tau-\xi;\phi-i,\psi), \qquad (42)$$

*and $f_2(x,v,\tau-\xi;\phi,\psi)$ is given in Proposition 4.2.*

**Proof:** Refer to Appendix 4.

□

Referring to equation (17), all that remains is to determine the European call price, $U(x, v, \tau)$. While Heston (1993) derives this using characteristic functions, we provide a slightly different derivation using Fourier transforms.

**Proposition 4.4.** *The price of the European call option, $U(x, v, \tau)$, is given by*

$$\begin{aligned} U(x, v, \tau) = e^x e^{(r-q)\tau} &\left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left( \frac{e^{-i\phi \ln K}}{i\phi} f_1(x, v, \tau; \phi, 0) \right) d\phi \right) \\ - K &\left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left( \frac{e^{-i\phi \ln K}}{i\phi} f_2(x, v, \tau; \phi, 0) \right) d\phi \right), \end{aligned} \tag{43}$$

*where $f_1(x, v, \tau; \phi, \psi)$ and $f_2(x, v, \tau; \phi, \psi)$ are given in Propositions 4.3 and 4.2.*

**Proof:** Refer to Appendix 5.

□

We can now combine the results of Propositions 4.3 and 4.4 to determine the integral equation for the American call option in terms of the original state variable $S$.

**Proposition 4.5.** *The price of an American call option written on $S$, where $S$ evolves according to the dynamics (1)-(2), is*

$$\begin{aligned} C_A(S, v, \tau) &= Se^{-q\tau} P_1(S, v, \tau, K; 0) - Ke^{-r\tau} P_2(S, v, \tau, K; 0) \\ &\quad + \int_0^\tau qSe^{-q(\tau-\xi)} P_1(S, v, \tau - \xi, e^{b_0(\xi)}; -b_1(\xi)\phi) d\xi \\ &\quad - \int_0^\tau rKe^{-r(\tau-\xi)} P_2(S, v, \tau - \xi, e^{b_0(\xi)}; -b_1(\xi)\phi) d\xi, \end{aligned} \tag{44}$$

*where*

$$P_j(S, v, \tau, K; \psi) \equiv \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left( \frac{e^{-i\phi \ln K}}{i\phi} f_j(\ln S, v, \tau; \phi, \psi) \right) d\phi, \tag{45}$$

*for $j = 1, 2$, and $f_j(x, v, \tau; \phi, \psi)$ is given by Propositions 4.3 and 4.2.*

**Proof:** Recall that $S = e^x$ and $C_A(S, v, \tau) = e^{-r\tau} U(x, v, \tau)$, and substitute into (43) and (41).

□

Comparing equation (44) with Kim's (1990) solution, we can readily draw parallels between the cumulative normal density functions appearing in the constant volatility case, and the $P_j$ terms occurring under stochastic volatility, here expressed as inverse Fourier transforms. We highlight, however, that analytic functional forms for the $P_j$ terms in the early exercise premium are only possible when we approximate the early exercise boundary $b(v, \tau)$ using the series expansion (26).

## 5. Early Exercise Boundary

While equation (44) provides us with an analytic expression for the American call option price, it depends upon the unknown free boundary $b(v, \tau)$. By evaluating (44) at $S = b(v, \tau)$ we can determine an integral equation for $b(v, \tau)$. Using the boundary condition (5), we have

$$C_A(b(v, \tau), v, \tau) = b(v, \tau) - K \approx e^{b_0(\tau)+vb_1(\tau)} - K. \tag{46}$$

Since there are two unknown time-dependent functions, $b_0(\tau)$ and $b_1(\tau)$, we must evaluate (46) at two distinct values of $v$, thus forming a linked system of integral equations for $b_0$ and $b_1$. If we let $v_0$ and $v_1$ denote distinct values of $v$, then the system of integral equations for $b(v, \tau)$ is

$$C_A(b(v_0(\tau), \tau), v, \tau) = e^{b_0(\tau)+v_0(\tau)b_1(\tau)} - K \tag{47}$$

$$C_A(b(v_1(\tau), \tau), v, \tau) = e^{b_0(\tau)+v_1(\tau)b_1(\tau)} - K. \tag{48}$$

We propose solving this system using an iterative method based on numerical techniques frequently applied to Volterra integral equations. We discretise the time domain into $N$ subintervals of length $\Delta\tau$, such that $T = N\Delta\tau$. At step $n = 0$ we know from Kim (1990) that the early exercise boundary is

$$b(v, 0) = \max\left(\frac{r}{q}K,\ K\right), \tag{49}$$

a result which also holds true under stochastic volatility, since the payoff for the American call does not explicitly depend upon $v$. It follows that for the linear approximation of the free boundary

$$b_0(0) = \max\left(\ln\left[\frac{r}{q}K\right],\ \ln K\right), \quad b_1(0) = 0. \tag{50}$$

This provides us with the initial values of $b_0$ and $b_1$ for the time-stepping procedure.

At each subsequent time step, $n = 1, 2, ..., N$ we must determine two unknowns, namely $b_0^n = b_0(n\Delta\tau)$ and $b_1^n = b_1(n\Delta\tau)$, whose values depend upon each other, as well as on previous values of $b_0(n\Delta\tau)$ and $b_1(n\Delta\tau)$ for $n = 0, 1, 2, ..., N-1$. When iterating for $b_0^n$ and $b_1^n$ we take as our initial approximations $b_{0,0}^n = b_0^{n-1}$, and $b_{1,0}^n = b_1^{n-1}$. We then solve the linked system of integral equations according to

$$b_{1,j}^n = \frac{1}{v_0^n}\left(\ln\left[C_A(\exp(b_{0,j-1}^n + v_0^n b_{1,j}^n), v_0^n, \tau) + K\right] - b_{0,i-1}^n\right) \tag{51}$$

$$b_{0,j}^n = \ln\left[C_A(\exp(b_{0,j}^n + v_0^n b_{1,j}^n), v_1^n, \tau) + K\right] - v_1^n b_{1,j}^n. \tag{52}$$

We continue sequentially, solving (51)-(52) for $j = 1, 2, ...$ until both $|b_{0,i}^n - b_{0,i-1}^n| < \epsilon_0$ and $|b_{1,i}^n - b_{1,i-1}^n| < \epsilon_1$, for arbitrary values of $\epsilon_0$ and $\epsilon_1$. Once the solutions have converged to the desired level of tolerance, the method advances to the next time step.

Since we are approximating the true early exercise boundary with a linear function of $v$, we seek values of $v_0$ and $v_1$ that will maximise the accuracy of this estimate. Tzavalis & Wang (2003) suggest using $\mathbb{E}_T[v(0)]$ and $\mathbb{E}_T[v(\tau)]$, since these values represent expected values of $v$ over the time intervals applicable to the early exercise premium[2]. Dufresne (2001) shows that these expectations are given by

$$\mathbb{E}_T[v(\tau)] = \theta + (v(T) - \theta)e^{-\kappa(T-\tau)}. \tag{53}$$

To ensure that the iterative scheme converges, we specify $v_0$ and $v_1$ according to

$$v_0 = \max\left(\mathbb{E}_T[v(\tau)],\ \mathbb{E}_T[v(0)]\right) \tag{54}$$

$$v_1 = \min\left(\mathbb{E}_T[v(\tau)],\ \mathbb{E}_T[v(0)]\right). \tag{55}$$

[2]Recall that $\tau$ is time remaining until maturity. Hence $\mathbb{E}_T[v(0)]$ represents the expected value of the volatility at the expiry date, taken at time $T$ until maturity.

