What Really Matters for Multivariate VaR Forecasting? An Empirical Analysis within a Unified Copula Framework

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Abstract

This paper examines different multivariate models to evaluate what are the main determinants when doing VaR forecasts for a portfolio of assets. To achieve this goal, we unify past multivariate models by using a general copula framework and we propose many new extensions. We differentiate the models according to the choice of the marginals distribution, the specification of the conditional moments of the marginals, the choice of the type of copula, the specification of the conditional copula parameters. Besides, we consider also the effects of the degree of assets' riskiness, the portfolio dimensionality and the time sample used for VaR backtesting. The calculated VaR values are then compared using three different testing procedures, including Kupiec's unconditional coverage test, Christoffersen's conditional coverage test and a recent bootstrap test of Superior Predicting Ability proposed by Hansen (2005) and Hansen and Lunde (2005).

Keywords: Multivariate modelling, Copulas, VaR, Forecasting

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1 Introduction

The goal of this paper is to examine and compare different multivariate models to forecast the distribution of log returns and estimate the Value at Risk for a portfolio of assets. To achieve this purpose, we make use of Copula theory to consider a wide range of multivariate models. Using copulas allows us to separate the marginals from the dependence structure, where the marginals need not be the same.

The main contribution of this work is to propose a unified framework that includes previous standard models like the CCC model by Bollerslev (1990) and the DCC model by Engle (2002), but allows us to consider much more general cases than the multivariate normal distribution. For example, since the marginals distributions of asset returns are often asymmetric and fat-tailed, we can use a Skew-T distribution with a threshold-GARCH model for the conditional volatility to take the leverage effect into account. The conditional third and fourth moments can be made time-varying, too. Moreover, we can consider different kinds of dependence structure, ranging from the normal copula till more sophisticated copulas like the Grouped-T or the Clayton copula, that are able to model also dependence in the tails. Again, the vector of copula parameters can be made time-varying.

Copulae have been successively used for measuring portfolio Value at Risk by Bouye', Durrleman, Nikeghbali, Riboulet, and Roncalli (2001), Embrechts, Lindskog, and McNeil (2003) and Cherubini, Vecchiato, and Luciano (2004). However, the applications made so far dealt with unconditional distributions, only, and the portfolios included a small number of assets in all cases. Furthermore, the effect of assets' riskiness and of different time samples for VaR backtesting were not considered at all.

Therefore, in order to evaluate what are the main determinants when doing VaR forecasts for a portfolio of assets, we compare different distribution assumptions for the marginals, as well as different dynamic specifications for their moments, to understand whether the proper modelling of the latter is more important than the type of distribution. In a similar fashion, we consider different type of copulas and different conditional specifications for their parameters. Besides, we consider portfolios of different sizes, ranging from two assets up to 100 assets, and of different riskiness, both investment grade and high-yield assets with lower credit rating. Finally, we consider different time samples for VaR backtesting, too.

The rest of the paper is organized as follows. Section 2 defines the Value at Risk and reviews the main back-testing techniques proposed so far. In Section 3 we provide an outline of multi- variate modelling, proposing a unified approach by means of copula theory. Section 4 describes the models used for the analysis and presents the main empirical results. We conclude in Section 5.

2 VaR: Definition and Evaluation

2.1 VaR Definition

Banks and financial institution face the everyday problem of measuring the market risk exposure of their assets: if we use a probabilistic framework and we suppose to be at time t, we want to assess the risk of a financial position for the next l periods. The Value at Risk is the most widely used measure of risk and it has become the benchmark risk measure thanks to the Basel II agreements. VaR is simply defined as the worst expected loss of a financial position over a target horizon with a given confidence level.

Computing VaR requires the actual amount of money invested, a given holding period (one day, one week, one month, etc.), the choice of the density probability for the returns and the confidence level. Let $\Delta V(l)$ be the change in the value of the assets in the financial position from t to $t+l$, while $F_l(x)$ be the cumulative distribution function (c.d.f.) of $\Delta V(l)$. Formally,

Definition 2.1 (VaR for a Long Position) *We define the VaR of a long position over time horizon l with probability p as*

$$p = \Pr\left[\Delta V(l) \leq \mathrm{VaR}(p,l)\right] = F_l\left(\mathrm{VaR}(p,l)\right)$$

where VaR is defined as a negative value.

