Option Pricing and the Implied Tail Index with the Generalized Extreme Value (GEV) Distribution[1]

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Abstract

The 1987 stock market crash, the LTCM debacle, the Asian Crisis, the bursting of the high technology Dot-Com bubble of 2001-2 with 30% losses of equity values, events such as 9/11 and sudden corporate collapses of the magnitude of Enron - have radically changed the view that extreme events have negligible probability. The well known drawback of the Black-Scholes model is that it cannot account for the negative skewness and the excess kurtosis of asset returns. Since the work of Jackwerth and Rubinstein (1996) which demonstrated the discontinuity in the implied skewness and kurtosis across the divide of the 1987 stock market crash - a large literature has developed, which aims to extract the risk neutral probability density function from traded option prices so that the skewness and fat tail properties of the distribution are better captured than in the case of lognormal models. This paper argues that the use of the Generalized Extreme Value Distribution (GEV) for asset returns provides not just a flexible framework that subsumes as special cases a number of classes of distributions that have been assumed to date in more restrictive settings – but also delivers the market implied tail index for the assets returns. Under the postulation of the GEV distribution in the Risk Neutral Density (RND) function for the asset returns, we obtain an original analytical closed form solution for the Harrison and Pliska (1981) no arbitrage equilibrium price for the European call option. The implied GEV parameters and RND are estimated from traded option prices for the period from 1997 to 2003. The pricing performance of the GEV option pricing model is compared to the benchmark Black-Scholes model and found to be superior at all time horizons and at all levels of moneyness. We explain how the implied tail index extracted from traded put prices is efficacious at identifying the fat tailed behaviour of losses and hence of the skew in the left tail of the RND function for the underlying price. The GEV implied RNDs before and after special events, such as the Asian Crisis and the LTCM crisis, are also analyzed.

Keywords: Risk neutral probability density function; Generalized Extreme Value Distribution; Implied Tail Index.

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1. Introduction

The 1987 stock market crash, the Asian Crisis (July–October 1997), the Sept. 1998 LTCM debacle, the bursting of the high technology Dot-Com bubble of 2000-02 with 30% losses of equity values, events such as 9/11, sudden corporate collapses of the magnitude of Enron - have radically changed the view that extreme events have negligible probability. In mainstream financial theory extreme events which occur with small probabilities have not been a matter of concern as in the dominant model of log normal asset prices the probability of extreme events is negligible. As noted by Jackwerth and Rubinstein (1996) in a lognormal model of assets prices, the probability of a stock market crash with some 28% loss of equity values is 10^{-160}, an event which is unlikely to happen even in the life time of the universe. However, recently there has been a growing pragmatic and theoretical interest in the shape and fatness of the tails of the distributions of stock returns.

Extreme value theory is a robust framework to analyse the tail behaviour of distributions. Extreme value theory has been applied extensively in hydrology, climatology and also in the insurance industry (see, Embrechts et. al. 1997). Despite early work by Mandelbrot (1963) on the possibility of fat tails in financial data and evidence on the inapplicability of the assumption of log normality in option pricing, a systematic study of extreme value theory for financial modelling and risk management has only begun recently. Embrechts et. al. (1997) is a comprehensive source on extreme value of theory and applications.[2] Also, Bali (2003) has used the GEV distribution to model the empirical distribution of returns, and Gilli and Kellezi (2003) have used extreme value theory for measuring risk.

The objective of this paper is to use the Generalized Extreme Value (GEV) distribution in the context of European option pricing with the view to overcoming the problems associated with existing option pricing models. Within the Harrison and Pliska (1981) asset pricing framework, the risk neutral probability density function (RND, for short) exists under an assumption of no arbitrage. By definition of a no arbitrage equilibrium, the current price of an asset is the present discounted value of its expected future payoff given a risk-free interest rate where the expectation is evaluated by the RND function. Breeden and Litzenberger (1978) were first to show how the RND function can be extracted from traded option prices.

The Black-Scholes (1973) and lognormal based RND models have well known drawbacks. First, the implied volatility smiles or smirks are inconsistent with the constancy required in the lognormal case for volatility across different strikes for options with the same maturity date. Further, this class of models cannot explicitly account for the negative skewness and the excess kurtosis of asset returns. Since, Jackwerth and Rubinstein (1996) demonstrated the discontinuity in the implied skewness and kurtosis across the divide of the 1987 stock market crash - a large literature has developed which aims to extract the RND function from traded option prices so that the skewness and fat tail properties of the distribution are better captured than is the case in lognormal models.

[2] Embrechts (1999, 2000) considers the potential limitations of extreme value theory. Without being exhaustive here, De Haan et. al. (1994) and Danielsson and de Vries (1997) study quantile estimation, Mc Neil (1999) gives an extensive overview of extreme value theory for risk management. See also Dowd (2002, pp.272-2845). The EVIM package of Gencay et. al. (2001) for extreme value analysis in Matlab has become a user friendly tool for financial data analysis.

Pricing biases caused by left skewness of asset returns that cannot be captured in the implied log-normal asset pricing models are now well understood (see, Savickas, 2002, 2004). Typically, in periods when the left skewness of asset prices increases, the Black-Scholes option prices for them will overprice out of the money call options and underprice in the money call options relative to when there is greater symmetry in the distribution function. Further, there is evidence that the option price is highly sensitive to the tail shape change which is a matter that is distinct to its sensitivity to the variance of the price distribution. However, the lack of closed form solutions to the option pricing model, the large number or parameters needed or the lack of easy interpretation of implied parameters have prevented many of the proposed models intended to deal with both the fat tails and the skew in asset prices from being of practical use in pricing and hedging.

This paper argues for the use of the Generalized Extreme Value Distribution (GEV) for asset returns in call option pricing models for the following reasons :

(i) It can provide a closed form solution for the European option price.

(ii) It yields a parsimonious European option pricing model with only three parameters defining the tail shape, location and scale.

(iii) It provides a flexible framework that subsumes as special cases a number of classes of distributions that have been assumed to date in more restrictive settings.

(iv) It most significantly, can deliver the market implied tail index for the asset returns. The latter is found to be time varying in a way that mirrors the lack of invariance in the recursively estimated tail index of asset returns (see, Quintos, Fan and Phillips, 2001) with jumps in the fat tailedness in crisis periods.

(v) Following the convention (see Dowd 2002, p.272) that asset returns are modelled as losses, and when using put options, the GEV model for negative returns yields a Fréchet type implied RND for returns. The corresponding implied RND for the price is left skewed and exhibits a fat tail on the left.

(vi) The success of the GEV based RND for the asset price in removing well known pricing biases associated with the Black Scholes model can be now established precisely in terms of changes in tail shape.

(vii) Having obtained a closed form solution for the option pricing model, we can also obtain a closed form solution for the new "greek" in the lexicon of option pricing, which measures the sensitivity of the option price to the tail index.

(viii) The closed form delta hedging formulation can be given.

This paper covers the first (vi) features listed above of the GEV RND model of option pricing and leave the last two for further work.

We will briefly now comment on how the GEV RND based option pricing model fits into the large edifice, given in Figure 1 below, built of the different methods used for the extraction of the implied distributions and their respective option pricing models that have arisen since the work of Breeden and Litzenberger (1978). See Jackwerth (1999) for an extensive survey. Based on this survey, the different methods can be classified into three main categories: parametric, semi parametric and non-parametric. Parametric methods can be divided into three sub-categories: generalized distribution methods, specific distributions and mixture methods. Generalized distribution methods introduce more flexible distributions with additional parameters beyond the two parameters of the normal or lognormal distributions. Within this subcategory, Aparicio and Hodges (1998) use generalized beta functions of the second kind, which are described by four parameters, and Corrado (2001) uses the generalized Lambda distribution. Under the specific distributions being assumed for the RND function, we find the Weibull distribution by Savickas (2002 and 2004), the skewed Student-t by de Jong and Huisman (2003). The Variance Gamma distribution used by Madan, Carr and Chang (1998), and Levy processes used among others by Matache, Nitsche and Schwab (2004) are more recent specifications with these methods having parameters that can control fat tails and skewness of the of the asset price. Up to seven parameters are associated with these models. Finally, the third sub-category within parametric methods is the mixture methods, which achieve greater flexibility by combining with different probabilities several simple distributions. The most popular method here is mixture of lognormals, which has been used by Ritchey (1990) and Gemmill and Saflekos (2000) using two lognormals, and Melick and Thomas (1997) using three lognormals. One problem associated with the mixture of distributions is that the number of parameters is usually large, and thus they may overfit the data. For example, the mixture of lognormals needs to estimate five parameters.

Under the category of semi parametric methods we find the Hypergeometric function used by Abarid and Rockinger (1997), and expansion methods such as the Gram-Charlier expansions used by Corrado and Su (1997) and Edgeworth expansions used by Jarrow and Rudd (1982). On the other hand, non-parametric methods attempt to achieve greater flexibility in fitting the risk-neutral distribution to option prices. Rather than requiring a parametric form of the distribution, they allow more general functions. The non-parametric methods can be again divided in three groups: kernel methods, maximum-entropy methods, and curve fitting methods. Kernel methods are related to regressions since they try to fit a function to observed data, without specifying a parametric form. Second, the methods based on maximum-entropy find a non-parametric probability distribution that tries to match the information content, while at the same time satisfying certain constraints, such as pricing observed options correctly. In the third group in this category, there are the curve fitting methods that try to fit the implied volatilities or the risk-neutral density with some flexible function. The model presented in this paper, as highlighted in Figure 1, falls in the general category of parametric models, and more specifically, within the sub-category of generalized distributions. In Section 2 of the paper we give a brief introduction on Extreme Value Theory and present the Generalized Extreme Value (GEV) distribution and its properties to indicate how the flexibility of this three parameter class of distribution can capture skew and fat tailedness as and when dictated by the data with no *a priori* restrictions.

