Complete Markets, Enforcement Constraints and Intermediation[*]

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ABSTRACT.   Alvarez and Jermann (2000) show that the constrained efficient allocations of endowment economies with complete markets and limited commitment can be decentralized with endogenous borrowing limits on the Arrow securities. In a model with capital accumulation, aggregate risk and competitive intermediaries, we show that such a decentralization is not possible unless one imposes an upper limit on the intermediaries' capital holdings. Since there is no empirical evidence of such restrictions, we also characterize the equilibrium with no capital accumulation constraints. We show that this allocation solves a similar system of equations to the one of the constrained optimal solution, a result which considerably simplifies the equilibrium computation. In addition, capital accumulation is higher in this case, since the intermediaries do not internalize that fact that a higher aggregate capital increases the incentives to default. Finally, this also implies that agents may enjoy a higher welfare in the long run in spite of the fact that this allocation is not constrained efficient.

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JEL Classification: E44, D52, G12

1. Introduction

During the recent years, models with limited commitment have been used to analyze several important economic issues. Among others, Thomas and Worrall (1988) study efficient wage contracts, Kocherlakota (1996) analyzes optimal risk sharing and Alvarez and Jermann (2001) study asset returns. Whereas the previous literature mostly focuses on closed economy endowment models, several authors, such as Krueger and Perri (2005), Wright (2005) and Bai and Zhang (2005), have recently incorporated capital accumulation into such a context to study inequality and open economy issues. On the other hand, limited commitment economies with both capital accumulation and aggregate uncertainty have received less attention. One exception is the work by Kehoe and Perri (2002a), where the authors use an open economy model with production to analyze stylized facts in international macroeconomics.

In the present paper, we study the consequences of introducing endogenous production and aggregate uncertainty into a closed economy framework that is similar to the one studied by some of the previous authors. In particular, we focus on the relationship between constrained efficient allocations and competitive equilibria with endogenous borrowing limits, providing versions of the two fundamental theorems of welfare economics for economies with production and limited enforcement. In our environment, limited commitment arises because agents have the option to default on their financial liabilities every period. In particular, we assume that all their assets are seized in the default period, after which they are excluded from future asset trade (risk sharing) permanently. This implies that they have to solely rely on their labour income, which depends on the aggregate state of the economy (capital stock and aggregate productivity). Given this, their outside option (autarky value) also depends on the aggregate states.

As shown by Alvarez and Jermann (2000), the constrained efficient allocations of exchange economy models with complete markets and limited commitment can be decentralized as a competitive equilibrium with debt constraints on the Arrow securities that are not too tight. These are the loosest possible borrowing limits that do not allow for equilibrium default. We first show that this decentralization is not possible if one introduces capital accumulation and aggregate uncertainty into such a framework. The reason is that, in the presence of binding enforcement constraints, a higher capital accumulation has two additional effects on the Euler condition that determines aggregate investment. On the one hand, it increases consumption and output next period, decreasing the incentives to default and raising therefore the benefits of a higher aggregate capital. On the other hand, a higher capital tightens the enforcement constraints through an increase in the outside option or autarky value, reducing the benefits of more capital. Since the previous two effects drive a wedge between the marginal rates of substitution and transformation, the optimal allocations cannot be decentralized as a competitive equilibrium, even in the presence of endogenous debt constraints.

This result has also been shown by Kehoe and Perri (2002b and 2004) for an open

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economy model where the agents are interpreted as countries and have separate production technologies. Further, the authors show that the constrained efficient allocations in their setup can be decentralized with either debt constraints on the Arrow securities and capital accumulation constraints or with government default on foreign loans and capital income taxes. In the present paper, we focus on a decentralization with borrowing constraints, since our agents cannot be interpreted as countries and sovereign default would therefore make no sense. On the other hand, one of our key extensions is the introduction of a competitive financial intermediation sector that operates the investment technology and sets the endogenous trading limits. In contrast to the findings of the previous authors, we show that a decentralization of the constrained efficient allocations in such a setting is not possible *only* due to the second effect of capital on the autarky value described above. Moreover, we show that the optimal allocations can be decentralized with endogenous debt constraints and with capital accumulation constraints on the capital holdings of the intermediaries.

Note that there is virtually no empirical evidence of the presence of capital accumulation constraints, and it is also difficult to provide equilibrium micro-foundations for these type of restrictions. Given this, we also characterize the equilibrium allocations with endogenous borrowing constraints on the Arrow security holdings but no capital accumulation constraints. In particular, we show that these allocations solve almost the same system of equations as the constrained efficient allocations, with the key difference that the autarky effects of aggregate capital accumulation are not internalized by the intermediaries. In addition, we provide micro-foundations for the endogenous borrowing constraints on Arrow securities by showing that they would arise as an equilibrium outcome if chosen by the intermediation sector. These characterization results provide a relatively simple solution method for a potentially very complicated equilibrium problem.

We also compare the competitive equilibria (with and without capital accumulation constraints) quantitatively. We find that the equilibrium allocations are qualitatively similar in both cases. In particular, they exhibit perfect risk sharing in the long run with our benchmark calibration. However, important differences arise in the short run. First, the economy with no capital accumulation constraints accumulates more capital because the constraints bind occasionally. Second, since a higher capital accumulation increases the autarky value and the incentives to default, the model with capital accumulation constraints leads to a bigger range of initial wealth distributions where the participation constraints are not binding in equilibrium. Finally, although agents can enjoy a higher consumption in the constrained optimal allocation, the fact that capital accumulation is lower affects their lifetime utilities negatively. We find that this last effect dominates for the more wealthy agents, since a higher consumption is less important for them. Given this, the allocation of the economy without capital accumulation constraints is not Pareto dominated by the constrained efficient allocation, where these constraints are imposed in equilibrium.

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Finally, we study the sensitivity of these results to alternative model formulations. First, we modify the autarky penalties by allowing agents to save in physical capital after default. We find that this modification does not alter any of the qualitative findings described above, although less risk sharing is obviously supported in this case. Second, we choose a different calibration where agents are more impatient and where the weight of capital income in their total income is lower. Under this scenario, the long run equilibrium allocations are not characterized by perfect risk sharing any more. In this case, the short run differences that we described above also hold in the long run. This implies that capital accumulation in the stationary distribution tends to be higher in the economy without capital accumulation constraints. More surprisingly, we find that the economy without capital accumulation constraints is actually experiencing a higher expected (average) welfare in the stationary distribution due to the higher aggregate capital.

Our results contribute to an increasingly growing literature studying models with limited commitment and capital accumulation. In particular, it provides a tractable framework that can be used to directly study the competitive equilibria in such a context, both along the transition and in the stationary distribution, without having to impose capital accumulation constraints. As we have seen, the general equilibrium capital accumulation effects that we have identified may play an important role when these models are applied to study important issues such as inequality or the welfare impact of government policies.

The paper is organized as follows. Section 2 introduces the model economy. Section 3 discusses the competitive equilibrium with endogenous borrowing limits and financial intermediaries that may be subject to capital accumulation constraints. Section 4 characterizes the constrained efficient allocations of the benchmark economy and shows that a decentralization as a competitive equilibrium with endogenous borrowing limits is only possible in the presence of accumulation constraints on the capital holdings of the intermediaries. The competitive equilibrium with no such constraints is characterized in Section 5, where we also provide micro foundations for the endogenous borrowing limits on Arrow securities. Finally, Section 6 compares the two competitive equilibria quantitatively and Section 7 summarizes and concludes.

2. THE ECONOMY

We consider an infinite horizon economy with aggregate uncertainty, endogenous production, idiosyncratic risk and participation constraints.[1] Time is discrete and indexed by $t = 0, 1, 2....$ Further, the resolution of uncertainty is represented by an information structure or event-tree N. Each node $s^t \in N$, summarizing the history until date t, has a finite number of immediate successors, denoted by $s^{t+1}|s^t$. We use the notation $s^r|s^t$ with $r \geq t$ to indicate that node s^r belongs to the sub-tree with root s^t. Further, with the exception of the unique

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[1]Our model economy extends the models in Kocherlakota (1996) and Alvarez and Jermann (2000) to a context with endogenous production.

root node s^0 at $t = 0$, each node has a unique predecessor, denoted by s^{t-1}. The probability of s^t as of period 0 is denoted by $\pi(s^t)$, with $\pi(s^0) = 1$. Moreover, we denote by $\pi(s^r|s^t)$ the conditional probability of s^r given s^t. For notational convenience, we let $\{x\} = \left\{x\left(s^t\right)\right\}_{s^t \in N}$ represent the entire state-contingent sequence for any variable x throughout the paper.

At each node s^t, there exists a spot market for a single consumption good $y(s^t) \in \mathbb{R}_+$, produced with the following aggregate technology:

$$y(s^t) = f(z(s^t), K(s^{t-1}), L(s^t)). \tag{1}$$

In the previous equation, $K(s^{t-1}) \in \mathbb{R}_+$ and $L(s^t) \in \mathbb{R}_+$ denote the aggregate capital and labor respectively, with $K\left(s^{-1}\right) \in \mathbb{R}_{++}$ given. Further, $z(s^t) \in \mathbb{R}_{++}$ is a productivity shock that follows a stationary Markov chain with N_z possible values. Given z, the production function $f(z, \cdot, \cdot) : \mathbb{R}_+^2 \to \mathbb{R}_+$ is assumed to be continuously differentiable on the interior of its domain, strictly increasing, strictly concave in K, and homogeneous of degree one in the two arguments. Capital depreciates at a constant rate δ. We also assume that $f_{LK}(z, K, L) > 0$, $\lim_{K \to 0} f_K(z, K, L) = \infty$ and $\lim_{k \to \infty} f_K(z, K, L) = 0$ for all $K > 0$ and $L > 0$.

The economy is populated by two types of households that are indexed by $i \in \{1, 2\} \equiv I$, with a continuum of identical consumers within each type.[2] Households have additively separable preferences over sequences of consumption $\{c_i\}$ of the form:

$$U(\{c_i\}) = \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u\left(c_i\left(s^t\right)\right) = E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_i\left(s^t\right)\right) \tag{2}$$

where $\beta \in (0, 1)$ is the subjective discount factor and E_0 denotes the expectation conditional on information at date $t = 0$. The period utility function u is strictly increasing, strictly concave, unbounded below and continuously differentiable, with $\lim_{c \to 0} u'(c) = \infty$ and $\lim_{c \to \infty} u'(c) = 0$.

At each date-state s^t, households receive a stochastic labour endowment $\epsilon_i(s^t)$, following a stationary Markov chain with N_ϵ possible values. Households supply labor inelastically, implying that $L(s^t) = \sum_{i \in I} \epsilon_i(s^t)$. Further, they have a potentially history dependent outside option of $V_i\left(s^t\right)$. Thus, they are subject to a participation constraint of the form:

$$\sum_{r=t}^{\infty} \sum_{s^r|s^t} \beta^{r-t} \pi\left(s^r\right) u\left(c_i\left(s^r\right)\right) \geq V_i(s^t) \text{ for } i \in I. \tag{3}$$

Finally, the resource constraint of the economy is given by:

$$\sum_{i \in I} c_i(s^t) + K(s^t) = y\left(s^t\right) + (1 - \delta)K(s^{t-1}). \tag{4}$$

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[2] All the results in the paper hold for any arbitrary finite number of types, and the assumption of two types is therefore without loss of generality. On the other hand, it simplifies both the notation and the computations.

3. Competitive Equilibrium

This section defines a competitive equilibrium with endogenous borrowing limits and a competitive intermediation sector for the framework described in section 2. To do this, we assume that the economy is populated by a representative firm that operates the production technology and by a risk neutral and competitive financial intermediation sector that operates the investment technology. Since we want to focus on symmetric equilibria where all intermediaries hold the same portfolio, we focus on the representative intermediary.

Each period, after observing the realization of the productivity shock, the representative firm rents labor from the households and physical capital from the intermediary to maximize the period profits:

$$\max_{K(s^{t-1}), L(s^t)} f(z(s^t), K(s^{t-1}), L(s^t)) - w\left(s^t\right) L(s^t) - r(s^t) K(s^{t-1}).$$

Profit maximization implies that factor prices are given by the following expressions:

$$w(s^t) = f_L(s^t) \equiv f_L(z(s^t), K(s^{t-1}), L(s^t)) \tag{5}$$

$$r(s^t) = f_K(s^t) \equiv f_K(z(s^t), K(s^{t-1}), L(s^t)). \tag{6}$$

The representative intermediary lives for two periods. An intermediary born at node s^t first decides how much capital $k(s^t)$ to purchase subject to the following capital accumulation constraint $k\left(s^t\right) \leq B\left(s^t\right)$. This constraint will play a central role when we decentralize the constrained efficient allocations. The capital is rented to the firm, earning a rental revenue of $r(s^{t+1})k(s^t)$ and a liquidation value of $(1 - \delta)k(s^t)$ the following period. Further, to finance the capital purchases, the intermediary sells the future consumption goods in the spot market for one period ahead contingent claims, which are traded at price $q(s^{t+1}|s^t)$. The intermediary solves:

$$\max_{k(s^t)} \left\{ -k(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)[r(s^{t+1}) + (1 - \delta)]k(s^t) \right\} \text{ s.t.}$$

$$k\left(s^t\right) \leq B\left(s^t\right) \tag{7}$$

If $\psi(s^t)$ is the multiplier on the capital accumulation constraint in (7), optimality requires that:

$$1 = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)[r(s^{t+1}) + (1 - \delta)] - \psi(s^t). \tag{8}$$

Here, it is important to note that $1 \leq \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)[r(s^{t+1}) + (1 - \delta)]$ due to the fact that $\psi(s^t) \geq 0$. In other words, if the savings constraint is not binding ($\psi(s^t) = 0$), the intermediary makes zero profits. Otherwise, the non-negative profits at node $s^t \in N$ are given by:

$$d\left(s^t\right) = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)[r(s^{t+1}) + (1 - \delta)]k(s^t) - k(s^t) = \psi(s^t)k(s^t). \tag{9}$$

We assume that profits are distributed to the households when they are realized, i.e. during the first period of the intermediary's life-cycle. The period before an intermediary starts its business, households own $\theta_i^0\left(s^{t-1}\right)$ shares of it, which they can immediately trade at a price $p\left(s^t\right)$. Note that this price represents the value of an intermediary that will pay dividends next period. At each s^t, households can also trade in a complete set of state contingent claims to one period ahead consumption. The budget constraint of household $i \in I$ is therefore given by:

$$\overline{c}_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)a_i(s^{t+1}) + p\left(s^t\right)\theta_i\left(s^t\right) \leq d(s^t)\theta_i\left(s^{t-1}\right) + a_i(s^t). \tag{10}$$

In the previous equation, $\overline{c}_i(s^t) = c_i(s^t) - p\left(s^t\right)\theta_i^0\left(s^{t-1}\right) - w\left(s^t\right)\epsilon_i\left(s^t\right)$ represents the individual consumption net of the value of initial shares in the intermediaries and of the individual labor income. In addition, $a_i(s^{t+1})$ and $\theta_i\left(s^t\right)$ represent the amount of state contingent claims and shares in the intermediary held by $i \in I$ at the end of period t.