The holder of short position suffers a loss when the value of the asset increases [i.e. $\Delta V(l) > 0$]. Hence,

Definition 2.2 (VaR for a Short Position) *The VaR of a short position is defined as*

$$p = \Pr\left[\Delta V(l) \geq \mathrm{VaR}(p,l)\right] = 1 - \Pr\left[\Delta V(l) \leq \mathrm{VaR}(p,l)\right] = 1 - F_l\left(\mathrm{VaR}(p,l)\right)$$

If the c.d.f. is known, then the VaR is simply a specified quantile of a portfolio's potential loss distribution over a given holding period. Particularly, the previous definitions show that VaR is concerned with the tail behavior of the c.d.f., where for a long position the left tail is important, yet a short position focuses on the right tail of the distribution.
We will propose and compare different models to forecast the multivariate distribution of log-returns and so to calculate VaR. Given the widespread use of VaR by banks and regulators, it is of sure interest to evaluate the accuracy of the different models used to estimate VaR.

2.2 VaR Evaluation

We will assess the performance of the competing multivariate models using the following back-testing techniques:

- Kupiec (1995) unconditional coverage test;
- Christoffersen (1998) conditional coverage test;
- Loss functions to evaluate VaR forecasts accuracy;
- Hansen and Lunde (2005) and Hansen's (2005) Superior Predictive Ability (SPA) test.

2.2.1 Unconditional Model Evaluation: Kupiec's test

This test is based on binomial theory and tests the difference between the observed and expected number of VaR exceedances of the effective portfolio losses. Since VaR is based on a confidence level p, when we observe N losses in excess of VaR out of T observations, hence we observe N/T proportion of excessive losses: the Kupiec's test answers the question whether N/T is statistically significantly different from p.

Following binomial theory, the probability of observing N failures out of T observations is $(1-p)^{T-N}p^N$, so that the test of the null hypothesis that the expected exception frequency $N/T = p$ is given by a likelihood ratio test statistic:

$$LR_{UC} = -2 \cdot \ln[(1-p)^{T-N}p^N] + 2 \cdot \ln[(1-N/T)^{T-N}(N/T)^N] \tag{2.1}$$

which is distributed as χ^2 (1) under the H_0 . This test can reject a model for both high and low failures but, as stated in Kupiec (1995), its power is generally poor.

2.2.2 Conditional Model Evaluation: Christoffersen's test

A more complete test was made by Christoffersen (1998), who developed a likelihood ratio statistic to test the joint assumption of unconditional coverage and independence of failures. Its main advantage over the previous statistic is that it takes account of any conditionality in our forecast: for example, if volatilities are low in some period and high in others, the VaR forecast should respond to this clustering event. The Christoffersen procedure enables us to separate clustering effects from distributional assumption effects. His statistic is computed as follows:

$$LR_{CC} = -2\ln[(1-p)^{T-N}p^N] + 2\ln[(1-\pi_{01})^{n00}\pi_{01}^{n01}(1-\pi_{11})^{n10}\pi_{11}^{n11}] \tag{2.2}$$

where n_{ij} is the number of observations with value i followed by j for $i, j = 0, 1$ and

$$\pi_{ij} = \frac{n_{ij}}{\sum_j n_{ij}} \tag{2.3}$$

are the corresponding probabilities. Under the H_0, this test is distributed as a $\chi^2(2)$.
If the sequence of N losses is independent, then the probabilities to observe or not observe a VaR violation in the next period must be equal, which can be written more formally as $\pi_{01} = \pi_{11} = p$. The main advantage of this test is that it can reject a VaR model that generates either too many or too few clustered violations, although it needs several hundred observations in order to be accurate.

2.2.3 Measures of Accuracy: Loss functions

The previous tests do not show any power in distinguishing among different, but close, alternatives. Moreover, as noted by the Basle Committee on Banking Supervision (1996), the magnitude as well as the number of exceptions are a matter of regulatory concern. This concern can be readily incorporated into a so called "loss function" by introducing a magnitude term. This was first accomplished by Lopez (1998). The general form of these loss functions is:

$$C_{t+1} = \begin{cases} f\left(L_{t+1}, VaR_{t+1}\right) & \text{if } L_{t+1} > VaR_{t+1} \\ g\left(L_{t+1}, VaR_{t+1}\right) & \text{if } L_{t+1} \leq VaR_{t+1} \end{cases}$$

where $f(x, y)$ and $g(x, y)$ are arbitrary functions such that $f(x, y) \geq g(x, y)$ for a given y, while L is the portfolio loss. The numerical scores C_{t+1} are constructed with a negative orientation, that is, lower values of C_{t+1} are preferred since exceptions are given higher scores than non-exceptions. Numerical scores are generated for individual VAR estimates, and the score for the complete sample is:

$$C = \sum_{i=1}^{T'} C_{t+i} \tag{2.4}$$

where T' is the number of observations used for VaR backtesting. Under very general conditions, accurate VaR estimates will generate the lowest possible score.
Many loss functions can be constructed. Lopez (1998) has proposed the following quadratic loss function:

$$C_{t+1} = \begin{cases} 1 + (L_{t+1} - VaR_{t+1})^2 & if \quad L_{t+1} > VaR_{t+1} \\ 0 \quad if \quad L_{t+1} \leq VaR_{t+1} \end{cases} \tag{2.5}$$