This specification allows the $b_1$ term to capture the majority of the volatility dependence in the free boundary before estimating $b_0$. It is important to note that when $v(T)$ is close to $\theta$, or $\tau$ is close to zero, $|v_0 - v_1|$ will be very small. In these instances the iterative scheme (51)-(52) will converge very slowly. We also note that under this method a different $b_0(\tau)$ and $b_1(\tau)$ will be found for each value of $v(T)$. In the event that $v(0) = \theta$, we must find another way to select a second volatility value for fitting $b_0$ and $b_1$. The simplest solution is to instead use $v(T) = \theta + \Delta v$ where $\Delta v$ is a small arbitrary value.

## 6. Numerical Implementation and Results

Since the equations (51) and (52) for $b_1$ and $b_0$ are highly nonlinear, they must be solved using numerical techniques. We use the bisection method for root finding, and the time integrals in (44) are approximated using the compound trapezoidal rule, as considered by Kallast & Kivinukk (2003), with a time step size of $\Delta\tau = 0.01$. While the order of accuracy for the trapezoidal rule is not very large, the weights for the method do not vary for even and odd numbers of integration points. Since we add one additional point to the integration scheme at each increasing time step, the compound trapezoidal rule provides a consistent free boundary estimate for each time step. The integral terms in (45) are estimated numerically using the Laguerre integration scheme with 50 integration points. The tolerance levels for the algorithm were set at $5 \times 10^{-8}$ for the bisection method and $5 \times 10^{-4}$ for the iterative scheme solving (51)-(52). In the case when $v_0 = v_1$, we select $\Delta v = 1 \times 10^{-3}$.

Firstly we present the price and early exercise boundary estimates for an American call option with a particular set of parameters, found using the iterative scheme in Section 5. The parameter values are provided in Table 1.

Figure 2 presents the price profile for an American call option under stochastic volatility for a range of $v$ values. As one would expect, for any given value of $v$ the profile has the standard shape for an American call. The price increases as $v$ increases. The corresponding early exercise surface estimate is provided in Figure 3. Again, for a given value of $v$ the shape of the early exercise boundary is typical for an American call option. The free boundary increases as $v$ increases, which is an intuitive result.

| Parameter | Value | Parameter | Value |
|---|---|---|---|
| $T$ | 0.25 | $\kappa$ | 4.00 |
| $r$ | 0.03 | $\theta$ | 0.09 |
| $q$ | 0.05 | $\sigma$ | 0.10 |
| $K$ | 100 | $\lambda$ | 0 |
| $\rho$ | 0 | | |

TABLE 1. Parameter values used to generate the price profile and early exercise surface in figures 2 and 3.

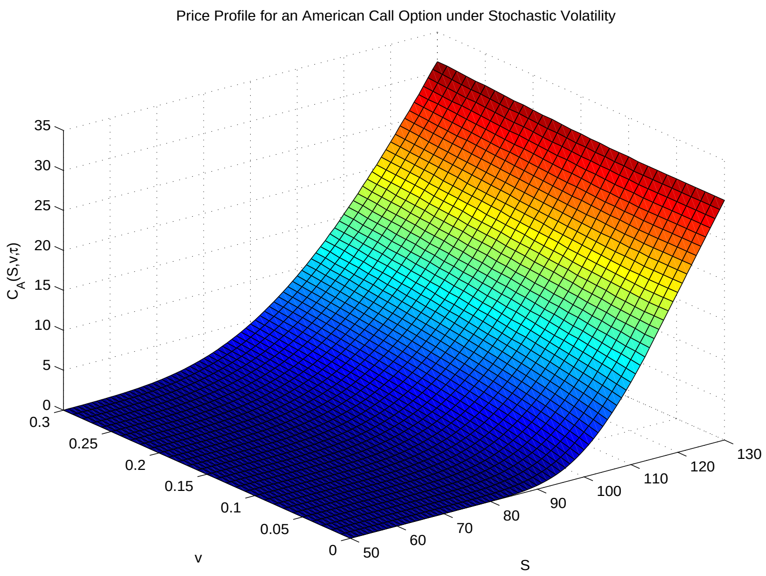


FIGURE 2. Price profile for a 3-month American call option. Parameter values are as listed in Table 1.

Next we shall examine the impact of stochastic volatility on the American call price relative to the constant volatility Black-Scholes model. We select $v(\tau) = 0.04$ and compute the early exercise boundary for the American call for $\rho = -0.5,\ 0,\ 0.5$, with all other parameter values as given in Table 1. To maintain consistency for the comparison, we select the volatility for the Black-Scholes model according to

$$v_{BS} = \theta - \frac{1}{\tau}(v_0 - \theta)(e^{-\kappa\tau} - 1),$$

and so in this case $v_{BS} = 0.05839$.

Figure 4 presents the early exercise boundary for the different values of $\rho$, as well as the constant volatility case. In all cases the early exercise boundary is lower than under constant volatility. It is interesting to note that near $\tau = 0.25$ the boundary for negative

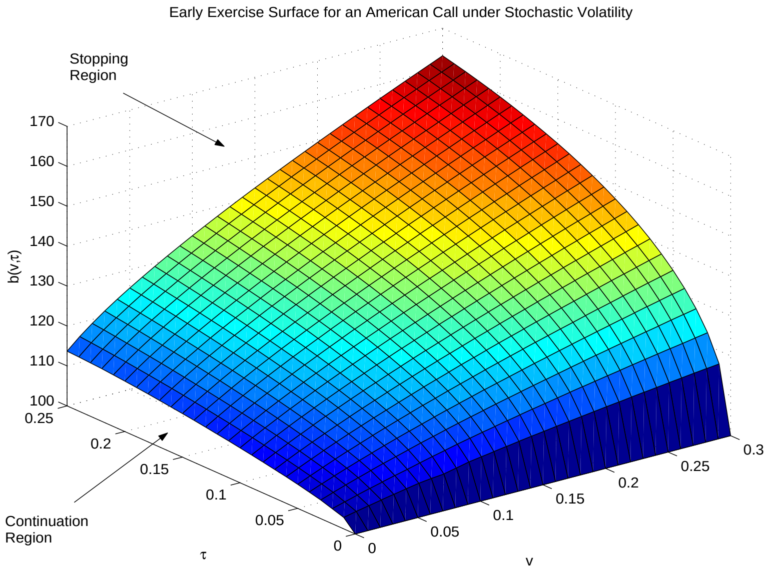


FIGURE 3. Early exercise surface for a 3-month American call option. Parameter values are as listed in Table 1.

correlation begins to rise above the constant volatility result. This indicates that under stochastic volatility, the option holder is likely to exercise the call early for lower values of $S$ for options with a time to maturity of up to 3 months.

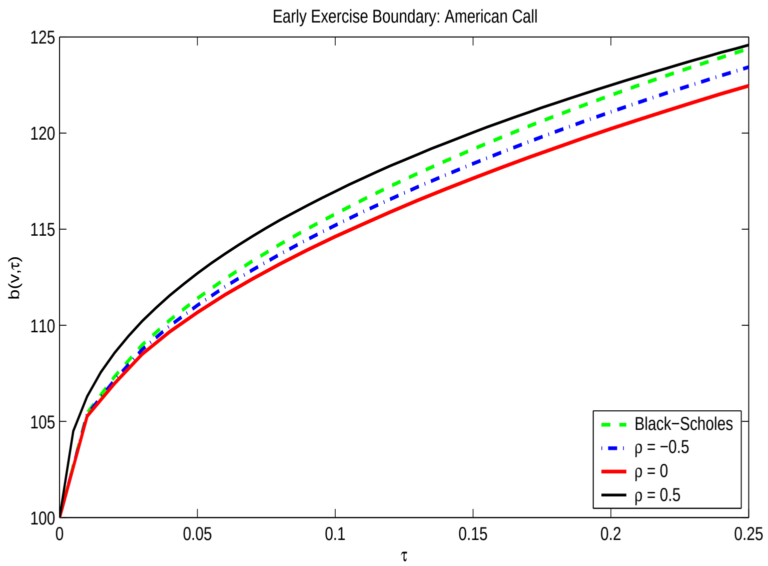


FIGURE 4. Comparing the early exercise boundary of a 3-month American call option for a range of correlation values. The constant volatility case is also provided. For the stochastic volatility model $v(\tau) = 0.04$, and for the Black-Scholes model $v_{BS} = 0.05839$. Other parameter values are as listed in Table 1.