Market clearing for the state contingent securities requires that the debt issued by the intermediaries matches the demand of the households, that is, $\sum_i a_i\left(s^{t+1}\right) = [r(s^{t+1}) + (1 - \delta)]K\left(s^t\right)$. Further, $\theta_i^0\left(s^{t-1}\right)$ is given for $i = 1, 2$ while $\sum_i \theta_i\left(s^t\right) = \sum_i \theta_i^0\left(s^{t-1}\right) = 1$. If we denote by $\omega_i\left(s^t\right) \equiv d(s^t)\theta_i\left(s^{t-1}\right) + a_i(s^t)$ the total asset wealth of the household at the beginning of period t, his optimization problem can be written as:

$$\max_{\{c_i, a_i, \theta_i\}} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t)\beta^t u\left(c_i\left(s^t\right)\right) \text{ s.t.}$$

$$\overline{c}_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)\omega_i\left(s^{t+1}\right) \leq \omega_i\left(s^t\right) \tag{11}$$

$$\omega_i\left(s^{t+1}\right) \geq A_i\left(s^{t+1}\right). \tag{12}$$

As reflected by the equation (12), the individual asset wealth is subject to a borrowing constraint of $A_i(s^{t+1})$. The equilibrium determination of these limits will be discussed below and in Section 6. If $\gamma_i(s^{t+1}) \geq 0$ is the multiplier on this constraint, the necessary and sufficient first order conditions with respect to $a_i(s^{t+1})$ and $\theta_i(s^t)$ from the maximization problem of household $i \in I$ imply that:

$$q(s^{t+1}|s^t) = \beta\pi(s^{t+1}|s^t)\left\{\frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)}\right\} + \frac{\gamma_i(s^{t+1})}{u'\left(c_i(s^t)\right)} \forall s^{t+1}|s^t \tag{13}$$

and

$$p(s^t) = \beta \sum_{s^{t+1}|s^t}\left\{\pi(s^{t+1}|s^t)\frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)}d(s^{t+1}) + \frac{\gamma_i(s^{t+1})}{u'\left(c_i(s^t)\right)}d(s^{t+1})\right\}.$$

Combining the above two first-order conditions yields the pricing equations for the shares of the intermediaries:

$$p\left(s^t\right) = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)d(s^{t+1}). \tag{14}$$

The previous equation can also be obtained using no arbitrage arguments. Further, it allows us to rewrite the agents problem as if the decision variable was the individual next period wealth $\omega_i\left(s^{t+1}\right)$ instead of $a_i(s^{t+1})$ and $\theta_i(s^t)$ separately. We will use this below in our definition of a competitive equilibrium. This result also implies that there are a continuum of possible combinations of $a_i(s^{t+1})$ and $\theta_i(s^t)$ that will yield the same allocations, since the share in the intermediaries is a "redundant" asset in spite of markets being endogenously incomplete. Finally, the transversality condition in terms of wealth is given by:

$$\lim_{t\to\infty} \sum_{s^t} \beta^t \pi(s^t) u'\left(c_i(s^t)\right) \left[\omega_i(s^t) - A_i(s^t)\right] \leq 0. \tag{15}$$

Definition 1. *A competitive equilibrium with borrowing constraints* $\{A_i\}_{i\in I}$, *capital accumulation constraints* $\{B\}$ *and initial conditions* $K(s^{-1})$ *and* $\{\omega_i(s^0)\}_{i\in I}$ *is a vector of quantities* $\{(c_i, \omega_i)_{i\in I}, k, K, d\}$ *and prices* $\{w, r, q\}$ *such that (i) given prices,* $\{c_i, \omega_i\}$ *solves the problem for each household* $i \in I$; *(ii) the factor prices* $\{w, r\}$ *satisfy the optimality conditions of the firm; (iii)* q, r *and* d *satisfy the optimality condition of the intermediary; (iv) all markets clear, i.e., for all* $s^t \in N$, $k(s^t) = K(s^t)$, $\sum_i \omega_i(s^{t+1}) = [r(s^{t+1}) + 1 - \delta]K(s^t) + d\left(s^t\right)$, $\sum_i \epsilon_i(s^t) = L(s^t)$ *and* $\sum_{i\in I} c_i(s^t) + K(s^t) = y\left(s^t\right) + (1 - \delta)K(s^{t-1})$.

As stated in Section 2, each household has an outside option of $V_i\left(s^t\right)$. In the present setting, we assume that households can leave the risk sharing arrangement at any date-state to go to financial autarky. In this case, they will only be able to consume their labour income, while they are excluded from financial markets forever.[3] Given this, we choose limits that are *not too tight,* in the sense that a looser limit would imply that an agent with that level of debt prefers to leave the trading arrangement. In order to determine these limits, we can define $S_i\left(s^t\right) = \left(\epsilon_i(s^t); \epsilon_{-i}(s^t), z\left(s^t\right), K\left(s^{t-1}\right)\right)$ and write the value of the trading arrangement recursively as follows:

$$W^{ce}(\omega_i(s^t), S_i(s^t)) = u\left(c_i\left(s^t\right)\right) + \beta \sum_{s^{t+1}} \pi(s^{t+1}|s^t) W^{ce}(\omega_i(s^{t+1}), S_i(s^{t+1})) \tag{16}$$

Definition 2. *The borrowing constraints* $\{A_i\}_{i\in I}$ *are* not too tight *if they satisfy the following condition for all* $i \in I$ *and all nodes* $s^t \in N$:

$$W^{ce}(A_i(s^t), S_i(s^t)) = V^{ce}(S_i(s^t)) \tag{17}$$

where the value of the outside option is given by:

$$V^{ce}(S_i(s^t)) = \sum_{r=t}^{\infty} \sum_{s^r|s^t} \beta^{r-t} \pi\left(s^r|s^t\right) u\left(w\left(s^r\right) \epsilon_i(s^r)\right). \tag{18}$$

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[3] In Section 7, we also consider the case where households are excluded from trade in Arrow securities but can still save by accumulating physical capital.

Note that, in the present setting, the value of staying in the trading arrangement W^{ce} is strictly increasing in wealth, whereas the autarky value V^{ce} is not a function of $\omega_i(s^t)$. This implies that the limits defined by (17) exist and they are unique under our assumptions on the utility function. In addition, since $W^{ce}(0, S_i(s^t)) \geq V^{ce}(S_i(s^t))$ and W^{ce} is increasing in ω_i, equation (17) implies that $A_i(s^t) \leq 0$. Intuitively, no agent would default with a positive level of wealth, since he could then afford a higher current consumption than in autarky and at least as high of a life-time utility as in autarky from next period on.

Before discussing the determination of the constraint efficient allocations, it important to note that all our results can be applied to an alternative setting where the intermediaries are infinitely lived. In this case, the intermediary decides how much capital $k(s^t)$ to purchase at each node subject to the capital accumulation constraint in (7). Further, the capital is rented to the firm and it is financed by selling the next period consumption goods in the spot market for one period ahead contingent claims. If we let $Q(s^{t+j}|s^t) = q(s^{t+j}|s^{t+j-1})q(s^{t+j-1}|s^{t+j-1})...q(s^{t+1}|s^t)$ be the state s^t price of consumption delivered at state s^{t+1}, the infinitely lived intermediary maximizes:

$$\max_{\{k\}} \left\{ \sum_{j=0}^{\infty} \sum_{s^{t+j}} Q(s^{t+j}|s^t) \left([r(s^{t+j}) + (1-\delta)]k(s^{t+j-1}) - k(s^{t+j}) \right) \right\}$$

It is easy to see that equations (8) and (9) still hold in the present setting. Further, we can assume that profits are distributed to the households every period according to their beginning of period ownership shares $\theta_i\left(s^{t-1}\right)$, where $\theta_i\left(s^{-1}\right)$ if given for $i = 1, 2$. If we let $\omega_i\left(s^t\right) \equiv \left[d(s^t) + p\left(s^t\right)\right]\theta_i\left(s^{t-1}\right) + a_i(s^t)$ and $\overline{c}_i(s^t) = c_i(s^t) - w\left(s^t\right)\epsilon_i\left(s^t\right)$, the price of the shares and the budget constraint of the households are given by:

$$p\left(s^t\right) = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)\left[d(s^{t+1}) + p\left(s^{t+1}\right)\right]$$

$$\overline{c}_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)a_i\left(s^{t+1}\right) + p\left(s^t\right)\theta_i\left(s^t\right) \leq \left[d(s^t) + p\left(s^t\right)\right]\theta_i\left(s^{t-1}\right) + a_i(s^t).$$

While this alternative setting might be more appealing, since it only requires setting $\theta_i\left(s^{-1}\right)$, note that it might lead to the typical shareholder disagreement problem under incomplete markets.[4] In other words, when the trading constraints are binding, different household types will typically value future output differently, since their marginal rates of substitution are not equalized. Note that this is not an issue if the intermediary lives for two periods, in which case, a household who holds the majority of shares at s^t will agree on using $q(s^{t+1}|s^t)$ as a discount factor. On the other hand, a currently unconstrained agent

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[4]For a good review of this literature see Grossman and Stiglitz (1977, 1980) or Dreze(1985). For a more formal discussion of this issue in a model where markets are exogenously incomplete see Carceles-Poveda (2005) and Coen-Pirani (2005).

may prefer a different discount rate if the intermediary is infinitely lived, since she may get constrained in some future contingency. Given this, we chose to work with the two period formulation. Both settings lead to the same qualitative and quantitative findings and this is therefore without loss of generality.

4. Constrained Efficient Allocations

This section characterizes the constrained efficient allocations of the economy in Section 2. As usual, the optimal allocations solve a central planning problem where the planner takes into account both the resource constraint and the participation constraints of the two households. If α_i is the initial Pareto weight assigned by the planner to each household, the problem of the planner can be written as follows:

$$\max_{\{c_i, K\}} \sum_{i \in I} \alpha_i \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u\left(c_i(s^t)\right) \text{ s.t.}$$

$$\sum_{i \in I} c_i(s^t) + K(s^t) = F(z(s^t), K(s^{t-1}), L(s^t)) \tag{19}$$

$$\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi(s^r) u(c_i(s^r)) \geq V(S_i\left(s^t\right)) \text{ for } i \in I. \tag{20}$$

Note that we have set $V_i\left(s^t\right) = V(S_i\left(s^t\right))$ by assuming that the outside option value for $i \in I$ depends on $S_i\left(s^t\right) = \left(\epsilon_i(s^t); \epsilon_{-i}(s^t), z\left(s^t\right), K\left(s^{t-1}\right)\right)$. Whereas standard dynamic programming is inapplicable to the previous setup, we can follow Marcet and Marimon (1999) and rewrite the Lagrangian of the above problem as follows:

$$\inf_{\{\gamma_i\}} \sup_{\{c_i, K\}} H \equiv \sum_{i \in I} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t \left\{ u(c_i(s^t))(\mu_i(s^t) + \alpha_i) - \gamma_i(s^t) V(S_i\left(s^t\right)) \right\}.$$

where $\beta^t \gamma_i(s^t)$ is the Lagrange multiplier of the time t participation constraint for household $i \in I$. Further, $\mu_i(s^t)$ is a pseudo state variable that is defined recursively as follows:

$$\mu_i(s^t) = \mu_i(s^{t-1}) + \gamma_i(s^t), \ \mu_i(s^{-1}) = 0 \text{ for } i = 1, 2. \tag{21}$$

It is easy to see that the solution to the previous problem can be characterized by the resource and participation constraints in (19)-(20) and by the following first order conditions:[5]

$$\frac{u'\left(c_1(s^t)\right)}{u'\left(c_2(s^t)\right)} = \lambda(s^t) = \frac{(1 + v_2(s^t))}{(1 + v_1(s^t))} \lambda(s^{t-1}) \tag{22}$$

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[5] The above first-order conditions for this problem are only necessary but not sufficient in general. The reason is that the constraint set defined by (19) and (20) is not necessarily convex. However, this does not seem to cause any problem in our model. In particular, when we impose the appropriate Kuhn-Tucker conditions, the above system of equations always yields a unique solution, implying that these conditions are also sufficient.

$$1 \quad = \quad \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} (1 + v_i(s^{t+1})) F_K\left(s^{t+1}\right) \right\} \tag{23}$$

$$- \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \sum_{j=1,2} \frac{v_j(s^{t+1})}{u'\left(c_j(s^t)\right)} V_K(S_j\left(s^{t+1}\right)) \right\} \text{ for } i \in I.$$

The terms $F_K\left(s^{t+1}\right) = f_K\left(z\left(s^{t+1}\right), K\left(s^t\right), L\left(s^{t+1}\right)\right) + 1 - \delta$ and $\left\{V_K(S_i\left(s^{t+1}\right))\right\}_{i \in I}$ on the right hand side of the previous equation represent the derivatives of total output F and of the outside option value V with respect to the aggregate capital stock K. Further, we have expressed the first order conditions in terms of the normalized multipliers λ and v_i, which simplify the system of equilibrium equations and are given by:

$$v_i(s^t) = \frac{\gamma_i(s^t)}{\mu_i(s^{t-1}) + \alpha_i} \text{ for } i \in I \tag{24}$$

$$\lambda(s^t) = \frac{\mu_2(s^t) + \alpha_2}{\mu_1(s^t) + \alpha_1}, \text{ with } \lambda(s^{-1}) = \frac{\alpha_2}{\alpha_1}. \tag{25}$$

Several remarks are worth noting. First, since $\mu_i(s^{t-1}) + \alpha_i > 0$, it follows that $v_i(s^t) > 0$ only if $\gamma_i(s^t) > 0$. This implies that $v_i(s^t)$ is positive only when the participation constraint of type $i \in I$ is binding. Second, λ represents a the time varying relative Pareto weight of type 2 households relative to type 1 households. Thus, as usual in models with endogenously incomplete markets, condition (22) implies that the relative consumptions of the two types are determined by their time varying relative Pareto weights. Third, as in other models with commitment (see e.g. Thomas and Worrall (1988) and Kocherlakota (1996)) whenever households of type 1 have a binding participation constraint $(v_1(s^t) > 0)$, λ will decrease, and their relative Pareto weight will therefore increase. The opposite happens when the participation constraint of type 2 household is binding. Finally, since the aggregate technology and the idiosyncratic income shocks are Markovian, the optimal allocation of this problem is recursive in $(\epsilon_1, \epsilon_2, z, K, \lambda)$.