Thus, as before, a score of one is imposed when an exception occurs, but now, an additional term based on its magnitude is included. The numerical score increases with the magnitude of the exception and can provide additional information on how the underlying VaR model forecasts the lower tail of the underlying L_{t+1} distribution.
Blanco and Ihle (1999) suggest an alternative way to deal with the size of exceptions by focusing on the average size of the exceptions:

$$C_{t+1} = \begin{cases} \frac{L_{t+1} - VaR_{t+1}}{VaR_{t+1}} & if \quad L_{t+1} > VaR_{t+1} \\ 0 \quad if \quad L_{t+1} \leq VaR_{t+1} \end{cases} \tag{2.6}$$

2.2.4 A Test for Superior Predictive Ability

Hansen and Lunde (2005) and Hansen (2005) propose a test for Superior Predictive Ability (SPA), which compares the performances of two or more forecasting models. The forecasts are evaluated using a prespecified loss function, like the previous two by Lopez (1998) and Blanco and Ihle (1999), and the best forecast model is the model that produces the smallest expected loss. The SPA tests for the best standardized forecasting performance relative to a benchmark model, and the null hypothesis is that none of the competing models is better than the benchmark. However, testing multiple inequalities is more complicated than testing equalities (or a single inequality) because the distribution is not unique under the null hypothesis. Nevertheless, a consistent estimate of the p-value can be obtained by using a bootstrap procedure, as well as an upper and a lower bound[1].
A possible strategy that can be implemented with the previous approaches is the following:

1. Apply Kupiec's and Christoffersen's tests at a first stage to choose the best models;

2. Then use the loss functions and Hansen's test to compare the costs of different admissible choices and refine the selection, respectively.

The first step is required since the loss functions by Lopez (1998) and Blanco and Ihle (1999) tend to favor more conservative models by construction: a model with no exceedances at all would be considered the best, but such a choice could be very expensive for a financial institution.

[1]The authors would like to thank Peter Hansen for supplying the Ox code that calculates the SPA test statistics and associated p-values.

3 Multivariate Modelling

While univariate VaR estimation has been widely investigated, the multivariate case has been dealt only by a smaller and more recent literature, regarding the forecasting of correlations between assets, too: empirical works which deal with this issue are those of Engle and Sheppard (2001), Giot and Laurent (2003) and Bauwens and Laurent (2004). When we use parametric methods, VaR estimation for a portfolio of assets can become very difficult due to the complexity of joint multivariate modelling. Moreover, computational problems arise when increasing the number of assets[2]. As a consequence of these difficulties, two models seem to have gained the major attention by practitioners and researchers so far:

- The "Constant Conditional Correlation" (CCC) model by Bollerslev (1990);
- The "Dynamic Conditional Correlation" (DCC) model by Engle (2002);

In short, let X_t be a vector stochastic process of dimension $n \times 1$ and θ a finite vector of parameters, then the two models can be expressed as follows

$$X_t = \mathrm{E}\left[X_t|\mathfrak{F}_{t-1}\right] + \varepsilon_t \tag{3.1}$$

$$\varepsilon_t = H_t^{1/2}\eta_t, \quad \eta_t \sim \text{i.i.d}(0, I_n) \tag{3.2}$$

where $H_t^{1/2}$ is a $n \times n$ positive definite matrix, I_n is the identity matrix of order n, while $\mathfrak{F}_t$ is the information set available at time t. For both these two models H_t can generally be written as

$$H_t = D_t R_t D_t \tag{3.3}$$

$$D_t = diag(h_{11,t}^{1/2} \dots h_{nn,t}^{1/2}) \tag{3.4}$$

$$R_t = (\rho_{ij,t}), \quad \text{with} \quad \rho_{ii,t} = 1 \tag{3.5}$$

where R_t is the $n \times n$ matrix of conditional correlations (constant or time-varying), and $h_{ii,t}$ is defined as a univariate GARCH model. Positivity of H_t follows from positivity of R_t and of each $h_{ii,t}$.

The *CCC model* by Bollerslev (1990) is defined as:

$$H_t = D_t R D_t \tag{3.6}$$

while h_{iit} can be defined as any univariate GARCH model and $R_t = R$ is a symmetric positive definite matrix with $\rho_{ii} = 1, \forall i$.Therefore, the conditional correlations are constant. Hence,

$$h_{ij,t} = \rho_{ij}\sqrt{h_{ii,t}h_{jj,t}} \quad i \neq j \tag{3.7}$$

and thus the dynamics of the covariance is determined only by the dynamics of the two conditional variances.