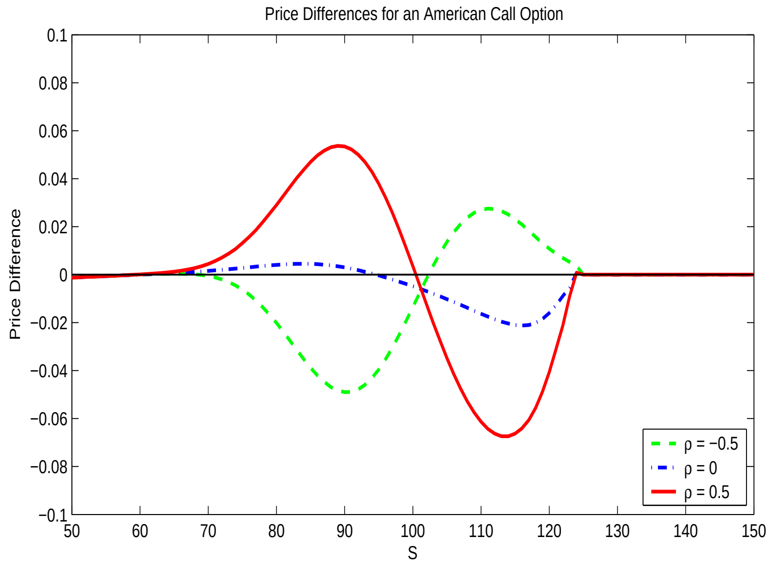


FIGURE 5. Comparing the price differences between the stochastic volatility and constant volatility American call option for a range of correlation values. For the stochastic volatility model $v(\tau) = 0.04$, and for the Black-Scholes model $v_{BS} = 0.05839$. Other parameter values are as listed in Table 1.

Finally, Figure 5 presents the price differences for the American call option under stochastic volatility and constant volatility, for the same range of $\rho$ values shown in Figure 4. The differences are computed as the stochastic volatility price less the Black-Scholes model price. The results for $\rho = -0.5$ and $\rho = 0.5$ are very similar to those presented by Heston (1993). We can see that under positive correlation, the American call is more expensive out-of-the-money than for the constant volatility case, and vice versa in-the-money. Negative correlation produces an inverse impact on the price differences. For zero correlation, it is interesting to note that the differences are similar in structure to the positive correlation case, but are much less pronounced. We also note that the differences rapidly become zero near the early exercise boundary for the American call. The early exercise feature effectively truncates the typical price differences observed in the European case, since early exercise immediately sets the option value equal to the payoff function. Thus Figure 5 implies that stochastic volatility can be used to model the volatility skews observed in financial market option prices, with the correlation parameter determining the direction of the skew.

With regards to the efficiency of this approach, we can make several comments. The computational speed is largely dependent upon the proximity of the "fitting volatilities", $v_0$ and $v_1$, to each other. As mentioned in Section 5, when $v(\tau)$ is close to $\theta$, $v_0$ and $v_1$ are also close, making equations (51) and (52) almost coincident. In this case the system is essentially ill-conditioned, and the basic iterative scheme converges very slowly to the true solution, and in some cases the scheme fails to converge at all. To generate the free boundary surface in Figure 3, the runtime is 43.92 minutes[3], or an average of 1.42 minutes per volatility grid point. However, it should be noted that for $v(\tau) = 0.04$, the computation time is 3.89 minutes, demonstrating that the time required depends greatly on the relative values of $v(\tau)$ and $\theta$. One possible solution is to expand on the selection process for $v_0$ and $v_1$, with a view to finding values that will be consistently well-spaced whilst still providing a good approximation for the true early exercise boundary.

## 7. Conclusion

In this paper we have presented an analytical integral equation for the price and early exercise boundary of an American call option under the square root process of Heston (1993). As suggested by Tzavalis & Wang (2003), we expanded the logarithm of the early exercise premium in a Taylor series about the long-run mean volatility level. Under this assumption we were able to derive the integral equations for the price and free boundary in analytic form using a combination of McKean's (1965) incomplete Fourier transform approach, and the volume potential method of Jamshidian (1992).

Under this Taylor series expansion, one must numerically determine two time-dependent functions in order to estimate the free boundary. We presented an iterative scheme for solving the linked system of integral equations for these functions, combining the volatility fitting methodology proposed by Tzavalis & Wang (2003) with the numerical integration scheme suggested by Kallast & Kivinukk (2003). We implemented this scheme, providing a sample price profile and early exercise surface for an American call option. In addition, we demonstrated the impact of stochastic volatility to the free boundary

[3]All code was implemented using LAHEY$^{\text{TM}}$FORTRAN 95 running on a PC with a Pentium 4 2.40 GHz processor, 521MB RAM, and running the Windows XP Professional operating system.

and price profile of an American call option relative to the constant volatility model of Black & Scholes (1973) for a range of correlation values. We found similar results to Heston (1993), with the added feature that the price differences in-the-money decay rapidly around the early exercise boundary.

There are several directions for future research based on these findings. The accuracy of the price and free boundary estimate under the series expansion for the early exercise boundary needs to be examined. In particular, this estimate should be compared with a proxy for the "true" free boundary, found using finite differences or the method of lines in two dimensions. This will allow further analysis of the accuracy and efficiency of the iterative solution technique presented in this paper. The proposed algorithm could be made more efficient by considering alternative choices for the "fitting volatilities" in the iterative scheme. A wider range of parameter values should also be explored, in particular the case where the risk-free rate is greater than the dividend yield, and different values for the time to expiry and rate of mean reversion.

Furthermore, these results could be expanded upon by considering the American call option under stochastic volatility in the case where there are jumps in the price process for the underlying asset. We infer that under such a model will not be possible to obtain an analytical integral expression for the early exercise premium, even when using the Taylor series expansion of Tzavalis & Wang (2003) for the early exercise boundary. This suggests that it is impossible to analyse American options under such a model using characteristic functions or Fourier transforms, preventing the use of solution techniques based on numerical integration, and implies the need to explore other numerical solution methods for such a problem.

