Behavioral Consistent Market Equilibria under Procedural Rationality *

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Abstract

In this paper we analyze a dynamic, asset pricing model where an arbitrary number of heterogeneous, procedurally rational investors divide their wealth between two assets. Both fundamental dividend process and behavior of traders are modeled in a very general way. In particular, agents' choices are described by means of the generic smooth functions defined on a commonly available information set. The choices are consistent with (but not limited to) the solutions of the expected utility maximization problems.

As a natural rest point of the corresponding dynamics we propose the notion of the Behavioral Consistent Equilibria (BCE) where the aggregate dynamics are consistent with the agents' investment choices. We show that provided that the dividend process is given, all possible equilibria of the system can be characterized by means of one-dimensional Equilibrium Market Line (EML). This geometric tool allows to separate the effects of dividend process and agents' behaviors on the aggregate dynamics. Namely, the precise shape of this line depends on the character of the dividend process, but the realized equilibrium, i.e. a point on the line, is determined by the ecology of agents' behaviors. We argue that the EML can be useful in investigation of the questions of existence, multiplicity and stability of the BCE and provide corresponding examples. The EML also allows to make the comparative static exercises in a framework with heterogeneous agents and discuss the relative performances of different strategies.

The notion of BCE can be considered as a generalization of the Rational Expectations Equilibrium on the framework with heterogeneous traders. It can be, therefore, useful also in other fields of economics where heterogeneity of actors plays an important role for the aggregate outcome.

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1 Introduction

This paper is devoted to the analytic investigation of an asset pricing model where an arbitrary number of heterogeneous generic traders participate in a speculative activity. The main purpose of the paper is to put forward the notion of the Behavioral Consistent Equilibria which can be considered as a generalization of the Rational Expectations Equilibria for an arbitrary dynamic framework with generic heterogeneous agents. We introduce this concept inside a simple model of speculative market, where such classical economic questions as origin of the equity premium or the survivance of irrational traders can be addressed.

We consider a pure exchange economy where one asset is a riskless security yielding a constant return on investment, and another asset is a risky equity paying a stochastic dividend. Trading takes place in discrete time and in each trading period the relative price of the risky asset is fixed through a market clearing condition. Agents participation to the market is described in terms of their individual demand for the risky asset. We impose only one restriction on the way in which the individual demands of traders are formed. Namely, the amount of the risky security demanded by any trader is assumed to be proportional to his current wealth. Corresponding *investment shares* of the agents' wealth are chosen at each period on the basis of the commonly available information.

This behavioral assumption is consistent with a number of strategies based on optimization, and in particular on the maximization of expected utility function with constant relative risk aversion (CRRA). However, the framework is not limited to such rational behaviors. Works of Herbert Simon (see e.g. Simon (1976)) emphasize that agents operating in the markets may not be optimizers, but still avoid to behave in completely random or irrational manner. That is even if they are not "rational" in the sense as this word is widely used in economics, the traders can follow some deliberately chosen or invented procedures. Such agents can be called *procedurally* rational to stress the difference with respect to the smaller class of substantively rational optimizers. We model procedural rationality by means of smooth *investment functions* which map the information set to the present investment share.

The presence of procedural rationality in the market naturally leads to the idea of heterogeneity of agents. There are no doubts that even rational agents differ in terms of preferences and implied actions. In the last years, many contributions emphasize an importance of the heterogeneity in expectations for explanation of observed "anomalies" of financial markets (i.e. those facts that cannot be explained by classic financial models) like huge trading volume or excess volatility, see e.g. Brock (1997). In this paper investment functions are agent-specific and, thus, describe the outcome of an idiosyncratic procedures which can be defined as the collective description of the preferences, beliefs and implied actions.

Using assumption of CRRA-type of behavior but avoiding the precise specification of investment functions and dividend process we derive the stochastic dynamical system governing asset price and agents' wealths. In the rest of the paper we study the natural rest points of the *deterministic skeleton* of this system, i.e. we consider the situation without exogenous noise (due to dividend) when only agents' heterogeneity drives the dynamics. Such skeleton, which gives an "average" dynamics, can be investigated using the analytic tools of the theory of dynamical systems. We introduce the notion of *Behavioral Consistent Equilibrium* (BCE) which incorporates two ideas. First, in the long-run agents should be able to find a behavioral rule, which would be constant in the absence of shock. In the presence of random dividends, therefore, the behavior of any agent would look like fluctuations around this constant benchmark caused by past dividend realizations. Second, these choices should be consistent with aggregate dynamics, which both affect and are affected by the agents' choices. Thus, our BCE can be considered as a generalization of the rational expectations hypothesis to the framework with generic procedurally rational heterogeneous agents.

It turns out that all the BCE can be characterized in an informative way even when both dividend process and set of behaviors of market participants kept general. In particular, there exist two types of equilibria. In one of the type the risky asset provides different expected return with respect to the riskless asset, so-called *equity premium*, while in the other type the equity premium is absent. Furthermore, we show that on the one hand, not any combination of the dividend process and investment functions can bring the equity premium. On the other hand, if such situation happened, then there exists a simple relation between equilibrium price return and investment fractions, irrespectively of the number of agents and the shapes of their investment functions. This equilibrium relation can be described by means of one-dimensional function, the *Equilibrium Market Line* (EML), previously introduced in Anufriev et al. (2006). The usefulness of this tool follows from its ability to separate the influence of the dividend process from the influence of the agents' behavior on the equilibrium relations. Indeed, the precise shape of the EML depends on the dividend process, while the precise location of equity-premium BCE on the given EML depends on the ecology of traders.

For illustrative purposes, we perform the detailed analysis of two particular specifications for the dividend process. Such examples allow us to visualize the EML, show how the EML can be used for the comparative statics exercises and also illustrate that the EML can also be used for the exploration of stability conditions.

Our model can be confronted with the last contributions in the field of the Heterogeneous Agent Models (HAMs) extensively reviewed in Hommes (2006). First, the majority of the HAMs are built assuming independence of the agents' demands from their wealths. In terms of expected utility theory, it amounts to consider constant absolute risk averse (CARA) traders. CARA-type of behavior is assumed for the sake of simplicity, in order to decouple wealth evolution from the system and concentrate on the analysis of price dynamics. Our choice of the CRRA framework is motivated, instead, by the empirical and experimental evidence in its favor (see the discussions in Levy et al. (2000) and Campbell and Viceira (2002)) and also by a relative rarity of corresponding HAMs (exceptions are Chiarella and He (2001), Chiarella et al. (2006), Anufriev et al. (2006)). Second, since HAMs are concentrated on the heterogeneity in expectations, it is typical to work with common demand functions (i.e. preferences, attitude towards risk, etc.). Furthermore, the expectations are modeled in the simplest possible way sufficient to reflect different stylized behaviors, like "fundamental", "trend chaser" or "contrarian" attitude. Keeping investment functions generic, we intend to avoid unrealistic simplicity of the agents' expectations and assumption of fixed preferences. Consequently, the role of such parameters as the length of memory or coefficient of trend extrapolation can be analyzed *without* changing the model set-up. Third, HAMs usually deal either few types of investors (e.g. two types in DeLong et al. (1991) and Chiarella and He (2001), three in Day and Huang (1990) and up to four types in Brock and Hommes (1998)) or with the limiting properties of the market when the number of types is large enough (Brock et al., 2005) to apply some variation of the Central Limit Theorem. Instead, our results are valid for any finite number of investors.

At the same time, the current model can also be compared with so-called, evolutionary finance literature (see, e.g. Blume and Easley (1992), Sandroni (2000) and Hens and Schenk-Hoppé (2005)). These are analytic investigations of the market with many assets populated by the agents of CRRA-type behavior. One important drawback of these contributions is the assumption of short life of the assets leading to the ignorance of the capital gain on the agents' wealths. Our work can be seen, thus, as an extension of the evolutionary finance analysis in this direction, even if we consider simpler market setting with only one risky asset.

Finally, one can consider this model as an analytic counterpart of the numerous simulations of the artificial financial markets with CRRA agents (see, e.g. Levy et al. (1994), Levy et al. (2000) and Zschischang and Lux (2001) and recent review in LeBaron (2006)). The need of such analytic investigation seems apparent because of the inherent difficulty to interpret the results of simulations in a systematic way. Model with *generic* agents' behaviors is especially useful given the tendency to simulate markets with many different types of behavior.

There is one common question which unify all three streams of the literature mentioned above. Is the Milton Friedman's hypothesis about impossibility for the non-rational agents to survive in the market valid? Our general results provide a simple and clear answer to this question. Indeed, we show that the survivors in the market are determined not only by their strategies, but instead by the total behavioral ecology coupled with the nature of the dividend process. Consequently, the Friedman's hypothesis is not valid in our framework, even if it can hold for some particular cases. E.g. applying our results to the set of expected utility maximizing behaviors one can re-obtain findings of Blume and Easley (1992) that the survivor is rational agent but not any rational agent will survive.

The rest of the paper is organized as follows. In Section 2 we describe our economy. First, we explicitly write the traders' inter-temporal budget constraints. Second, we derive the resulting dynamics in terms of returns and wealth shares. Finally, we introduce the dividend specification and agent-specific investment functions. In Section 3 we define the Behavioral Consistent Equilibria, discuss this notion and give general characterization of the corresponding dynamics. We devote Section 4 to the more detailed discussion of that equilibria in two specific examples of the dividend process. The Equilibrium Market Line is introduced for each case and compared. In Section 5 we address the question of stability of the Behavioral Consistent Equilibria. The discussion is concentrated around one special case of the dividend process, namely when the dividend yield is constant. The general case is rather impossible to address but the particular example gives some important hints about what can happen in the general model. Our conclusions, and the directions our work will plausibly take in the future, are briefly mentioned in Section 6.

2 Definition of the Model

Consider a simple pure exchange economy, where trading activities take place in discrete time. The economy is composed by a riskless asset (bond) giving in each period a constant interest rate $r_f > 0$ and a risky asset (equity) paying a random dividend D_t at the beginning of each period t. The riskless asset is considered the numéraire of the economy and its price is fixed to 1. The ex-dividend price P_t of the risky asset is determined at each period, on the basis of the aggregate demand, through market-clearing condition. The resulting intertemporal budget constraint is obtained below. It allows to derive an explicit dynamical system governing the evolution of the economy. The assumptions on the nature of the dividend process and of the investment choices are discussed at the end of this Section.

2.1 Intertemporal budget constraint

The economy is populated by a fixed number N of traders[1] who do not consume but reinvest their total wealths each period. In the beginning of time t each trader n gets dividends D_{t+1} and interest rate r_f per each share of the corresponding asset. He then determines a new portfolio deciding to invest a certain fraction $x_{t,n}$ of the current wealth $W_{t,n}$ into the risky asset. Thus, after the trading session agent n possesses $x_{t,n}\, W_{t,n}/P_t$ shares of the risky asset and $(1-x_{t,n})\, W_{t,n}$ shares of the riskless asset, and wealth evolution reads

$$W_{t+1,n}(P_{t+1}) = (1-x_{t,n})\, W_{t,n}\,(1+r_f) + \frac{x_{t,n}\, W_{t,n}}{P_t}\,(P_{t+1}+D_{t+1}) \quad . \tag{2.1}$$

Therefore, the wealth at any period depends on the current price of the risky asset. Price at time t is fixed so that aggregate demand equals aggregate supply. Assuming a constant supply of the risky asset, whose quantity can then be normalized to 1, price P_t satisfies to

$$\sum_{n=1}^{N} x_{t,n}\, W_{t,n}(P_t) = P_t \quad . \tag{2.2}$$

Solving the last equation together with N equations (2.1) written for the previous period, one can find P_t and $W_{t,n}$ for all n simultaneously. Once the price is fixed and the new portfolios and wealths are determined, the economy is ready for the next round.