As reflected by the Euler equation in (23), when the participation constraints are not binding for any household at any continuation history $s^{t+1}|s^t$, implying that $v_i(s^{t+1}) = 0$ for $i = 1, 2$, the equation reduces to the standard capital Euler condition of the stochastic growth model. On the other hand, the presence of binding enforcement constraints at s^{t+1} introduces two additional effects on the inter-temporal allocation of consumption and capital.

First, it increases the planner's marginal rate of substitution between period t and $t + 1$ goods, raising the benefits of a higher aggregate capital at $t + 1$, since this increases future consumption and decreases the default incentives. This is reflected by *the presence of* $v_i\left(s^{t+1}\right)$ on the first part of the right hand side of the equation. Second, it tightens the enforcement constraints through an increase in the autarky value, reducing the benefits of more capital at $t + 1$. This is reflected by *the autarky effects* on the second part of the right hand side of the equation.

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The following property will prove to be useful later on. *If $\{c_1, c_2\}$ is constrained efficient and $W(S_j(s^t)) > V(S_j(s^t))$, it has to be the case that:*

$$\frac{u'(c_j(s^t))}{u'(c_j(s^{t-1}))} = \max_{i=1,2} \frac{u'(c_i(s^t))}{u'(c_i(s^{t-1}))}.$$

Essentially, this states that unconstrained agents have the maximal marginal rate of substitution in the constrained efficient equilibrium. Note that this can be easily checked if we rewrite equation (22) as follows:

$$\frac{u'(c_1(s^t))}{u'(c_2(s^t))} = \lambda(s^t) = \frac{(1 + v_2(s^t))}{(1 + v_1(s^t))} \frac{u'(c_1(s^{t-1}))}{u'(c_2(s^{t-1}))}.$$

In addition, it implies that, for all s^t:

$$\max_{i=1,2} \frac{u'(c_i(s^t))}{u'(c_i(s^{t-1}))} = \frac{u'(c_1(s^t))(1 + v_1(s^t))}{u'(c_1(s^{t-1}))} = \frac{u'(c_2(s^t))(1 + v_2(s^t))}{u'(c_2(s^{t-1}))}. \tag{26}$$

In what follows, we consider allocations that have high implied interest rates, in the sense that they have a finite present value when discounted with the appropriate present value prices.[6]

Definition 3. A (total) consumption allocation $\{c\} \equiv \{c_1 + c_2\}$ has high implied interest rates if:

$$\sum_{t=0}^{\infty} \sum_{s^t} Q_p(s^t|s^0) c(s^t) < \infty \text{ for } \forall s^t$$

where

$$q_p(s^{t+1}|s^t) = \max_{i=1,2} \beta \pi(s^{t+1}|s^t) \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right\} \tag{27}$$

$$Q_p(s^t|s^0) = q_p(s^t|s^{t-1}) q_p(s^{t-1}|s^{t-2}) ... q_p(s^1|s^0). \tag{28}$$

5. Decentralization with Capital Accumulation Constraints

This section shows that a decentralization of the constrained efficient allocations with debt constraints is only possible if one imposes the savings constraint on the capital holdings of the intermediary. In a related model, Kehoe and Perri (2002b, 2004) show that a decentralization is possible if one introduces either debt constraints and a savings constraint on the individual capital holdings or government default on foreign loans together with capital income taxes. In the present framework, however, it is difficult to imagine that governments would default on behalf of some of the households against some other households in the same economy. Given this, we focus on a decentralization with borrowing constraints.[7]

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[6] This assumption is not very restrictive in the present setting, since it will be satisfied whenever consumption is bounded away from zero.

[7] An alternative decentralization based upon our intermediation structure with borrowing constraints and capital income taxes is analyzed by Chien and Lee (2005).

We start by showing that constrained efficient allocations with an outside option of financial autarky cannot be decentralized as a competitive equilibrium in the presence of binding capital accumulation constraints.

Proposition 1. *Let* $\{c_1, c_2, K\}$ *be a constrained efficient allocation. Then, it cannot be decentralized as a competitive equilibrium with only borrowing constraints that are not too tight unless the participation constraints in the constrained efficient allocation never bind.*

Proof of Proposition 1. To prove the proposition, we can use equation (26) to rewrite the Euler condition of the planner in (23) as follows:

$$1 = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \max_{i=1,2} \left\{ \frac{u'\left(c_i\left(s^{t+1}\right)\right)}{u'\left(c_i\left(s^t\right)\right)} \right\} F_K\left(s^{t+1}\right) - \sum_{j=1,2} \frac{v_j(s^{t+1})}{u'\left(c_j\left(s^t\right)\right)} V_K\left(S_j\left(s^{t+1}\right)\right). \tag{29}$$

Consider now the case where there are no capital accumulation constraints. In this case, the intermediaries always make zero profits, implying that $d\left(s^t\right) = 0$ and $p\left(s^t\right) = 0$ for all $s^t \in N$. Hence, households only trade in Arrow securities subject to the following budget and portfolio constraints:

$$c_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)a_i(s^{t+1}) \leq a_i(s^t) + w_i\left(s^t\right) \tag{30}$$

$$a_i(s^{t+1}) \geq A_i\left(s^{t+1}\right) \tag{31}$$

Since the portfolio constraint in (31) can only be binding for one of the two households, it follows that $\gamma_i(s^{t+1}) = 0$ for at least one household. Given this, equations (13) and (8) of the competitive equilibrium can be rewritten as:

$$q(s^{t+1}|s^t) = \beta\pi(s^{t+1}|s^t) \max_{i=1,2} \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \right\} \tag{32}$$

$$1 = \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t)\beta \max_{i=1,2} \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \right\} F_K\left(s^{t+1}\right) \tag{33}$$

where we have substituted for $r\left(s^{t+1}\right)$ from (6). Clearly, the two Euler equilibrium conditions in (29) and (33) cannot be satisfied by the same allocation $\{c_1, c_2, K\}$ if the participation constraint is ever binding, that is, if $v_j(s^{t+1}) > 0$ for some $j \in I$ and some $s^{t+1}|s^t$ with $\pi(s^{t+1}|s^t) > 0$. Given this, the constrained efficient allocations cannot be decentralized as a competitive equilibrium with borrowing constraints that are not too tight.∎

Several remarks are worth noting. First, this result is in contrast to the one obtained by Alvarez and Jermann (2000), who show that a decentralization of the constrained efficient allocations with borrowing constraints that are not too tight is possible in the absence of capital accumulation. Second, a similar result has also been shown by Kehoe and Perri (2002b,

2004), who study an economy with no financial intermediaries and with two production sectors that can be interpreted as countries. In their environment, however, a decentralization is not possible due to the two different effects on the standard Euler equation discussed earlier. In contrast to this, Proposition 1 illustrates that the optimal allocations in our set-up and in the presence of financial intermediaries cannot be decentralized solely due to the *autarky effects*, an observation that will prove to be useful in the next section.

To better illustrate the previous statement, consider a setting with no financial intermediaries and with two production sectors, as in Kehoe and Perri. In such a framework, households invest in zero net supply Arrow securities and in capital holdings subject to the following constraints:

$$c_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)a_i(s^{t+1}) + k_i\left(s^t\right) \leq w(s^t)\epsilon_i(s^t) + a_i(s^t) + r\left(s^t\right)k_i\left(s^{t-1}\right)$$

$$k_i\left(s^t\right) \geq 0 \text{ and } a_i(s^{t+1}) \geq A_i\left(s^{t+1}\right).$$

Further, the first order conditions with respect to the individual capital holdings imply that:[8]

$$1 = \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t)\beta \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \right\} F_K^i\left(s^{t+1}\right) \text{ for } i \in I.$$

Comparing the previous equation to the Euler equation of the planner in (23), it is easy to see that both cannot be satisfied by the same allocation, even if $V_K(S_i\left(s^t\right)) \equiv 0$ for all s^{t+1} and $j = 1, 2$. The key difference is that, in the presence of intermediaries, the equilibrium prices of the Arrow securities given by (32) already take into account one of the effects of capital accumulation on the value of the risk-sharing arrangement.

To solve this problem, Kehoe and Perri (2002b) suggest to impose a savings constraint on the individual capital holdings $\{k_i\}_{i \in I}$ that takes care of the two effects on the capital Euler equation. In what follows, we show that a similar result can also be obtained in our setup. In particular, we show that the constrained efficient allocations can be decentralized with borrowing constraints on the Arrow securities that are not too tight if one also imposes a savings constraint on the capital holdings of the intermediary. This is stated by the following proposition, which is the second fundamental theorem of welfare economics for our environment. The proof of this and of all the remaining propositions throughout the text are relegated to Appendix 1.

Proposition 2. *Let $\{c_1, c_2, K\}$ be a constrained efficient allocation where $c\left(s^t\right) = \sum_i c_i(s^t)$ has high implied interest rates. Further, assume that the intermediary in the decentralized economy is subject to capital accumulation constraints of the form $k(s^t) \leq B(s^t)$. Then, the constrained efficient allocations can be decentralized as a competitive equilibrium with borrowing constraints that are not too tight.*

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[8]Note that the restriction on capital holdings will not be binding in equilibrium.

The proof of this proposition extends the ones in Alvarez and Jermann (2000) and Kehoe and Perri (2002b) to the presence of production and financial intermediaries that are subject to capital accumulation constraints.

First, we show that $\{B\}$ **c**an be set so that a constrained efficient allocation that satisfies the planner's capital Euler equation also satisfies the optimality condition of the intermediary in the competitive equilibrium. Second, the allocations of the planner's problem can be used to construct the dividends d and share prices p, as well as the factor prices (r, w) and the Arrow security prices q that satisfy the optimality conditions of the firms and the households. Further, we can iterate on the budget constraints in the competitive equilibrium to obtain the wealth levels $\{\omega_i\}_{i \in I}$ that support the optimal allocations at every node. It is then easy to see that the constructed allocations clear the markets and satisfy the transversality condition. In addition, we first set the borrowing limits $\{A_i\}_{i \in I}$ equal to $\{\omega_i\}_{i \in I}$ whenever the participation constraints in the planner's problem are binding and to the natural borrowing limit otherwise.

Finally, we can construct the value functions in the competitive equilibrium from the value functions of the planner's problem and redefine the borrowing limits so that they are not too tight for the cases where the participation constraint in the planner's problem is not binding. This way, the constructed allocations with the new borrowing limits are still be feasible and optimal.

6. Characterization of the CE without Capital Accumulation Constraints

The previous section has shown that a decentralization of the constrained efficient allocations with borrowing constraints that are not too tight is possible in the presence of financial intermediaries that are subject to accumulation constraints on their capital holdings. However, there is no evidence for these type of constraints in the data. In addition, it is difficult to imagine how these upper bounds would arise as an equilibrium outcome. Given this, the present section characterizes the equilibrium allocations with borrowing constraints that are not too tight and with no binding capital accumulation constraints. In particular, we show that these allocations satisfy the same system of equations as the constrained efficient problem except the Euler condition in (23), which is replaced by:

$$1 = \beta \sum_{s^{t+1} | s^t} \pi(s^{t+1} | s^t) \left\{ \frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)} \left(1 + v_i\left(s^{t+1}\right)\right) \left[f_K(s^{t+1}) + 1 - \delta\right] \right\}. \qquad (34)$$

The previous result is stated by Propositions 4 and 5. In addition, Proposition 3 below shows that, if we allow the intermediaries to set the borrowing constraints on the households, they will choose the ones which are not too tight, providing a micro foundation for our endogenous borrowing limits.

Proposition 3. *(i) The CE with borrowing constraints that are not too tight remains a competitive (Nash) equilibrium if the intermediaries are allowed set the borrowing limits.*

(ii) No symmetric competitive (Nash) equilibrium exists for limits that are looser than the ones that are no too tight.