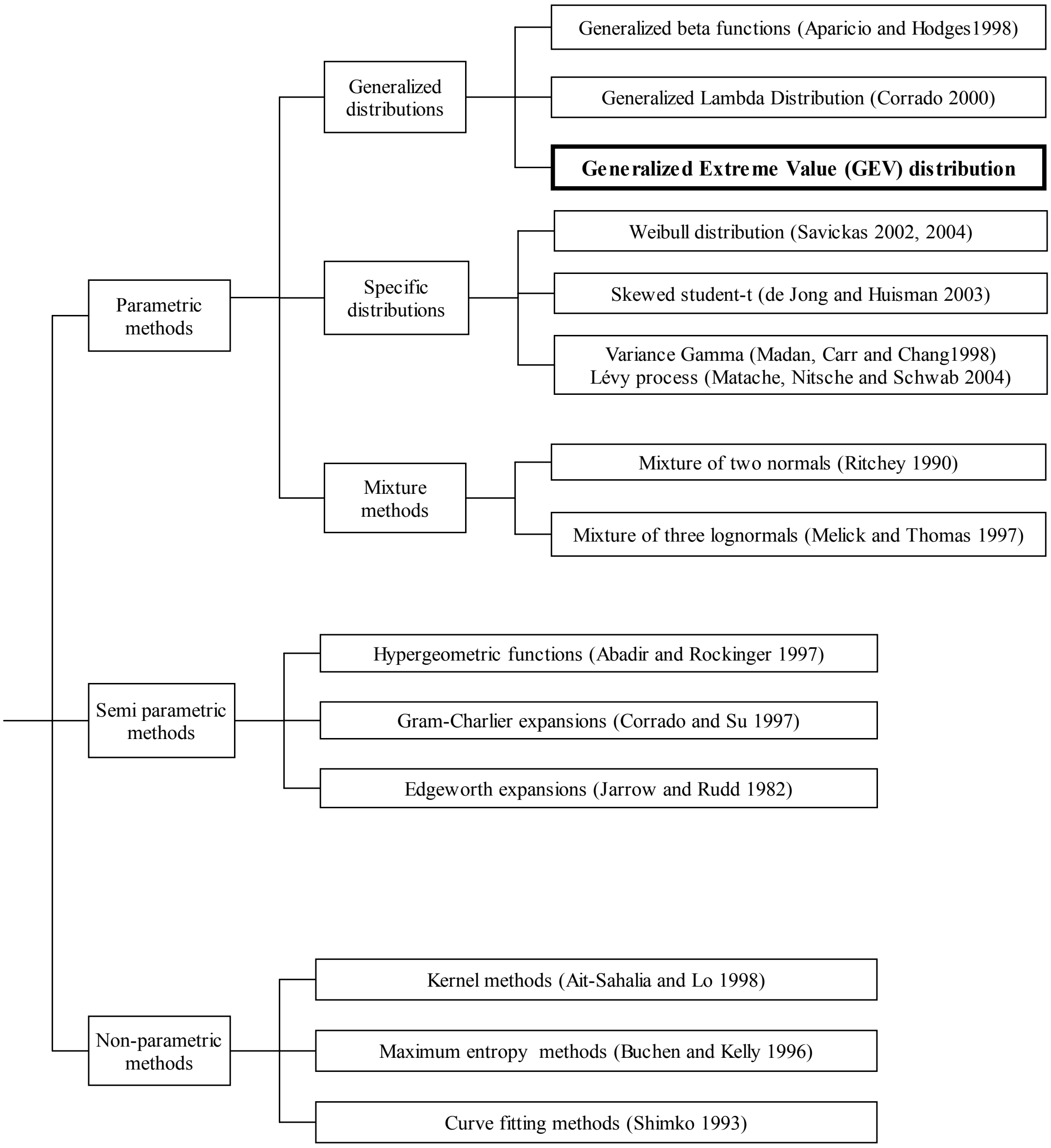


Figure 1: Classification of most common RND estimation methods

The rest of the paper is organized as follows. In section 3.1 the use of the GEV distribution for the RND returns in the context of the Harrision and Kreps (1991) no arbitrage pricing model is discussed. In Section 3.2 the closed form solutions for the arbitrage free European call and put option price equations are derived in the case of the GEV distribution for the RND function. Section 4 reports the empirical results on the estimated implied GEV parameters and RND function for the FTSE 100 European options from 1997 to 2003. In Sections 4.3 and 4.4 the fit of the postulated GEV option pricing model is compared to the benchmark Black-Scholes and is found to be superior at all levels of moneyness and at all time horizons, removing the well known price bias of the Black-Scholes model. In Section 4.6, the analysis of the implied tail indexes is given is given and their role in the event studies surrounding periods of extreme falls in the FTSE-100 index is given. Finally, Section 5 gives the conclusions of the paper.

2. Extreme Value Theory and the GEV distribution

Unlike normal distributions that arise from the use of the central limit theorem on sample averages, extreme value distributions arise from the limit theorem of Fisher and Tippet (1928) on extreme values or maxima in sample data. The class of GEV distributions is very flexible with the shape parameter ξ (and hence the tail index defined as $\alpha = \xi^{-1}$) controlling the shape and size of the tails of the different subclasses of distributions subsumed under it. The distributions associated with $\xi > 0$ are called Fréchet and these include well known fat tailed distributions such as the Pareto, Cauchy, Student-t and mixture distributions. It is generally held that $\xi > 0$ for financial returns data and barring the controversy of the lack of finiteness of higher moments in some cases of this class of distributions3, financial returns data are considered to belong to the Fréchet class of GEV distributions. If $\xi = 0$, the GEV distribution is the Gumbel class and includes the normal, exponential, gamma and lognormal distributions where only the lognormal distribution has a moderately heavy tail. Finally, in the case where $\xi < 0$, the distribution is Weibull. These are short tailed distributions with finite lower bounds and include distributions such as uniform and beta distributions. Figure 2 below illustrates the GEV density function for each the three types of distributions that the GEV can take based on the shape parameter ξ. Note that the three graphs only differ on the value of ξ, having the same value for location and scale parameters. The three families of extreme value distributions can be nested into a single parametric representation, as shown by Jenkinson(1955) and von Mises(1936). This representation is known as the "Generalized Extreme Value" (GEV) distribution and is given by:

$$G(x) = \exp\left(-(1+\xi x)^{-1/\xi}\right) \qquad \text{with} \qquad 1+\xi x > 0, \quad \xi \neq 0 \qquad (1)$$

where ξ is know as the shape parameter and determines the tail-thickness, introduced by Mises von (1936). Recall that, in general, the distribution function G of a random variable X is given by $G(x) = P\{X \leq x\}$. The density g is the derivative of the distribution function, that is $g = G'$, and for the GEV function is found to be:

$$g(x) = \left((1+\xi x)^{-1-1/\xi}\right)\exp\left(-(1+\xi x)^{-1/\xi}\right) \qquad (2)$$

[3] For $\xi > 0$, the expectation $E(X^k)$ for the Generalized Pareto Distribution stochastic process, X, is infinite for $k \geq 1/\xi$. For instance the variance is infinite for $\xi = .5$. Estimates of the tail index from the returns distribution of financial time series has been found to be between 2 and 4 (Lux and Sornette, 2002).

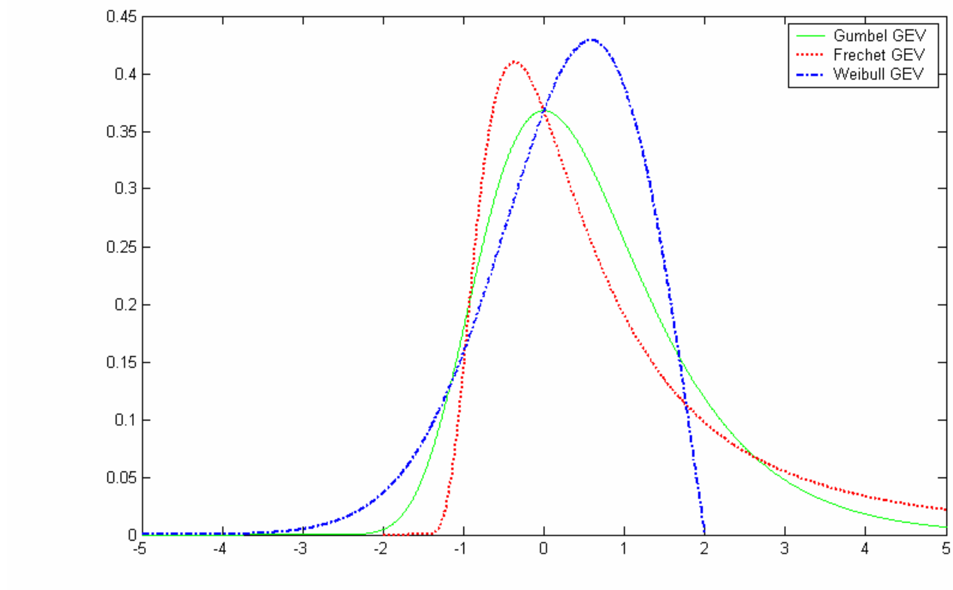


Figure 2: The GEV-Gumbel (ξ=-0.5), GEV-Fréchet (ξ=0.5) and GEV-Weibull (ξ=0) densities

3. The GEV Option Pricing Model

3.1 Arbitrage Free Option Pricing and the Risk Neutral Density

Let S_t denote the underlying asset price at time t. The European call option C_t is written on this asset with strike K and maturity T. We assume the interest rate r is constant. Following the Harrison and Pliska (1981) result on the arbitrage free European call option price, there exists a risk neutral density (RND) function, $g(S_T)$, such that the equilibrium call option price can be written as:

$$C_t(K) = E_t^Q\left(e^{-r(T-t)}\max(S_T - K,0)\right) = e^{-r(T-t)}\int_K^\infty (S_T - K)g(S_T)dS_T \tag{3}$$

where $E_t^Q[\cdot]$ is the risk-neutral expectation operator, conditional on all information available at time t, and $g(S_T)$ is the risk-neutral density function of the future underlying. In an arbitrage-free economy we also have the martingale condition, where the expectation is taken with respect to the RND g(.):

$$S_t = e^{-r(T-t)}E_t^Q(S_T) \tag{4}$$

Similarly, for a put option the arbitrage free option pricing equation is give by:

$$P_t(K) = e^{-r(T-t)}E_t^Q[\max(K - S_T,0)] = e^{-r(T-t)}\int_0^K (K - S_T)g(S_T)dS_T \tag{5}$$

3.2 European Call and Put Option Price with GEV returns

In this paper, we assume that the distribution of asset returns is represented by the GEV distribution, and we derive closed form solutions for the call and put option pricing equations by analytically solving the integrals in (3) and (5).Asset returns are defined as simple returns in order to make possible the derivation of the closed form solutions for the option price:

$$R_T = \frac{S_T - S_0}{S_0} = \frac{S_T}{S_0} - 1 \tag{6}$$

Here R_T is distributed as a GEV density function $f(R_t)$ given in (2). The RND function $g(S_T)$ for the stock price S_t in (3) is then given by the general formula :

$$g(S_T) = f(R_T)\left|\frac{\partial R_t}{\partial S_t}\right| = f(R_T)\frac{1}{S_0} \tag{7}$$

The Fisher-Tippet Theorem deals with the convergence or the limit law of maxima. Suppose the observations $R_1, R_2, R_3, \ldots, R_m$ on the stock returns series R is a sequence of iid random variables from an unknown distribution $F(R^{\sim}) = Pr(R \leq R^{\sim})$ and m is the sample size. Denote the maximum of the first $n<m$ observations of R by:

$M_n = max\ (R_1, R_2, \ldots\ldots R_n)$.