## Appendix 1. Deriving the Jamshidian Formulation under Stochastic Volatility

Here we make use of McKean's (1965) incomplete Fourier transform method to derive a generalisation of Jamshidian's (1992) representation of the free boundary value problem for American call options in the case of stochastic volatility. To apply the Fourier transform (13) to the PDE (8), we extend the domain to $-\infty < x < \infty$ by multiplying

(8) by $H(\ln b(v,\tau) - x)$, where $H(x)$ is the Heaviside step function given by (15). The incomplete Fourier transform of (8) with respect to $x$ is therefore

$$\int_{-\infty}^{\ln b(v,\tau)} e^{i\phi x} \left\{ \frac{v}{2}\frac{\partial^2 U_A}{\partial x^2} + \rho\sigma v \frac{\partial^2 U_A}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 U_A}{\partial v^2} + \left(r - q - \frac{v}{2}\right)\frac{\partial U_A}{\partial x} \right.$$
$$\left. + (\alpha - \beta v)\frac{\partial U_A}{\partial v} - \frac{\partial U_A}{\partial \tau} \right\} dx = 0,$$

which, using the properties of integral calculus, can be rewritten as

$$\int_{-\infty}^{\infty} e^{i\phi x} \left\{ \frac{v}{2}\frac{\partial^2 U_A}{\partial x^2} + \rho\sigma v \frac{\partial^2 U_A}{\partial x \partial v^2} + \left(r - q - \frac{v}{2}\right)\frac{\partial U_A}{\partial x} + (\alpha - \beta v)\frac{\partial U_A}{\partial v} - \frac{\partial U_A}{\partial \tau} \right\} dx$$
$$= \int_{\ln b(v,\tau)}^{\infty} e^{i\phi x} \left\{ \frac{v}{2}\frac{\partial^2 U_A}{\partial x^2} + \rho\sigma v \frac{\partial^2 U_A}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 U_A}{\partial v^2} + \left(r - q - \frac{v}{2}\right)\frac{\partial U_A}{\partial x} \right.$$
$$\left. +(\alpha - \beta v)\frac{\partial U_A}{\partial v} - \frac{\partial U_A}{\partial \tau} \right\} dx.$$

Taking the inverse Fourier transform according to (14), and noting that when $x \geq \ln b(v,\tau)$, the American call option price is simply given by the payoff function, $U_A(x,v,\tau) = (e^x - K)e^{r\tau}$, we have

$$\frac{v}{2}\frac{\partial^2 U_A}{\partial x^2} + \rho\sigma v \frac{\partial^2 U_A}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 U_A}{\partial v^2} + \left(r - q - \frac{v}{2}\right)\frac{\partial U_A}{\partial x} + (\alpha - \beta v)\frac{\partial U_A}{\partial v} - \frac{\partial U_A}{\partial \tau}$$
$$= H(x - \ln b(v,\tau)) \left\{ \frac{v}{2} e^x e^{r\tau} + \left(r - q - \frac{v}{2}\right) e^x e^{r\tau} - (e^x - K) r e^{r\tau} \right\},$$

which simplifies to

$$\frac{\partial U_A}{\partial \tau} = \frac{v}{2}\frac{\partial^2 U_A}{\partial x^2} + \rho\sigma v \frac{\partial^2 U_A}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 U_A}{\partial v^2} + \left(r - q - \frac{v}{2}\right)\frac{\partial U_A}{\partial x}$$
$$+ (\alpha - \beta v)\frac{\partial U_A}{\partial v} + H(x - \ln b(v,\tau))\{e^{r\tau}(qe^x - rK)\},$$

which must now be solved in the domain $0 \leq \tau \leq T$, $-\infty < x < \infty$, and $0 \leq v < \infty$, subject to the initial condition (9). This is a generalisation of the Jamshidian (1992) formulation of the problem, which requires us to solve an inhomogeneous PDE for the American call option price in an unrestricted domain.

## Appendix 2. Proof of Proposition 4.1

From equation (28), we can linearly decompose $V(x, v, \tau)$ into two integral terms, such that

$$V(x, v, \tau) = I_1(x, v, \tau) - I_2(x, v, \tau).$$

Firstly we define $I_1$ as

$$\begin{aligned}
I_1(x, v, \tau) &= \int_0^\tau \int_0^\infty \int_0^\infty \frac{qe^{y+b_0(\xi)}}{2\pi} e^{r\xi} \int_{-\infty}^\infty e^{wb_1(\xi)} e^{-i\phi(y+b_0(\xi))} e^{-iwb_1(\xi)\phi} \\
&\qquad \times \hat{f}_2(x, v, \tau - \xi; \phi, w) d\phi dy dw d\xi \\
&= \int_0^\tau \int_0^\infty \frac{qe^{y+b_0(\xi)e^{r\xi}}}{2\pi} \int_{-\infty}^\infty \int_0^\infty e^{-i\phi(y+b_0(\xi))} e^{iwb_1(\xi)[-\phi-i]} \\
&\qquad \times \hat{f}_2(x, v, \tau - \xi; \phi, w) dw d\phi dy d\xi \\
&= \int_0^\tau \int_0^\infty \frac{qe^{y+b_0(\xi)} e^{r\xi}}{2\pi} \int_{-\infty}^\infty e^{-i\phi(y+b_0(\xi))} \\
&\qquad \times \int_{-\infty}^\infty H(w) e^{iwb_1(\xi)[-\phi-i]} \hat{f}_2(x, v, \tau - \xi; \phi, w) dw d\phi dy d\xi \\
&= \int_0^\tau \int_0^\infty \frac{qe^{y+b_0(\xi)} e^{r\xi}}{2\pi} \int_{-\infty}^\infty e^{-i\phi(y+b_0(\xi))} f_2(x, v, \tau - \xi; \phi, -b_1(\xi)(\phi + i)) d\phi dy d\xi,
\end{aligned}$$

where

$$f_2(x, v, \tau; \phi, \psi) \equiv \int_{-\infty}^\infty H(w) \hat{f}_2(x, v, \tau - \xi\phi, w) e^{i\psi w} dw.$$

Thus

$$I_1(x, v, \tau) = \int_0^\tau \int_{-\infty}^\infty \frac{qe^{(1-i\phi)b_0(\xi)} e^{r\xi}}{2\pi} \int_0^\infty e^{y(1-i\phi)} dy f_2(x, v, \tau - \xi; \phi, -b_1(\xi)(\phi + i)) d\phi d\xi.$$

Changing the integration variable from $\phi$ to $\phi - i$ gives

$$I_1(x, v, \tau) = \int_0^\tau \int_{-\infty}^\infty \frac{qe^{-b_0(\xi)i\phi} e^{r\xi}}{2\pi} \left( \int_0^\infty e^{-yi\phi} dy \right) f_2(x, v, \tau - \xi; \phi - i, -b_1(\xi)\phi) d\phi d\xi.$$

Similarly for $I_2(x, v, \tau)$ we have

$$
\begin{aligned}
I_2(x, v, \tau) &= \int_0^\tau \int_0^\infty \int_0^\infty \frac{rKe^{r\xi}}{2\pi} \int_{-\infty}^\infty e^{-i\phi(y+b_0(\xi))} e^{-iwb_1(\xi)\phi} \hat{f}(x, v, \tau - \xi; \phi, w) d\phi dy du d\xi \\
&= \int_0^\tau \frac{rKe^{r\xi}}{2\pi} \int_0^\infty \int_{-\infty}^\infty e^{-i\phi(y+b_0(\xi))} \int_{-\infty}^\infty H(w) e^{-iwb_1(\xi)\phi} \\
&\qquad \times \hat{f}(x, v, \tau - \xi; \phi, w) dw d\phi dy d\xi \\
&= \int_0^\tau \frac{rKe^{r\xi}}{2\pi} \int_{-\infty}^\infty e^{-i\phi b_0(\xi)} \left( \int_0^\infty e^{-i\phi y} dy \right) f_2(x, v, \tau - \xi; \phi, -b_1(\xi)\phi) d\phi d\xi.
\end{aligned}
$$

## Appendix 3. Solving for $f_2$

We begin by assuming that the solution to (33) is of the form

$$f_2(x, v, \tau - \xi; \phi, \psi) = \exp\{g_0(\phi, \psi, \tau - \xi) + g_1(\phi, \psi, \tau - \xi)x + g_2(\phi, \psi, \tau - \xi)v\}.$$