The previous proposition shows first that no intermediary has incentives to loosen or tighten the limits individually when they are not too tight, since these deviations are not profitable. This implies that these constraints arise as an equilibrium decision of the intermediaries. Intuitively, since the intermediaries make zero profits with any limits which do not allow for default, they have no incentive to tighten them. On the other hand, since they are price-takers, they cannot break even with looser limits. Further, the proposition also shows that no symmetric equilibrium exists where some or all of the limits are looser than the ones dictated by (17). This result is due to the fact that, if there is default, the intermediaries can always increase their profits by not buying Arrow securities from households with a positive probability of default next period. We are now ready to state our equivalence results.

Proposition 4. *Let $\{c_1, c_2, K\}$ be a solution to equations (19), (20), (22), (24), (25) and (34) where $\{c\} = \sum_i \{c_i\}$ has high implied interest rates. Then, this allocation can be decentralized as a competitive equilibrium with borrowing constraints that are not too tight.*

Proposition 5. *Let $\left\{(c_i, a_i)_{i \in I}, K, q, r, w\right\}$ be a competitive equilibrium with borrowing constraints $\{A_i\}_{i \in I}$ that are not too tight. Then $\left\{(c_i)_{i \in I}, K\right\}$ is a solution to equations (19), (20), (22), (24), (25) and (34). Further, $c = \sum_i c_i$ satisfies the high implied interest rates condition with respect to the price $Q(s^t|s^0)$ defined by:*

$$Q(s^t|s^0) = q(s^t|s^{t-1})q(s^{t-1}|s^{t-2})...q(s^1|s^0).$$

Several remarks are worth noting. First, whereas the competitive equilibrium without capital accumulation constraints solves a system of equations that is very similar to the optimal planner's problem, considerably simplifying the equilibrium computations, the solution is different to the optimal allocation due to the fact that it ignores the autarky effects. In other words, the financial intermediaries do not internalize the effect of capital accumulation on the agents' autarky valuations, whereas the planner internalizes this effect in the (constrained) optimal allocation. The qualitative and quantitative differences between the two allocations are studied in section 7 below.

Second, it is important to note that $\lambda(s^t)$ measures the relative wealth of the two types of households in the competitive equilibrium. To see this, we can define the Lagrange multipliers of (30) by $\beta^t \xi_i(s^t)$. In equilibrium, we must have that:

$$\frac{\xi_1(s^t)}{\xi_2(s^t)} = \frac{u'\left(c_1(s^t)\right)}{u'\left(c_2(s^t)\right)} = \lambda(s^t),$$

where the second inequality is a consequence of Propositions 4 and 5. Since $\xi_i(s^t)$ measures the marginal utility of wealth, a bigger $a_i(s^t)$ corresponds to a smaller is $\xi_i(s^t)$. Consequently, a higher $\lambda(s^t)$ implies that agent 1 has a smaller initial wealth compared to type 2 households.

7. Quantitative Comparison of the Competitive Equilibria

This section compares the two competitive equilibrium allocations (with and without capital accumulation constraints) numerically. The parameters of the economy are calibrated following the asset pricing and real business cycle literature. The time period is assumed to be one quarter, and the discount factor and depreciation rate are therefore set to $\beta = 0.99$ and $\delta = 0.025$. The first parameter is chosen to generate an annual average interest rate of approximately 4% in the stationary distribution, whereas the second replicates the US average investment to capital ratio during the postwar period.

Concerning the functional forms, we assume that the production function is Cobb-Douglas, with a constant capital share of $\alpha = 0.36$. Further, the utility function of the households is assumed to be $u(c) = \log(c)$. Finally, the exogenous shock processes are assumed to be independent. In particular, the aggregate technology shock follows a two state Markov chain with $z \in \{z_l, z_h\} = \{0.99, 1.01\}$, and its transition matrix is given by:

$$\Pi_z = \begin{bmatrix} \pi_{ll} & \pi_{lh} \\ \pi_{hl} & \pi_{hh} \end{bmatrix} = \begin{bmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{bmatrix}.$$

The aggregate labor supply is constant and we normalize it to 1. As to the idiosyncratic income process, it is assumed to follow a seven state Markov chain. The values and transition matrix are obtained by using the Hussey and Tauchen (1991) procedure to discretize the following process:

$$\epsilon^{i\prime} = (1 - \psi_\epsilon)\mu_\epsilon + \psi_\epsilon \epsilon^i + u, \; u \sim N(0, \sigma_u^2).$$

The shock parameters are set to $\psi_\epsilon = 0.956$ and $\sigma_u^2 = 0.082$, corresponding to quarterly adjusted estimates from annual idiosyncratic earnings data. Further, since a constant aggregate labor supply implies that $\epsilon^{-i} = 1 - \epsilon^i$, the values for ϵ^1 were chosen to be symmetric around $\mu_\epsilon = 0.5$. This implies that the idiosyncratic productivity of the two types follows the same process and the shocks are perfectly negatively correlated across the two types.

Finally, note that Proposition 4 and 5 provide us with a relatively easy and analogous computational method for both models. In the competitive equilibrium with capital accumulation constraints (*autarky effects*), we use equations (19), (20), (22), (24), (25) and (23). Further, to solve for the competitive equilibrium with no capital accumulation constraints (*no autarky effects*), we use the same system of equations but (23) is replaced by (34).

In what follows, we let $s_1 = [\epsilon, \lambda; z, K]$ and $s_2 = [1 - \epsilon, 1/\lambda; z, K]$. Under our Markovian assumption on the shocks, the previous set of equations implies that we can describe the optimal allocations in both models by the consumption functions $\{c_i(s_i)\}_{i=1,2}$, the normalized multipliers on the participations constraints $\{\nu_i(s_i)\}_{i=1,2}$ and the laws of motion for the relative wealth $\lambda'(s_1)$ and the aggregate capital $K'(s_1)$. To solve for these functions, we use policy functions iterations in both models.

Our numerical results for this benchmark parametrization are presented in Figures 1 to 6 of Appendix 2. All the optimal policies are conditioned on the low aggregate technology shock $z = 0.99$ and on $K = 38.6$, which is the mean of the stationary distribution of capital, but similar pictures can be obtained for the high technology shock. For expositional convenience, we have plotted the results for only three levels of the labour endowment, where ϵ_1 is the lowest and ϵ_7 is the highest labor endowment. Recall that type 2 households have the highest labor endowment when type 1 households have the lowest. Note also that both types have equal endowments when $\epsilon_4 = 1 - \epsilon_4 = 0.5$.

Figure 1 displays $\lambda' \equiv \lambda(s^{t+1})$ as a function of $\lambda \equiv \lambda(s^t)$ for the three different levels of the idiosyncratic income shocks. The first important observation based on this figure is that agents enjoy permanent perfect risk sharing in the long run in both models. To see this, assume first that our initial λ is inside its ergodic set, which is equal to $\lambda \in [0.8368, 1.195]$ and $\lambda \in [0.8366, 1.1953]$ for the models without and with the savings constraint respectively. As we see on the graph, $\lambda' = \lambda$ inside this region, independently of the labor income shocks. Condition (22) then implies that this can only happen if neither agent's participation constraint is binding. In addition, the ratio of marginal utilities remains constant over time. The last result, however, is the defining feature of a perfect risk sharing allocation.

Assume now that we start with $\lambda > 2.5$, implying that type 1 households hold significantly lower initial assets, and they are therefore entitled to less consumption than type 2 households. In this case, Figure 1 implies that λ' depends on the idiosyncratic income of the agent, and that it will drop to a new level depending on the shock realization. In particular, the higher the idiosyncratic income, the lower will be the new level of the relative wealth λ'. This is due to the fact that type 1 agents will then enjoy a higher autarky value and require therefore a higher compensation for staying in the risk sharing arrangement.

Here, it is important to note that, whenever λ jumps, type 1 agents' participation constraint is binding, and this new level of λ' pins down the borrowing constraint A_i of the competitive equilibrium faced by type 1 households in the previous period. This process will go on until the highest income (ϵ^7) is experienced by the type 1 agents. In this case, λ will enter the stationary distribution[9] ($\lambda = 1.195$) and remain constant forever. Thus, agents will enjoy permanent perfect risk sharing from that period on. In addition, a symmetric argument implies that whenever $\lambda < 0.83$, λ will become 0.83 and remain constant forever after finite number of periods. Finally, whereas agents will obtain full insurance in the long-run for any initial wealth distribution, note that the economy may experience movements in consumption and in λ in the short run.

The second important observation is that two economies are qualitatively very similar.

As stated above, the long-run behavior is practically identical, in the sense that there is perfect risk sharing in the long run. In addition, if $\lambda(s^0) \in [0.8368, 1.195]$, the long-run allocations will be identical. This is due to the fact that the borrowing constraints (and therefore the savings constraint of the intermediary) will never bind in this case. Thus, the individual consumptions will be determined by $\lambda(s^0)$ and the capital accumulation will be (unconstrained) efficient. On the other hand, if $\lambda(s^0)$ is outside the above interval, the long-run allocations will be somewhat different due to the fact that the bounds of the stationary distribution are slightly different in the two models. As we see, the model with savings constraint allows for a slightly wider range of λ (the wealth distribution) where the participation constraints are not binding. As we will see below, this is the consequence of the different capital accumulation pattern in the two economies.

Figure 2, shows the optimal consumption of type 1 households in the two economies as a function of λ for different levels of the labor endowment. Obviously, as the relative wealth of type 1 households decreases (λ increases) their consumption decreases. Also, since we have perfect risk sharing in the stationary distribution, consumption does not depend on the idiosyncratic labour endowment there. For the same reason, the optimal consumption allocations are identical across the two models in this range. Outside the stationary distribution, as expected, consumption is increasing in the labour endowment. We also observe that the model with autarky effects allows for a higher consumption for every λ and ϵ outside the stationary distribution. As explained below, this is the consequence of higher capital accumulation in the economy with capital accumulation constraints.

Figure 3 displays the next period's aggregate capital K' as a function of λ and ϵ. Again, aggregate capital is independent of both the wealth distribution and the labour endowments in the stationary distribution, where it is at its efficient level. On the other hand, markets are effectively incomplete outside the stationary distribution, where we see a higher capital accumulation. This result is well-documented in models with exogenously incomplete markets (see e.g. Aiyagari (1994) for a model without aggregate uncertainty and Ábrahám and Cárceles-Poveda (2005) for a model with a similar set-up but trade in physical capital only). As reflected by the figure, a similar behavior arises in the present setting. In particular, capital accumulation is higher when the low idiosyncratic labour endowment coincides with low wealth (high λ). This is the case for type 1 households on the upper right corner of the figure and for type 2 households in the upper left corner.

To see why this happens, we can look at Figure 1 and at the Euler equation of the constrained efficient problem in (23). It is clear from Figure 1 that, when type 1 households have a labour endowment of ϵ_7 and low λ (high wealth), the participation constraint of type 2 households is going to be binding in many continuation states ($v_i(s^{t+1}) > 0$). In turn, this implies that the return of investment is higher, and more capital will be accumulated.

In the decentralized problem this is equivalent to an increase of most of the Arrow security

prices $q(s^{t+1}|s^t)$, implying that intermediaries have to pay a lower return to the agents and can therefore invest more. This is the only effect in the model without autarky effects. On the other hand, this over accumulation is mitigated by the autarky effects in the constrained efficient allocation. In this case, the planner internalizes that a higher capital will increase the autarky values, leading to a lower capital accumulation than in the economy with no capital accumulation constraints. In the decentralized optimal solution, this is internalized with a binding upper limit on capital accumulation, which deters intermediaries from excessively overinvesting. In this case, households will also have less incentives to default, since the value of their outside option is lower due to a lower capital accumulation. As a consequence, we obtain perfect risk sharing for a higher range of the wealth distribution (a higher range of λ) in the model with capital accumulation constraints.

Using the results stated in Propositions 2, 4 and 5, we have also depicted the individual consumptions c_i and the next period capital stock K' as a function of the initial Arrow security holdings a_1 and the same levels of idiosyncratic shocks in Figures 4 and 5.[10] As already documented above, Figure 5 illustrates that capital accumulation is always higher in the economy with no capital accumulation constraints. In particular, capital accumulation is the highest when the low idiosyncratic shock for the type 1 households ϵ_1 is combined with a low level of initial asset holdings a_1, or when the high idiosyncratic shock for the type 1 households ϵ_7 is combined with a high level of initial asset holdings a_1. In this latter case, the borrowing constraint will be binding for the type 2 households. We also note that the difference between the two economies is significant. In terms of the average investment, the economy without autarky effects invest 15% more than the one with autarky effects when the lowest wealth coincides with the lowest income. Consequently, consumption will be higher in the constrained efficient allocation, especially with these combinations of idiosyncratic income and initial asset holdings. This is reflected in figure 4.

Finally, Figure 6 shows the life-time utilities of the agents for different initial wealth levels. Obviously, welfare is identical across the two economies in the stationary distribution, since the allocations are identical. Outside the stationary distribution, however, agents gain some utility in the allocation with no capital accumulation constraints compared to the allocation with autarky effects if they are relatively wealthy ($a_1 > 30$), and they loose some utility when they are less wealthy ($a_1 < 10$). The reason for the utility loss in the constrained efficient allocation is that, although agents can enjoy a higher current consumption, there is also less capital accumulation, affecting their life-time utility negatively. Since the higher consumption is more important in utility terms for the low wealth agents, this second effect dominates only for relatively wealthy households.

Overall, we conclude that both economies have very similar allocations in the long run (stationary distribution), and they exhibit some important differences in the short run. As

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[10] For these calculations, we have set $\theta_1^0(s^{t-1}) = \theta_2^0(s^{t-1}) = 0.5$.

we have seen, the model without capital accumulation constraints leads to higher short run capital accumulation and consequently to a lower current consumption. A key question is how robust these properties are to some key features of our model and calibration. In order to check this, we have also investigated several variations of the above model and calibration in what follows.