Consider standardized extreme values:

$Y_n = (M_n - \mu_n)/\sigma_n,$

where μ_n is the location parameter in the sample n and σ_n is the scale parameter. Setting $R_t = R^{\sim}$, the distribution of the standardized extreme values of R, as $n \to \infty$, converges to the GEV distribution function $F(R_t)$ with its density function $f(R_t)$ given here in the form in von Mises (1936) (see, Reiss and Thomas, 2001, p. 16-17):

$$f(R_T) = \frac{1}{\sigma}\left(1 + \frac{\xi(R_T - \mu)}{\sigma}\right)^{-1-1/\xi} \exp\left(-\left(1 + \frac{\xi(R_T - \mu)}{\sigma}\right)^{-1/\xi}\right) \tag{8}$$

Note that when the shape parameter $\xi > 0$ the fat tailed distribution is of the type Fréchet, being held to arise in the case of losses and hence negative returns (see Dowd 2002, p.272). On assuming this and substituting (8) into (7), we have the RND function of the stock price:

$$g(S_T) = \frac{1}{S_0\sigma}\left(1 + \frac{\xi(-R_T - \mu)}{\sigma}\right)^{-1-1/\xi} \exp\left(-\left(1 + \frac{\xi(-R_T - \mu)}{\sigma}\right)^{-1/\xi}\right) \tag{9}$$

where the following condition needs to be satisfied $1 + \xi(-R_T - \mu)/\sigma > 0$. When $\xi > 0$, the distribution of S_T is truncated on the right, and therefore, the upper limit of integration for the call option price in (3) is $S_T < S_0(1 - \mu + \sigma/\xi)$ instead of infinity. On the other hand, when $\xi < 0$ the distribution of S_T is truncated on the left, and therefore, the lower limit of integration for the call option price in (3) becomes $S_T > \max[K, S_0(1-\mu+\sigma/\xi)]$.

Consider the change of variable:

$$y = 1 + \frac{\xi}{\sigma}(-R_T - \mu) = 1 + \frac{\xi}{\sigma}\left(1 - \frac{S_T}{S_0} - \mu\right). \tag{10a}$$

The density function in (9) for the underlying price at maturity becomes:

$$g(S_T) = g(y) = \frac{1}{S_0 \sigma}\left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right). \tag{10b}$$

Similarly, the stock price S_T and dS_T can be written in terms of y as follows:

$$S_T = S_0\left(1 - \mu - \frac{\sigma}{\xi}(y-1)\right) \qquad \text{and} \qquad dS_T = -S_0 \frac{\sigma}{\xi}\, dy\,, \tag{10c}$$

The lower limit of integration of the call option equation in (3) becomes: $H = 1 + \frac{\xi}{\sigma}\left(1 - \frac{K}{S_0} - \mu\right)$.

Similarly, the upper limit of integration becomes 0 when substituting $S_0(1-\mu+\sigma/\xi)$ into (10a).

Substituting for $g(S_T)$, as defined in (10b), and for S_T and dS_T ,as defined in (10c), into (3) we have:

$$C_t e^{r(T-t)} = \int_H^0 \left(S_0\left(1 - \mu - \frac{\sigma}{\xi}(y-1)\right) - K\right) \frac{1}{S_0 \sigma}\left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right)\left(-S_0 \frac{\sigma}{\xi}\right) dy \tag{11}$$

Simplifying and rearranging (11) we have:

$$\begin{aligned} C_t e^{r(T-t)} &= -\frac{1}{\xi}\int_H^0 \left(S_0\left(1 - \mu - \frac{\sigma}{\xi}(y-1)\right) - K\right)\left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right) dy \\ &= \frac{1}{\xi}\left[\frac{S_0 \sigma}{\xi}\int_H^0 y\left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right) dy - \left(S_0\left(1 - \mu + \frac{\sigma}{\xi}\right) - K\right)\int_H^0 \left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right) dy\right] \\ &= \frac{1}{\xi}\left[\frac{S_0 \sigma}{\xi}\psi_1 - \left(S_0\left(1 - \mu + \frac{\sigma}{\xi}\right) - K\right)\psi_2\right] \end{aligned} \tag{12}$$

The integral ψ_1 in (12) above can be evaluated in terms of the incomplete Gamma function (see Appendix A for proof), and its solution is:

$$\psi_1 = \int_H^0 y^{-1/\xi} \exp\left(-y^{-1/\xi}\right) dy = -\xi\,\Gamma\left(1-\xi, H^{-1/\xi}\right) \tag{13}$$

The solution of integral ψ_2 in (12) is:

$$\psi_2 = \int_H^0 \left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right) dy = \left[\xi \exp\left(-y^{-1/\xi}\right)\right]_H^0 = \xi\left(-\exp\left(-H^{-1/\xi}\right)\right) \tag{14}$$

Combining results for ψ_1 and ψ_2, we obtain a closed form for the GEV call option price $\xi > 0$

$$C_t(K) = e^{-r(T-t)}\left\{\frac{-S_0\,\sigma}{\xi}\Gamma\left(1-\xi, H^{-1/\xi}\right)-\left(S_0\left(1-\mu+\frac{\sigma}{\xi}\right)-K\right)\left(-e^{-H^{-1/\xi}}\right)\right\} \tag{15}$$

Grouping the terms with S_0 together we have:

$$C_t(K) = e^{-r(T-t)}\left\{S_0\left(\left(1-\mu+\frac{\sigma}{\xi}\right)\left(e^{-H^{-1/\xi}}\right)-\frac{\sigma}{\xi}\Gamma\left(1-\xi, H^{-1/\xi}\right)\right)-K\left(e^{-H^{-1/\xi}}\right)\right\} \tag{16}$$

Similarly, the closed form solution for the put option price under GEV returns in (5) is found to be (details in Appendix A for the proof):

$$P_t(K) = e^{-r(T-t)}\left\{K\left(e^{-h^{-1/\xi}}-e^{-H^{-1/\xi}}\right)-S_0\left(\left(1-\mu+\sigma/\xi\right)\left(e^{-H^{-1/\xi}}-e^{-h^{-1/\xi}}\right)-\frac{\sigma}{\xi}\Gamma\left(1-\xi, h^{-1/\xi}, H^{-1/\xi}\right)\right)\right\} \tag{17}$$

4. Results

4.1 Data description

The data used in this study are the daily settlement prices of the FTSE 100 index call and put options published by the London International Financial Futures and Options Exchange (LIFFE). These settlement prices are based on quotes and transactions during the day and are used to mark options and futures positions to market. Options are listed at expiry dates for the nearest four months and for the nearest June and December. FTSE 100 options expire on the third Friday of the expiry month. The FTSE 100 option strikes are in intervals of 50 or 100 points depending on time-to-expiry, and the minimum tick size is 0.5. There are four FTSE 100 Futures contracts a year, expiring on the third Friday of March, June, September and December.

To synchronize the maturity dates for Futures and Options we only consider options with the same four maturity dates as the FTSE 100 futures contracts. The period of study was from 1997 to 2003, so there were 32 expiration dates (7 years with 4 contracts per year). This period includes some events, such as the Asian crisis, the LTCM crisis and the 9/11 attacks, which will be used to analyze how the implied RNDs behave before/after such events, which resulted in a sudden fall of the underlying FTSE 100 index. Table 1 below summarizes the average number of different daily strikes for each of the years in the period under study, including both call and put options.

Period	Number of daily strikes
1997	20
1998	26
1999	31
2000	35
2001	39
2002	38
2003	34
All Years	32

Table 1: Average number of strikes per year

The European-style FTSE100 options, even though they are options on the FTSE 100 index, can be considered as options on the futures on the index, because the futures contract expires at the same date as the option. Therefore, the futures will have the same value as the index at maturity, and can be used as a proxy of the underlying FTSE 100 index. By using this method, we avoid having to use the dividend yield of the FTSE 100 index, and the martingale condition in (4) becomes:

$$F_t = E_t^Q(S_T) \tag{18}$$

where F_t is the price of the FTSE 100 futures contract at t, and S_T is the FTSE 100 index at maturity T.

The LIFFE exchange quotes settlement prices for a wide range of options, even though some of them may have not been traded on a given day. In this study we only consider prices of traded options, that is, options that have a non-zero volume. The data were also filtered to exclude days when the cross-sections of options had less than three option strikes, since a minimum of three strikes is required to estimate the three parameters of the GEV model. Also, options whose prices were quoted as zero or that had less than 2 days to expiry were eliminated. Finally, option prices were checked for violations of the monotonicity condition.[4]

The risk-free rates used are the British Bankers Association's 11 a.m. fixings of the 3-month Short Sterling London InterBank Offer Rate (LIBOR) rates from the website www.bba.org.uk. Even though the 3-month LIBOR market does not provide a maturity-matched interest rate, it has the advantages of liquidity and of approximating the actual market borrowing and lending rates faced by option market participants (Bliss and Panigirtzoglou 2004).

The option data used in this study can be divided into 6 moneyness categories, following the classification in Bakshi, Cao and Chen (1997). Note that moneyness is defined as S/K. A call option is *out-of-the-money* (OTM) if S/K is smaller than 0.97; at the money (ATM) if S/K is greater than 1.03, and *at-the-money* when S/K is in the range (0.97, 10.3). On the other hand, a put option is *out-of-the-money* (OTM) if S/K is greater than 1.03, *in-the-money* if S/K is smaller than 0,97, but *in-the-money* is defined as with call options. An additional classification is done in terms of days to expiration: short term (less than 30 days to expiration), medium term (30-60 days), and long term (60-90). There are options data available for time to expiration longer than 90 days, but the number of prices available for such long time horizons is smaller. Tables 2a and 2b below report the average option price for each category, and the number of option observations in that range (shown in braces).

[4] Monotonicity requires that the call (put) prices are strictly decreasing (increasing) with respect to the exercise price.

	Moneyness S/K	Days to Expiration <30	30-60	60-90	Subtotal
OTM	<0.94	£6.47	£20.27	£35.61	
		{1327}	{2599}	{2219}	{6145}
	0.94-0.97	£19.66	£59.75	£100.55	
		{1416}	{1494}	{915}	{3825}
ATM	0.97-1	£51.85	£117.94	£170.36	
		{1655}	{1436}	{895}	{3986}
	1-1.03	£130.15	£192.56	£250.25	
		{1370}	{1046}	{695}	{3111}
ITM	1.03-1.06	£248.87	£293.68	£355.07	
		{819}	{562}	{303}	{1684}
	>1.06	£584.89	£631.84	£702.52	
		{993}	{715}	{493}	{2201}
Subtotal		{7580}	{7852}	{5520}	{20952}

Table 2a: Sample properties of call options in period 1997-2003.