The dynamics defined by (2.1) and (2.2) describes an exogenously growing economy due to the continuous injections of new riskless assets, whose price remains, under the assumption of totally elastic supply, unchanged. It is convenient to remove this exogenous economic expansion from the dynamics of the model. To this purpose we introduce rescaled variables

$$w_{t,n} = \frac{W_{t,n}}{(1+r_f)^t} \quad , \qquad p_t = \frac{P_t}{(1+r_f)^t} \quad , \qquad e_t = \frac{D_t}{P_{t-1}\,(1+r_f)} \quad , \tag{2.3}$$

denoted with a lower case letters. The last quantity, e_t, represents (to within the factor) the dividend yield. Rewriting (2.1) and (2.2) for the same time period $t+1$ in terms of these new variables one obtains

$$\begin{cases} p_{t+1} = \displaystyle\sum_{n=1}^{N} x_{t+1,n}\, w_{t+1,n} \\ w_{t+1,n} = w_{t,n} + w_{t,n}\, x_{t,n} \left(\dfrac{p_{t+1}}{p_t} - 1 + e_{t+1}\right) \qquad \forall n \in \{1,\ldots,N\} \quad . \end{cases} \tag{2.4}$$

These equations represent an evolution of state variables $w_{t,n}$ and p_t over time, provided that stochastic process $\{e_t\}$ is given and the set of investment shares $\{x_{t,n}\}$ is specified. Under the assumption that the investment shares $x_{t,n}$ do not depend on the contemporaneous price and wealth[2], equations (2.4) imply a simultaneous determination of the equilibrium price p_{t+1} and of the agents' wealths $w_{t+1,n}$, so that the state of the system at time $t+1$ is defined only implicitly. For analytical purposes, one has to derive the explicit equations that govern the system dynamics.

[1] All traders with identical investment behavior will have constant relative wealths in our framework, so that only the total wealth of all such traders matters for the dynamics. Thus, one can think about N different *strategies* present in the market, each of which is employed by an arbitrary number of traders. (cf. Hens and Schenk-Hoppé (2005)).

[2] In other words, the agents' behavior is consistent with the CRRA framework. This crucial for our analysis assumption will be formalized in Section 2.4.

2.2 Dynamical system for wealth and return

The transformation of the implicit dynamics (2.4) into explicit form with positive prices is not generally possible and entails some restriction on the possible market positions available to agents. The following notation is useful to present the dynamics in a more compact form. Let a_n be an agent specific variable, dependent or independent from time t. We denote with $\langle a \rangle_t$ the *wealth weighted average* of this variable at time t on the population of agents,

$$\langle a \rangle_t = \sum_{n=1}^{N} a_n\, \varphi_{t,n} \ , \quad \text{where} \quad \varphi_{t,n} = \frac{w_{t,n}}{w_t} \quad \text{and} \quad w_t = \sum_{n=1}^{N} w_{t,n} \quad . \tag{2.5}$$

The next result gives the condition for which the dynamical system implicitly defined in (2.4) can be made explicit without a violation of the requirement of positiveness of prices

Proposition 2.1. *Let us assume that the initial price p_0 is positive. From equations* (2.4) *it is possible to derive a map $\mathbb{R}^N \to \mathbb{R}^N$ that describes the evolution of traders' wealth $w_{t,n}$ with positive prices $p_t \in \mathbb{R}^+ \ \forall t$ provided that*

$$\Big(\langle x_t \rangle_t - \langle x_t\, x_{t+1} \rangle_t \Big) \Big(\langle x_{t+1} \rangle_t - (1 - e_{t+1}) \langle x_t\, x_{t+1} \rangle_t \Big) > 0 \qquad \forall t \quad . \tag{2.6}$$

If previous condition is met, the growth rate of (rescaled) price $r_{t+1} = p_{t+1}/p_t - 1$ reads

$$r_{t+1} = \frac{\langle x_{t+1} - x_t \rangle_t + e_{t+1} \langle x_t\, x_{t+1} \rangle_t}{\langle x_t\, (1 - x_{t+1}) \rangle_t} \quad , \tag{2.7}$$

while the agents' (rescaled) wealth shares $\varphi_{t,n}$ evolve according to

$$\varphi_{t+1,n} = \varphi_{t,n} \frac{1 + (r_{t+1} + e_{t+1})\, x_{t,n}}{1 + (r_{t+1} + e_{t+1}) \langle x_t \rangle_t} \qquad \forall n \in \{1, \ldots, N\} \quad . \tag{2.8}$$

Moreover, the individual growth rates of (rescaled) wealth $\rho_{t+1,n} = w_{t+1,n}/w_{t,n} - 1$ are given by

$$\rho_{t+1,n} = x_{t,n} \big(r_{t+1} + e_{t+1} \big) \qquad \forall n \in \{1, \ldots, N\} \quad . \tag{2.9}$$

Proof. See appendix A. □

The market evolution is now *explicitly* described by the system of $N+1$ equations in (2.7) and (2.8). The dynamics of rescaled price p_t can be derived from (2.7) in a trivial way, but price will remain positive only if condition (2.6) is satisfied[3]. Finally, using (2.4), one can easily obtain the evolution of unscaled price P_t.

Expression (2.7) for the return determination stresses the role of the relative agents' wealth. In this set-up, more wealthy agents have a higher influence on the market price. Return r_{t+1} depends on the investment decisions from two periods, $t+1$ and t. High value of the agent's investment choice immediately increases total demand for the risky asset and pushes the return up. Consequently, the return is an increasing function of the contemporaneous investment shares. On the other hand, the high investment shares from the previous period lead to the high past price and, therefore, to the low level of the current return. In addition, there is an opposite affect due to the dividend payment. The dividends paid to the agent at time $t+1$

[3]In general, it may be quite difficult to check the validity of this condition at each time step. However, if agents are diversifying and do not go short, then inequality (2.6) is satisfied (Anufriev et al., 2006).

are proportional to his demand for the risky asset in the previous period t. At the same time, these dividends, through the agents' wealth, contribute to the increase of the current demand and price.

Equation (2.8) describes the evolution of the relative wealth shares. We do not impose a restriction about positiveness of the agent's wealth, so that agents are allowed to be in debt. Thus, the wealth shares $\varphi_{t,n}$ are not assumed to belong to $[0,1]$ interval, but their sum must be equal to 1. One can interpret (2.8) as a *replicator dynamics*, initially used in mathematical biology and then in evolutionary economics. Indeed, using (2.9), this equation can be written as follows

$$\varphi_{t+1,n} = \varphi_{t,n} \, \frac{1+\rho_{t+1,n}}{1+\big\langle \rho_{t+1} \big\rangle_t} \quad .$$

Thus, the market influence of any agent changes according to the agent's *relative* performance with respect to the average performance, where one has to take the (rescaled) wealth return as a measure of performance.

2.3 Dividend process

An exogenous dividend process is described in the following

Assumption 1. The dividend process has a multiplicative structure, so that

$$D_{t+1} = D_t \, (1+r_f) \, \big(1+g(r_t)+\epsilon_{t+1}\big) \quad , \tag{2.10}$$

where $g(\cdot)$ is a deterministic function of past price return describing an average growth rate of the dividend *rescaled* by the risk-free interest rate, and ϵ_{t+1} is a random variable with zero mean.

Thus, we assume that the growth rate of the dividend is a noisy function of the past price return. Such assumption is reasonable, for instance, for a business firm whose current financial situation depends on the past history and where the dividend is paid as a part of the revenue. Notice that $g(r_t)$ provides an average growth rate of the dividend with respect to the risk-free interest. Therefore, this function takes negative values each time when unscaled dividend D_t grows with the rate lesser than r_f, and their value is positive if D_t grows with the rate higher than r_f.

In the current paper our main interest lies in studying of the *deterministic skeleton* of the dynamics, i.e. the system without random disturbances which describes, in a sense, the average dynamics. Thus, we substitute the random variable ϵ_{t+1} by its mean value. Together with (2.10) it implies the following evolution of the yield process

$$e_{t+1} = e_t \, \frac{1+g(r_t)}{1+r_t} \quad . \tag{2.11}$$

The case with general dividend growth function will be examined in Section 3, and then two cases with the different specifications of function g will be considered in more details. We analyze the case of *constant dividend yield* when $g(r_t) = r_t$, and the case of *constant dividend growth* in which function g is constant: $g(r_t) = g$.

2.4 Procedurally rational agents

The last component of the model, the agents' behavior, will be defined as general as possible inside this framework. Substantially, the only imposed restriction will be an independence of the investment shares of the contemporaneous price and wealth levels[4]. Recall that only under such condition Proposition 2.1 properly defines the dynamics of price and wealth.

We assume that agents base their investment decisions at time t exclusively on the public and commonly available information set formed by past prices (up to and including P_{t-1}) and realized dividends (up to and including D_t). Equivalently, the same set can be composed from the past price returns and dividend yields. We make, therefore, the following

Assumption 2. For each agent n there exists a differentiable *investment function* f_n which maps the present information set into his investment share:

$$x_{t,n} = f_n\big(r_{t-1}, r_{t-2}, \dots ; e_t, e_{t-1}, \dots\big) \quad . \tag{2.12}$$

Function f_n in the right-hand side of (2.12) gives a complete description of the investment decision of the n-th agent. The generality of the investment functions implies a big flexibility in the modeling of the agents' behaviors. Formulation (2.12) includes as special cases both technical trading, when agents' decisions are mostly driven by the observed price fluctuations, and also more fundamental attitude, when agents make the decision on the basis of the past dividends or price-dividend ratio. At the same time, Assumption 2 rules out other possible dependencies of the investment choices, like an explicit relation with past investment choices or with investment choices of other traders. The assumption is also violated for the wealth independent demand functions as in the case of the constant absolute risk aversion behavior.

The results of the present paper are stated in terms of general investment functions. Anufriev (2005) discusses the application of these general results to a number of special classes of the investment behavior. He shows, in particular, that the current framework can be applied both to the expected utility maximizers and also to the mean-variance maximizers.

To stress the generality of our analysis even further, we provide a possible interpretation of our investment function, used in the majority of the models with heterogeneous agents[5]. The investment choice described by (2.12) can be obtained as an outcome of two distinct steps. In the first step agent n, using a set of estimators $\{g_{n,1}, g_{n,2}, \dots\}$, forms his expectation at time t about the behavior of future prices, $\theta_{n,j} = g_{n,j}(\mathbb{I}_{t-1})$, where $\mathbb{I}_{t-1}$ denotes the information set and $\theta_{\cdot,j}$ stands for some statistics of the return distribution at time $t+1$ (e.g. average return, variance or the probability that a certain return threshold is crossed). With these expectations, using a choice function h_n, possibly derived from some optimization procedure, agent computes the fraction of the wealth invested in the risky asset $x_{t+1,n} = h_n(\theta_{n,1}, \theta_{n,2}, \dots)$. The investment function f_n defined in Assumption 2 would be the composition of estimators $\{g_{n,\cdot}\}$ and choice function h_n.