Relaxing the Autarky Punishment. In the first experiment, we allow agents to accumulate physical capital in autarky, increasing the value of the outside option and limiting the scope of risk sharing in both economies. Formally, the autarky value at state-date s^t solves the following problem:

$$V^{CE}(s^t) \equiv \max_{\{c_i(s^{t+\tau}), \kappa_i(s^{t+\tau})\}_{\tau=0}^{\infty}} \sum_{\tau=0}^{\infty} \sum_{s^{t+\tau}} \pi(s^t) \beta^t u \left(c_i \left(s^{t+\tau} \right) \right) \text{ s.t.}$$

$$c_i(s^{t+\tau}) + \kappa_i \left(s^{t+\tau} \right) \leq w(s^{t+\tau}) \epsilon_i(s^{t+\tau}) + r \left(s^{t+\tau} \right) \kappa_i \left(s^{t+\tau-1} \right) \text{ for} \forall \tau \geq 0 \tag{35}$$

$$\kappa_i \left(s^{t+\tau} \right) \geq 0 \text{ for } \forall \tau \geq 0 \text{ and } \kappa_i \left(s^{t-1} \right) \equiv 0. \tag{36}$$

where $\kappa_i \left(s^{s^{t+\tau}} \right)$ represents the individual capital holdings of type $i \in I$ households. Note that the budget constraint in (35) implies that households face (exogenously) incomplete asset markets after default. Further, the first constraint in (36) reflects that households can only save but not borrow (short-sell) physical capital after default. Finally, we assume that they take the aggregate capital accumulation and therefore the current and future prices ($w(s^{t+\tau})$ and $r \left(s^{t+\tau} \right)$) as given. Since we only consider individual (Nash) deviations and there is no default in equilibrium, these expectations are indeed rational.

Whereas we obtain a narrower range of λ in the stationary distribution, all the key qualitative findings of our original model are robust to this extension. In particular, we still find a perfect risk sharing in the long-run in both economies, while there is higher capital accumulation and a lower consumption in the short run with no capital accumulation constraints.[11] We can therefore conclude that neither the qualitative differences between the two equilibria nor the long-run perfect risk sharing property is a consequence of the tight autarky penalty that we have assumed in the benchmark model.

Using Different Parameterizations. To see if our results are robust to different parameter values, we have also studied a significantly different parametrization of the benchmark model. First, it is clear that a lower individual discount factor will make default more attractive in this environment. For this reason, we have set β to 0.65. This relatively low value of the discount factor was used by Alvarez and Jermann (2001), who study asset pricing implications of limited commitment in an endowment economy. Since this parametrization is more consistent with an annual model, we have also increased δ to 0.1. Second, it is clear

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[11] More detailed results are available from the authors upon request.

that our economy is approaching a pure exchange economy as the one studied by Alvarez and Jermann (2000) as α goes to 0. In addition, the higher α is, the more important capital income becomes for the determination of the agents' consumption. In other words, a lower capital share will make default ceteris paribus more attractive. Given this, we have reduced α to 0.20.[12]

Some of the key results resulting from this parametrization are shown on Figures 7 to 9. As shown by Figure 7, the long-run stationary distribution of λ is not degenerate with the new parameterization, implying that the individual shares of aggregate consumption are fluctuating in the long run. First, this shows that the full risk sharing result obtained with the benchmark parametrization is due to the specific parameter values we have chosen before. On the other hand, our results illustrate that the qualitative differences between the equilibria (with and without capital accumulation constraints) remain the same with the new parameterization. In particular, the competitive equilibrium without capital constraints is accumulating more capital, whereas the constrained efficient economy (with capital accumulation constraints) does not Pareto dominate the economy without accumulation constraints. Since this last economy does not exhibit full risk sharing in the long run, we can also study the differences between the two equilibria in the stationary distribution.

Figure 8 displays the path for the aggregate capital stock in the stationary distribution and along some (artificial) business cycle simulations. On the second panel of the figure, the aggregate productivity shock alternates between 10 low and 10 high values. At the same time, we draw 1000 independent samples of the idiosyncratic process of the agents for the same time horizon and we average out the results across these independent samples. Both the time series and the "business cycle" figures show that the aggregate capital stock is indeed higher in the economy without capital accumulation constraints. Finally, Figure 9 shows how the expected welfare of an agent changes during these artificial business cycles. Note that, by the law of large numbers, this expected welfare can be interpreted as the aggregate (social) welfare in the stationary distribution that arises if we assign equal weights to both types. Strikingly, we see that welfare is higher under the no capital accumulation constraint equilibrium throughout the business cycle. This result suggests that, on average, the higher income in this economy due to a higher capital accumulation offsets the welfare loss due to less risk sharing. Of course, since this allocation is not constrained efficient but satisfies the constraints of the planner's problem (20) by construction, agents will suffer welfare losses during the transition towards the higher capital accumulation that will more than offset the long run gains.

The previous welfare result has several important implications. First, it is related to the results of Davila et al. (2005) who study exogenously incomplete market economies

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[12]This value is actually consistent with the estimates of Lustig (2004), who classifies proprietor's income from farms and partnerships as labor income.

with heterogeneous agents and show that these economies may benefit from a higher capital accumulation in the long run for similar general equilibrium reasons. Further, this result indicates that less risk sharing can have non-trivial benefits in production economies due to precautionary capital accumulation.

8. CONCLUSIONS

The present paper has shown that, in contrast to the findings in exchange economies, the constrained efficient allocations of a model with limited commitment and capital accumulation cannot be decentralized as a competitive equilibrium with borrowing constraints that are not too tight. Our first key result is that, with the introduction of financial intermediaries, this decentralization becomes possible by imposing an upper limit on the intermediary's capital holdings. In addition, we characterize the competitive equilibrium with borrowing constraints that are not too tight without imposing any capital accumulation constraints. In particular, we show that this allocation solves a similar system of equations to the one of the constrained optimal solution, whereas the only inefficiency in this economy is coming from the fact that intermediaries do not internalize the effects of the aggregate capital on the autarky value of the agents. Moreover, the borrowing limits are micro founded in such a setting, since the intermediaries have no incentives to loosen or tighten them.

We think that the last set of results are particularly important, since they characterize an empirically plausible competitive equilibrium which can be used to analyze several applied questions where capital accumulation and limited commitment are both important. As an example, one could study consumption and wealth inequality along the growth path, where capital accumulation could play an important role in determining the incentives to default. In these cases, the computation of the equilibrium allocation would not require any extra burden as compared to the computation of the optimal solution due to our main characterization result.

Finally, we also show that, using a standard calibration, there are small qualitative differences between the equilibrium allocations with and without the capital accumulation constraint, especially in the long run. This is mostly due to the fact that, under standard macroeconomic calibration, agents are relatively patient, whereas capital income is a relatively important source of income. In this case, autarky is not an attractive enough outside option, even if agents can save after default.[13] We then show that, with a different parameterization such that default becomes more attractive, we obtain much less risk sharing in the long run.

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[13] Note that this is in contrast to the results of Kehoe and Perri (2002, 2004), who study an open economy with complete markets and production, obtaining an imperfect risk sharing allocation in the long run. First, whereas their idiosyncratic shocks, are interpreted and calibrated as country specific *aggregate* productivity shocks, they are shocks to *individual* labour productivity in our economy. Second, Bai and Zhang (2005) calibrate a similar economy differently and they find extensive risk sharing in the long run.

Our numerical examples also highlight the fact that the equilibrium where no capital constraints are imposed may deliver a higher welfare than the constrained efficient allocation to some agents. Further, agents may enjoy a higher expected welfare in the stationary distribution in this economy. This implies that policies (capital accumulation constraints or capital taxes) that are designed to eliminate the autarky effects are not necessarily desirable for the society, especially if the objective is to maximize welfare of future generations. Moreover, this effect identifies a non-trivial benefit from less risk sharing (tighter borrowing constraints) which arises only in production economies. In sum, our paper points out that a production economy with aggregate uncertainty can give a significantly different answer than an exchange economy when models with limited commitment are used to evaluate the welfare implications of different policies.

APPENDIX 1

Proof of Proposition 2. To prove the proposition, we first note that the capital accumulation constraint $B(s^t)$ can be set so that a constrained efficient allocation that satisfies the planner's capital Euler equation in (29) also satisfies the optimality condition of the intermediary in (8). In particular, when the enforcement constraint in the planner's problem does not bind for any household at period $t+1$, implying that $v_i\left(s^{t+1}\right) = 0$ for $i = 1, 2$ and all $s^{t+1}|s^t$, $B(s^t)$ is set to an arbitrary large number so that $B(s^t) > K(s^t)$, where $K(s^t)$ is capital stock in the planner's problem. In this case, $\psi(s^t) = 0$. Further, when the enforcement constraint in the planner's problem is binding for any of the two households, $B(s^t)$ is set to the level of capital that solves the optimal allocation. In this case, equations (29) and (8) imply that the multiplier of the capital accumulation constraint is given by:

$$\psi(s^t) = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \sum_{i=1,2} \frac{v_i(s^{t+1})}{u'\left(c_i(s^t)\right)} V_K(S_i\left(s^{t+1}\right)) \right\}$$

Whereas $v_i(s^{t+1}) \geq 0$ and $u'\left(c_i(s^t)\right) \geq 0$, we have that $V_K(S_i\left(s^{t+1}\right)) \geq 0$ for $i = 1, 2$, since our assumptions on the production function imply that the marginal product of labor is increasing in capital. Given this and that $v_i(s^{t+1}) \geq 0$ and $u'\left(c_i(s^t)\right) \geq 0$, it follows that $\psi(s^t) \geq 0$.

The factor prices $w\left(s^t\right)$ and $r\left(s^t\right)$ that satisfy the optimality conditions of the firm in the competitive equilibrium can be constructed from the capital allocation of the planner's problem using equations (5)-(6). Further, the consumption allocations from the planner's problem and equations (27) and (28) can be used to define the prices $q(s^{t+1}|s^t) = q_p(s^{t+1}|s^t)$ and $Q(s^{t+1}|s^t) = Q_p(s^{t+1}|s^t)$. In addition, $q(s^{t+1}|s^t)$ can be used to define the multiplier $\gamma_i(s^{t+1})$ so that the asset Euler condition of the agents in equation (13) is satisfied. It is easy to check that the multiplier will have the desired properties. In particular, if $v_i = 0$,

$\gamma_i(s^{t+1}) = 0$. Further, if $v_i(s^{t+1}) > 0$, it follows that $\gamma_i(s^{t+1}) > 0$. To see this, suppose that $v_j(s^{t+1}) > 0$ for some $j = 1, 2$. Then,

$$\beta\pi(s^{t+1}|s^t)\frac{u'\left(c_j(s^{t+1})\right)}{u'\left(c_j(s^t)\right)} < \max_{i=1,2}\left\{\pi(s^{t+1}|s^t)\beta\frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)}\right\}$$

and

$$q(s^{t+1}|s^t) = \beta\pi(s^{t+1}|s^t)\max_{i=1,2}\left\{\frac{u'\left(c_i(s^{t+1})\right)}{u'\left(c_i(s^t)\right)}\right\} = \beta\pi(s^{t+1}|s^t)\frac{u'\left(c_j(s^{t+1})\right)}{u'\left(c_j(s^t)\right)} + \frac{\gamma_i(s^{t+1})}{u'\left(c_j(s^t)\right)}.$$

Since the high implied interest rate condition holds, we can then use the budget constraint of the households in the competitive equilibrium to construct the wealth levels $\omega_i(s^t)$ that support the constrained efficient consumption allocations at every node. To do this, we first construct the profits $d\left(s^t\right)$ from (9), the share price $p\left(s^t\right)$ from (14) and the individual labor incomes from $w_i\left(s^t\right) = w\left(s^t\right)\epsilon_i\left(s^t\right)$. Further, we iterate on the budget constraint of each household to obtain that:

$$\omega_i(s^t) = \sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t}Q(s^{t+n}|s^t)\overline{c}_i(s^{t+n}) \tag{37}$$

and we let:

$$\omega_i(s^0) = \sum_{t=0}^{\infty}\sum_{s^t|s^0}Q(s^t|s^0)\overline{c}_i(s^t). \tag{38}$$

Concerning the trading limits, if $v_i(s^t) = 0$ for agent i, we set the limits at the natural borrowing limit, which is given by:

$$A_i(s^{t+1}) = -\sum_{n=1}^{\infty}\sum_{s^{t+n}|s^t}Q(s^{t+n}|s^t)\left[w_i(s^{t+n})\epsilon_i(s^{t+n}) + \theta_i^0(s^{t+n-1})p\left(s^{t+n}\right)\right]$$

and we will redefine the limit for these cases later. In addition, if $v_i(s^t) > 0$, we set $A_i(s^{t+1}) = \omega_i(s^{t+1})$, implying that it will be binding when the participation constraint in the planner's problem is binding. The transversality condition is satisfied, since:

$$\lim_{t\to\infty}\sum_{s^t}\beta^t\pi(s^t)u'\left(c_i(s^t)\right)\left[\omega_i(s^t) - A_i(s^t)\right]$$

$$\leq \lim_{t\to\infty}\sum_{s^t}\beta^t\pi(s^t)u'\left(c_i(s^t)\right)\left[\sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t}Q(s^{t+n}|s^t)c_i(s^{t+n})\right]$$

$$\leq u'\left(c_i(s^0)\right)\lim_{t\to\infty}\sum_{s^t}\beta^t\pi(s^t)\frac{u'\left(c_i(s^t)\right)}{u'\left(c_i(s^0)\right)}\left[\sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t}Q(s^{t+n}|s^t)\sum_i c_i(s^{t+n})\right]$$

$$\leq u'\left(c_i(s^0)\right)\lim_{t\to\infty}\sum_{s^t}Q(s^t|s^0)\left[\sum_{n=0}^{\infty}\sum_{s^{t+n}|s^t}Q(s^{t+n}|s^t)\sum_i c_i(s^{t+n})\right] = 0.$$

The first inequality follows from the fact that $[\omega_i(s^t) - A_i(s^t)]$ is equal to zero if the participation constraint is binding and it is equal to $\sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) c_i(s^{t+n}) \geq 0$ otherwise, since $\omega_i(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \bar{c}_i(s^{t+n})$. The second follows from the fact that $c_i(s^t) \leq \sum_i c_i(s^t)$. The third inequality follows from the the definition of $Q(s^t|s^0)$ and from the fact that $Q(s^t|s^0) \geq \beta^t \pi(s^t) \frac{u'(c_i(s^t))}{u'(c_i(s^0))}$ by construction. Finally, the last equality follows form the high implied interest rate condition.