As can be seen, the medium term time to maturity category has the greatest number of data points, followed by the short term and then long term, for both puts and calls. In terms of moneyness, the largest number of observations is found in the OTM category (47% of total number of observations), and with an average price of . For put options, the largest number of observations is also found in the OTM category, accounting for 60% of the total number of observations.

	Moneyness S/K	Days to Expiration <30	30-60	60-90	Subtotal
ITM	<0.94	£704.87	£640.78	£775.36	
		{1128}	{579}	{552}	{2259}
	0.94-0.97	£255.18	£297.75	£348.61	
		{746}	{558}	{280}	{1584}
ATM	0.97-1	£129.78	£193.59	£248.69	
		{1362}	{987}	{578}	{2927}
	1-1.03	£58.70	£129.17	£184.05	
		{1583}	{1352}	{813}	{3748}
OTM	1.03-1.06	£30.06	£82.74	£133.13	
		{1378}	{1225}	{744}	{3347}
	>1.06	£11.96	£29.19	£50.13	
		{3416}	{5375}	{3880}	{12671}
Subtotal		{9613}	{10076}	{6847}	{26536}

Table 2b: Sample properties of put options in period 1997-2003.

4.2 Methodology

For each expiry date listed in Table 4 below, a target observation date was determined with horizons of 90, 60, 30 and 10 days to maturity. If no options were traded on the target observation date, the nearest date with traded options was used. The estimation of the implied RND was conducted using the GEV model and the Black-Scholes model, for each of these dates, separately for calls and for puts.

The structural GEV parameters ξ, μ and σ were estimated by minimizing the sum of squared errors (SSE) between the analytical solution of the GEV option pricing equations in (16) and (17) and the observed traded call prices with strikes K_i. The optimization was performed using the non-linear least squares algorithm from the Optimization toolbox in Matlab.

$$SSE(t) = \min_{\zeta,\mu.\sigma} \sum_{i=1}^{N} \left(C_t(K_i) - \widetilde{C}_t(K_i) \right)^2 \qquad (19)$$

4.3 Pricing performance

The pricing performance of the GEV and Black-Scholes model is reported in Tables 3 and 4 below in terms of the root mean square error RMSE, which represents the average pricing error in pence per option.

$$RMSE(t) = \frac{1}{N}\sqrt{SSE(t)} \qquad (20)$$

	90		60		30		10	
	BS	GEV	BS	GEV	BS	GEV	BS	GEV
Mar-97	20.47	0.25	9.13	1.19	7.96	0.2	3.74	0.58
Jun-97	7.16	0.38	1.4	0.21	8.68	0.83	5.72	0.42
Sep-97	4.97	0.35	5.67	0.47	7.57	0.49	3.19	0.61
Dec-97	37.63	0.38	19.09	1.5	5.69	1.48	2.72	0.6
Mar-98	7.36	2.02	6.84	2.69	5.17	0.45	6.85	1.25
Jun-98	47.86	0.16	20.24	1.47	3.33	0.6	4.62	0.94
Sep-98	3.73	0.31	3.47	0.7	4.63	1.07	5.21	1.74
Dec-98	23.63	3.14	14.07	1.46	8.16	1.48	2.66	0.29
Mar-99	20.34	7.23	13.8	3.08	7.66	2.3	4.53	0.68
Jun-99	9.55	1.52	13.13	1.13	5.33	0.83	2.47	0.6
Sep-99	10.85	1.62	10.37	1.55	7.89	0.63	2.99	0.28
Dec-99	5.38	0.57	17.62	3.7	6.58	2.29	3.13	0.51
Mar-00	7.65	0.69	6.23	0.67	7.35	1.36	2.21	0.6
Jun-00	7.31	2.49	10.1	0.13	5.35	0.4	4.45	0.35
Sep-00	7.48	0.49	14.53	0.51	18.63	0.57	5.92	0.23
Dec-00	2.39	0.54	11.58	1.24	2.7	0.28	4.8	0.24
Mar-01	5.58	1.05	3.6	0.36	3.84	0.32	3.62	0.99
Jun-01	5.93	0.56	10.12	0.48	3.15	0.56	1.55	0.32
Sep-01	3.19	0.19	8.82	0.73	2.33	0.25	7.68	0.82
Dec-01	9.87	1.37	11.72	3.22	4.62	1.24	1.19	1.26
Mar-02	22.26	0.78	2.72	0.27	2.28	0.5	2.28	0.21
Jun-02	1.24	0.38	1.77	0.45	3.23	0.67	0.94	0.31
Sep-02	9.3	0.95	9.61	0.25	7.22	1.16	4.58	0.57
Dec-02	17.01	1.62	11.58	0.93	4.75	0.93	3.77	1.25
Mar-03	17.51	0.91	10.15	0.93	4.29	0.61	1.68	0.89
Jun-03	6.1	0.5	4.9	1.02	4.53	0.2	2.27	0.67
Sep-03	3.89	0.22	2.54	0.24	1.21	0.63	1.36	0.8
Dec-03	2.49	0.91	7.46	2.93	2.62	1.35	1.2	0.65
Average	**11.72**	**1.13**	**9.37**	**1.20**	**5.60**	**0.85**	**3.48**	**0.67**

Table 3: RMSE for call options in pence

The GEV option pricing model outperforms the Black-Scholes model at all time horizons. Both models consistently display an improvement in performance as time to maturity decreases. Specially, the GEV model removes the pricing bias that the Black-Scholes (BS) model exhibits for options far from maturity, with the BS model having an average error of 11.72 pence per option at 90 days to maturity, while the GEV model has an average error of 1.13 pence per option, representing a 96% reduction. For options very close to maturity, the average price per option with the BS model is 3.48 pence, while with the GEV model is 0.67 pence. Even though the BS model improves considerable for options close to maturity, the GEV model still represents an 80% reduction in pricing error.

A similar result is obtained for put options, shown in Table 4 below, even though the average error per option is slightly greater for puts than for calls, at all times to maturity and for both models. The pricing performance improves in both models as time to maturity decreases, with the GEV model removing the pricing bias that the BS model exhibits for options far from maturity.

	90		60		30		10	
	BS	GEV	BS	GEV	BS	GEV	BS	GEV
Mar-97	1.11	0.14	2.99	1.27	1.88	0.31	1.66	0.31
Jun-97	5.27	0.24	4.96	0.15	3.78	0.38	1.03	0.41
Sep-97	3.08	0.36	3.45	0.02	3.9	0.2	1.69	0.55
Dec-97	6.31	0.37	6.04	0.28	10.59	0.91	3.16	0.44
Mar-98	20.38	3.85	11.27	0.24	6.08	0.54	2.43	0.6
Jun-98	11.57	0.13	13.29	2.32	10.01	0.93	3.41	0.45
Sep-98	13.52	3.02	10.4	1.19	9.89	1.58	6.83	1.5
Dec-98	38.22	4.35	28.57	1.49	13.82	1.42	4.64	0.78
Mar-99	29.75	0.41	19.36	2.4	11.53	1.84	3.5	0.48
Jun-99	22.64	2.97	15.14	1.32	10.25	1.38	3.12	0.58
Sep-99	24.35	2.19	17.97	0.91	11.37	1.06	3.56	0.47
Dec-99	22.25	2.57	25.68	4.67	7.79	1.39	2.79	0.24
Mar-00	15.9	1.04	14.41	1.31	9.81	1.53	2.1	0.84
Jun-00	13.99	0.81	17.35	1.07	6.53	0.88	2.72	0.33
Sep-00	8.33	0.62	4.37	0.49	1.14	0.38	5.19	0.3
Dec-00	6.72	0.97	9.04	0.78	8.45	0.97	5.38	0.29
Mar-01	15.48	1.08	9.76	0.57	4.81	0.87	5.71	1.21
Jun-01	12.7	0.81	8.17	1.57	5.79	2.25	4.94	0.62
Sep-01	7.08	0.1	10.75	0.71	4.45	0.2	21.16	5.15
Dec-01	23.11	3.17	18.89	1.75	8.22	1.96	4.86	1.63
Mar-02	16.19	1.47	6.76	0.65	7.42	1.55	1.68	0.24
Jun-02	6.46	0.41	9.02	1.84	8.57	2.01	4.26	1.08
Sep-02	13.99	1.16	18.34	1.07	9.58	1.64	4.86	0.47
Dec-02	24.63	1.54	15.98	1.78	8.97	0.78	4.98	0.79
Mar-03	20.72	1.49	14.05	1.07	8.25	1.28	3.34	1.43
Jun-03	14.04	1.07	9.87	0.74	6.68	0.64	1.54	0.24
Sep-03	8.76	0.67	6.41	0.62	4.25	1.24	2.39	1.16
Dec-03	9.84	0.85	10.89	1.48	5.17	0.6	1.52	0.85
Average	**14.87**	**1.35**	**12.26**	**1.21**	**7.46**	**1.10**	**4.09**	**0.84**

Table 4: RMSE for put options in pence

4.4 Removal of pricing bias

It has been well documented that the Black-Scholes model exhibits a pricing bias for out of the money and in the money options, while pricing quite accurately in the money options (Rubinstein, 1985). The pricing bias was calculated for each of the 32 maturities, both for the Black and Scholes and the GEV model at two time horizons: 90 days and 10 days to maturity. The pricing bias is defined in equation (29) as the deviation of the calculated price with respect to the observed market price:

$$\textit{Price bias} = \textit{Market price} - \textit{Calculated price} \qquad (21)$$

To summarize the results, the pricing bias was averaged across the 32 maturities at each level of moneyness. The average pricing bias for call options are plotted below in Figure 3a for a 90 days time horizon and in Figure 3b for a 10 days time horizon. In keeping with the results obtained in the previous section, the Black-Scholes model shows deterioration in pricing accuracy for far from maturity contracts. It can be seen that the GEV model outperforms the Black-Scholes model at all levels of moneyness, and for both close to maturity and far to maturity. At far from maturity, the Black-Scholes model overprices OTM options, and underprices ATM and ITM options. The Black-Scholes model displays the highest pricing bias for deep OTM options, with an average error of 20 pence per option. In the same figure we can see that the GEV model greatly removes this pricing bias, giving an error of 2 pence for deep OTM options, and around 1 pence at the other levels of moneyness.