This interpretation is intuitive but is not required by our framework, however. In our model agents are not forced to use some specific predictors, rather they are allowed to map the past return history into the future investment choice, with arbitrary smooth function. Thus, agents may not be optimizers, i.e. "rational" in the sense as this word is widely used in economics. Nevertheless, they do not behave completely randomly or irrational. Using the terminology coined by Herbert Simon (see e.g. Simon (1976)), the traders modeled in our

[4]This is equivalent to the requirement on the demand function to increase linearly with wealth and have a constant price elasticity.

[5]See, e.g. Brock and Hommes (1998); Chiarella and He (2001); Anufriev et al. (2006).

paper are *procedurally* rational. Investment functions describe the outcome of an idiosyncratic procedures which can be defined as the collective description of the preferences, beliefs and implied actions.

Description of the agents' behaviors concludes the specification of our asset-pricing model. In the deterministic skeleton the complete system reads

$$\begin{cases} r_{t+1} = \dfrac{\big\langle x_{t+1} - x_t \big\rangle_t + e_{t+1} \big\langle x_t \, x_{t+1} \big\rangle_t}{\big\langle x_t \, (1 - x_{t+1}) \big\rangle_t} \\ e_{t+1} = e_t \, \dfrac{1 + g(r_t)}{1 + r_t} \\ x_{t+1,n} = f_n\big(r_t, r_{t-1}, \ldots; e_{t+1}, e_t, \ldots\big) \\ \varphi_{t+1,n} = \varphi_{t,n} \, \dfrac{1 + (r_{t+1} + e_{t+1}) \, x_{t,n}}{1 + (r_{t+1} + e_{t+1}) \, \big\langle x_t \big\rangle_t} \quad . \end{cases} \tag{2.13}$$

It is composed by the aggregate dynamics of price return (2.7) and dividend yield (2.11), and also of the agent-specific dynamics of the investment shares (2.12) and the wealth shares (2.8).

3 Static Behavioral Consistent Equilibria

According to the tradition well-established in the economic literature, any reasonable notion of an "economic equilibrium" should imply that no agent makes systematic mistakes. The introduction of "rational expectations" in works of Muth (1961) and Lucas (1978) made the notion of economic equilibrium much stronger. The hypothesis of the rational expectations is developed within the representative agent framework and says that the dynamics generated by the actions of such an agent are consistent with agents' *a priori* perceptions about this dynamics.

In the framework with heterogeneous and procedurally rational agents, such concept of equilibrium is inappropriate. The natural generalization of this concept would be a kind of "dynamic behavioral consistent equilibrium" in which the agents' actions (based upon beliefs, preferences and procedures) generate dynamics consistent with the co-evolving agents' actions. In formal terms, any trajectory of system (2.13) corresponds to an "equilibrium" in this sense. In this paper we concentrate on the long-run characteristics of such equilibria and confine an analysis on the more simple situation, where the emerging dynamics do not enforce agents to modify their choices of the desirable portfolio at given prices. Thus, in such equilibria the agents are not changing their behaviors and, at the same time, the aggregate dynamics is consistent with the procedures used by any agent to compute his action. We call such concept a *Behavioral Consistent Equilibrium.* In the setup outlined in Section 2 such equilibria can be defined as follows

Definition 3.1. Behavioral Consistent Equilibria (BCE) are the trajectories of system (2.13) with fixed investment shares $x_{t,n} = x^*_n$ for all n.

Many heterogeneous agent models, e.g. Day and Huang (1990), Chiarella (1992) or Brock and Hommes (1998), analyze the dynamics of the markets populated by agents having simple, stylized behaviors. The emerging complex dynamics are characterized in some cases by the cyclic or even chaotic motion of the market positions for individual agents. Apparently, such dynamic scenarios do not satisfy to the above definition, even if the implied agents' behaviors are not necessarily inconsistent. Nevertheless, the situation in which an agent changes his position each period in a chaotic manner neither seems realistic, nor can play a role of the generalization of the notion of equilibrium for multi-agent setting. Defined above *static* concept of the BCE is, arguably, the largest possible deviation from the standard representative agent equilibrium, where substantial analysis in terms of the properties of general dividend process and agents' behaviors is still feasible. The main goal which we pursue in the rest of the paper will be to study the local properties of the BCE of system (2.13) which are characterized by the stationary wealth distribution with $\varphi_{t,n} = \varphi^*_n$ for all n. In contrast with the works cited above we are not interested here in the investigation of the global dynamics in the market with a limited number of the agents with simplest behaviors.

One of the questions of our interest concerns the survival of the agents in the market. We adopt the deterministic version of the concepts of survival and dominance used in DeLong et al. (1991) and Blume and Easley (1992) and introduce

Definition 3.2. Agent n is said to *"survive"* in the BCE if his equilibrium wealth share differs from zero, $\varphi^*_n \neq 0$. Agent n is said to *"dominate"* the economy, if he is the only survivor in the economy, so that $\varphi^*_n = 1$.

General structure of the BCE introduced in Definition 3.1 is summarized in the following

Proposition 3.1. *Let us consider Behavioral Consistent Equilibrium of the economy with arbitrary dividend growth function g, vector of investment shares $\boldsymbol{x}^* = (x^*_1, \dots, x^*_n)$ and wealth distribution $\boldsymbol{\varphi}^* = (\varphi^*_1, \dots, \varphi^*_n)$. Then:*

(i) If $e_t = 0$ for some t, then in the equilibrium both dividend yield and price return are zero, agent's investment shares are $x^*_n = f_n\big(0, 0, \dots; 0, 0, \dots\big)$, while the wealth shares are arbitrary numbers summed to 1.

(ii) If $e_t \neq 0$ for all t, then ratio of the price return to dividend yield, $y_t = r_t/e_t$, takes a constant value y^*.

If $\big\langle x^* \big\rangle = 0$ (where the weighted average of investment shares $\boldsymbol{x}^*$ is taken with respect to the wealth distribution $\boldsymbol{\varphi}^*$), then $y^* = -1$.

If $\big\langle x^* \big\rangle \neq 0$, then all survivors have the same investment share $x^*_\diamond$, and

$$y^* = \frac{x^*_\diamond}{1 - x^*_\diamond} \quad . \tag{3.1}$$

Furthermore, the evolution of the yield process $\{e_t\}$ is described as

$$e_{t+1} = e_t \, \frac{1 + g(y^* \, e_t)}{1 + y^* \, e_t} \quad , \tag{3.2}$$

while the equilibrium investment shares satisfy to

$$x^*_n = f_n\big(e_t \, y^*, e_{t-1} \, y^*, \dots; e_{t+1}, e_t, \dots\big) \quad . \tag{3.3}$$

Proof. See appendix B. □

This Proposition formally describes the conditions which should be satisfied in any BCE in the setting with the CRRA agents. Notice that, in general, BCE are not the steady-states of dynamical system (2.13), since both the price return and dividend yield can evolve in equilibrium. The ratio of these variables must be constant in this case, however.

Equilibrium with zero yield from item *(i)* of the Proposition does not exist in our economy with positive dividend but can be reached asymptotically. We, therefore, extend the system for the case of zero dividend values. Any other BCE is represented, given arbitrary investment functions f_n and dividend growth function g, by the triple $(y^*, \boldsymbol{x^*}, \boldsymbol{\varphi^*})$. For given y^*, corresponding yield process $\{e_t\}$ is defined by (3.2). Description of the agents' behaviors then implies that investment shares satisfy to (3.3). If $y^* = -1$ then the wealth shares of the agents should satisfy to $\langle x^* \rangle = 0$. Otherwise, consistency condition (3.1) should hold for the investment shares of all survivors. It is useful at this point to distinguish between two types of equilibria. Notice that relation $y^* = -1$ is equivalent to the zero value of the total return for the risky asset, $r_t + e_t = 0$. In terms of the unscaled variables, it means that the risky asset gives the same total return r_f, as the risk-less asset. It motivates the following definition.

Definition 3.3. The BCE is called an *equity-premium* if ratio $y^* \neq -1$. Otherwise, i.e. when either $y^* = -1$ or $e^* = r^* = 0$, the BCE is called a *no-equity-premium* equilibrium.

Next two corollaries describe the conditions for the existence of these two types of the BCE, given the general investment functions f_n and dividend growth function g. We start with the equity-premium equilibria, which are more interesting from an economic viewpoint, since an equity premium is observed in the real markets. According to Proposition 3.1, all survivors in such equilibrium have a homogeneous investment share $x^*_\diamond$. Inverting (3.1), we immediately get the following

Corollary 3.1 (Existence of the Equity-Premium BCE). *Starting with an arbitrary dividend growth function g, let $\{e_t(y)\}$ be a sequence recursively defined as follows*

$$e_{t+1} = e_t \, \frac{1 + g(y\, e_t)}{1 + y\, e_t} \quad .$$

If there exist such y^* that for any surviving agent n, investment function f_n evaluated on the sequence $\{e_t(y)\}$ satisfies to

$$f_n\big(e_t\, y^*, e_{t-1}\, y^*, \dots; e_{t+1}, e_t, \dots\big) = \frac{y^*}{1+y^*} \qquad \forall t \quad , \tag{3.4}$$

then an Equity-Premium BCE exists. Assuming without loss of generality that the first k agents survive ($1 \le k \le N$), corresponding equilibrium is composed of

- *y^* which solves (3.4) for all $n \in \{1, \dots, k\}$,*
- *vector $\boldsymbol{x^*} = (x^*_1, \dots, x^*_n)$ defined by (3.3) with yield process $\{e_t(y^*)\}$ and*
- *vector $\boldsymbol{\varphi^*}$ whose components satisfy to*

$$\varphi^*_n = 0 \quad \textit{if } n > k \qquad \textit{and} \qquad \sum_{n=1}^{k} \varphi^*_n = 1 \quad . \tag{3.5}$$

This Corollary shows that existence of an equity-premium equilibrium depends on whether implicit (with respect to the variable y) equation (3.4) possesses some solution. The right-hand side of this equation is independent both of the dividend specification and the agents' behaviors. It leads us to

Definition 3.4. The Equilibrium Market Line (EML) is function $l(y)$ defined according to

$$l(y) = \frac{y}{1+y} \quad . \tag{3.6}$$

Corollary 3.1 says that all possible equity-premium equilibria can be obtained as intersections of the EML with the corresponding projection of an investment function of any survivor. Such projection is given by the left-hand side of (3.4) and, through the yield process $\{e_t\}$, is ultimately defined by means of the dividend growth function g. If for some investment function intersection y^* exists, then the equilibrium exists as soon as the investment function of any agent is constant on the corresponding yield trajectory $\{e_t(y^*)\}$.