To show that markets clear, we can sum the total asset wealth in (37) and (38), obtaining that:

$$\sum_i \omega_i(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \sum_i \bar{c}_i(s^{t+n}) = \left[f_K(s^t) + (1 - \delta) \right] K(s^{t-1}) + d\left(s^t\right)$$

$$\sum_i \omega_i(s^0) = \sum_{t=0}^{\infty} \sum_{s^t} Q(s^t|s^0) \sum_i \bar{c}_i(s^t) = \left[f_K(s^0) + (1 - \delta) \right] K(s^0) + d\left(s^0\right).$$

Here, we have used the definitions of $r\left(s^t\right)$ and $p\left(s^t\right)$ and the fact that $\sum_i a_i\left(s^t\right) = r\left(s^t\right) K\left(s^{t-1}\right)$ and $\sum_i \theta_i\left(s^t\right) = 1$. In addition, summing the two budget constraints, we have that:

$$\begin{aligned} \sum_i c_i(s^t) &= \sum_i \omega_i\left(s^t\right) + p\left(s^t\right) + w\left(s^t\right) - \sum_{s^{t+1}|s^t} q\left(s^{t+1}|s^t\right) \sum_i \omega_i\left(s^{t+1}\right) \\ &= y\left(s^t\right) + (1 - \delta) K(s^{t-1}) - K\left(s^t\right). \end{aligned}$$

where we have used the definitions of $p\left(s^t\right)$ and $d\left(s^t\right)$ in (14) and (9) and the homogeneity of degree 1 property of the production function.

It only remains to redefine the borrowing limits so that they are not too tight. To do this, we first construct the autarky values at each node using the allocations of the planner:

$$V^{ce}(S_i(s^t)) = \sum_{r=t}^{\infty} \sum_{s^r|s^t} \beta^{r-t} \pi\left(s^r\right) u\left(f_L\left(s^r\right) \epsilon_i\left(s^r\right)\right).$$

Similarly, we can construct the value function $W^{ce}(\omega_i(s^t), S_i(s^t))$ and use the two functions to redefine the borrowing constraints for the nodes where the limit is not binding. In particular, we can iterate on the constraint $A_i(s^t)$ until we find the one that satisfies $W^{ce}(A_i(s^t), S_i(s^t)) = V^{ce}(S_i(s^t))$. Since the new set of constraints constraint is (weakly) tighter than before, the new value of $\omega_i - A_i$ still satisfies the transversality condition. Further, since, these constraints do not bind for any household for whom the participation constraint is not binding in the planner's solution, the allocation derived above with the original constraints is still feasible and optimal.∎

Proof of Proposition 3. (i) We first show that there are no profitable deviations from the equilibrium allocation with limits that are tighter or looser then the ones defined

by (17). To see this, first notice that tightening the limits will not increase the profits of any intermediary. Further, we now show that no intermediary can make positive profits by loosening the limits, that is, by setting $\overline{A}_i(s^t) \leq A_i(s^t) < 0$ for all s^t. To do this, assume (without a loss of generality) that $\overline{A}_1(\widehat{s}) < A_1(\widehat{s})$ for some $\widehat{s}|\widetilde{s}$ where the borrowing constraint is binding for type 1 agents at the level of wealth $A_1(\widehat{s})$. Under the original prices $q(s^{t+1}|\widetilde{s})$, this implies that type 1 agents would default next period if node $\widehat{s}|\widetilde{s}$ occurs. Since type 1 households would choose $a_1(\widehat{s}) < A_i(\widehat{s}) < 0$ and default if $\widehat{s}$ occurs, it is easy to see that the intermediary would make negative profits. First define $\overline{a}_1(s^{t+1}|\widetilde{s})$ as the asset decision of type 1 households under the new limits and observe that $\overline{a}_1(\widehat{s}) < A_i(\widehat{s}) \leq 0$ under $q(\widehat{s}|\widetilde{s})$. Then, default of type 1 households imply that the profits of the intermediary are given by:

$$
\begin{aligned}
\overline{d}(\widetilde{s}) &= -k(\widetilde{s}) + \sum_{s^{t+1}|\widetilde{s}} q(s^{t+1}|\widetilde{s})[r(s^{t+1}) + (1-\delta)]k(\widetilde{s}) + q(\widehat{s}|\widetilde{s})\overline{a}_1(\widehat{s}) \\
&< -k(\widetilde{s}) + \sum_{s^{t+1}|\widetilde{s}} q(s^{t+1}|\widetilde{s})[r(s^{t+1}) + (1-\delta)]k(\widetilde{s}) = 0.
\end{aligned}
$$

The second equality follows from the equilibrium condition of the intermediaries in (8).

(ii) We now show that there does not exist any symmetric equilibrium with limits that are looser than the limits that are not too tight. To do this, we assume there exists an equilibrium with prices q and limits $\{A_i\}_{i=1,2}$ such that agents of type 1 would default under some continuation history $s^{t+1}|s^t = \widehat{s}|s^t$ if the current history is $s^t = \widetilde{s}$. First, notice that perfect competition would still require that intermediaries will make zero profits, which would be given by:

$$
d(\widetilde{s}) = -k(\widetilde{s}) + \sum_{s^{t+1}|\widetilde{s}} q(s^{t+1}|\widetilde{s})[r(s^{t+1}) + (1-\delta)]k(\widetilde{s}) + q(\widehat{s}|\widetilde{s})a_1(\widehat{s}) = 0.
$$

Since a household would only default at node $\widehat{s}$ if $a_1(\widehat{s}) < 0$, the previous equation implies that:

$$
-k(\widetilde{s}) + \sum_{s^{t+1}|\widetilde{s}} q(s^{t+1}|\widetilde{s})[r(s^{t+1}) + (1-\delta)]k(\widetilde{s}) > 0.
$$

Thus, in any symmetric equilibrium with default, it must be the case that:

$$
\sum_{s^{t+1}|\widetilde{s}} q(s^{t+1}|\widetilde{s})[r(s^{t+1}) + (1-\delta)] - 1 > 0.
$$

The previous condition implies that any intermediary could make arbitrarily positive profits by trading only with agents of type 2 and by demanding arbitrary large amounts of total deposits ($\sum_{s^{t+1}|\widetilde{s}} q(s^{t+1}|\widetilde{s})a_2(s^{t+1}|\widetilde{s})$) from them. However, this contradicts the fact that the original portfolio was optimal for the intermediaries under $q(s^{t+1}|s^t)$. $\blacksquare$

Proof of Proposition 4. The proof follows the same arguments as the proof of proposition 2, and we therefore only sketch it in what follows. First, given the consumption

allocations $\{c_i\}_{i=1,2}$ from the planner's problem, we can use (27) and (28) to define the prices $q(s^{t+1}|s^t) = q_p(s^{t+1}|s^t)$ and $Q(s^{t+1}|s^t) = Q_p(s^{t+1}|s^t)$ for all nodes. Further, since the high implied interest rate condition holds, we can then use the prices and the budget constraint of the households in (30) to construct the holdings $\{a_i\}_{i=1,2}$ so that the constrained efficient consumption allocations $\{c_i\}_{i=1,2}$ are feasible at every node. Note that, in the absence of a capital accumulation constraint, the profits of the intermediary are always equal to zero. Concerning the trading limits, if $v_i(s^t) = 0$ for agent i, we first set $A_i(s^{t+1}) = -\sum_{n=1}^{\infty}\sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t)w_i(s^{t+n})$ and we will redefine this limit later. Further, if $v_i(s^t) > 0$, we set $A_i(s^{t+1}) = a_i(s^{t+1})$, implying that it will be binding when the participation constraint in the planner's problem is binding. To make sure that the sufficient Euler equations are satisfied, we can first use $q(s^{t+1}|s^t)$ to define the multiplier $\gamma_i(s^{t+1})$ so that the Euler condition of the agents in (13) is satisfied. It is easy to see that an allocation that satisfies (34) also satisfies the equilibrium condition of the intermediary in (8) with $\psi = 0$. Further, using the same arguments as in the proof of proposition 2, we can check that the transversality condition $\lim_{t\to\infty}\sum_{s^t}\beta^t\pi(s^t)u'\left(c_i(s^t)\right)[a_i(s^t) - A_i(s^t)] \leq 0$ is satisfied. Finally, we can construct the value functions $W(a_i(s^t); S_i(s^t))$ and $V(S_i(s^t))$ from the value functions of the planner's problem and redefine the borrowing constraints on Arrow security holdings so that they satisfy $W(A_i(s^{t+1}); S_i(s^{t+1})) = V(S_i(s^{t+1}))$ at every node. Since these limits do not bind for the originally unconstrained consumers, the constructed allocations are still feasible and optimal.∎

Proof of Proposition 5. To prove the proposition, we first note that the resource constraint in (19) is satisfied by the competitive equilibrium allocations. Since the asset holdings are subject to portfolio restrictions $\{A_i\}_{i\in I}$ that are not too tight, the value functions in the competitive equilibrium satisfy:

$$W^{ce}(a_i(s^t), S_i(s^t)) \geq V^{ce}(S_i(s^t))$$

for all $i = 1,2$ and all $s^t \in N$, where $W^{ce}(a_i(s^t), S_i(s^t)) = \sum_{r=t}^{\infty}\sum_{s^r}\beta^{r-t}\pi(s^r|s^t)u(c_i(s^r))$ and $V^{ce}(S_i(s^t)) = \sum_{r=t}^{\infty}\sum_{s^r}\beta^{r-t}\pi(s^r|s^t)u(w(s^r)\epsilon_i(s^r))$. Given this, the functions defined by $W(S_i(s^t)) = W^{ce}(a_i(s^t), S_i(s^t))$ and $V(S_i(s^t)) = V^{ce}(S_i(s^t))$ satisfy the participation constraints in (20). We also note that the competitive equilibrium allocations still solve the same problem if the borrowing constraints on the Arrow securities of the unconstrained households are substituted for the natural borrowing limits defined by:

$$A_i(s^{t+1}) = -\sum_{n=1}^{\infty}\sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t)w_i\left(s^{t+n}\right).$$

Optimality implies that the previous limit is finite.[14] In addition, since the shocks z and ϵ lie in a compact set, the present values of K and $f_L\left(s^t\right)$ are finite, we can use the

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[14]In an exchange economy context with sequential trade and potentially incomplete financial markets,

resource constraint to show that the competitive equilibrium allocation satisfies the high implied interest rate condition.

To recover the multipliers in the planner's problem, we can first use the equilibrium consumption allocations to define $\lambda(s^t) = \frac{u'(c_1(s^t))}{u'(c_2(s^t))}$. Further, $\{v_i\}_{i=1,2}$ can be recovered as follows. If the portfolio constraint is not binding for household i at node s^t in the decentralized problem, we set $v_i(s^t) = 0$. Otherwise, if it is binding for agent two, we set $v_1(s^t) = 0$ and $v_2(s^t)$ is recovered from:

$$\frac{u'(c_1(s^t))}{u'(c_2(s^t))} = (1 + v_2(s^t))\frac{u'(c_1(s^{t-1}))}{u'(c_2(s^{t-1}))}$$

Similarly, if it is binding for agent one, we set $v_2(s^t) = 0$ and $v_1(s^t)$ is recovered from:

$$\frac{u'(c_1(s^t))}{u'(c_2(s^t))} = \frac{1}{(1 + v_1(s^t))}\frac{u'(c_1(s^{t-1}))}{u'(c_2(s^{t-1}))}.$$

Clearly, this implies that equations (22) and (24)-(25) are satisfied. In addition, the zero profit condition in equation (8) of the decentralized solution with $\psi = 0$ can be rewritten as:

$$
\begin{aligned}
1 &= \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t)\left\{\max_{i=1,2}\left[\frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))}\right]F_K(s^{t+1})\right\} \\
&= \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t)\left\{\left[\frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))}(1 + v_i(s^{t+1}))\right]F_K(s^{t+1})\right\}
\end{aligned}
$$

Given this, the Euler equation of the planner in (34) is also satisfied.∎

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Santos and Woodford (1997) show that the natural borrowing limit implied by the optimal allocations has to be finite. Otherwise, one can construct a portfolio that yields more utility than the optimal allocation. The same proof can be used in the present setup.