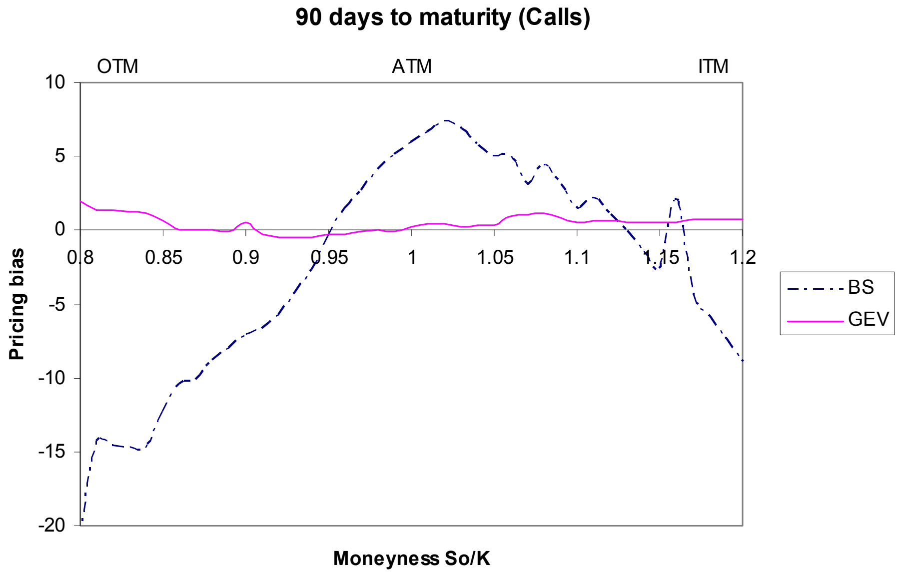


Figure 3a: Average price deviations in terms of moneyness for call options 90 days to maturity

In Figure 3b, we can see that for close to maturity options, the BS model overprices OTM and ITM options, and underprices ATM options, but with the pricing bias oscillating only between +2 pence and -5 pence. The GEV model displays a pricing bias that oscillates between ±1 pence, underpricing OTM and ITM options, and overpricing ATM options. Both models display a reduction in pricing bias as time to maturity decreases.

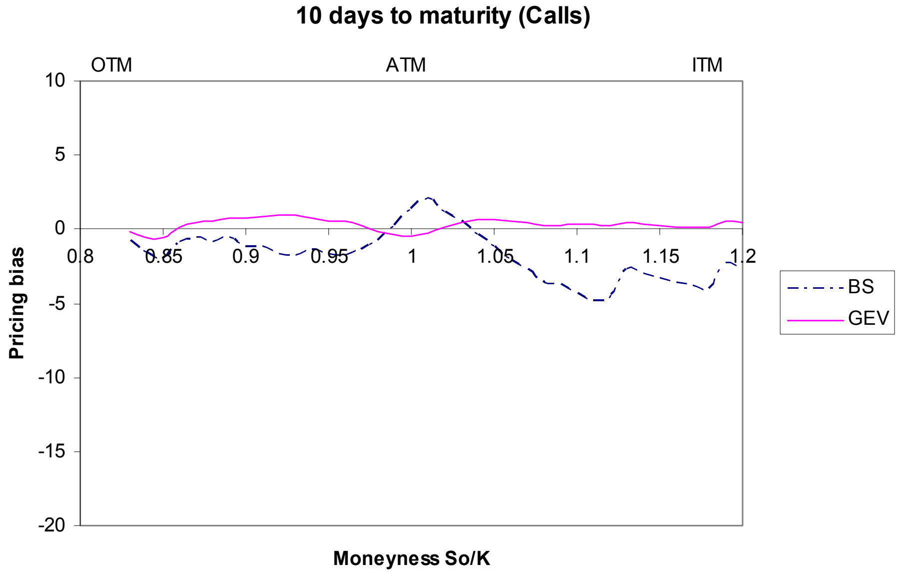


Figure 3b: Average price deviations in terms of moneyness for call options 10 days to expiration

Figures 4a and 4b display the pricing bias for put options. From Figure 4a we can see that for far from maturity put options (90 days to maturity) the BS model overprices ITM and ATM put options, and underprices OTM put options, with pricing bias oscillating between -25 pence for ITM options, and +20 pence for OTM options. On the other hand, the GEV model slightly overprices ITM options by a maximu of 1.14 pence, and slightly underprices OTM put optios by a maximum of 0.82 pence. For close to maturity options, Figure 4b shows that BS model underprices ITM and OTM put options by a maximum of around 5 pence, and overprices ATM put options by 3.5 pence. The GEV model displays a pricing error that oscillates between ±1.3 pence, overpricing ITM put options and overpricing OTM put options .

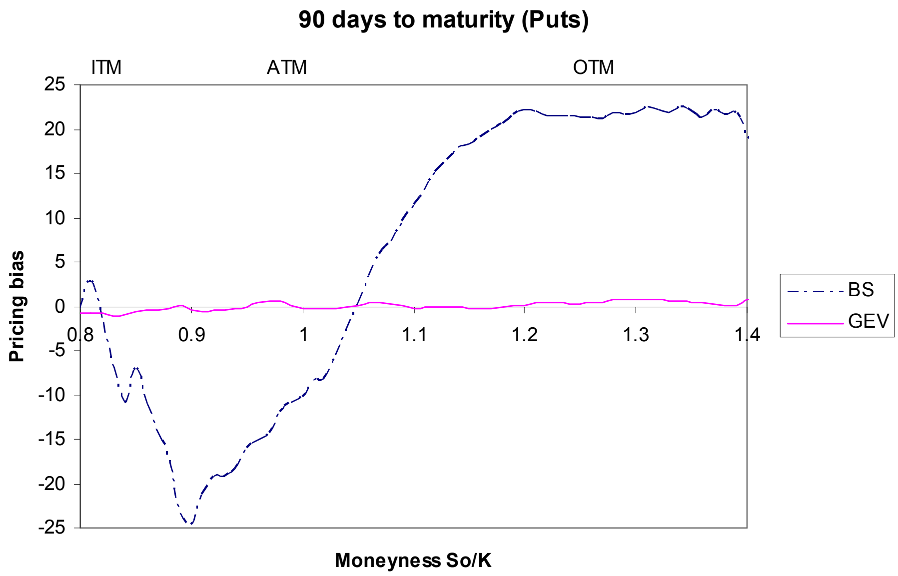


Figure 4a: Average price deviations in terms of moneyness for puts 90 days to expiration

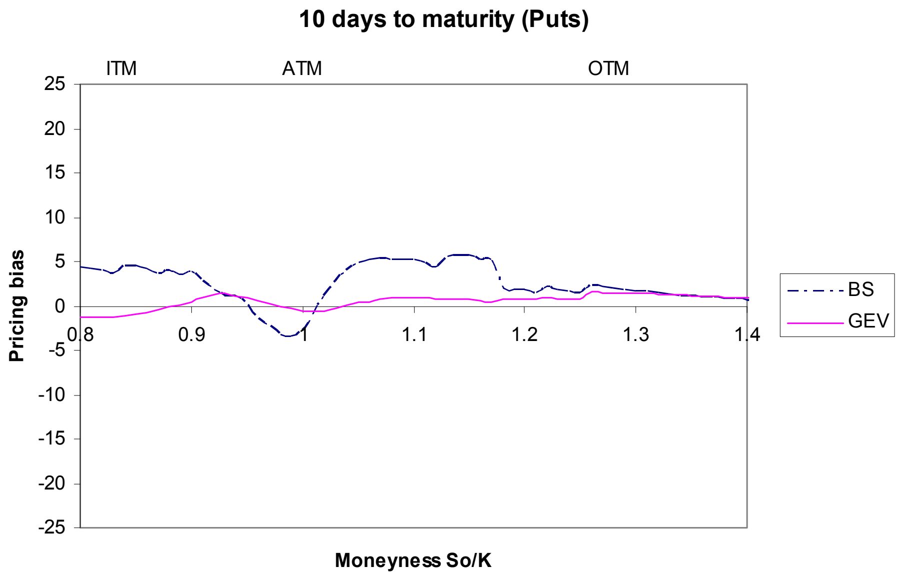
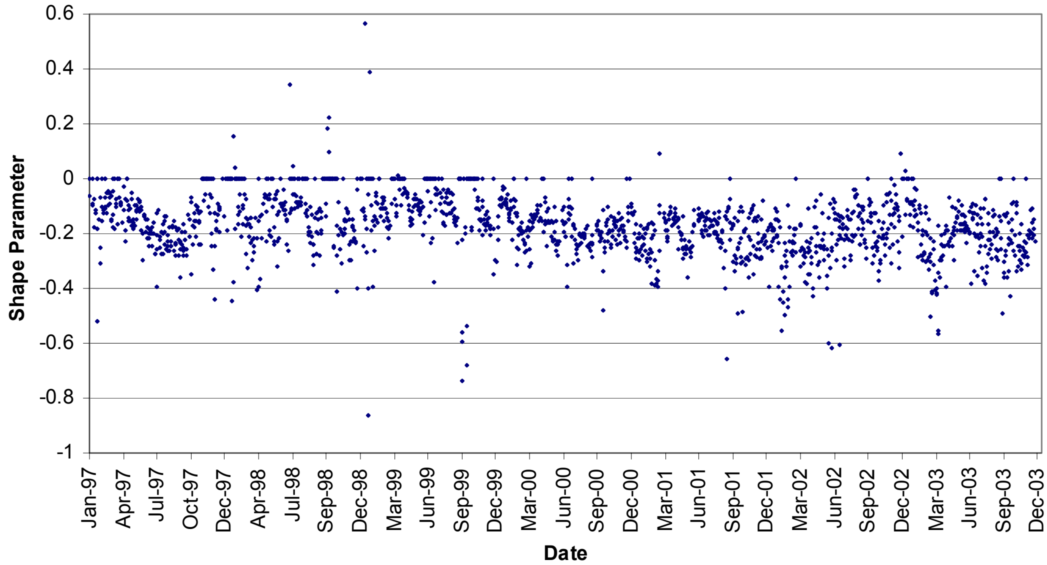


Figure 4b: Average price deviations in terms of moneyness for puts 10 days to expiration

4.5 Implied tail indices

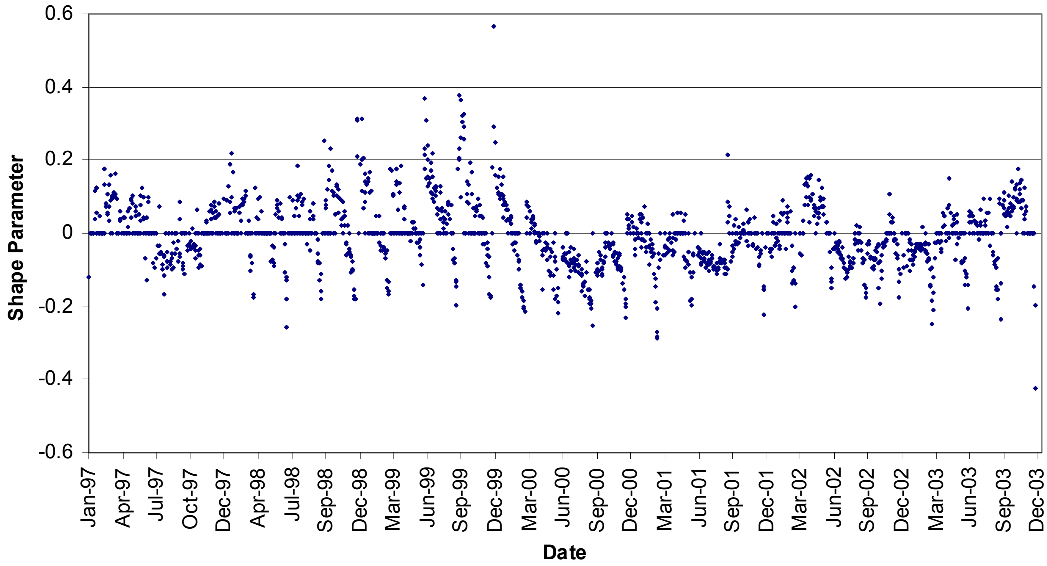
The time series of the implied GEV shape parameter ξ for negative stock returns defined in equation (8) are displayed in Figure 5a for puts options and Figure 5b for call options. Only the values of ξs that are statistically significant different from zero at 95% confidence interval are displayed.