Notice that for the existence of equilibrium with $k > 1$ survivors, (3.4) should be satisfied for k agents. Thus, projections of the investment functions of all survivors should intersect the EML in the same point. Consequently, equilibria with more than one survivor are non-generic, in a sense that an economy with arbitrary, so to say "randomly defined", investment functions has probability zero of displaying equilibrium with multiple survivors. This statement will be illustrated in the examples of the next Section. There is also a second difference between equity-premium equilibria with one and many survivors. In the former equilibrium Corollary 3.1 defines a precise value for each component (y^*, $\boldsymbol{x}^*$ and $\boldsymbol{\varphi}^*$), so that a single point is uniquely determined. Instead, in the equilibrium with $k > 1$ survivors, there is a residual degree of freedom: while y^* and investment shares x^*'s are uniquely defined, the only requirement on the equilibrium wealth shares of the surviving agents is the fulfillment of the second equality in (3.5). Consequently, a whole hyperplane of the equity-premium PRE with many survivors exists. The particular fixed point eventually chosen by the system will depend on the initial conditions.

It follows from Proposition 3.1 that there are no special conditions for the existence of no-equity-premium equilibria with zero yield and return. For those no-equity-premium equilibria where the dividend yield is positive we have the following

Corollary 3.2 (Existence of the No-Equity-Premium BCE). *Starting with an arbitrary dividend growth function g, let us define recursively the following yield process*

$$e_{t+1} = e_t \, \frac{1+g(-e_t)}{1-e_t} \quad .$$

If there exist a vector $\boldsymbol{x}^*$ with components $x^*_n = f_n(-e_t, -e_{t-1}, \dots; e_{t+1}, e_t, \dots)$, then any vector $\boldsymbol{\varphi}^* = (\varphi^*_1, \dots, \varphi^*_n)$ such that

$$\sum_{n=1}^{N} x^*_n \varphi^*_n = 0 \qquad \text{and} \qquad \sum_{n=1}^{N} \varphi^*_n = 1 \tag{3.7}$$

define some No-Equity-Premium BCE.

No-equity-premium equilibria with positive dividend yield do not exist if at least one investment function f_n is not constant on the sequence $\{e_t\}$. Furthermore, any no-equity-premium equilibrium should have at least two survivors, because otherwise condition (2.6) is violated. In all other cases, (3.7) defines a *manifold* of the no-equity-premium equilibria.

In this Section we put forward a notion of the Behavioral Consistent Equilibrium, which generalizes an idea of the rational expectations on the framework with many procedurally rational agents. We provided a general (independent both on the dividend process and on the agents' behaviors) characterization of such equilibria. In particular, we established that so called equity-premium equilibria (with one or many survivors) are possible and introduced geometric locus, called the Equilibrium Market Line, for characterization of such equilibria. Notice that a nature of the equity premium in our framework is completely different from the one described in the limits of the classical financial paradigm, see e.g. Lucas (1978); Mehra and Prescott (1985). In that studies an equity premium is considered as a monetary incentive paid in the equilibrium to the optimizing, risk-averse representative agent for possessing an asset with positive covariance with his consumption. Instead, in our framework an equity premium is endogenously generated through dynamic behavior of heterogeneous, procedurally rational agents.

4 Equilibrium Market Line as Locus of BCE

We illustrate below the use of the EML for the two special cases of the dividend process. We specify the dividend growth function $g(\cdot)$ from Assumption 1 and keep a total generality in what concerns the agents' investment functions.

4.1 Economy with constant dividend yield

We start with the case when the dividend growth function reads

$$g(r_t) = r_t \quad .$$

Thus, price and dividend grow at the same rate, so that from (2.11) it follows that dividend yield e_t takes some constant value $\bar{e}$. Simple derivation shows that constant dividend yield characterizes the fundamental price under the assumption of the geometric random walk of the dividends. Notice, however, that in our model price is determined through the market clearing condition and is not necessary fixed on the fundamental level. Therefore, assumption of constant yield is rather restrictive. We make it because of three reasons. First, implied simplification of the dynamical system allows us to discuss application of the EML tool in an illustrative way. Second, the annual historical data for the S&P 500 index suggest that yield can be reasonably described as a bounded positive random variable whose behavior is roughly stationary. Third, the assumption of the constant dividend yield is common to several works in literature (Chiarella and He, 2001, 2002; Anufriev et al., 2006).

Deterministic skeleton (2.13) of the dynamics simplifies to

$$\begin{cases} r_{t+1} = \dfrac{\langle x_{t+1} - x_t \rangle_t + \bar{e} \, \langle x_t \, x_{t+1} \rangle_t}{\langle x_t \, (1 - x_{t+1}) \rangle_t} \\[2ex] x_{t+1,n} = f_n\big(r_t, r_{t-1}, \ldots ; \bar{e}\big) \\[2ex] \varphi_{t+1,n} = \varphi_{t,n} \, \dfrac{1 + (r_{t+1} + \bar{e}) \, x_{t,n}}{1 + (r_{t+1} + \bar{e}) \, \langle x_t \rangle_t} \quad . \end{cases} \tag{4.1}$$

Simulations show that this dynamical system represents a reasonable approximation to the dynamics in the stochastic case, when e_t are *i.i.d.* random variables obtained from a common distribution with positive mean $\bar{e}$.

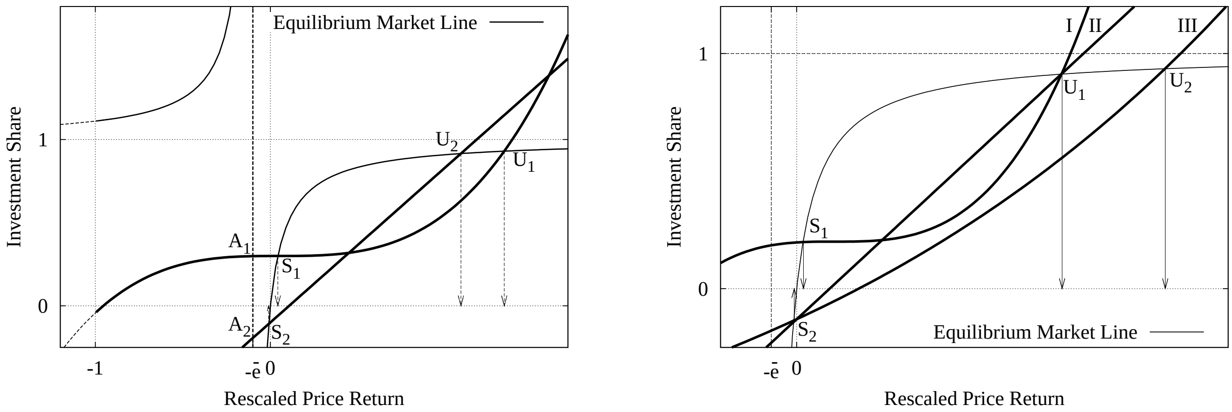

Figure 1: Investment functions (thick curves) based on the last realized return. The equilibria are found as intersections with the EML (thin curve). **Left panel**: There are 4 equilibria with one survivor, S_1, U_1, S_2 and U_2. In the former two an agent with nonlinear investment function survives, while in the other two an agent with linear function survives. Points A_1 and A_2 represents the no-equity-premium equilibrium. **Right panel**: Non-generic equity-premium PRE. In equilibrium S_2 agents II and III survives, while in U_1 agents I and II survive.

From Proposition 3.1 it follows that price return r_t is constant in all BCE with stationary wealth distribution. Thus, analysis of the BCE is reduced to the analysis of the fixed points of system (4.1). Two Corollaries from Section 3 immediately lead to

Proposition 4.1. *Let $(r^*, \boldsymbol{x^*}, \boldsymbol{\varphi^*})$ be a BCE of the dynamics with constant yield $\bar{e}$. Then*

$$x^*_n = f_n\big(r^*, r^* \ldots ; \bar{e}, \bar{e}, \ldots\big) \qquad \forall n \in \{1, \ldots, N\} \quad , \tag{4.2}$$

and the following three mutually exclusive cases are possible:

(i) ***"Equity Premium" BCE with a single survivor.*** *In the BCE only one agent (say, first) survives and, therefore, dominates the economy. Thus, $\varphi^*_1 = 1$, while all other equilibrium wealth shares are zero. The equilibrium return r^* satisfies to*

$$f_1\big(r^*, r^*, \ldots ; \bar{e}, \bar{e}, \ldots\big) = l(r^*/\bar{e}) \quad , \tag{4.3}$$

where the EML $l(\cdot)$ was introduced in Definition 3.4.

(ii) ***"Equity Premium" BCE with many survivors.*** *In the BCE more than one agent survive. Assuming that the first k agents survive, the equilibrium wealth shares satisfy* (3.5)*. The equilibrium return r^* satisfies the following k equations*

$$f_n\big(r^*, r^*, \ldots ; \bar{e}, \bar{e}, \ldots\big) = l(r^*/\bar{e}) \qquad \forall n \in \{1, \ldots, k\} \quad . \tag{4.4}$$

(iii) ***"No Equity Premium" with many survivors.*** *Investment shares and wealth shares of the agents satisfy* (3.7)*, while equilibrium return $r^* = -\bar{e}$.*

The last Proposition is illustrated in Fig. 1. Let us fix $\bar{e}$ and express the EML as a function of one variable r as follows

$$l_e(r) = \frac{r}{\bar{e} + r} \quad . \tag{4.5}$$

For exogenously given $\bar{e}$ relations (4.3) and (4.4) provide the equations with respect to the rescaled price return r. They can be visualized. Assuming for the moment that agents base their decisions exclusively on the return of the last period, we draw the plots of the investment functions as thick curves and plot of the EML (4.5) as a thin curve. The intersections of the investment function with the EML represent all possible equity-premium equilibria of the system. In the situation in the left panel there are 4 such equilibria denoted as S_1, S_2, U_1 and U_2. Since in all these points only one investment function intersects the EML, all such equilibria have a single survivor and are described by Proposition 4.1*(i)*. In S_1 and U_1 the agent with non-linear investment function survives, while in S_2 and U_2 the survivor has linear investment function. Equilibrium return r^* is determined as an abscissa of the corresponding intersection, while the investment share of the non-surviving agent can be found, in accordance with (4.2), as an intersection of his investment function with the vertical line passing through the equilibrium return.

In the right panel of Fig. 1 we illustrate Proposition 4.1*(ii)* and demonstrate a non-generic situation of the equity-premium BCE with many survivors. In U_1, for instance, two agents with strategies marked I and II survive. The equilibrium return and investment shares are defined in the same way as in the previous case. However, point U_1 represents now a whole manifold of equilibria, since equilibrium wealth shares of two survivors can be two arbitrary numbers which are summed to 1.

Notice that all points of the EML (4.5) with $r < -1$ are meaningless from economic point of view, since they lead to negative prices. The vertical asymptote of the EML at $r = -\bar{e}$ can be used to illustrate the no-equity-premium equilibria from Proposition 4.1*(iii)*. Points A_1 and A_2 in the left panel of Fig. 1 represent investment shares of two agents. Their wealth shares can be computed from (3.7).