[2]See the review of multivariate GARCH models by Bauwens, Laurent, and Rombouts (2006) for a treatment of these issues.

Alternatively, the *DCC model* by Engle (2002) (see also Engle and Sheppard (2001)) is defined as:

$$H_t = D_t R_t D_t \tag{3.8}$$

where D_t is defined as in (3.4), $h_{ii,t}$ can be defined again as any univariate GARCH model, and

$$R_t = (diagQ_t)^{-1/2} Q_t (diagQ_t)^{-1/2} \tag{3.9}$$

where the $n \times n$ symmetric positive definite matrix Q_t is given by:

$$Q_t = \left(1 - \sum_{l=1}^{L} \alpha_l - \sum_{s=1}^{S} \beta_s\right) \bar{Q} + \sum_{l=1}^{L} \alpha_l \eta_{t-l} \eta'_{t-l} + \sum_{s=1}^{S} \beta_s Q_{t-s} \tag{3.10}$$

where $\eta_{it} = \varepsilon_{it}/\sqrt{h_{ii,t}}$, $\bar{Q}$ is the $n \times n$ unconditional variance matrix of η_t, α_l (≥ 0) and β_s (≥ 0) are scalar parameters satisfying $\sum_{l=1}^{L} \alpha_l + \sum_{s=1}^{S} \beta_s < 1$. These conditions are needed to have $Q_t > 0$ and $R_t > 0$. Q_t is the covariance matrix of η_t , since $q_{ii,t}$ is not equal to 1 by construction. Then, it is transformed into a correlation matrix by (3.9). If $\theta_1 = \theta_2 = 0$ and $\bar{q}_{ii} = 1$ the *CCC* model is obtained.

Since η_t is assumed to be i.i.d. $N(0, I_n)$ and $H_t = D_t R_t D_t$, after some algebraic manipulation and neglecting the constant part, the sample log-likelihood becomes:

$$\begin{aligned} L_T(\theta) &= -\frac{1}{2}\sum_{t=1}^{T}(\log|D_t R_t D_t| + u'_t R_t^{-1} \eta_t) && (3.11) \\ &= -\frac{1}{2}\sum_{t=1}^{T} 2\log|D_t| + \eta'_t \eta_t \leftarrow L_{1T}(\theta_1^*) && (3.12) \\ & \quad -\frac{1}{2}\sum_{t=1}^{T}(\log|R_t| + \eta'_t R_t^{-1} \eta_t - \eta'_t \eta_t) \leftarrow L_{2T}(\theta_2^* | \theta_1^*) && (3.13) \end{aligned}$$

where θ_1^* are the parameters of the conditional variances D_t estimated with GARCH models of different type in the first step, while θ_2^* are the parameters of the conditional correlations R_t estimated in the second step: if the CCC model is involved, $R_t = R$ is simply the sample correlation matrix of the standardized residuals, while if the DCC is involved the model is estimated by maximizing L_{2T} with numerical methods.
We will show in the next two sections that these models can be presented as special cases within a more general copula approach.

3.1 Copula Modelling

Copula theory provides an easy way to deal with (otherwise) complex multivariate modeling. The essential idea of the copula approach is that a joint distribution can be factored into the marginals and a dependence function called a copula. The term copula comes the Latin language and means link: the copula couples the marginal distributions together to form a joint distribution. The dependence relationship is entirely determined by the copula, while scaling and shape (mean, standard deviation, skewness, and kurtosis) are entirely determined by the marginals.

Copulas can be useful for combining risks when the marginal distributions are estimated individually: marginals are initially estimated separately and can then be combined in a joint density by a copula that preserves the characteristics of the marginals.
Copulas can also be used to obtain more realistic multivariate densities than the traditional joint normal, which is simply the product of a normal copula and normal marginals (as we will see in the next section 3.1.1: for example, the normal dependence relation can be preserved using a normal copula, but marginals can be entirely general (for example, a normal copula with one T-student marginal and one logistic marginal). In this case we have the so-called meta joint distribution functions, whose theoretical background has been studied recently by Embrechts (2001) and Fang and Fang (2002).