Figure 5a: Shape parameter ξ for call options from GEV returns 1997 – 2003



Thus, as can be see from the Figures 5a and 5b, fat tails in negative returns are best captured when using put options, the reason being is that there is a larger number of traded contracts at low strike prices (the OTM puts see Table 2b), which are used to protect portfolios against downside moves of the index. Figure 5a shows that in contrast, for call options there are relatively few statistically significant values of ξ >0.

Figure 5b: Shape parameter ξ for put options from GEV returns 1997 - 2003



We will now further explain the difference of estimating the RND function when using put options or when using call options. We choose a typical day, in terms of the number of options traded, range of moneyness, and value of the implied ξs , to illustrate the difference in the estimated density functions. Figure 6 below shows a composite picture of the implied RNDs extracted from traded options for calls (solid line) and puts (dotted line) separately. The implied shape parameter ξ was -0.348 when using call options, and 0.031 when using put options. Note how when ξ is positive (for put options) the density of the price is of Weibull type, implying a fat tail on the left side (the losses), and a truncation on the right side (the gains). On the other hand, when ξ is negative (for call options) the density of the price is of Fréchet type, implying a fat tail on the right side (the gains), and a truncation point on the left side (the losses). To obtain the asymmetry of the right and left tails of the RND distribution, we recommend the use of this composite method as tail sensitive information for the left tail is effectively obtained from the heavily traded out of the money puts (Table 2b) and similarly the bulk of the information for the implied right tail comes from the out of the money calls (Table 2a). For other RND extraction methods given in Figure 1 that are not so reliant on an explicit tail shape parameter and where the use of extrapolation methods are in vogue, we can expect to find coarser information on the tail shape.

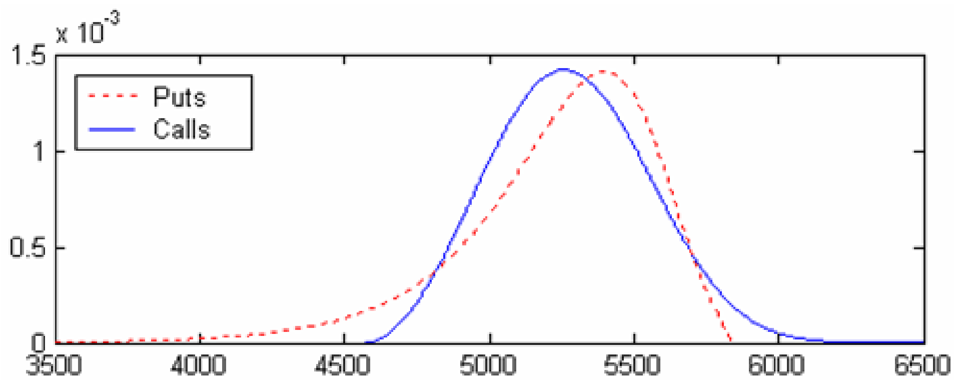


Figure 6: Implied RNDs of the price using calls and puts on 22 April 2002

The number and range of traded options for 22 April 2002 are displayed in Table 5 below. The value of the FTSE 100 index was 5236.5, which is in between both ranges of put and call strikes. However, there were put options traded at much lower strikes, the out of the money put options trading down to 4125. These put options with such low strikes are the ones that determine the exact shape of the implied value of the shape parameter and hence of the market implied the left tail of the underlying. Put options with high strikes give very little information in this regard, since its price is usually the intrinsic value of the option (i.e. $K - S_T$). If we compare the range of strikes traded on that day, with the RNDs from Figure 6, we can see that call options can not be used to estimate the left tail of the price distribution because the lowest strike available is 5125, which is quite close to the central part of the distribution. Similarly, the call options trading at high strikes are the ones that determine the shape parameter of the right hand side.

Contracts	Number of traded strikes	Minimum strike	Maximum strike
Call options	13	5125	5825
Put options	19	4125	5825

Table 5: Summary of call and put options traded on 22 April 2002

4.6 Event study

In this section the changes in implied RNDs before and after special events are compared. The major events that occurred within the period of study (1997 – 2003) are the Asian Crisis, the LTCM crisis and 9/11 events. The implied distribution are obtained by using put options only, because, as explained in the previous section, the implied tail index is best captured when using put options, since put options are used for protecting against downside risk.

The Asian Crisis

The Asian Crisis has been pint pointed to happen around 20th October 1997. Figure 7 below displays the implied RNDs using the GEV model and using the Black-Scholes model. Note that in both cases the GEV density exhibits a fatter than normal left tail. A higher kurtosis indicates a fatter tails; therefore probabilities of extreme events are larger. Both statistics reflect an increased fear of further market declines. Note that the two right hand side figures display the implied return distributions, and since we have modelled negative returns, losses are on the positive side, and profits are on the negative side of the x axis. In both cases, before and after the Asian crisis, the implied densities show a fat tail on the losses, implying higher than normal probabilities of downside moves, with these probabilities increasing after the Asian crisis.

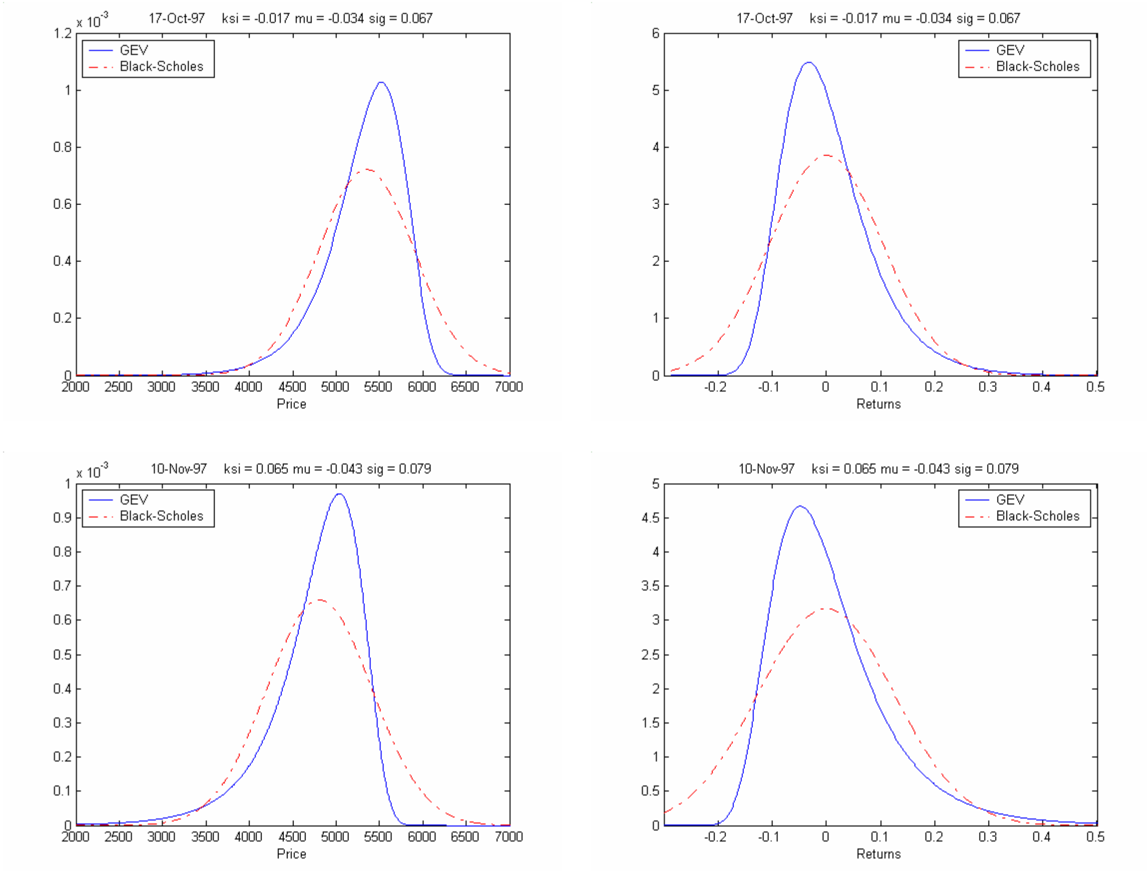


Figure 7: Implied RNDs before and after the Asian Crisis

The LTCM crisis

In early 1998. LTCM controlled over \$100 billion with its portfolio, while the net asset value was only \$4 billion. Also, LTCM's swaps position was 5% of the entire global market, with its portfolio valued at \$1.25 trillion (Risk, October 1998). On 17 August 1998 Russia devaluated the rouble. This resulted in a liquidity crisis which severely affected LTCM's portfolio. LTCM's equity had decreased to \$2.3 billion by 1 September of that year, and by 22 September was only one forth of that at \$600 million. On 23 September 1998, the Federal Reserve Bank of New York organized a rescue package to prevent a . A group of major banks and LTCM's creditors inject \$3.5 billion into the fund, received 90% of LTCM's equity and take over its management. Figure 8 below shows the implied RNDs extracted from option prices before the major events happened, on 14 September 1998, and after, on 24 September 1998. The options used had expiration in December 1998. The densities extracted from both the GEV model and the corresponding Black-Scholes density. The shape parameter ξ changed from 0.0191 before the LTCM crisis, to 0.254 on the 24^{th} of September. The skewness of the GEV distribution changed from 1.25 to 5.70, and the kurtosis from 6.01 to 188.92 the day of the event. This indicates that the market expectations changed considerably after the 24^{th} of September, expecting more downward moves of the index.