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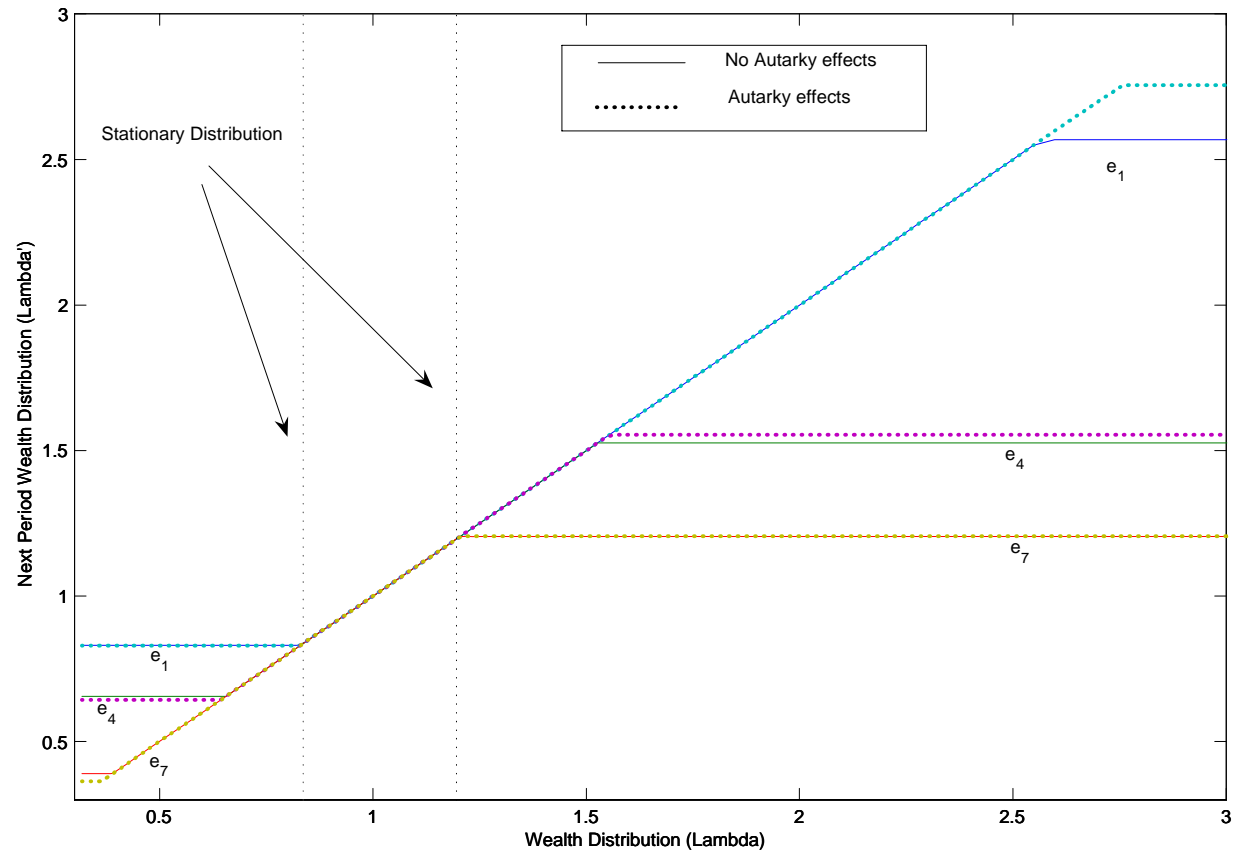**Figure 1: Next Period Wealth Distribution (λ') as a Function of λ and ϵ**

Figure 2: Optimal Consumption (c_1) as a Function of λ and ϵ
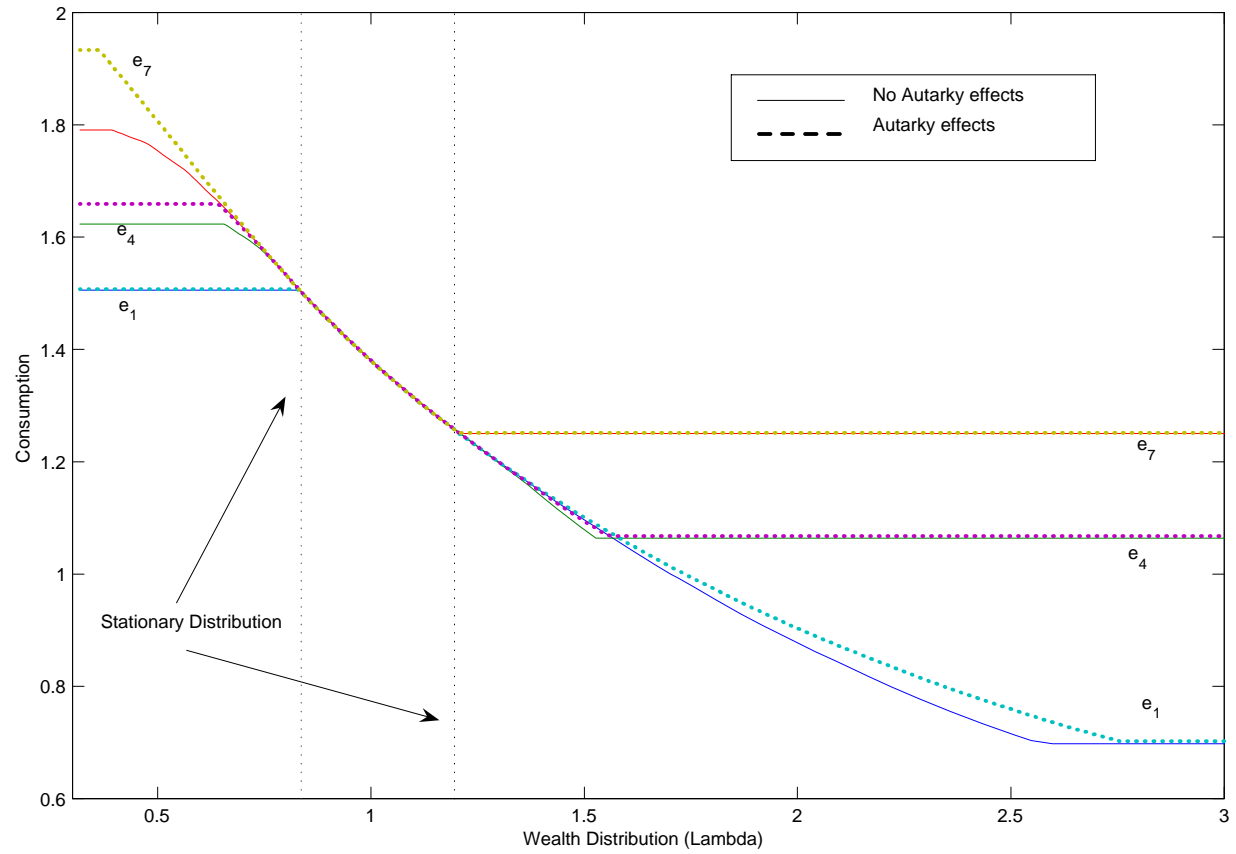
Figure 3: Next Period Capital Stock (K') as a Function of λ and ϵ
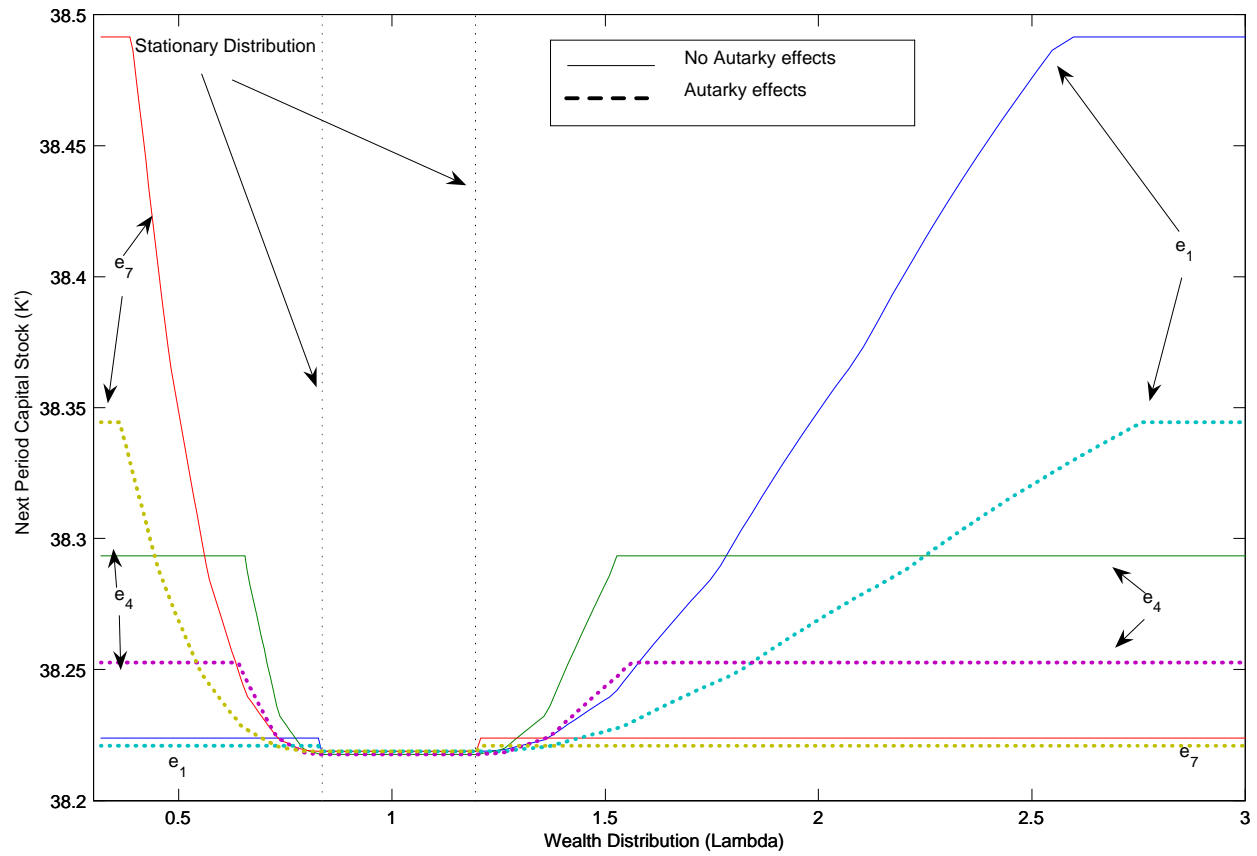
Figure 4: Optimal Consumption (c_1) as a Function of a_1 and ϵ
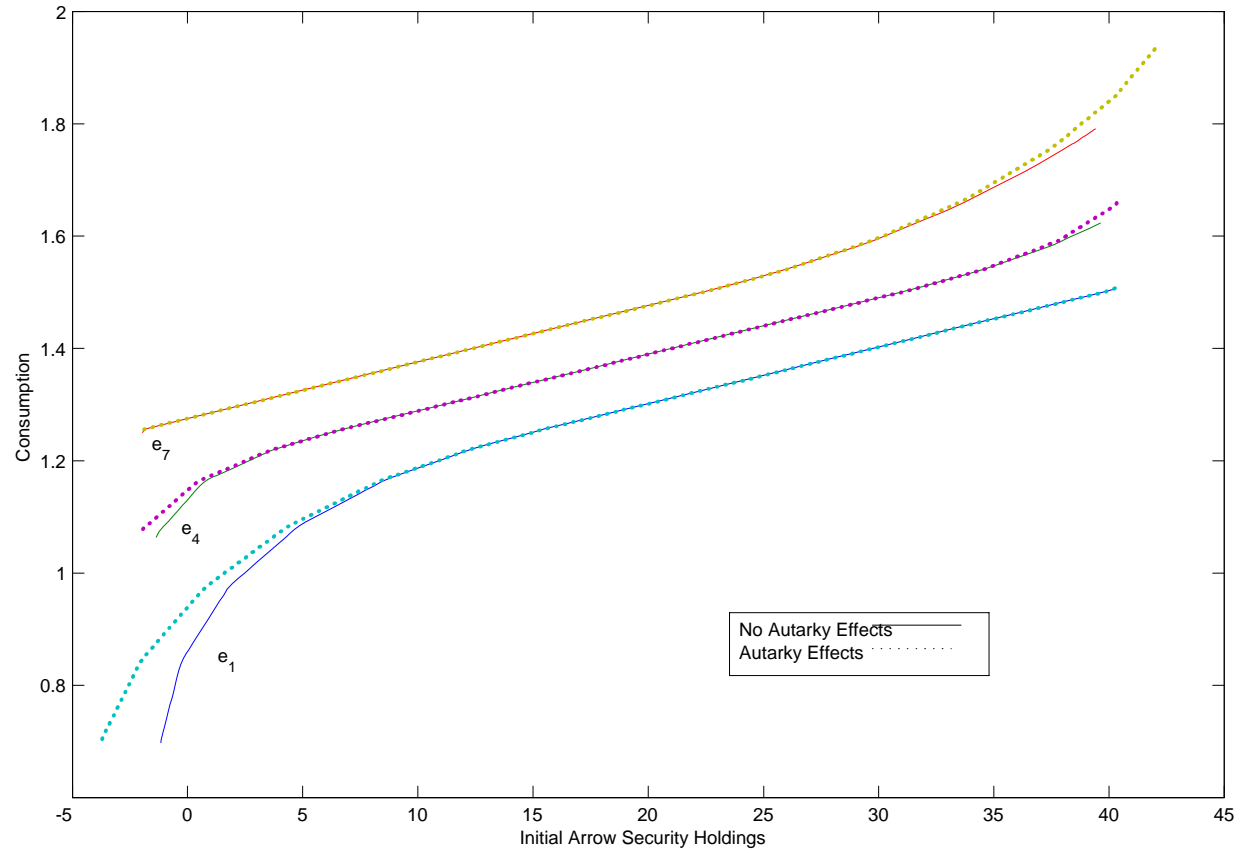
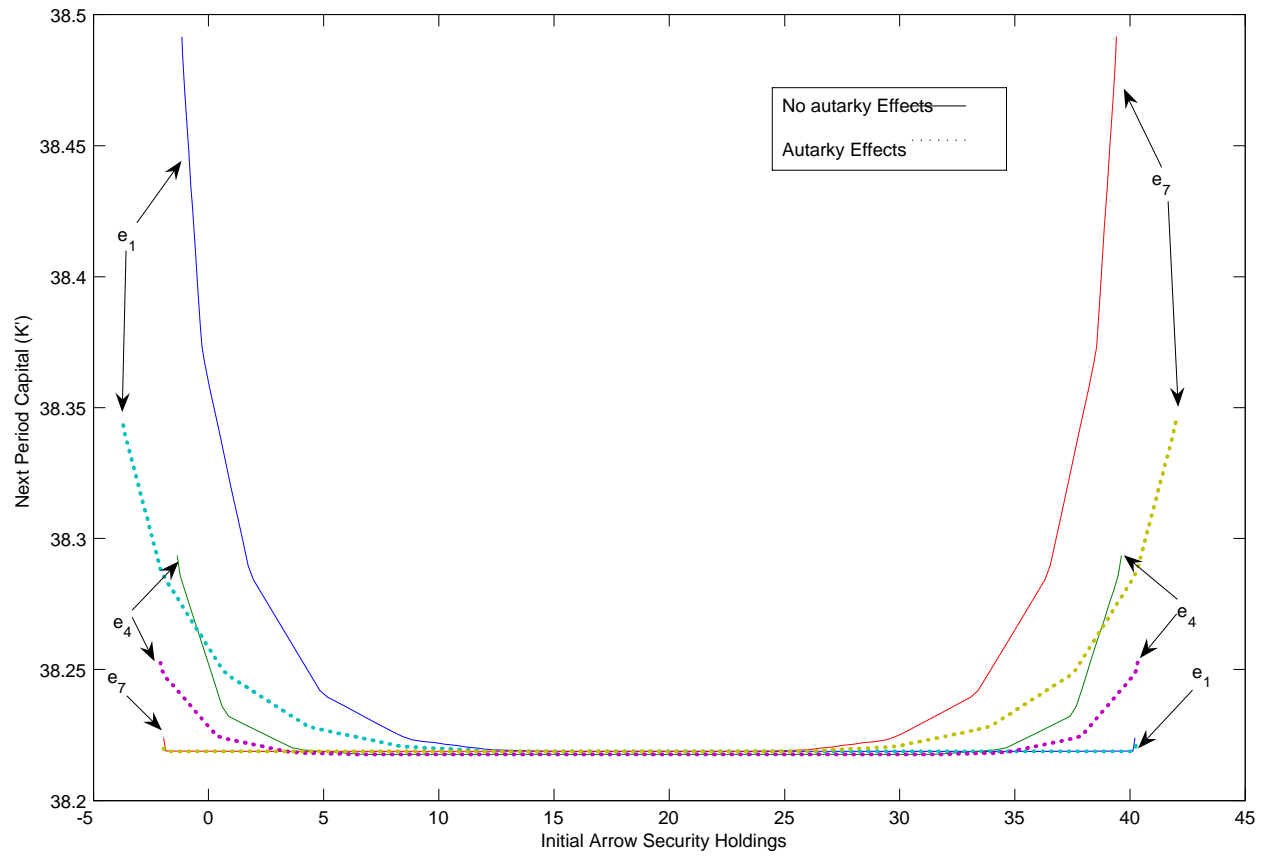**Figure 5: Next Period Capital Stock (K') as a Function of a_1 and ϵ**