Substituting this into (33) yields

$$\left\{ \frac{g_1^2}{2} + g_1 g_2 \rho\sigma + \frac{\sigma^2 g_2^2}{2} - \frac{g_1}{2} - \beta g_2 - \frac{\partial g_2}{\partial \tau} \right\} v - \left\{ \frac{\partial g_1}{\partial \tau} \right\} x + \left\{ (r - q) g_1 + \alpha g_2 - \frac{\partial g_0}{\partial \tau} \right\}.$$

Equating coefficients of $x$ and $v$, we arrive at a system of three ordinary differential equations (ODEs) for the functions $g_0$, $g_1$, and $g_2$, given by

$$\frac{\partial g_2}{\partial \tau} = \frac{g_1^2}{2} - \frac{g_1}{2} + g_1 g_2 \rho\sigma + \frac{\sigma^2}{2} g_2^2 - \beta g_2, \tag{56}$$

$$\frac{\partial g_1}{\partial \tau} = 0, \tag{57}$$

$$\frac{\partial g_0}{\partial \tau} = (r - q) g_1 + \alpha g_2. \tag{58}$$

The initial conditions for this system are

$$g_2(\phi, \psi, 0) = i\psi, \quad g_1(\phi, \psi, 0) = i\phi, \quad g_0(\phi, \psi, 0) = 0.$$

It is trivial to show that the solution to (57) is

$$g_1(\phi, \psi, \tau - \xi) = i\phi.$$

Substituting for $g_1$ in (56) we have

$$\frac{\partial g_2}{\partial \tau} = \frac{\sigma^2}{2} g_2^2 - \beta g_2 - \frac{\phi^2}{2} - \frac{i\phi}{2} + \rho\sigma i\phi g_2,$$

which we recognise as being a Ricatti equation with solution[4]

$$g_2(\phi, \psi, \tau - \xi) = \frac{\beta - \rho\sigma i\phi}{\sigma^2} - \frac{D_2[a_1 e^{\frac{D_2}{2}(\tau-\xi)} - a_2 e^{-\frac{D_2}{2}(\tau-\xi)}]}{\sigma^2[a_1 e^{\frac{D_2}{2}(\tau-\xi)} + a_2 e^{-\frac{D_2}{2}(\tau-\xi)}]}$$

where we define

$$D_2^2 \equiv (\rho\sigma i\phi - \beta)^2 + \sigma^2 \phi(\phi + i),$$

and $a_1, a_2$ are constants. From the initial condition we find that

$$\frac{a_1 - a_2}{a_1 + a_2} = \frac{\beta - \rho\sigma i\phi - \sigma^2 i\psi}{D_2},$$

and thus we have $a_1 = 1 + \gamma$ and $a_2 = 1 - \gamma$, where we set

$$\gamma \equiv \frac{\beta - \rho\sigma i\phi - \sigma^2 i\phi}{D_2}.$$

Substituting for $a_1$ and $a_2$ we obtain

$$g_2(\phi, \psi, \tau - \xi) = \frac{\beta - \rho\sigma i\phi}{\sigma^2} - \frac{D_2}{\sigma^2} \frac{[e^{D_2(\tau-\xi)} - 1 + \gamma(e^{D_2(\tau-\xi)} + 1)]}{[e^{D_2(\tau-\xi)} + 1 + \gamma(e^{D_2(\tau-\xi)} - 1)]}.$$

Using the definition of $\gamma$, we can rewrite $g_2$ as

$$g_2(\phi, \psi, \tau - \xi) = i\psi + \frac{\beta - \rho\sigma i\phi - \sigma^2 i\psi + D_2}{\sigma^2} \left[ \frac{1 - e^{D_2(\tau-\xi)}}{1 - G_2(\psi) e^{D_2(\tau-\xi)}} \right],$$

where we set

$$G_2(\psi) \equiv \frac{\beta - \rho\sigma i\phi - \sigma^2 i\psi + D_2}{\beta - \rho\sigma i\phi - \sigma^2 i\psi - D_2}.$$

Now we turn to solving equation (58). Integrating with respect to $\tau$, and making use of the initial condition for $g_0$, we have

$$\int_\xi^\tau \frac{\partial g_0}{\partial s}(\phi, \psi, s - \xi) ds = (r - q) i\phi(\tau - \xi) + \alpha \int_\xi^\tau g_2(\phi, \psi, s - \xi) ds.$$

[4]In Appendix 5 we provide further details on how to solve Ricatti equations such as (56).

Consider the integral term

$$\int_{\xi}^{\tau} g_2(\phi, \psi, s-\xi)ds = i\psi(\tau-\xi) + \frac{\beta - \rho\sigma i\phi - \sigma^2 i\psi + D_2}{\sigma^2} \int_{\xi}^{\tau} \frac{1 - e^{D_2(s-\xi)}}{1 - G_2(\psi)e^{D_2(s-\xi)}} ds.$$

After making the change of integration variable $\zeta = G_2(\psi)e^{D_2(s-\xi)}$, extensive manipulations result in

$$\begin{aligned} g_0(\phi, \psi, \tau-\xi) &= (r-q)i\phi(\tau-\xi) \\ &\quad + \frac{\alpha}{\sigma^2}\left\{(\beta - \rho\sigma i\phi + D_2)(\tau-\xi) - 2\ln\left[\frac{1 - G_2(\psi)e^{D_2(\tau-\xi)}}{1 - G_2(\psi)}\right]\right\}. \end{aligned}$$

It should be noted that while our final expressions for $g_0$ and $g_2$ differ in appearance from those presented by Tzavalis & Wang (2003), both solutions have been verified as identical[5]. We present these alternative forms for $g_0$ and $g_2$ because they generalise the solution for the European call presented by Heston (1993).

## Appendix 4. Deriving a Generalisation of Kim's Expression for the Early Exercise Premium

To express $V(x, v, \tau)$ in a way that generalises Kim (1990) to the stochastic volatility case, we must focus on the term $I_1(x, v, \tau)$ from equation (30). From $I_1(x, v, \tau)$ we have

$$f_2(x, v, \tau-\xi; \phi-i, \psi) = \exp\{g_0(\phi-i, \psi, \tau-\xi) + g_1(\phi-i, \psi, \tau-\xi)x + g_2(\phi-i, \psi, \tau-\xi)\}.$$

Firstly consider $g_1(\phi - i, \psi, \tau - \xi)$ which is simply

$$g_1(\phi - i, \psi, \tau-\xi) = i(\phi - i) = g_1(\phi, \psi, \tau-\xi) + 1.$$

For $g_2(\phi - i, \psi, \tau - \xi)$ we have

$$\begin{aligned} g_2(\phi - i, \psi, \tau-\xi) &= i\psi + \frac{\beta - \rho\sigma(1 + i\phi) - \sigma^2 i\psi + D_2}{\sigma^2}\left[\frac{1 - e^{D_1(\tau-\xi)}}{1 - G_1(\psi)e^{D_1(\tau-\xi)}}\right] \\ &\equiv \bar{g}_2(\phi, \psi, \tau-\xi), \end{aligned} \tag{59}$$

[5]This verification involves a tedious amount of straightforward algebra which we have chosen to omit.

where

$$D_1^2 \equiv (\rho\sigma i\phi - \beta + \rho\sigma)^2 + \sigma^2\phi(\phi - i), \tag{60}$$

and

$$G_1(\psi) \equiv \frac{\beta - \rho\sigma i\phi - \rho\sigma - \sigma^2 i\psi - D_1}{\beta - \rho\sigma i\phi - \rho\sigma - \sigma^2 i\psi + D_1}. \tag{61}$$