3.1.1 A Brief Review of Copula Theory

In what follows, the definition of a copula function and some of its basic properties are given, while the reader interested in a more detailed treatment is referred to Nelsen (1999) and Joe (1997).
An n-dimensional copula is a multivariate cumulative distribution function with uniform distributed margins in [0,1]. We now recall its definition, following Joe(1997) and Nelsen (1999).
Let consider $X_1, \ldots X_n$ to be random variables, and H their joint distribution function, then we have:

Definition 3.1 (Copula) *A copula is a multivariate distribution function H of random variables $X_1 \ldots X_n$ with standard uniform marginal distributions $F_1, \ldots, Fn$, defined on the unit n-cube $[0,1]^n$, with the following properties:*

1. *The range of $C\ (u_1, u_2, \ldots, u_n)$ is the unit interval [0,1];*
2. *$C\ (u_1, u_2, \ldots, u_n) = 0$ if any $u_i = 0$, for $i = 1, 2, \ldots, n$.*
3. *$C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$, for all $u_i \in [0,\ 1]$*

The previous three conditions provides the lower bound on the distribution function and ensures that the marginal distributions are uniform.
The Sklar's theorem justifies the role of copulas as dependence functions.

Theorem 3.1 (Sklar's theorem) *Let H denote a n-dimensional distribution function with margins $F_1 \ldots F_n$. Then there exists a n-copula C such that for all real ($x_1, \ldots, x_n$)*

$$H(x_1, \ldots, x_n) = C(F(x_1), \ldots, F(x_n)) \tag{3.14}$$

If all the margins are continuous, then the copula is unique; otherwise C is uniquely determined on $RanF_1 \times RanF_2 \ldots RanF_n$, where Ran is the range of the marginals. Conversely, if C is a copula and $F_1, \ldots F_n$ are distribution functions, then the function H defined in (2.2) is a joint distribution function with margins $F_1, \ldots F_n$.

Proof: See Sklar (1959), Joe(1997) or Nelsen (1999). ■

The last statement is the most interesting for multivariate density modelling, since it implies that we may link together any $n \geq 2$ univariate distributions, of any type (not necessarily from the same family), with any copula in order to get a valid bivariate or multivariate distribution.

Corollary 3.1 *Let $F_1^{(-1)}$, ..., $F_n^{(-1)}$ denote the generalized inverses of the marginal distribution functions, then for every $(u_1$, ..., $u_n)$ in the unit n-cube, exists a unique copula C : [0,1] $\times \ldots \times$ [0,1] $\rightarrow$ [0,1] such that*

$$C(u_1, \ldots, u_n) = H(F_1^{(-1)}(u_1), \ldots, F_n^{(-1)}(u_n)) \tag{3.15}$$

Proof: See Nelsen (1999), Theorem 2.10.9 and the references given therein. ■
From this corollary we know how to extract a copula out of a given joint distribution.
By applying Sklar's theorem and the previous corollary, we can derive the multivariate copula density $c(F_1(x_1),, \ldots, F_n(x_n))$, associated to a copula function $C(F_1(x_1),, \ldots, F_n(x_n))$:

$$f(x_1, ..., x_n) = \frac{\partial^n \left[C(F_1(x_1), \ldots, F_n(x_n))\right]}{\partial F_1(x_1), \ldots, \partial F_n(x_n)} \cdot \prod_{i=1}^{n} f_i(x_i) = c(F_1(x_1), \ldots, F_n(x_n)) \cdot \prod_{i=1}^{n} f_i(x_i)$$

and therefore we get

$$c(F_1(x_1),\ ...,\ F_n(x_n)) = \frac{f(x_1, ..., x_n)}{\prod_{i=1}^{n} f_i(x_i)} \cdot\ , \tag{3.16}$$

3.1.2 Families of Copulas: Elliptical Copulas

The class of elliptical distributions provides useful examples of multivariate distributions because they share many of the tractable properties of the multivariate normal distribution. Furthermore, they allow to model multivariate extreme events and forms of non-normal dependencies. Elliptical copulas are simply the copulas of elliptical distributions (see Fang, Kotz, and Hg (1987) for a detailed treatment of Elliptical distributions).
We present two copulae belonging to the elliptical family and that will be later used in empirical applications, the Gaussian and T-copula. By using the procedure outlined in (3.16), we can derive their density functions.

1. The copula of the multivariate Normal distribution is the *Normal-copula*, whose probability density function is:

$$c(\Phi(x_1),\ \ldots,\ \Phi(x_n)) = \frac{f^{Normal}(x_1, \ldots, x_n)}{\prod_{i=1}^{n} f_i^{Normal}(x_i)} = \frac{1}{|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\zeta'(\Sigma^{-1} - I)\zeta\right) \tag{3.17}$$

   where $\zeta = (\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n))'$ is the vector of univariate Normal inverse distribution functions, $u_i = \Phi\ (x_i)$, while Σ is the correlation matrix.