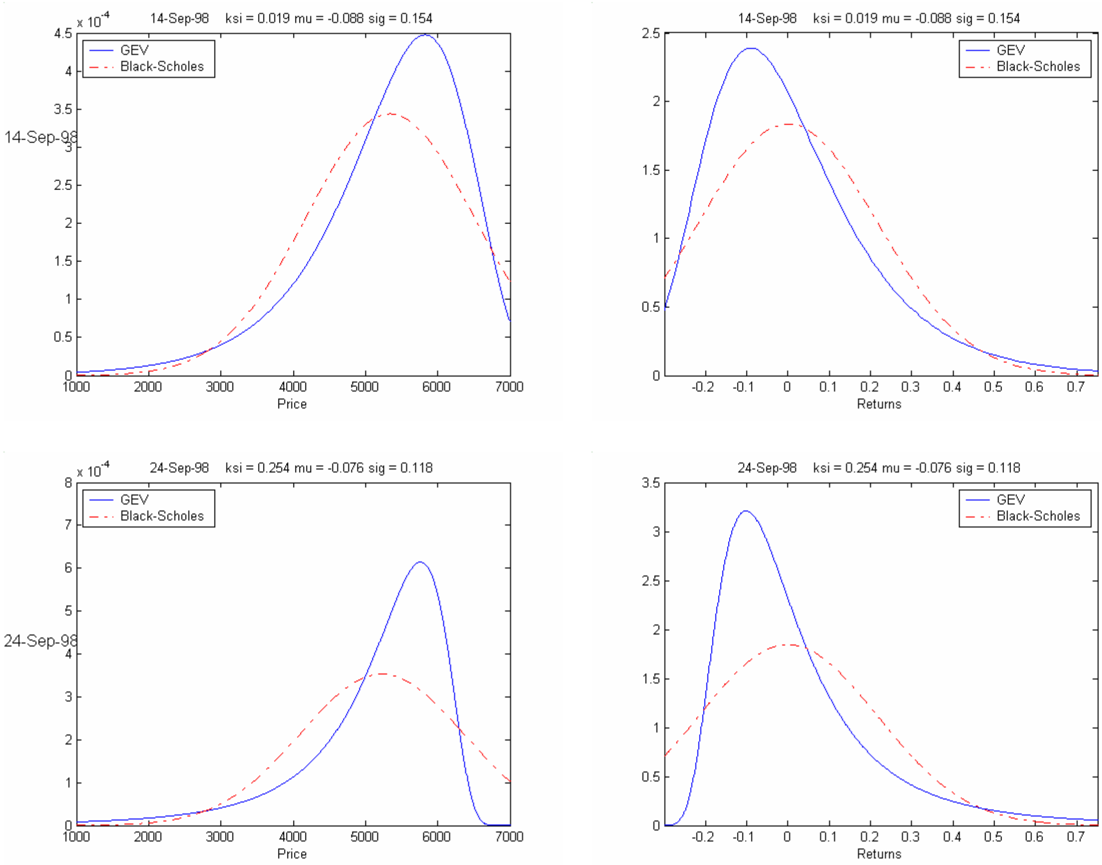


Figure 8: Implied RNDs before and after the LTCM Crisis

The 11th September 2001 events

Figure 9 shows the implied RND extracted from option prices with expiration on September 2001, for the GEV model, and the corresponding Black-Scholes density. The shape parameter ξ changed from -0.0953 the day before the events, implying no fat tails, to 0.2151. As can be seen in Figure 9 below, the tail of the GEV distribution is fatter than the equivalent BS distribution for the 11 of September. The skewness of the GEV distribution changed from 0.65 to 3.98, and the kurtosis from 3.62 the day before the events (almost equal to the kurtosis of the normal distribution) to 69.53 the day of the event. This indicates that the market expectations changed immediately, expecting more downward moves of the index.

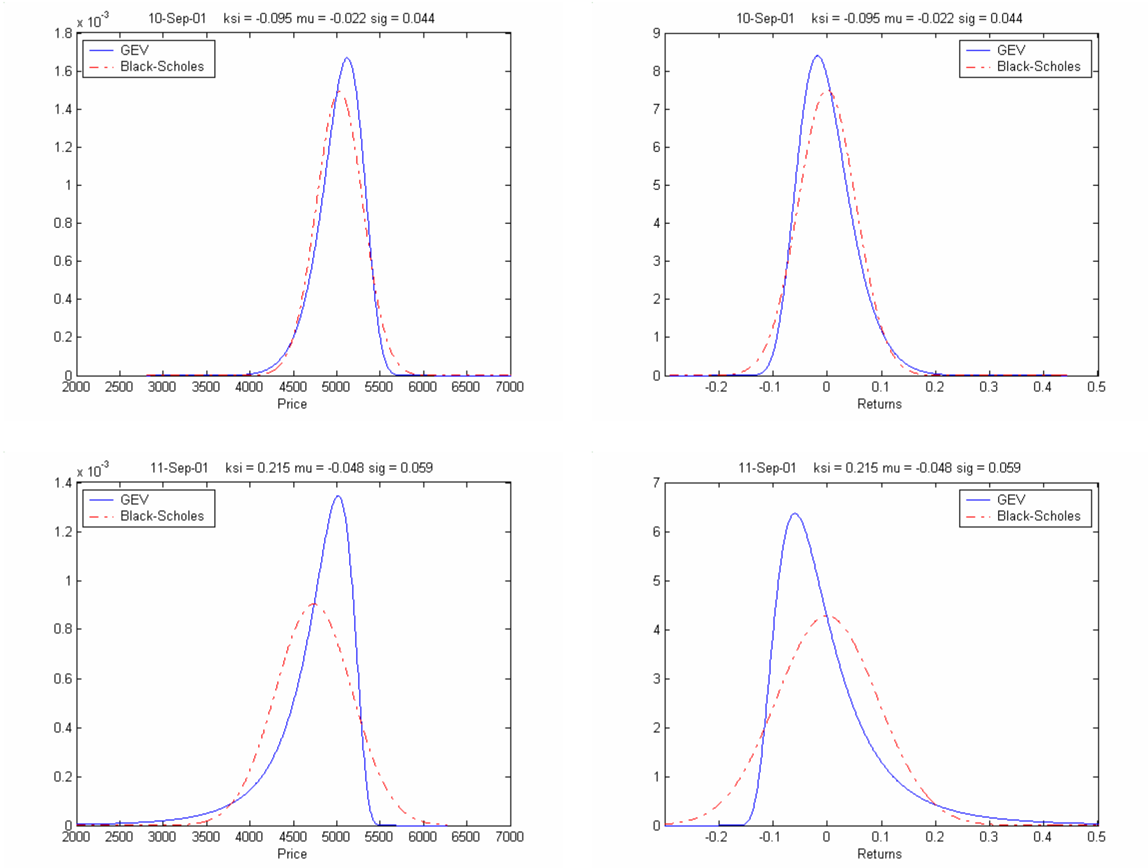


Figure 9: Implied RNDs before and after the 9/11 events

Implied Moments of distributions around special events (using put options only)

Table 6 below summarizes the higher moments of the GEV distribution and the BS distribution for each of the 3 events, before and after. Skewness is the third central moment of the implied pdf standardised by the third power of the standard deviation. It provides a measure of asymmetry. It measures the relative probabilities above and below the mean outcome. Kurtosis is the fourth central moment of the implied pdf standardised by the fourth power of the standard deviation. It provides a measure of the degree of 'fatness' of the tails of the implied pdf. Fatter tails in pdfs are usually associated with a greater degree of 'peakedness' in the centre of the pdf.

Event	Expiry	Date	ξ	Mean	Variance	Skewness	Kurtosis	BS var
Asian Crisis	Dec-97	17-Oct-97	-0.0168	0.0040	0.0071	1.0425	4.9539	0.0107
		10-Nov-97	0.0650	0.0079	0.0122	1.5916	8.1784	0.0158
LTCM	Dec-98	14-Sep-98	0.0191	0.0045	0.0413	1.2584	6.0132	0.0472
		24-Sep-98	0.2540	0.0304	0.0613	5.7034	188.9237	0.0467
9/11	Sep-01	10-Sep-01	-0.0953	0.0001	0.0026	0.6578	3.6224	0.0028
		11-Sep-01	0.2151	0.0017	0.0123	3.9833	69.5357	0.0087

Table 6: Implied moments of distributions around special events

In all three events, the shape parameter ξ increases after the event, which indicates that the implied distributions reflect the market sentiment of increased fear of downward moves. However, as noted in previous studies (Gemmill and Saflekos, 2000), the implied RNDs do not predict the downward moves. Additionally, after each of the three events, skewness increases, indicating a higher asymmetry in the returns distribution. Similarly, kurtosis also increases, becoming much larger than 3 (the kurtosis of the normal distribution), indicating that the increase in probability density, in the tails of the distribution.

5. Conclusions

We proposed a new option pricing model that is based on the GEV distribution of the returns. Closed form solutions for the Harrison and Pliska (1981) no arbitrage equilibrium price for the European call and put options were derived. When applying the solutions of the GEV option pricing model to price options on the FTSE 100, it was found that the price deviations from market prices were substantially smaller than the ones from the Black-Scholes, at all times to maturity and at all level of moneyness. The implied RNDs exhibited a consistent negative skewness and leptokurtosis. As discussed in Melick and Thomas (1997 and 1999), outside the lowest and highest available strike prices there is an infinite variety of probability mass that can be consistent with the observed option prices. However, the shape-differences on the implied RNDs of the GEV model with the lognormal density of the Black-Scholes model are within the range of available strikes. Given that the GEV model has smaller price deviations than the Black-Scholes model, we can conclude that the GEV option pricing model presented in paper substantially reduces the bias introduced by the Black-Scholes model. A more rigorous comparison of the results obtained with the GEV option pricing model against the other parametric models of option pricing that have been proposed (mixture of lognormals, Weibull, Gamma distributions, etc) and against non-parametric models should be done. However, this comparison is likely to conclude that the accuracy of the pricing model is directly related to the number of parameters of the model. A model that has as many parameters as prices to estimate can easily produce a zero error. As previously said, one of the main advantages of the GEV model is the small number of parameters that need to be estimated, which is only three.

Traded put prices help estimate the tail parameter for the left tail of the price distribution, while traded call prices help estimate the tail parameter for the right tail of the price distribution. As argued in Section 4.5, to obtain the asymmetry of the right and left tails of the RND distribution, we recommend the use of this composite method as tail sensitive information for the left tail is effectively obtained from the heavily traded out of the money puts (Table 2b) and similarly the bulk of the information for the implied right tail comes from the out of the calls (Table 2a).