Finally, $g_0(\phi - i, \psi, \tau - \xi)$ is given by

$$\begin{aligned} g_0(\phi - i, \psi, \tau - \xi) &= (r - q)(1 + i\phi)(\tau - \xi) \\ &\quad + \frac{\alpha}{\sigma^2}\left\{(\beta - \rho\sigma(1 + i\phi) + D_1)(\tau - \xi) - 2\ln\left[\frac{1 - G_1(\psi)e^{D_1(\tau-\xi)}}{1 - G_1(\psi)}\right]\right\} \\ &\equiv (r - q)(\tau - \xi) + \bar{g}_0(\phi, \psi, \tau - \xi). \end{aligned} \tag{62}$$

This leads us to the definition

$$\begin{aligned} & f_2(x, v, \tau - \xi; \phi - i, \psi) \\ & = \exp\{(r - q)(\tau - \xi) + \bar{g}_0(\phi, \psi, \tau - \xi) + x + g_1(\phi, \psi, \tau - \xi)x + \bar{g}_2(\phi, \psi, \tau - \xi)v\} \\ & \equiv e^x e^{(r-q)(\tau-\xi)} f_1(x, v, \tau - \xi; \phi, \psi). \end{aligned}$$

We can re-express the integrals $I_1$ and $I_2$ using results from Shephard (1991)[6], combined with the properties of complex conjugates. Beginning with $I_1$,

$$\begin{aligned} I_1(x, v, \tau) = & \int_0^\tau \int_{-\infty}^\infty \frac{q e^{-b_0(\xi)i\phi} e^{r\tau}}{2\pi} \left[\frac{e^{-yi\phi}}{-i\phi}\right]_0^\infty \\ & \qquad \times e^x e^{(r-q)(\tau-\xi)} f_1(x, v, \tau - \xi; \phi, -b_1(\xi)\phi) d\phi d\xi \\ = & \; e^x e^{r\tau} \int_0^\tau q e^{-q(\tau-\xi)} \left(\frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{-b_0(\xi)i\phi}}{i\phi} f_1(x, v, \tau - \xi; \phi, -b_1(\xi)\phi) d\phi \right. \\ & \qquad \left. - \lim_{y\to\infty} \frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{-(b_0(\xi)+y)i\phi}}{i\phi} f_1(x, v, \tau - \xi; \phi, -b_1(\xi)\phi) d\phi \right) d\xi \\ = & \; e^x e^{r\tau} \int_0^\tau q e^{-q(\tau-\xi)} \\ & \qquad \times \left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-b_0(\xi)i\phi}}{i\phi} f_1(x, v, \tau - \xi; \phi, -b_1(\xi)\phi)\right) d\phi\right) d\xi, \end{aligned}$$

[6]See Appendix 5 for further details.

where Re( ) denotes the real-valued component of a complex function. For $I_2$ similar steps yield

$$I_2(x,v,\tau) = \int_0^\tau rKe^{r\xi}\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-b_0(\xi)i\phi}}{i\phi}f_2(x,v,\tau-\xi;\phi,-b_1(\xi)\phi)\right)d\phi\right)d\xi,$$

and substituting $I_1$ and $I_2$ into (29) results in equation (41).

## Appendix 5. Deriving Heston's Solution using Fourier Transforms

The European call price $U(x,v,\tau)$ satisfies the PDE (22) subject to the initial condition

$$U(x,v,0) = \max(e^x - K, 0) \equiv h(x).$$

In keeping with our derivation for the early exercise premium, we shall solve this problem using Fourier transforms. Define the Fourier transform of $U(x,v,\tau)$ with respect to $x$ as

$$\mathcal{F}\{U(x,v,\tau)\} \equiv \int_{-\infty}^{\infty} U(x,v,\tau)e^{i\phi x}dx = \hat{U}(\phi,v,\tau).$$

Applying this transform[7] to the PDE (22), we have

$$\frac{\partial \hat{U}}{\partial \tau} = \frac{\sigma^2 v}{2}\frac{\partial^2 \hat{U}}{\partial v^2} + (\alpha - (\beta + \rho\sigma i\phi)v)\frac{\partial \hat{U}}{\partial v} - \left(\frac{v}{2}\phi^2 + i\left(r - q - \frac{v}{2}\right)\phi\right)\hat{U}.$$

Thus we now need to solve a 1-dimensional PDE for $\hat{U}(\phi,v,\tau)$ subject to the transformed initial condition $\hat{U}(\phi,v,0) \equiv \hat{h}(\phi)$.

We assume that the form of the solution is

$$\hat{U}(\phi,v,\tau) = \hat{h}(\phi)\exp\{A(-\phi,\tau) + B(-\phi,\tau)v\} \tag{63}$$

[7]In applying the transform we assume that $U$, $\partial U/\partial x$ and $\partial^2 U/\partial x^2$ all tend to zero as $x$ tends to $\pm\infty$. This is not true in practice for $U$ and $\partial U/\partial x$ in the case where $U$ is a European call. To validate this assumption, one must instead take the complex Fourier transform of $U$, by allowing $\phi$ to be a complex valued parameter, and then consider a strip in the complex plane for which the transforms of $U$, $\partial U/\partial x$ and $\partial^2 U/\partial x^2$ will be finite. Lewis (2000) shows that such a strip exists, and our assumptions lead to the correct solution by virtue of this result.

where $A(-\phi, 0) = B(-\phi, 0) = 0$. Substituting this into the PDE we have

$$\left[\frac{\partial B(-\phi,\tau)}{\partial \tau} + (\beta + \rho\sigma i\phi)B(-\phi,\tau) \ - \frac{\sigma^2}{2}B^2(-\phi,\tau) + \frac{\phi^2}{2} - \frac{i\phi}{2}\right] v$$
$$= -\frac{A(-\phi,\tau)}{\partial \tau} + \alpha B(-\phi,\tau) - i\phi(r-q).$$

Equating coefficients for powers of $v$ we have the system of ODEs

$$\frac{\partial B(\phi,\tau)}{\partial \tau} = \frac{\sigma^2}{2}B^2(\phi,\tau) - (\beta - \rho\sigma i\phi)B(\phi,0,\tau) - \frac{\phi^2}{2} - \frac{i\phi}{2} \tag{64}$$
$$\frac{\partial A(\phi,\tau)}{\partial \tau} = \alpha B(\phi,\tau) + i\phi(r-q), \tag{65}$$

solved subject to the initial condition $A(\phi, 0) = B(\phi, 0) = 0$.