2. On the other hand, the copula of the multivariate Student's T-distribution is the Student's *T-copula*, whose density function is:

$$\begin{aligned} c(t_\upsilon(x_1), \ldots, t_\upsilon(x_n)) &= \frac{f^{Student}(x_1, \ldots, x_n)}{\prod_{i=1}^{n} f_i^{Student}(x_i)} = \\ &= |\Sigma|^{-1/2} \frac{\Gamma\left(\frac{\upsilon+n}{2}\right)}{\Gamma\left(\frac{\upsilon}{2}\right)} \left[\frac{\Gamma\left(\frac{\upsilon}{2}\right)}{\Gamma\left(\frac{\upsilon+1}{2}\right)}\right]^n \frac{\left(1 + \frac{\zeta'\Sigma^{-1}\zeta}{\upsilon}\right)^{-\frac{\upsilon+n}{2}}}{\prod_{i=1}^{n}\left(1 + \frac{\zeta_i^2}{2}\right)^{-\frac{\upsilon+1}{2}}} \end{aligned} \tag{3.18}$$

where $\zeta = (t_\upsilon^{-1}(u_1), \ldots, t_\upsilon^{-1}(u_n))'$ is the vector of univariate Student's T inverse distribution functions, ν are the degrees of freedom, $u_i = t_\nu(x_i)$, while Σ is the correlation matrix.

Both these copulae belong to the class of Elliptical copulae [3]. An interesting extension that we will consider in our empirical analysis is the *Grouped-T copula* introduced in Daul, Giorgi, Lindskog, and McNeil (2003). The Grouped-T copula can be considered as a copula imposed by a kind of multivariate-T distribution where m distinct groups of assets have m different degrees of freedom. Like the previous two copulas, it can be easily applied to high dimensional portfolios. See Daul, Giorgi, Lindskog, and McNeil (2003) and Demarta and Mcneil (2005) for further details.

3.1.3 Families of copulas: Archimedean Copulas

An alternative to Elliptical copulae is given by *Archimedean copulae*: Archimedean copulae provide analytical tractability and a large spectrum of different dependence measure. These copulae can be used in a wide range of applications for the following reasons: a) The ease with which they can be constructed; b) The many parametric families of copulas belonging to this class; c) The great variety of different dependence structures. An Archimedean copula can be defined as follows:

Definition 3.2 (Archimedean copula) *Let consider a function φ : [0; 1] $\rightarrow$ [0;1] which is continuous, strictly decreasing φ'(u) < 0, convex φ"(u) > 0, and for which φ (0) = ∞ and φ (1) = 0. We then define the pseudo inverse of $\varphi^{[-1]}$: [0; ∞] $\rightarrow$ [0;1] such that :*

$$\varphi^{[-1]}(t) = \left\{ \begin{array}{ll} \varphi^{-1}(t) & for\ 0 \le t \le \varphi(0) \\ 0 & for\ \varphi(0) \le t \le \infty \end{array} \right\}$$

As φ is convex , the function C: [0; 1]^2 $\rightarrow$ [0;1] defined as

$$C(u_1, u_2) = \varphi^{-1}[\varphi(u_1) + \varphi(u_2)] \tag{3.19}$$

is an Archimedean copula and φ is called the "generator" of the copula. Moreover, if φ (0) = ∞ , the pseudo inverse describes an ordinary inverse function (that is $\varphi^{[-1]} = \varphi^{(-1)}$) and we call φ and C, a strict generator and a strict Archimedean copula, respectively.

The multivariate extension can be found in Embrechts, Lindskog, and McNeil (2003) as well as in Joe (1997): for all $n \geq 2$, the function C: [0; 1]^n $\rightarrow$ [0;1] defined as

$$C(u_1, \ldots, u_n) = \varphi^{-1}[\varphi(u_1) + \ldots + \varphi(u_n)] \tag{3.20}$$

is an n-dimensional Archimedean copula if and only if φ^{-1} is completely monotone on $[0, \infty)$.
Among the different Archimedean copulas, we will make use of the *Clayton (or Cook Johnson) copula*, which corresponds to copula B4 in Joe(1997). We do this choice for the following reasons:

[3]See Cherubini et al. (2004) for more details

- It possess positive lower "tail dependence". This is a measure of dependence between random variables in the extreme lower joint tails. Informally, lower tail dependence measure the probability of an extremely large negative return on one asset, given that another asset has yielded an extremely large negative return[4].
- The simulation algorithm can be easily implemented for high dimensional portfolios, differently from all other Archimedean copulas[5].