We assumed that investment functions depend only on the last return but our illustration does not depend on this assumption. In general case, we have to interpret the plots in Fig. 1 as plots of *sections* of general investment functions by the hyperplane defined as

$$\{r = r_t = r_{t-1} = \dots \quad , \qquad \bar{e} = e_t = e_{t-1} = \dots \quad \} \quad .$$

Fixing exogenous yield $\bar{e}$, we draw in Fig. 2 the two-dimensional surface representing some investment function which depends on the past return r_0 and the return r_1 of one period before the last. Thick curve on the surface is the intersection of the investment function with the "symmetric" plane $r_0 = r_1$. The curve representing the EML $l_e(r)$ is drawn on the same plane, so that the picture on that plane is completely analogous to the previous examples from Fig. 1.

4.2 Economy with constant rate of the dividend growth

We turn now to the case when the dividend grows with exogenously given constant rate g, i.e.

$$g(r_t) = g \quad .$$

This special case is more suitable for exogenous modeling of the production side of the economy. The deterministic skeleton approximates in this case a geometric random walk of the dividend, which is common assumption in the economic literature.

It is important to stress that growth rate g is not restricted to be positive. Recall from (2.10) that function $g(\cdot)$ represents the growth rate of the dividend *after* rescaling. Thus, positive values of g correspond to the case when the (unscaled) dividend grows faster than the

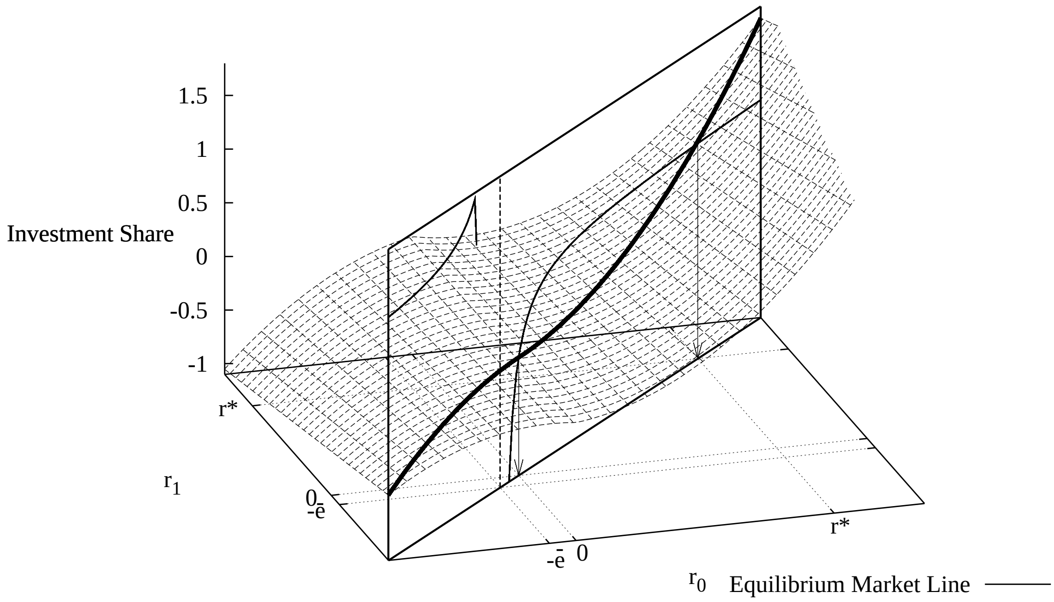

Figure 2: Investment function illustrated an investment decision based on the last two realized returns $x_t = f(r_{t-1}, r_{t-2})$ and its intersection (thick curve) with the plane $r_{t-1} = r_{t-2}$. Equilibria are found on this plane as intersections with the EML (thin curve).

risk-free interest rate r_f, while negative g implies that growth rate of the (unscaled) dividend is smaller than r_f.

According to Proposition 3.1 any BCE has constant value y^* of the ratio of the price return to the dividend yield. In this case, evolution of the yield (3.2) reads

$$e_{t+1} = e_t \, \frac{1+g}{1+y^* \, e_t} \quad . \tag{4.6}$$

This dynamics can be easily characterized, as we prove in the following

Lemma 4.1. *For a given dividend growth rate g and equilibrium value y^* there are two fixed points of dynamics* (4.6)*: $e_1^* = g/y^*$ and $e_2^* = 0$.*

If $g > 0$ the dynamics is globally attracted by e_1^* (or globally diverges when $y^* = 0$), while for $g < 0$ the global attractor is e_2^*. In the point $g = 0$ system (4.6) *exhibits a transcritical bifurcation.*

Proof. See appendix C. □

Thus, in the case of constant dividend growth rate the yield dynamics have unique attractor. It implies that the BCE with stationary wealth distribution can be characterized by constant dividend yield e^* and, therefore, by constant price return r^*. In other words, all the BCE are the fixed points of system (2.13).

The last Lemma suggests that there exist a difference in economic interpretation between present case with constant dividend growth rate and the previous example of constant yield.

In the economy with exogenous dividend yield, the equilibrium price return was determined by the configuration of the agents' investment functions (possibly jointly with initial conditions). In the present case, however, the equilibrium price return is either equal to 0 or to g, i.e. it is determined independent of the agents' behavior. Instead, another component of the total return, the equilibrium dividend yield, depends on the agents' behaviors. Using the results from Section 3 we straight-forwardly derive the following

Proposition 4.2. *Let* $(r^*, e^*, \boldsymbol{x}^*, \boldsymbol{\varphi}^*)$ *be a BCE of the dynamics with constant dividend growth rate* g. *Then*

$$x_n^* = f_n\big(r^*, r^* \ldots ; e^*, e^*, \ldots\big) \qquad \forall n \in \{1, \ldots, N\} \quad , \tag{4.7}$$

and the following two cases are possible:

(i) ***"Equity Premium" BCE with one or more survivors.*** *In the equilibrium* k *agents survive (where* $k \geq 1$*). Assuming that the first* k *agents survive, the equilibrium wealth shares satisfy to* (3.5)*. The equilibrium return* $r^* = g$*, while the equilibrium dividend yield satisfies the following* k *equations*

$$f_n\big(g, g, \ldots ; e^*, e^*, \ldots\big) = l(g/e^*) \qquad \forall n \in \{1, \ldots, k\} \quad , \tag{4.8}$$

where the EML $l(\cdot)$ *was introduced in Definition 3.4.*

(ii) ***"No Equity Premium" BCE.*** *The equilibrium return and dividend yield are zero:* $r^* = e^* = 0$*, while the agents' wealth shares are arbitrary numbers summed to one.*

The formulation of the last Proposition is very similar to Proposition 4.1. From an economic point of view their results are quite different, however. As we mentioned above, in the present case the ecology of investors determines the dividend yield and not the price return. Relatedly, the presence of an equity premium in equilibrium now entirely depends on the sign of the exogenous parameter g, because for given g only one type of *stable* equilibria can present in the economy.

Indeed, from Lemma 4.1 it follows that when $g \leq 0$, the yield dynamics is attracted by $e_2^* = 0$ and so $r^* = 0$. Hence, in this case the economy can possess only no-equity-premium equilibria. Furthermore, the dividend yield in such equilibria will approach zero as opposite to the case from Section 4.1. On the other hand, when g is positive, the only attractor of (4.6) is $e_1^* = g/y^*$ so that $r^* = g$. Thus, in all stable equilibria

$$r^* = \max\,(0, g) \quad . \tag{4.9}$$

In terms of unscaled variable this relation stands that equilibrium return of price P is equal to maximum among risk-free interest rate r_f and growth rate of the dividend D.

Fig. 3 illustrates the first item of the last Proposition for the case when $g > 0$. Similarly to Fig. 1 we draw the cross sections of agents' investment functions by the hyperplane

$$\{g = r_t = r_{t-1} = \ldots \quad , \qquad e = e_t = e_{t-1} = \ldots \quad \} \quad ,$$

and the EML (3.6) which, for exogenously given g, becomes a function of the dividend yield:

$$l_g(e) = \frac{g}{e + g} \quad . \tag{4.10}$$

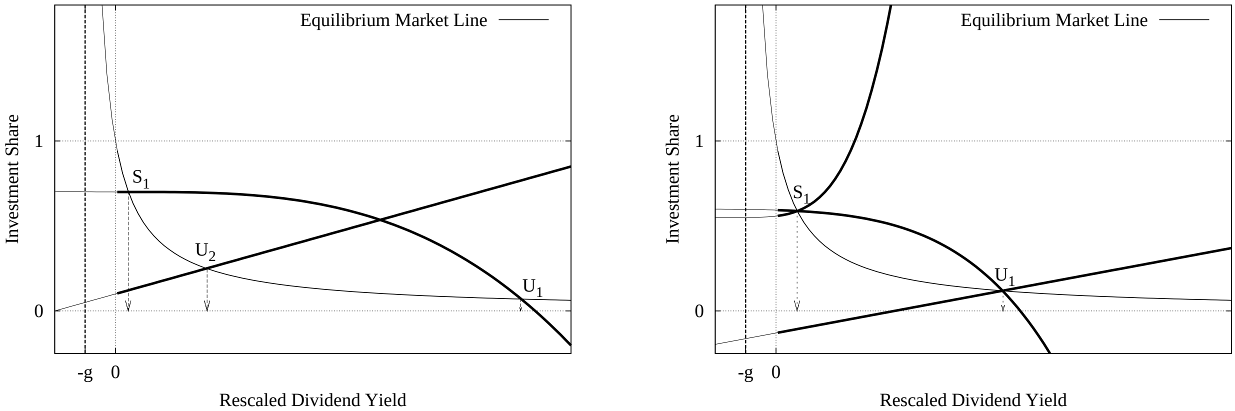

Figure 3: Investment functions (thick curves) based on the last dividend yield for fixed dividend growth rate $g > 0$. The equilibria are found as intersections with the EML (thin curve). **Left panel**: There are 3 equilibria with one survivor, S_1, U_1 and U_2. In the former two an agent with nonlinear investment function survives, while in the last one an agent with linear function survives. **Right panel**: Non-generic equity-premium PRE. In equilibrium S_1 two agents with non-linear functions survive, while in U_1 agent with non-linear function and agent with linear function survive.

Intersections of the investment functions with the EML provide the BCE and illustrate relation (4.8). The equilibrium yield and investment shares of the survivors are given, respectively, as abscissa and ordinate of the corresponding intersection. The non-surviving agents invest according to their rules for the given equilibrium dividend yield. Notice that the equilibria with one survivor illustrated in the left panel of Fig. 3 are generic, while the equilibria with many survivors illustrated in the right panel are not, since in the latter case many investment functions should intersect the EML in the same point.

In this Section we have considered two examples of our general model for special dividend processes. In both cases the equilibria without an equity premium can exist. In both cases also equity-premium equilibria are possible and can be easily illustrated by means of the EML introduced in Section 3. The EML is a function of the variable which can be conveniently chosen depending on the dividend specification. For the constant yield economy such variable was price return r, while for the economy with constant growth rate of the dividend the variable was dividend yield e. In general case, according to Proposition 3.1, the equilibrium ratio r_t/e_t would be similarly determined from (3.3). Figures 1 and 3 allow us to illustrate that existence and number of the Behavioral Consistent Equilibria crucially depends both on the dividend process (which determine the locus of possible equilibria as a corresponding representation of the general EML) and on the ecology of the agents present in the market (which determine the precise point on the EML).