Figure 6: Life-Time Utility (W_1) as a Function of a_1 and ϵ
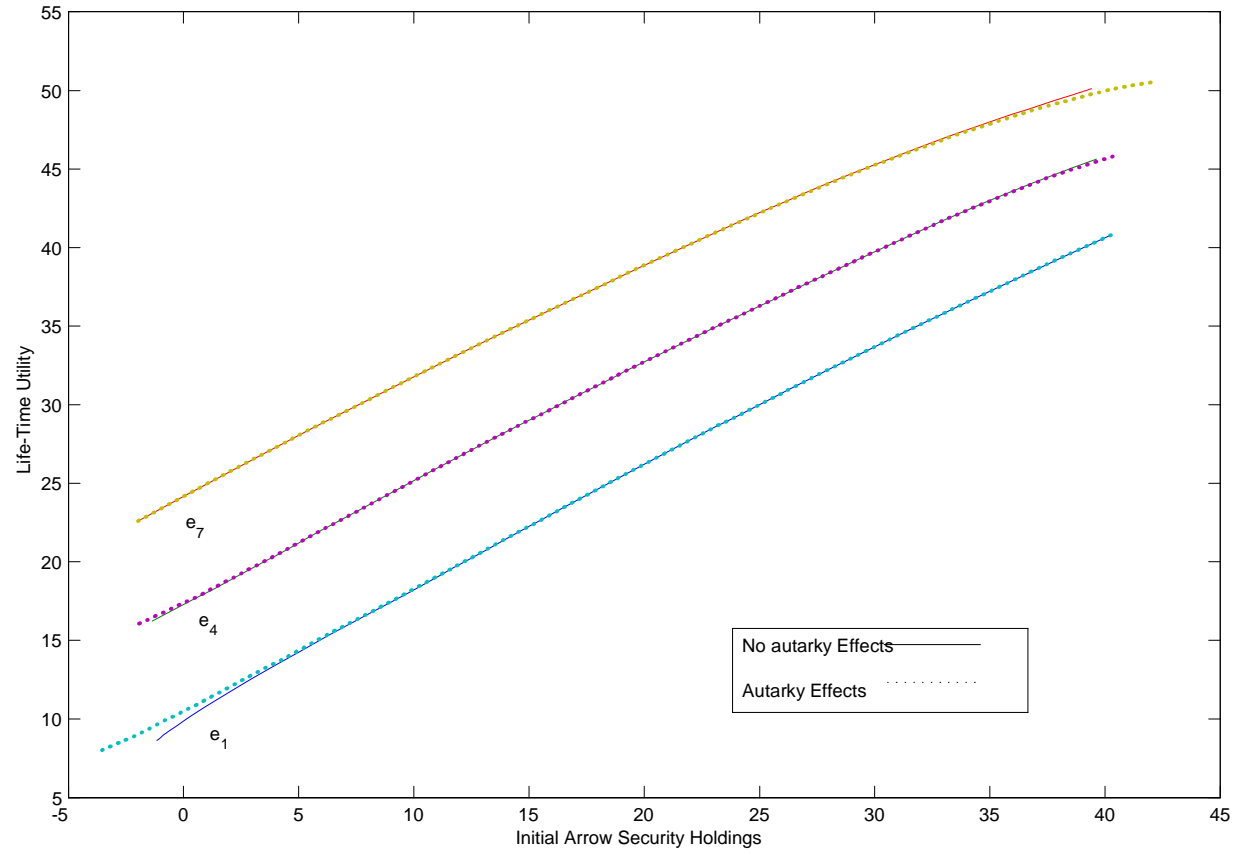
Figure 7: Next Period Wealth Distribution (λ') as a Function of λ and ϵ
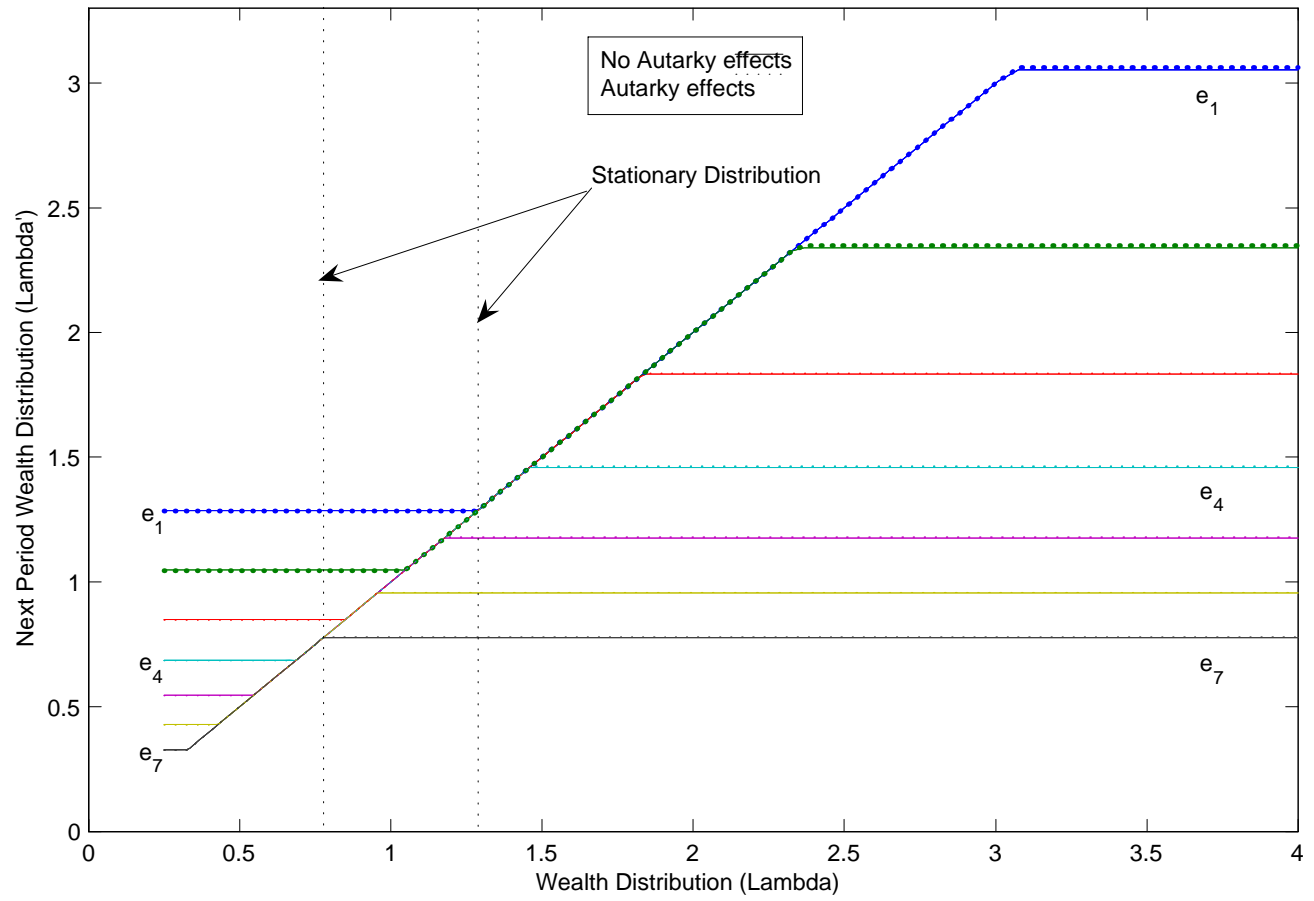
Figure 8: Next Period Capital Stock (K') from Time Series Simulations

Aggregate Capital Accumulation in a Long Run Simulation

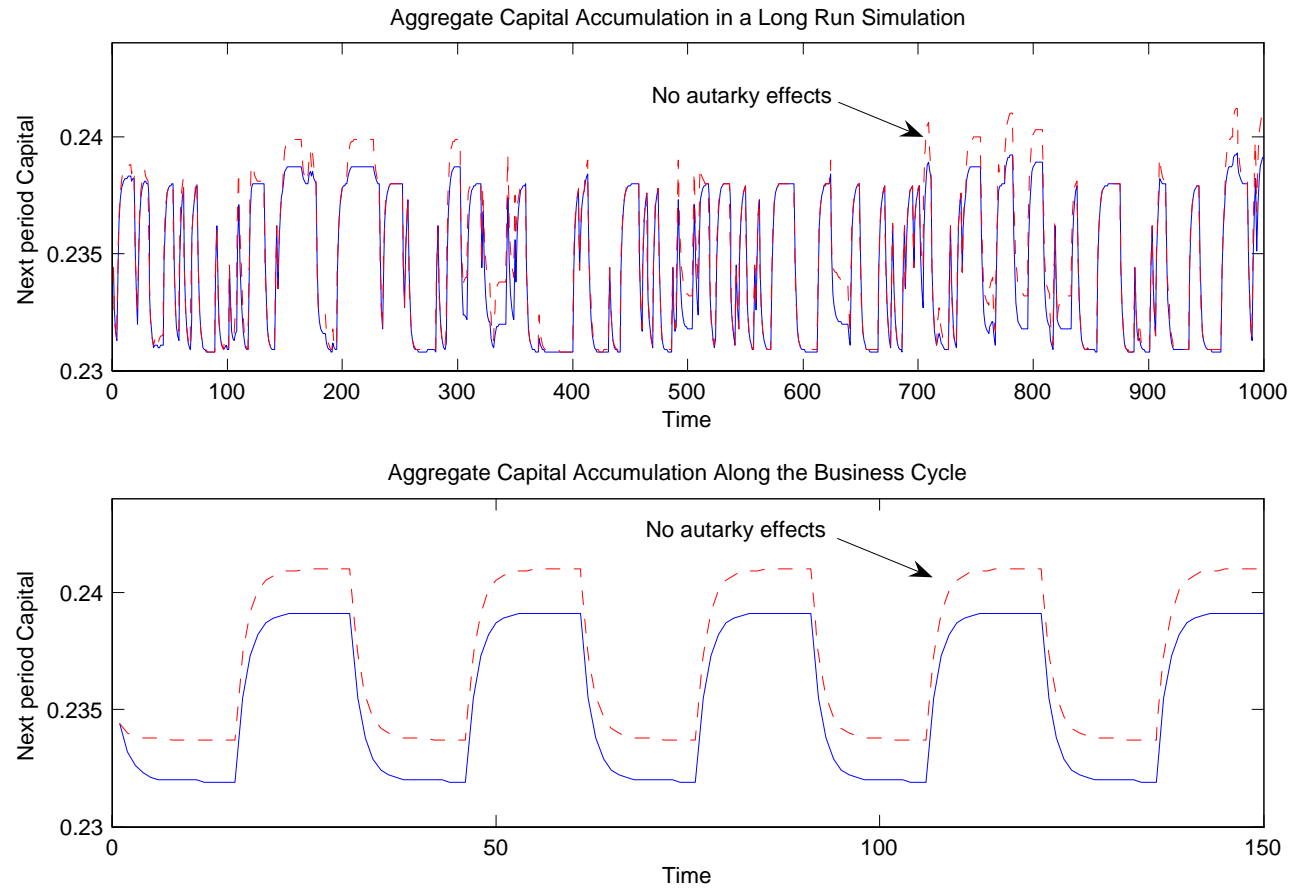Aggregate Capital Accumulation Along the Business Cycle

Figure 9: Average Life-Time Utility (W_1) from Time Series Simulations