If we consider the generator $\varphi(t) = (t^{-\alpha} - 1)/\alpha$, with $\alpha \in [-1, \infty)\backslash\{0\}$ and inverse $\varphi^{-1}(t) = (1+t)^{-1/\alpha}$, by using (3.20) we get,

$$C(u_1, \dots, u_n) = \max \left[\left(\sum_{j=1}^{N} u_j^{-\alpha} - n + 1 \right)^{-1/\alpha} , \quad 0 \right] \tag{3.21}$$

However, if $\alpha > 0$ then we have $\varphi(0) = \infty$, and the above expression becomes

$$C(u_1, \dots, u_n) = \left(\sum_{j=1}^{N} u_j^{-\alpha} - n + 1 \right)^{-1/\alpha} \tag{3.22}$$

3.2 A Unified Approach

3.2.1 The CCC and DCC models Restated

Our goal is to show that the CCC and DCC models can be easily represented as special cases within a more general copula framework.
Particularly, the multivariate normal likelihood can be decomposed in the same way as (3.12-3.13) by using the procedure outlined in (3.17), that is considering the joint normal density function as the by product of a *normal copula* with correlation matrix $\Sigma = R_t$ together with *normal marginals*:

$$f^{Normal}(x_1, \dots, x_n) = c^{Normal}(F_1^{Normal}(x_1), \ \dots, \ F_n^{Normal}(x_n); R_t) \cdot \prod_{i=1}^{n} f_i^{Normal}(x_i) \tag{3.23}$$

where F_i^{Normal} is the normal cumulative density function. If we use the notation in (3.1-3.2), the two models can be restated as follows:

$$X_t = \mathrm{E}\left[X_t | \mathfrak{F}_{t-1}\right] + D_t \eta_t \tag{3.24}$$

$$\eta_t \sim H(\eta_1, \dots, \eta_n) \equiv C^{Normal}(F_1^{Normal}(\eta_1), \dots, F_n^{Normal}(\eta_n); R_t) \tag{3.25}$$

where $D_t = diag(h_{11,t}^{1/2} \dots h_{nn,t}^{1/2})$ and we used Sklar's Theorem.
Furthermore the two step DCC estimation procedure highlighted in (3.12-3.13), corresponds exactly to the *Inference for Margins (IFM)* method first proposed by Joe and Xu (1996) for copula estimation. According to the IFM method, the parameters of the marginal distributions are estimated in a first stage, while the parameters of the copula

[4] The Students T copula generates symmetric tail dependence, i.e. both lower and upper tail dependence, while the normal copula generates zero tail dependence, instead. See Joe (1997) and Cherubini, Vecchiato, and Luciano (2004) for more details.

[5] See chapter 6 in Cherubini, Vecchiato, and Luciano (2004) for more details about copula simulation.

are estimated separately in a second stage. Like the one-step ML estimator it verifies the properties of asymptotic normality, but the covariance matrix must be modified (Joe and Xu, 1996, Joe (1997)):

$$\sqrt{T}(\hat{\theta}_{IFM} - \theta_0) \rightarrow N(0, V(\theta_0)) \tag{3.26}$$

where θ_0 is a vector of marginals and copula parameters, $V(\theta_0) = \mathbf{D}^{-1}\mathbf{M}\ (\mathbf{D}^{-1})^\top$ is the so called "Godambe" Information Matrix, where $\mathbf{D} = \mathbf{E}[\partial \mathrm{g}(\theta)^\top / \partial\ \theta]$, $\mathbf{M} = \mathbf{E}\ [\mathrm{g}(\theta)^\top \mathrm{g}(\theta)]$, and $\mathrm{g}(\theta)$ is the score function. It is no surprise that this asymptotic result corresponds to the one reported in Engle and Sheppard (2001) for the two step DCC estimation.

Therefore, if we consider the *CCC model*, this implies estimating n univariate GARCH models of any type with a normal distribution at a first stage. The *normal cumulative* distributions functions of the standardized residuals $u_{i,t} = \Phi\ (\eta_{i,t})$ are then used as arguments within the normal copula density (3.17) with constant correlation matrix $R_t = R$. However, since $\zeta_t = (\Phi^{-1}(u_{1,t}), \ldots, \Phi^{-1}(u_{n,t}))'$ in (3.17) is a vector of univariate *normal inverse* distribution functions, the estimated constant correlation matrix corresponds to the estimated correlation matrix of the standardized residuals of the CCC model.
In a similar fashion, if we consider a *DCC model* instead, the normal cumulative d.f. and inverse functions cancel each other and the log-likelihood of the copula density is maximized assuming a dynamic structure for the correlation matrix R_t given by (3.9-3.10).

3.2.2 Some extensions: Skew-T Marginals and Dynamic Copulas

It is clear from the previous section that the copula approach enable us to consider far more general cases than the normal CCC and DCC models.
Two well known deviations from normality are fat tails and asymmetry. One marginal distribution that is used to allow for excess kurtosis is the Student's T and it has been generalized to allow for skewness by Hansen (1994). Despite other generalizations have been proposed, we chose this one due to its simplicity and its past success in modelling economic variables (Patton (2004a) - Patton (2004b), Jondeau and Rockinger (2003)) .