5 Stability of the Behavioral Consistent Equilibria

An important question which should be addressed after the new conception of equilibrium has been introduced concerns the issue of stability of these equilibria. The generality of our framework does not allow to study the question of stability in total generality. Therefore, for illustrative purposes, we present here the stability analysis for the simplest possible dividend specification. Namely, as in Section 4.1, we assume that the dividend yield is constant. For other dividend specifications similar analysis should also be possible. For instance, in the case of constant growth rate of the dividend considered in Section 4.2, the dynamics of the yield is globally stable, according to Lemma 4.1. Therefore, one would expect the similar results also for the dynamics of complete system.

Let us assume that the dividend yield is constant in the market and equal to $\bar{e}$. Proposition 4.1 provide a complete description of location of equilibria for this case. Notice that for the purpose of stability analysis we can safely ignore the dependence of the investment functions on the yield. Furthermore, we will consider the case when the domain of any investment function is finite. Thus, we assume in this Section that investment function of agent n has the following form

$$x_{t,n} = f_n(r_{t-1}, \dots, r_{t-L}) \quad ,$$

where memory span L can be chosen the same for all the agents.

We start the analysis with the case of a single agent present in the market. Such case is more simple because the last set of equations in system (4.1) disappears. The stability conditions in the single agent case will turn out to be important in more general case which we will discuss in Section 5.2.

5.1 Single Agent Market

First of all, notice that only the equilibria discussed in Proposition 4.1*(i)* can occur in the market with single agent. The stability conditions are derived from the analysis of the roots of the characteristic polynomial associated with the Jacobian of corresponding system computed at equilibrium. The characteristic polynomial does, in general, depend on the behavior of individual investment function f in an infinitesimal neighborhood of point $(r^*, \dots, r^*)$. This dependence can be summarized with the help of the following

Definition 5.1. The *stability polynomial* $P(\mu)$ of the investment function f in the BCE is

$$P_f(\mu) = \frac{\partial f}{\partial r_{t-1}}\,\mu^{L-1} + \frac{\partial f}{\partial r_{t-2}}\,\mu^{L-2} + \dots + \frac{\partial f}{\partial r_{t-L-1}}\,\mu + \frac{\partial f}{\partial r_{t-L}} \ , \tag{5.1}$$

where all the derivatives are computed in point $(r^*, \dots, r^*)$.

Using this definition, the stability conditions can be formulated in terms of the equilibrium return r^*, and of the slope of the EML $l_e(r)$ defined in (4.5) computed in the BCE

$$l_e'(r^*) = \frac{\bar{e}}{(\bar{e} + r^*)^2} \ .$$

The following applies

Proposition 5.1. *The BCE of system* (4.1) *with single agent is (locally) asymptotically stable if all the roots of polynomial*

$$Q(\mu) = \mu^{L+1} - \frac{P_f(\mu)}{r^*\, l_e'(r^*)}\Big(\big(1 + r^*\big)\,\mu - 1\Big) \quad , \tag{5.2}$$

are inside the unit circle.

The equilibrium is unstable if at least one of the roots of $Q(\mu)$ lies outside the unit circle.

Proof. The condition above is a direct consequence of the characteristic polynomial of the Jacobian matrix at equilibrium. See appendix D for derivation. □

Once investment function f is known, polynomial $P_f(\mu)$ and, in turn, polynomial $Q(\mu)$ are explicitly derived. The analysis of $L+1$ roots of $Q(\mu)$, which are usually called *multipliers*, can be performed in order to reveal the role of the different parameters in stabilization of a given equilibrium. Such rigorous analysis is often unfeasible even for simple investment functions, so one should rely on the computational approach, mostly.

Notice, however, that when $L = 1$, the polynomial (5.2) has second degree. The roots of such polynomial can be computed and so the stability conditions can be derived explicitly. Namely, one has

Proposition 5.2. *The BCE of system* (4.1) *with single agent and one lag in the investment function ($L = 1$) is (locally) asymptotically stable if*

$$\frac{f'(r^*)}{l'_e(r^*)}\,\frac{1}{r^*} < 1\,, \qquad \frac{f'(r^*)}{l'_e(r^*)} < 1 \qquad \text{and} \qquad \frac{f'(r^*)}{l'_e(r^*)}\,\frac{2+r^*}{r^*} > -1 \quad . \tag{5.3}$$

Fixed point exhibits Neimark-Sacker*, fold or* flip *bifurcation if the first, second or third inequality in* (5.3) *turns to equality, respectively.*

For proof of this proposition and analysis of some other special cases where analytical results are available, see Anufriev et al. (2006). In the left panel of Fig. 4 we show the stability region derived in Proposition 5.2 in coordinates r^* and $f'(r^*)/l'_e(r^*)$. The second coordinate is the relative slope of the investment function at equilibrium with respect to the slope of the EML. If the slope of f at the equilibrium increases, the system tends to lose its stability. In particular, in the stable equilibrium the slope of investment function is smaller than the slope of the EML.

Let us consider, for example, the left panel of Fig. 1 and suppose for the moment that these are the investment functions of the agents with memory span equal to 1. One can immediately see that equilibria U_1 and U_2 are unstable, due to the violation of the second inequality in (5.3). On the contrary, the slope of the nonlinear investment function in S_1 is very small, so that, presumably, this equilibrium is stable.

5.2 Multi Agent Market

Three Propositions below provide the stability conditions for the cases enumerated in Proposition 4.1, i.e. for generic case of a single survivor, for non-generic case of many survivors and for generic case with many survivors and without the equity premium. The derivation of these Propositions requires quite cumbersome algebraic manipulations and we refer the reader to our working paper Anufriev and Bottazzi (2006) for the intermediate lemmas and final proofs.

For the generic case of a single survivor equilibrium one has the following

Proposition 5.3. *Let* $(r^*, \boldsymbol{x}^*, \boldsymbol{\varphi}^*)$ *be a BCE of system* (4.1) *associated with a single survivor equilibrium. Without loss of generality we can assume that the survivor is the first agent. Let* $P_{f_1}(\mu)$ *denote the* $(L-1)$*-dimensional stability polynomial associated with the investment function of the survivor.*

This equilibrium is (locally) asymptotically stable if the two following conditions are met:
1) *all the roots of polynomial*

$$Q_1(\mu) = \mu^{L+1} - \frac{(1+r^*)\,\mu - 1}{r^*\, l'_e(r^*)}\, P_{f_1}(\mu) \quad , \tag{5.4}$$

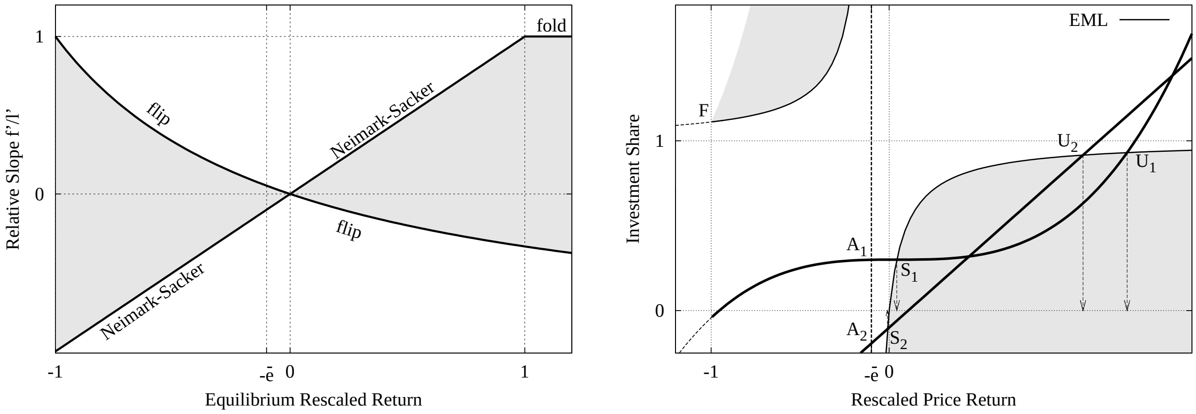

Figure 4: Equilibria and their stability for the many agent system. **Left panel**: Equilibrium stability region (gray) and the bifurcation types for the single agent case with $L = 1$ in coordinates r^* and $f'(r^*)/l'(r^*)$. **Right panel:** Stability conditions for multi-agent market in generic BCE. Region where investment shares of the non-surviving agents satisfy to the stability conditions (5.5) is shown in gray.

are inside the unit circle.

***2)** the equilibrium investment shares of the non-surviving agents satisfy to*

$$-2 - r^* < x_n^* \left(r^* + \bar{e}\right) < r^* \,, \quad 1 < n \leq N \,. \tag{5.5}$$

The equilibrium is unstable if at least one of the roots of polynomial in (5.4) *is outside the unit circle or if at least one of the inequalities in* (5.5) *holds with the opposite (strict) sign.*

In particular, the system exhibits a fold bifurcation if one of the $N-1$ right-hand inequalities in (5.5) *becomes an equality and a flip bifurcation if one of the $N-1$ left-hand inequalities becomes an equality.*

Thus, the stability condition for a generic fixed point in the multi-agent economies is twofold. First, equilibrium should be "self-consistent", i.e. remain stable even if any non-surviving agent would be removed from the economy. This very intuitive result strictly follows from the comparison between $Q_1(\mu)$ and polynomial $Q(\mu)$ in (5.2). This is however not enough. A further requirement comes from the inequalities in (5.5). In particular, according to the left-hand inequality, the wealth growth rate of those agents who do not survive in the stable equilibrium should be strictly less than the wealth growth rate of the survivors r^*. Thus, in those equilibria where $r^* > -\bar{e}$ the surviving agent must be the most aggressive and invest a higher wealth share in the risky asset. On the other hand, in those equilibria where $r^* < -\bar{e}$ the survivor has to be the least aggressive.

The EML plot can be used to obtain a geometric illustration of the previous Proposition. In the right panel of Fig. 4 we draw again the two investment functions discussed in Section 4.1. The region where the additional condition (5.5) is satisfied is reported in gray. We have found before four possible equilibria: S_1, S_2, U_1 and U_2. Proposition 5.3 states that, first, the dynamics cannot be attracted by an equilibrium which was unstable in the respective single-agent cases. And, second, it cannot be attracted by an equilibrium in which non-surviving agent invests in the point belonging to a white region. Points U_1 and U_2 will be, for example, unstable if the investment function of survivor has only one lag (cf. Proposition 5.2). Therefore, they are unstable also in the multi-agents market. From item **2)** of Proposition 5.3

it follows that S_1 is the only stable equilibrium of the system with two agents. Notice, indeed, that in the abscissa of S_1, i.e. for the equilibrium return, the linear investment function of the non-surviving agent II passes below the investment function of the surviving agent and belongs to the gray area. On the contrary, in the abscissa of S_2, the investment function of the non-surviving agent I has greater value and does not belong to the gray area. Consequently, this equilibrium is unstable.