Consider firstly the ODE (64). This is a Ricatti equation, and is solved by first setting

$$B(\phi,\tau) = -\frac{2z^{'}(\tau)}{\sigma^2 z(\tau)} + \frac{\beta - \rho\sigma i\phi}{\sigma^2},$$

and noting that

$$\frac{\partial B(\phi,\tau)}{\partial \tau} = -\frac{2}{\sigma^2}\left[\frac{zz'' - (z')^2}{z^2}\right],$$

where $z'(\tau) = \partial z/\partial \tau$. Substituting this into the ODE gives

$$z'' - \frac{1}{4}\left[(\beta - \rho\sigma i\phi)^2 + \sigma^2\phi(\phi + i)\right] z = 0,$$

which is a second order ODE with solution

$$z(\tau) = a_1 e^{\frac{D_2}{2}\tau} + a_2 e^{-\frac{D_2}{2}\tau},$$

where $a_1, a_2$ are constants, and $D_2$ is defined by equation (39). Thus $B(\phi, \tau)$ becomes

$$B(\phi,\tau) = \frac{\beta - \rho\sigma i\phi}{\sigma^2} - \frac{D_2[a_1 e^{\frac{D_2}{2}\tau} - a_2 e^{-\frac{D_2}{2}\tau}]}{\sigma^2[a_1 e^{\frac{D_2}{2}\tau} + a_2 e^{-\frac{D_2}{2}\tau}]}.$$

Applying the initial condition $B(\phi, 0) = 0$, we have

$$\frac{a_1 - a_2}{a_1 + a_2} = \frac{\beta - \rho\sigma i\phi}{D_2},$$

and thus $a_1 = 1+\gamma$ and $a_2 = 1-\gamma$, where we set $\gamma \equiv (\beta - \rho\sigma i\phi)/D_2$. Substituting for $a_1$ and $a_2$, extensive manipulations result in

$$B(\phi,\tau) = \frac{\beta - \rho\sigma i\phi + D_2}{\sigma^2}\left[\frac{1-e^{D_2\tau}}{1-G_2(0)e^{D_2\tau}}\right] = g_2(\phi,0,\tau),$$

where $G_2(\psi)$ is defined by equation (40), and $g_2(\phi,\psi,\tau)$ is given by (38).

For the second ODE (65), we can integrate with respect to $\tau$ and use the initial condition $A(\phi,0)=0$ to obtain

$$A(\phi,\tau) = \alpha\int_0^\tau B(\phi,s)ds + i\phi(r-q)\tau.$$

Consider the integral term, which is given by

$$\int_0^\tau B(\phi,s)ds = \int_0^\tau \frac{(\beta-\rho\sigma i\phi + D_2)}{\sigma^2}\left(\frac{1-e^{D_2 s}}{1-G_2(0)e^{D_2 s}}\right)ds.$$

Making a change of integration variable according to $\zeta = G_2(0)e^{D_2 s}$, we eventually find that

$$\int_0^\tau B(\phi,s)ds = \frac{(\beta-\rho\sigma i\phi + D_2)}{\sigma^2}\tau - \frac{2}{\sigma^2}\ln\left(\frac{1-G_2(0)e^{D_2\tau}}{1-G_2(0)}\right),$$

and hence

$$A(\phi,\tau) = i\phi(r-q)\tau + \frac{\alpha}{\sigma^2}\left\{(\beta-\rho\sigma i\phi + D_2)\tau - 2\ln\left(\frac{1-G_2(0)e^{D_2\tau}}{1-G_2(0)}\right)\right\} = g_0(\phi,0,\tau),$$

where $g_0(\phi,\psi,\tau)$ is given by equation (36).

Having found $\hat{U}(\phi,v,\tau)$, we can invert the Fourier transform to obtain $U(x,v,\tau)$. By use of the convolution theorem[8] we have

$$U(x,v,\tau) = \int_\infty^\infty h(\zeta)\left\{\frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\phi(x-\zeta)}e^{g_0(-\phi,0,\tau)+g_2(-\phi,0,\tau)v}d\phi\right\}d\zeta.$$

Changing the integration variable from $\phi$ to $-\phi$ we have

$$U(x,v,\tau) = \int_{-\infty}^\infty h(\zeta)\left\{\frac{1}{2\pi}\int_{-\infty}^\infty e^{i\phi(x-\zeta)}\exp\{g_0(\phi,0,\tau)+g_2(\phi,0,\tau)v\}d\phi\right\}d\zeta.$$

[8]We recall that the convolution theorem for Fourier transforms is

$$\mathcal{F}\left\{\int_{-\infty}^\infty f(u)g(x-u)du\right\} = F(\phi)G(\phi).$$

We recall from equation (35) that

$$f_2(x, v, \tau; \phi, \psi) = \exp\{g_0(\phi, \psi, \tau) + g_2(\phi, \psi, \tau)v + ix\phi\},$$

and thus using Fubini's theorem we have

$$U(x, v, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(x, v, \tau; \phi, 0)\left(\int_{-\infty}^{\infty} u(\zeta)e^{-i\phi\zeta}\right) d\phi.$$

Since we are dealing with a European call option, $h(\zeta) = \max(e^{\zeta} - K, 0)$, and thus

$$U(x, v, \tau) = \bar{I}_1(x, v, \tau) - K\bar{I}_2(x, v, \tau), \tag{66}$$

where

$$\bar{I}_1(x, v, \tau) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(x, v, \tau; \phi, 0)\int_{\ln K}^{\infty} e^{\zeta}e^{-i\phi\zeta}d\zeta d\phi,$$

and

$$\bar{I}_2(x, v, \tau) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(x, v, \tau; \phi, 0)\int_{\ln K}^{\infty} e^{-i\phi\zeta}d\zeta d\phi.$$

Consider firstly $\bar{I}_1(x, v, \tau)$ which can be expressed as

$$\bar{I}_1(x, v, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(x, v, \tau; \phi, 0)\int_{\ln K}^{\infty} e^{(l-i\phi)\zeta}d\zeta d\phi.$$

Changing the integration variable from $\phi$ to $\phi + i$ gives

$$\bar{I}_1(x, v, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(x, v, \tau; \phi - i, 0)\int_{\ln K}^{\infty} e^{-iw\zeta}d\zeta dw.$$

The kernal $f_2$ is given by

$$f_2(x, v, \tau; \phi - i, 0) = \exp\{g_0(\phi - i, 0, \tau) + g_2(\phi - i, 0, \tau)v + x(\phi - i)i\}.$$

Evaluating $g_0$ we have

$$\begin{aligned} g_0(\phi - i, 0, \tau) &= i(\phi - i)(r - q)\tau + \\ &\quad \frac{\alpha}{\sigma^2}\left\{(\beta - \rho\sigma i[\phi - i])\tau - 2\ln\left(\frac{1 - G_1(0)e^{D_1\tau}}{1 - G_1(0)}\right)\right\}, \\ &\equiv (r - q)\tau + \bar{g}_0(\phi, 0, \tau), \end{aligned}$$

where $D_1$ and $G_1(\psi)$ are given by equations (60) and (61) respectively. If we also define

$$g_2(\phi - i, 0, \tau) = \frac{\beta - \rho\sigma i\phi + \rho\sigma + D_1}{\sigma^2}\left[\frac{1 - e^{D_1\tau}}{1 - G_1(0)e^{D_1\tau}}\right] \equiv \bar{g}_2(\phi, 0, \tau),$$

we have

$$\begin{aligned} f_2(x, v, \tau; \phi - i, 0) &= \exp\{\bar{g}_0(\phi, 0, \tau) + \bar{g}_2(\phi, 0, \tau)v + ix\phi\}e^{x+(r-q)\tau} \\ &\equiv f_1(x, v, \tau; \phi, 0)e^{x+(r-q)\tau}, \end{aligned}$$

and hence

$$\bar{I}_1(x, v, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{x+(r-q)\tau} f_1(x, v, \tau; \phi, 0)\int_{\ln K}^{\infty} e^{-i\phi\zeta} d\zeta d\phi.$$