In all three event studies, the shape parameter ξ increases after the event, which indicates that the implied distributions reflect the market sentiment of increased fear of downward moves. However, as noted in previous studies (Gemmill and Saflekos, 2000), the implied RNDs do not predict the downward moves but only reflect it. Additionally (see, Table 6) after each of the three events, left skewness increases and kurtosis also increases, becoming much larger than 3 (the kurtosis of the normal distribution).

Future work will analyse the hedging properties of the GEV RND model of option pricing. Further the implied tail index and the GEV implied underlying price has uses in risk management especially in for calculating Value-at-Risk (VaR). Indeed, we can obtain the Ait-Sahalia and Lo (2000) Economic-VaR (E-VaR)[5] calculations under GEV assumption for returns and compare it with the statistical or historical VAR calculations for the same.

Appendix A: Proof of solutions for the call and put option price equation

For the call option price equation, when $\xi > 0$ we have:

$$\psi_1 = \int_H^0 y^{-1/\xi} \exp\left(-y^{-1/\xi}\right) dy \tag{A.1}$$

Consider the change of variable $t = y^{-1/\xi}$, and $y = t^{-\xi}$ $dy = -\xi t^{-1-\xi} dt$

The lower limit of integration becomes $\underline{t} = H^{-1/\xi}$

The upper limit of integration becomes $\bar{t} = 0^{-1/\xi} = \dfrac{1}{0^{1/\xi}} = \dfrac{1}{0} = \infty$

$$\psi_1 = \int_{H^{-/\xi}}^{\infty} t\, e^{-t}\, t^{-1-\xi} d\left(t^{-\xi}\right) = -\xi \int_{H^{-/\xi}}^{\infty} t^{-\xi}\, e^{-t}\, dt = -\xi \int_{H^{-/\xi}}^{\infty} t^{(1-\xi)-1}\, e^{-t}\, dt \tag{A.2}$$

We can solve this integral directly by using the definition of the Incomplete Gamma function:

$$\Gamma(a,b) = \int_b^{\infty} t^{a-1} e^{-t} dt \tag{A.3}$$

Thus:

$$\psi_1 = -\xi \Gamma\left(1-\xi, H^{-1/\xi}\right) \tag{A.4}$$

The solution of integral ψ_2 in (12) is:

$$\psi_2 = \int_H^0 \left(y^{-1-1/\xi}\right) \exp\left(-y^{-1/\xi}\right) dy = \left[\xi \exp\left(-y^{-1/\xi}\right)\right]_H^0 = \xi\left(-\exp\left(-H^{-1/\xi}\right)\right) \tag{A.5}$$

Since

$$\lim_{y \to 0} \exp\left(-y^{-1/\xi}\right) = \lim_{y \to 0} \frac{1}{\exp\left(\dfrac{1}{y^{1/\xi}}\right)} = \frac{1}{\exp\left(\dfrac{1}{0^{1/\xi}}\right)} = \frac{1}{\exp\left(\dfrac{1}{0}\right)} = \frac{1}{\exp(\infty)} = \frac{1}{\infty} = 0 \qquad \text{for } \xi > 0$$

We can show that the same result holds when $\xi < 0$. The upper limit of integration for the call price integral is ∞, since the price distribution is not truncated on the left.

Evaluating the first integral:

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[5] It has been argued that E-VaR is a more general measure risk since it incorporates the investor's risk preferences, the demand–supply effects, and the probabilities that correspond to extreme losses (Panigirtzoglou and Skiadopoulos 2004).

$$\psi_1 = \int_H^\infty \left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right)dy = \left[\xi \exp\left(-y^{-1/\xi}\right)\right]_H^\infty = \xi\left(-\exp\left(-H^{-1/\xi}\right)\right) \qquad \text{(A.6)}$$

Since

$$\lim_{y\to\infty} \exp\left(-y^{-1/\xi}\right) = \lim_{y\to 0} \frac{1}{\exp\left(\frac{1}{y^{1/\xi}}\right)} = \frac{1}{\exp\left(\infty^{1/\xi}\right)} == \frac{1}{\exp(\infty)} = \frac{1}{\infty} = 0 \qquad \text{for } \xi < 0$$

The second integral (14) also yields the same result when $\xi < 0$:

$$\psi_2 = \int_H^\infty y^{-1/\xi} \exp\left(-y^{-1/\xi}\right)dy$$

When applying the change of variable $t = y^{-1/\xi}$, the upper limit of integration becomes

$\bar{t} = \infty^{-1/\xi} = \infty$ since $\xi < 0$.

And thus, equation (19) is also found to be the solution for the call option price when $\xi < 0$.

Similarly, we can solve the put option price equation. The upper limit of integration *K* in the put option equation becomes $H = 1 + \frac{\xi}{\sigma}\left(1 - \frac{K}{S_0} - \mu\right)$. The lower limit of integration 0 in the put option equation becomes $h = 1 + \frac{\xi}{\sigma}(1-\mu)$. Substituting $g(S_T)$ and simplifying we have

$$P_t e^{r(T-t)} = \int_0^K (K - S_T) g(S_T) dS_T \qquad \text{(A.7)}$$

$$= \int_h^H \left(K - S_0\left(1 - \mu - \frac{\sigma}{\xi}(y-1)\right)\right)\frac{1}{S_0\sigma}\left(y^{-1-1/\xi}\right)e^{\left(-y^{-1/\xi}\right)}\left(-S_0\frac{\sigma}{\xi}\right)dy$$

$$= -\frac{1}{\xi}\int_h^H \left(K - S_0\left(1 - \mu - \frac{\sigma}{\xi}(y-1)\right)\right)\left(y^{-1-1/\xi}\right)e^{\left(-y^{-1/\xi}\right)} dy \qquad \text{(A.8)}$$

$$= \frac{1}{\xi}\left[\frac{-S_0\sigma}{\xi}\int_h^H y\left(y^{-1-1/\xi}\right)e^{\left(-y^{-1/\xi}\right)}dy - \left(K - S_0\left(1-\mu+\frac{\sigma}{\xi}\right)\right)\int_h^H \left(y^{-1-1/\xi}\right)e^{\left(-y^{-1/\xi}\right)}dy\right]$$

Evaluating the first integral:

$$\psi_1 = \int_h^H \left(y^{-1-1/\xi}\right)e^{\left(-y^{-1/\xi}\right)}dy = \left[\xi e^{\left(-y^{-1/\xi}\right)}\right]_h^H = \xi\left(e^{\left(-H^{-1/\xi}\right)} - e^{\left(-h^{-1/\xi}\right)}\right) \qquad \text{(A.9)}$$

For the second integral:

$$\psi_2 = \int_h^H y^{-1/\xi}\exp\left(-y^{-1/\xi}\right)dy \qquad \text{(A.10)}$$

Consider the change of variable $t = y^{-1/\xi}$, and $y = t^{-\xi}$ $dy = -\xi t^{-1-\xi}dt$

$$\psi_2 = \int_{h^{-/\xi}}^{H^{-/\xi}} t\, e^{-t}\, t^{-1-\xi} d\left(t^{-\xi}\right) = -\xi\int_{h^{-/\xi}}^{H^{-/\xi}} t^{-\xi}\, e^{-t}\, dt = -\xi\int_{h^{-/\xi}}^{H^{-/\xi}} t^{(1-\xi)-1}\, e^{-t}\, dt \qquad \text{(A.11)}$$

We can solve this integral directly by using the definition of the generalized Gamma function:

$$\Gamma(a,z_0)-\Gamma(a,z_1)=\Gamma(a,z_0,z_1)=\int_{z_0}^{z_1} t^{a-1}e^{-t}dt \tag{A.12}$$

Thus: $\psi_2 = -\xi\,\Gamma\left(1-\xi, h^{-1/\xi}, H^{-1/\xi}\right)$

Combining results for ψ_1 and ψ_2, we obtain a closed form solution for the put option equation:

$$P_t(K) = e^{-r(T-t)}\left\{\frac{S_0\sigma}{\xi}\Gamma\left(1-\xi, h^{-1/\xi}, H^{-1/\xi}\right) - \left(K - S_0\left(1-\mu+\frac{\sigma}{\xi}\right)\right)\left(e^{-H^{-1/\xi}} - e^{-h^{-1/\xi}}\right)\right\} \tag{A.13}$$

And rearranging (27), by grouping terms with K and S_0

$$P_t(K) = e^{-r(T-t)}\left\{K\left(e^{-h^{-1/\xi}} - e^{-H^{-1/\xi}}\right) - S_0\left(\left(1-\mu+\sigma/\xi\right)\left(e^{-H^{-1/\xi}} - e^{-h^{-1/\xi}}\right) - \frac{\sigma}{\xi}\Gamma\left(1-\xi, h^{-1/\xi}, H^{-1/\xi}\right)\right)\right\}$$

(A.14)

Appendix B: The Gamma function

The function $\Gamma(a)$ is the Euler gamma function, and is defined as: $\Gamma(a)=\int_0^{\infty} t^{a-1}e^{-t}dt$

When the lower limit of integration is greater than zero, it is know as the incomplete gamma function $\Gamma(a,z_0)$, and is defined as:

$$\Gamma(a,z_0)=\int_{z_0}^{\infty} t^{a-1}e^{-t}dt$$

The definition of the generalized incomplete gamma function $\Gamma(a,z_0,z_1)$ is given by the integral:

$$\Gamma(a,z_0)-\Gamma(a,z_1)=\Gamma(a,z_0,z_1)=\int_{z_0}^{z_1} t^{a-1}e^{-t}dt$$

The three versions of Gamma functions are summarized in the table below:

Name	Notation	Definition
Gamma function	$\Gamma(a)$	$=\int_0^{\infty} t^{a-1}e^{-t}dt$
Incomplete Gama function	$\Gamma(a,z_0)$	$=\int_{z_0}^{\infty} t^{a-1}e^{-t}dt$
Generalized incomplete Gamma function	$\Gamma(a,z_0,z_1)$	$=\int_{z_0}^{z_1} t^{a-1}e^{-t}dt$

Table A.1: Definitions of the Gamma functions

The Gamma function is related to the factorial function, and may be related to the fractional Brownian Motion.

The relationship of the Gamma function with the factorial is based on that the gamma function interpolates the factorial function. For an integer n:

$$\Gamma(n+1)=n! = product(1:n)$$

n: gamma(n+1) = n! = prod(1:n)

Recall from Reiss and Thomas 2001:19 that the Gamma function has the following two properties:

$$\Gamma(1-\xi)=-\xi\,\Gamma(-\xi)$$
$$\Gamma(1)=1$$

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