Let us move now to consider the non-generic case, when k different agents survive in the equilibrium. The equilibria have been found in Proposition 4.1*(ii)*. The following applies

Proposition 5.4. *Let* $(r^*, \boldsymbol{x}^*, \boldsymbol{\varphi}^*)$ *be a BCE of system* (4.1) *associated with k survivors defined by* (4.2) *and* (4.4).

The BCE as a fixed point of the corresponding dynamical system is never hyperbolic and, consequently, never (locally) asymptotically stable.

Let $P_{f_n}(\mu)$ *be the stability polynomial of investment function* f_n. *The BCE is (locally) stable if the two following conditions are met:*

1) *all the roots of polynomial*

$$Q_{1\diamond k}(\mu) = \mu^{L+1} - \frac{(1+r^*)\,\mu - 1}{r^*\, l_e'(r^*)} \sum_{n=1}^{k} \varphi_n^*\, P_{f_n}(\mu) \quad , \tag{5.6}$$

are inside the unit circle.

2) *the equilibrium investment shares of the non-surviving agents satisfy to*

$$-2 - r^* < x_n^*\,(r^* + \bar{e}) < r^* \,, \quad k < n \leq N \,. \tag{5.7}$$

The BCE is unstable if at least one of the roots of polynomial in (5.6) *is outside the unit circle or if at least one of the inequalities in* (5.7) *holds with the opposite (strict) sign.*

The non-hyperbolic nature of the equilibria with many survivors turns out to be a direct consequence of their non-unique specifications. The motion of the system along the $k-1$ dimensional subspace consisting of the continuum of equilibria leaves the aggregate properties of the system invariant so that all these equilibria can be considered equivalent. Proposition 5.4 also provides the stability conditions for perturbations in the hyperplane orthogonal to the non-hyperbolic manifold formed by equivalent equilibria. The polynomial $Q_{1\diamond k}(\mu)$ is quite similar to the corresponding polynomial in Proposition 5.3, except that one has to weight the stability polynomial of the different investment functions $P_{f_k}(\mu)$ with the weights corresponding to the relative wealth of survivors in the equilibrium. At the same time, the constraint on the investment shares in (5.7) is identical to the one obtained in (5.5). In particular, similar to the case with a single survivor, in those equilibria where $r^* > -\bar{e}$ all surviving agents must be more aggressive than those who do not survive, and *vice versa*.

Finally, let us analyze the “no equity premium” BCE, where $r^* = -\bar{e}$. We consider general situation and allow some agents to have zero wealth shares. Without loss of generality, we assume that first $k \leq N$ agents survive in the equilibrium. The following result characterizes the stability of such equilibria

Proposition 5.5. *Let* $(r^*, \boldsymbol{x}^*, \boldsymbol{\varphi}^*)$ *be a BCE identified in Proposition 4.1(iii).*

If $N \geq 3$, *the BCE as a fixed point of the corresponding dynamical system is non-hyperbolic and, consequently, is not (locally) asymptotically stable. The BCE is (locally) stable if all the* *roots of the following polynomial are inside the unit circle*

$$\mu^{L+1} + \frac{\mu - 1}{\langle x^2 \rangle} \sum_{j=1}^{k} \varphi_j^* \, P_{f_j}(\mu) \quad , \tag{5.8}$$

where $P_{f_j}(\mu)$ *is the stability polynomial of investment function* f_j *computed in point* $(-\bar{e}, \ldots, -\bar{e})$, *and* $\langle x^2 \rangle = \sum_{n=1}^{k} \varphi_n^* \, x_n^{*\,2}$.

The equilibrium is unstable if at least one of the roots of polynomial in (5.8) *is outside the unit circle.*

As in the case of Proposition 5.4, the "no-equity-premium" equilibria can be non-hyperbolic, due to possibility to change wealth between agents without changing the aggregate properties of the dynamics. For the complete stability analysis, the roots of polynomial (5.8) should be analyzed for specific investment functions. In particular, when all investment functions are horizontal in point $-\bar{e}$, the equilibrium (if it exists) is always stable.

5.3 Optimal selection and multiple equilibria

In this Section, using the geometric interpretation based on the EML we discuss some relevant implications of the previous Section about the asymptotic behavior of the model and its global properties. We confine the discussion to the generic case of equilibria with a single survivor.

The first implication concerns the aggregate dynamics of the economy. Let us consider a stable many-agent BCE equilibrium. Let us suppose that r^* is the equilibrium return and that the first agent survives. It is easy to see that his wealth return is equal to r^* and this is also the asymptotic growth rate of the *total* wealth. Then, we can interpret the second requirement of Proposition 5.3 as saying that, in the dynamic competition, those agent survives who allows the economy to have the highest possible rate of growth. Indeed, if any other agent $n \neq 1$ survived, the economy would have grown with rate $x_n^* \, (r^* + \bar{e})$, which is less than r^* according to (5.5). This result can be called an *optimal selection principle* since it clearly states the market endogenous selection toward the best aggregate outcome.

To be a bit more specific, notice that in equilibria with $r^* > -\bar{e}$ the overall wealth of the economy grows (in particular for $r^* < 0$ the negative wealth grows to 0), while in equilibria with $r^* < -\bar{e}$ the wealth of the economy falls. Thus, according to the optimal selection principle the surviving agent must be the most aggressive in equilibria where the economy grows and must be the least aggressive investor in equilibria where the economy shrinks.

Notice, however, that this selection does not apply to the whole set of equilibria, but only to the subset formed by equilibria associated with stable fixed points in the single agent case (c.f. (5.6)). For instance, with the investment functions shown in the right panel of Fig. 4, the dynamics will never end up in U_2, even if this is the equilibrium with the highest possible return. Furthermore, the variety of possible investment functions implies that the optimal selection principle has a local character. Indeed, even if we exclude all unstable single-agent equilibria, the market will not choose the equilibrium with the highest growth rate. Sometimes it can be the case like in the right panel of Fig. 4. However, it is often not the case and a simple counter-example is provided by a single investment function possessing *multiple stable equilibria* as shown in Fig. 5. For this investment function both S_L and S_H are stable equilibria. Now suppose that an agent possessing this function competes on the market with other agents which are more risk averse than him and always invest smaller shares of wealth in the risky asset. In this situation, these two equilibria, S_H and S_L, remain stable and the riskier agent

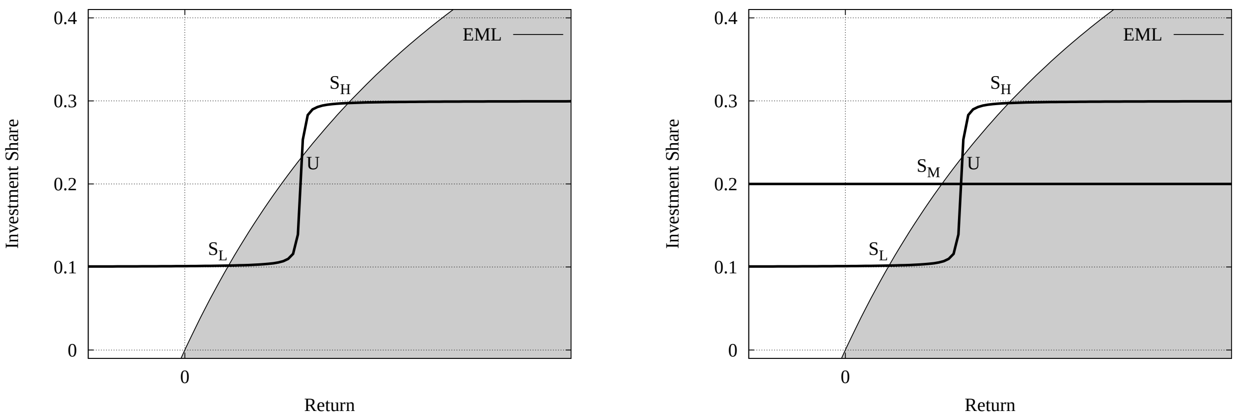

Figure 5: **Left panel:** A non-linear investment function leading to multiple equilibria. S_H and S_L are stable while U is unstable. **Right panel:** The same non-linear investment function together with a constant investment function. S_H and S_M are stable while S_L and U are unstable.

will ultimately dominate the market. The resulting market equilibrium will only depend on the initial conditions.

Possible presence of the BCE without equity premium, identified for the multi-agent market in Proposition 4.1*(iii)*, is another source of multiple equilibria. For example, in the situation depicted in the right panel of Fig. 4 there is a stable equilibrium where agent with non-linear strategy survives (point S_1) and no-equity-premium stable equilibrium where both agents survive (points A_1 and A_2) and agent with non-linear strategy possesses high enough wealth share φ_1^* (cf. polynomial (5.8) in Proposition 5.5).

The existence of multiple equilibria also leads to another interesting implication of Proposition 5.3. Consider again the investment function in the left panel of Fig. 5 and add a second agent with constant investment function, to obtain the situation shown in the right panel of Fig. 5. The entry of the new agent changes the possible equilibria, which become the points S and S_H. Notice, however, that in these two equilibria different agents dominate the market. If the market before the entry of the new agent was in S_H, the first agent still remains the more aggressive, and the entry of the new agent does not affect his dominant position. On the other hand, if the equilibrium before the entry was S_L, this equilibrium becomes unstable and the system will tend to move away from it. The ensuing dynamics could ultimately choose the investment function of the new entrant as the dominant one. This simple example suggests that, at least inside our framework, the definition of a dominance order relation on the space of trading strategies is impossible.

6 Conclusion

This paper introduces novel concept of the Behavioral Consistent Equilibrium which generalizes somehow the rational expectations equilibrium for the framework with arbitrary number of procedurally rational traders. We found the elegant, geometric way to characterize equilibria and their stability in a speculative pure exchange economies with heterogeneous CRRA traders. The framework is relatively general in terms of agents' behaviors and differs from most of the models with heterogeneous agents in two important respects.

First, we analyze the aggregate dynamics and asymptotic behavior of the market when *an arbitrary large number* of traders participate to the trading activity. Second, we do not restrict in any way the procedure used by agents in order to build their forecast about future prices, nor the way in which agents can use this forecast to obtain their present asset demand. In our terms, agent with any smooth investment function mapping the information set to the present investment choice can present in the model.

Even if consideration of an arbitrary number of generic agents' behaviors leads us to study dynamical systems of an arbitrarily large dimension, we are able to provide a complete characterization of market equilibria and, for some particular case of dividend process, a description of their stability conditions in terms of few parameters characterizing the traders investment strategies. In particular, we find that, irrespectively of the number of agents operating in the market and of the structure of their demand functions, only three types of equilibria are possible:

- generic equilibria, associated with isolated fixed points, where a single agent asymptotically possesses the entire wealth of the economy,
- non-generic equilibria, associated with continuous manifolds of fixed points, where many agents possess a finite shares of the total wealth,
- generic equilibria associated with many survivors, where the economy does not possess the equity premium.

Furthermore, we show in total generality that a simple function, the "Equilibrium Market Line", can be used to obtain a geometric characterization of the location of all these types of equilibria. Furthermore, some results about stability conditions can also be inferred from the same EML. The precise shape of the EML depends on the dividend process.