We can show that

$$\begin{aligned} \bar{I}_1(x, v, \tau) &= \lim_{\zeta\to\infty}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{x+(r-q)\tau} f_1(x, v, \tau; \phi, 0)\left[\frac{e^{-i\phi\zeta} - e^{-i\phi\ln K}}{-i\phi}\right] d\phi\right\} \\ &= \lim_{\zeta\to\infty} e^{x+(r-q)\tau}\frac{1}{2\pi}\int_0^{\infty}\left(\frac{e^{-i\phi\ln K} - e^{-i\phi\zeta}}{i\phi} f_1(x, v, \tau; \phi, 0)\right. \\ &\qquad\left. - \frac{e^{i\phi\ln K} - e^{i\phi\zeta}}{i\phi} f_1(x, v, \tau; -\phi, 0)\right) d\phi \\ &= e^{x+(r-q)\tau}\left(\frac{1}{2\pi}\int_0^{\infty}\frac{f_1(x, v, \tau; \phi, 0)e^{-i\phi\ln K} - f_1(x, v, \tau; \phi, 0)e^{i\phi\ln K}}{i\phi} d\phi\right) \\ &\quad - e^{x+(r-q)\tau}\lim_{\zeta\to\infty}\left(\frac{1}{2\pi}\int_0^{\infty}\frac{f_1(x, v, \tau; \phi, 0)e^{-i\phi\zeta} - f_1(x, v, \tau; -\phi, 0)e^{i\phi\zeta}}{i\phi} d\phi\right). \end{aligned}$$

In order to arrive at Heston's (1993) form for $\bar{I}_1$, we will now need to consider results that are unique to characteristic functions. Shephard (1991) shows that

$$F(x) = \frac{1}{2} - \frac{1}{2\pi}\int_0^{\infty}\frac{f(\phi)e^{-i\phi x} - f(-\phi)e^{i\phi x}}{i\phi} d\phi,$$

where $F(x)$ is a cumulative density function. Note also that the complex conjugate of $f_j(x, v, \tau; \phi, \psi)e^{-i\phi x}/i\phi$ is given by

$$\overline{\frac{f_j(x, v, \tau; \phi, \psi)e^{-i\phi x}}{i\phi}} = \frac{f_j(x, v, \tau; -\phi, -\psi)e^{i\phi x}}{-i\phi}$$

for $j = 1, 2$. This is easily proven by substitution using equations (42) and (35).

Thus we have

$$\begin{aligned}\bar{I}_1(x,v,\tau) &= e^{x+(r-q)\tau}\left(\frac{1}{2\pi}\int_0^\infty 2\mathrm{Re}\left(\frac{f_1(x,v,\tau;\phi,0)e^{-i\phi\ln K}}{i\phi}\right)d\phi + \lim_{\zeta\to\infty}F(\zeta) - \frac{1}{2}\right)\\ &= e^{x+(r-q)\tau}\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{f_1(x,v,\tau;\phi,0)e^{-i\phi\ln K}}{i\phi}\right)d\phi\right),\end{aligned}$$

and similarly for $\bar{I}_2(x,v,\tau)$ we have

$$\bar{I}_2(x,v,\tau) = \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{f_2(x,v,\tau;\phi,0)e^{-i\phi\ln K}}{i\phi}\right)d\phi.$$

Substituting for $\bar{I}_1$ and $\bar{I}_2$ into (66) yields the result in Proposition 4.4.

## References


Black, F. & Scholes, M. (1973), 'The Pricing of Corporate Liabilites', *Journal of Political Economy* **81**, 637–659.

Broadie, M., Detemple, J., Ghysels, E. & Torrès, O. (2000), 'American Options with Stochastic Dividends and Volatility: A Nonparametric Investigation', *Journal of Econometrics* **94**, 53–92.

Cakici, N. & Topyan, K. (2000), 'The GARCH Option Pricing Model: a Lattice Approach', *Journal of Computational Finance* **3**, 71–85.

Carr, P., Jarrow, R. & Myneni, R. (1992), 'Alternative Characterizations of American Put Options', *Mathematical Finance* **2**, 87–106.

Chiarella, C. & Ziogas, A. (2005), 'Evaluation of American Strangles', *Journal of Economic Dynamics and Control* **29**, 31–62.

Clarke, N. & Parrott, K. (1999), 'Multigrid for American Option Pricing with Stochastic Volatility', *Applied Mathematical Finance* **6**, 177–195.

Dufresne, D. (2001), 'The Integrated Square-Root Process', *Centre for Actuarial Studies, Department of Economics, University of Melbourne* . Research Paper No. 90.

Fouque, J.-P., Papanicolaou, G. & Sircar, K. R. (2000), *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge.

Heston, S. (1993), 'A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options', *Review of Financial Studies* **6**, 327–343.

Hull, J. & White, A. (1987), 'The Pricing of Options on Assets with Stochastic Volatility', *Journal of Finance* **42**, 281–300.

Ikonen, S. & Toivanen, J. (2004), 'Operator Splitting Methods for American Options with Stochastic Volatility', *Applied Mathematics Letters* **17**, 809–814.

Jacka, S. D. (1991), 'Optimal Stopping and the American Put', *Mathematical Finance* **1**, 1–14.

Jamshidian, F. (1992), 'An Analysis of American Options', *Review of Futures Markets* **11**, 72–80.

Johnson, H. & Shanno, D. (1987), 'Option Pricing when the Variance is Changing', *Journal of Financial and Quantitative Analysis* **22**, 143–151.

Kallast, S. & Kivinukk, A. (2003), 'Pricing and Hedging American Options using Approximations by Kim Integral Equations', *European Finance Review* **7**, 361–383.

Kim, I. J. (1990), 'The Analytic Valuation of American Options', *Review of Financial Studies* **3**, 547–572.

Lewis, A. L. (2000), *Option Valuation under Stochastic Volatility*, Finance Press, California.

McKean, H. P. (1965), 'Appendix: A Free Boundary Value Problem for the Heat Equation Arising from a Problem in Mathematical Economics', *Industrial Management Review* **6**, 32–39.

Oosterlee, C. W. (2003), 'On Multigrid for Linear Complementarity Problems with Application to American-Style Options', *Electronic Transactions on Numerical Analysis* **15**, 165–185.

Reisinger, C. & Wittum, G. (2004), 'On Multigrid for Anisotropic Equations and Variational Inequalities: Pricing Multi-Dimensional European and American options', *Computing and Visualization in Science* **7**, 189–197.

Ritchken, P. & Trevor, R. (1999), 'Pricing Options under Generalized GARCH and Stochastic Volatility Processes', *Journal of Finance* **54**, 377–402.

Scott, L. O. (1987), 'Option Pricing when the Variance Changes Randomly: Theory, Estimation, and an Application', *Journal of Financial and Quantitative Analysis* **22**, 419–438.

Shephard, N. G. (1991), 'From Characteristic Function to Distribution Function: A Simple Framework for the Theory', *Econometric Theory* **7**, 519–529.

Stein, E. M. & Stein, J. C. (1991), 'Stock Price Distributions with Stochastic Volatility: An Analytical Approach', *Review of Financial Studies* **4**, 727–752.

Touzi, N. (1999), 'American Options Exercise Boundary when the Volatility Changes Randomly', *Applied Mathematics and Optimization* **39**, 411–422.

Tzavalis, E. & Wang, S. (2003), 'Pricing American Options under Stochastic Volatility: A New Method using Chebyshev Polynomials to Approximate the Early Exercise Boundary', *Department of Economics, Queen Mary, University of London* . Working Paper, No. 488.

Wiggins, J. B. (1987), 'Option Values under Stochastic Volatility: Theory and Empirical Estimates', *Journal of Financial Economics* **19**, 351–372.

Zhu, J. (2000), *Modular Pricing of Options: An Application of Fourier Analysis*, Springer-Verlag, Berlin.

Zvan, R., Forsyth, P. A. & Vetzal, K. R. (1998), 'Penalty Methods for American Options with Stochastic Volatility', *Journal of Computational and Applied Mathematics* **91**, 199–218.