Our general results provide, we believe, a simple and clear description of the principles governing the asymptotic market dynamics resulting from the competition of different trading strategies. The optimizing agents may dominate non-optimizing agents but may also be dominated by them. In general, the ultimate result of competition between agents depends on the whole market ecology. The EML is a handy and useful tool for demonstration such phenomena as absence of equilibrium, presence of multiple equilibria, and also for comparative statics exercises. From this plot (and results of stability analysis) the following two "impossibility theorems" follow in an obvious way. First, there exist no "best" strategy, independently of what "best" means exactly, since any possible market equilibrium can be destabilized by some investment function. Second, it is impossible to build a dominance order relation inside the space of trading strategies, since two strategies may generate multiple stable equilibria with *different* survivors in them, so that the outcome will depend on the initial conditions or noise.

The present analysis can be extended in many directions. First of all, there is still open question what will be stability conditions for other dividend processes. Our preliminary results of the analytic investigation in this direction show that some results (like presence of stable equity premium in equilibria for "flat" investment functions) also hold for constant growth rate of the dividend case. Second, in the limits of our framework, one can wonder about other possible dynamics. For instance, we have shown that there is a theoretical possibility do not have any equilibrium at all. The dynamics in this case remain unknown. Also the dynamics after bifurcations, which is the key question in many heterogeneous agent models, were not investigated. Probably numerical methods can be effectively applied to study these questions and also clarify the role of initial conditions and the determinants of the relative size of the basins of attraction for multiple equilibria scenarios. Third, our general CRRA-framework led us in Proposition 2.1 to the system in terms of returns and wealth shares. There are numerous behavioral specifications which were not analyzed here and still consistent with such framework. These specifications range from the evaluation of the "fundamental" value of the asset, possibly obtained from a private source of information, to a strategic behavior that try to keep in consideration the reaction of other market participants to the revealed individual choices. Furthermore, one may ask what are the consequences of the optimal selection principle for a market in which the set of strategies is not "frozen", but instead is evolving in time, plausibly following some adaptive process. For instance, one can assume that agents imitate the behavior of other traders (see e.g. Kirman (1991)) or that they update strategies according to recent relative performances (see e.g. Brock and Hommes (1998)). The analysis of the consequences of the introduction of such strategies on the optimal selection principle may, ultimately, refute the statement about the impossibility of defining a dominance relation among strategies.

APPENDIX

A Proof of Proposition 2.1

Plugging the expression for $w_{t+1,n}$ from the second equation in system (2.4) into the right-hand side of the first equation of the same system, and assuming that $p_t > 0$ and, consistently with (2.6), $p_t \neq \sum x_{t+1,n}\, x_{t,n}\, w_{t,n}$ one gets

$$p_{t+1} = \left(1 - \frac{1}{p_t}\sum_{n=1}^{N} x_{t+1,n}\, x_{t,n}\, w_{t,n}\right)^{-1} \left(\sum_{n=1}^{N} x_{t+1,n}\, w_{t,n} + \big(e_{t+1} - 1\big)\sum_{n=1}^{N} x_{t+1,n}\, w_{t,n}\, x_{t,n}\right) =$$

$$= p_t \frac{\sum_n x_{t+1,n}\, w_{t,n} + (e_{t+1} - 1)\sum_n x_{t+1,n}\, w_{t,n}\, x_{t,n}}{\sum_n x_{t,n}\, w_{t,n} - \sum_n x_{t+1,n}\, x_{t,n}\, w_{t,n}} =$$

$$= p_t \frac{\big\langle x_{t+1}\big\rangle_t - \big\langle x_t\, x_{t+1}\big\rangle_t + e_{t+1}\big\langle x_t\, x_{t+1}\big\rangle_t}{\big\langle x_t\big\rangle_t - \big\langle x_t\, x_{t+1}\big\rangle_t} \quad ,$$

where we used the first equation of (2.4) rewritten for time t to get the second equality. Condition (2.6) is obtained imposing $p_{t+1} > 0$, and the dynamics of price return in (2.7) is immediately derived. From the second equation of (2.4) it follows that

$$w_{t+1,n} = w_{t,n}\big(1 + x_{t,n}\,(r_{t+1} + e_{t+1})\big) \qquad \forall n \in \{1, \dots, N\} \quad , \tag{A.1}$$

which leads to (2.9). To obtain the wealth share dynamics, divide both sides of (A.1) by w_{t+1} to have

$$\varphi_{t+1,n} = \frac{w_{t,n}}{\sum_m w_{t+1,m}}\big(1 + x_{t,n}\,(r_{t+1} + e_{t+1})\big) =$$

$$= \frac{w_{t,n}}{\sum_m w_{t,m} + (r_{t+1} + e_{t+1})\sum_m x_{t,m} w_{t,m}}\big(1 + x_{t,n}\,(r_{t+1} + e_{t+1})\big) =$$

$$= \frac{\varphi_{t,n}}{1 + (r_{t+1} + e_{t+1})\sum_m x_{t,m}\varphi_{t,m}}\big(1 + x_{t,n}\,(r_{t+1} + e_{t+1})\big) \quad ,$$

where (A.1) has been used to get the second line and we divided both numerator and denominator by the total wealth at time t to get the third.

B Proof of Proposition 3.1

According to Definition 3.1, in any BCE the first equation of (2.13) can be simplified to

$$r_{t+1} = e_{t+1} \left\langle {x^*}^2 \right\rangle \Big/ \left\langle x^*(1-x^*) \right\rangle \quad . \tag{B.1}$$

If $e_{t+1} = 0$, then from the second equation of (2.13) the dividend yield is zero for all subsequent periods. Therefore, also price return is zero in that periods. The equilibrium investment shares are computed straight-forwardly, while the stationarity of the wealth distribution follows from the last equation in (2.13).

If $e_{t+1} \neq 0$, (B.1) shows that the ratio of the return to the dividend is invariant over time. We denote this ratio as y^*.

If $\left\langle x^* \right\rangle = 0$ then (2.6) guarantees that $\left\langle {x^*}^2 \right\rangle \neq 0$, so that from (B.1) it immediately follows that $r_{t+1} = -e_{t+1}$, i.e. $y^* = -1$. Notice from the last equation in (2.13) that the stationarity of the wealth distribution is automatically satisfied in this case.

If $\left\langle x^* \right\rangle \neq 0$ then the wealth distribution will be stationary only if $\varphi_{t,n} = \varphi_{t+1,n}$ for any agent n. This condition apparently holds for any agent with zero wealth share. Let us impose this condition on an arbitrary surviving agent n. Since $r_{t+1} + e_{t+1} \neq 0$ (because otherwise $\left\langle x^* \right\rangle = 0$), his investment share x^*_n should be equal to $\left\langle x^* \right\rangle$, which is independent of n. Therefore, all survivors have the same investment share, which we denote as $x^*_\diamond$. Plugging $x^*_\diamond$ to (B.1), we get (3.1).

The remaining part of the Proposition is an obvious consequence of the definition of value y^*.

C Proof of Lemma 4.1

It is straight-forward to see that $e^*_1 = g/y^*$ and $e^*_2 = 0$ are the only fixed points of (4.6). In order to investigate a global dynamics we introduce variable $z_t = 1/e_t$ and rewrite (4.6) as follows

$$z_{t+1} = \frac{z_t}{1+g} + \frac{y^*}{1+g} \quad . \tag{C.1}$$

The last equation is linear and its dynamics are easy to grasp. When $g > 0$ the slope of the right-hand side of (C.1) is less than 1, so that the globally stable fixed point is $z^* = y^*/g$. Then, the original system is attracted by e^*_1.

If $g < 0$ the fixed point $z^* = y^*/g$ becomes unstable, and the dynamics (C.1) diverges. Therefore, in this case, as well as in the case with $g = 0$, the point e^*_2 is an global attractor for dynamics (4.6).

Thus, when the parameter g crosses zero value, an unstable and stable fixed points collide and exchange stability. This is typical phenomenon under transcritical bifurcation.

D Proof of Proposition 5.1

The $(L+1) \times (L+1)$ Jacobian matrix $\boldsymbol{J}$ of system (4.1) with a single agent reads

$$\begin{Vmatrix}
0 & \frac{\partial f}{\partial r_0} & \frac{\partial f}{\partial r_1} & \frac{\partial f}{\partial r_2} & \dots & \frac{\partial f}{\partial r_{L-2}} & \frac{\partial f}{\partial r_{L-1}} \\
R^x & R^f \frac{\partial f}{\partial r_0} & R^f \frac{\partial f}{\partial r_1} & R^f \frac{\partial f}{\partial r_2} & \dots & R^f \frac{\partial f}{\partial r_{L-2}} & R^f \frac{\partial f}{\partial r_{L-1}} \\
0 & 1 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 1 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 1 & 0
\end{Vmatrix} , \tag{D.1}$$

where for shortness we use r_l as a notation for r_{t-1-l} and

$$R^x = \frac{\partial R(x^*, x^*)}{\partial x} = -\frac{1}{x^*\,(1-x^*)}\,, \qquad R^f = \frac{\partial R(x^*, x^*)}{\partial x'} = \frac{1+r^*}{x^*\,(1-x^*)}\,. \tag{D.2}$$

The stability condition of equilibrium are provided by the following

Lemma D.1. *The characteristic polynomial $P_J(\mu)$ of system* (4.1) *with a single agent in the BCE $r^*, \boldsymbol{x^*}$ is*

$$P_J(\mu) = (-1)^{L-1}\left(\mu^{L+1} - \frac{(1+r^*)\mu - 1}{x^*(1-x^*)} P_f(\mu)\right) \tag{D.3}$$

where $P_f(\mu)$ denotes the stability polynomial of function f introduced in (5.1)*.*

Proof. Consider (D.1) and introduce $(L+1)\times(L+1)$ identity matrix $\boldsymbol{I}$. Expanding the determinant of $\boldsymbol{J} - \mu\,\boldsymbol{I}$ by the elements of the first column one has

$$\begin{aligned}\det\big(\boldsymbol{J} - \mu\,\boldsymbol{I}\big) = (-\mu)\,(-1)^{L-1}&\left(\left(R^f\frac{\partial f}{\partial r_0} - \mu\right)\mu^{L-1} + R^f\frac{\partial f}{\partial r_1}\,\mu^{L-2} + \cdots + R^f\frac{\partial f}{\partial r_{L-1}}\right) - \\ - R^x\,(-1)^{L-1}&\left(\frac{\partial f}{\partial r_0}\,\mu^{L-1} + \frac{\partial f}{\partial r_1}\,\mu^{L-2} + \cdots + \frac{\partial f}{\partial r_{L-2}}\,\mu + \frac{\partial f}{\partial r_{L-1}}\right) = \\ = (-1)^{L-1}&\left(\mu^{L+1} - \left(\mu R^f + R^x\right)\sum_{k=0}^{L-1}\frac{\partial f}{\partial r_k}\,\mu^{L-1-k}\right)\end{aligned}\,,$$

which, using relations in (D.2) and definition of stability polynomial in (5.1) reduces to (D.3). □

Using the relationship $l'_e(r^*) = x^*(1-x^*)/r^*$ it is immediate to see that, apart from irrelevant sign, (D.3) is identical to (5.2).

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