Definition 3.3 (Skew-T distribution) *Let y_t be a random variable which follows a conditional Skewed-T distribution with density function f(0, 1, ν_t, λ_t) with mean zero and variance one by construction, in order to be a suitable model for the standardized residuals of a conditional mean and variance model. The conditional parameters ν_t, λ_t control the kurtosis and skewness of the variable, respectively, and can be made time-varying. The density function is reported below (for more details, see Hansen 1994):*

$$\text{Skew}-\text{T} \quad f(y_t; 0, 1, \nu_t, \lambda_t) = \begin{cases} bc\left(1 + \frac{1}{\nu_t+2}\left(\frac{by_t+a}{1-\lambda_t}\right)^2\right)^{-(\nu_t+1)/2} & \text{for } y_t \leq -\frac{a}{b} \\ bc\left(1 + \frac{1}{\nu_t+2}\left(\frac{by_t+a}{1+\lambda_t}\right)^2\right)^{-(\nu_t+1)/2} & \text{for } y_t > -\frac{a}{b} \end{cases}$$

where,

$$c = \frac{\Gamma\left(\frac{\nu_t+1}{2}\right)}{\Gamma\left(\frac{\nu_t}{2}\right)\sqrt{\pi(\nu_t-2)}}$$

$$
\begin{aligned}
a &= 4\lambda_t c \left(\frac{\nu_t-2}{\nu_t-1}\right) \\
b &= \sqrt{1+3\lambda_t^2 - a^2}
\end{aligned}
$$

This density is defined for $2< \nu_t < \infty$ and $-1 < \lambda_t <+1$. Moreover, this density encompasses a large set of conventional densities:

i.) if λ_t=0, the Skew-t reduces to the traditional Student-T distribution.

ii.) If ν_t=∞ we have the skew-normal density.

iii.) If λ_t=0 and ν_t=∞ we have the normal density.

Similarly to Student's T, given the restriction $\nu_t > 2$, this distribution is well defined and its second moment exists, while skewness exists if $\nu_t > 3$ and kurtosis is defined if $\nu_t > 4$. The parameter λ_t controls for skewness: If it is bigger than zero, we have positive skewness, while if it is smaller than zero the distribution is negative skewed. The cumulative distribution function (c.d.f.), the inverse-c.d.f. and relative proofs are reported in Appendix A.

Therefore, a multivariate model that allows for marginal skewness, kurtosis and normal dependence can be expressed as follows:

$$
\begin{aligned}
X_t &= \mathrm{E}\left[X_t|\mathfrak{F}_{t-1}\right] + D_t\eta_t & (3.27)\\
\eta_t &\sim H(\eta_1,\ldots,\eta_n) \equiv C^{Normal}(F_1^{Skew-T}(\eta_1),\ldots,F_n^{Skew-T}(\eta_n);R_t) & (3.28)
\end{aligned}
$$

where F_i^{Skew-T} is the cumulative distribution function of the marginal Skew-T, and R_t can be made constant or time-varying, as in the standard CCC and DCC models, respectively. If the financial assets present symmetric tail dependence, we can use a Student's T copula, instead,

$$
\begin{aligned}
X_t &= \mathrm{E}\left[X_t|\mathfrak{F}_{t-1}\right] + D_t\eta_t & (3.29)\\
\eta_t &\sim H(\eta_1,\ldots,\eta_n) \equiv C^{Student's\,T}(F_1^{Skew-T}(\eta_1),\ldots,F_n^{Skew-T}(\eta_n);R_t,\nu) & (3.30)
\end{aligned}
$$

where ν are the Student's T copula degrees of freedom, while if the financial assets may be separated in m distinct groups we can use a Grouped T copula:

$$
\begin{aligned}
X_t &= \mathrm{E}\left[X_t|\mathfrak{F}_{t-1}\right] + D_t\eta_t & (3.31)\\
\eta_t &\sim H(\eta_1,\ldots,\eta_n) \equiv C^{Grouped\,T}(F_1^{Skew-T}(\eta_1),\ldots,F_n^{Skew-T}(\eta_n);R_t,\nu_1,\ldots,\nu_m) & (3.32)
\end{aligned}
$$

Finally, if they show lower tail dependence only, we can use a Clayton copula, instead,

$$
\begin{aligned}
X_t &= \mathrm{E}\left[X_t|\mathfrak{F}_{t-1}\right] + D_t\eta_t & (3.33)\\
\eta_t &\sim H(\eta_1,\ldots,\eta_n) \equiv C^{Clayton}(F_1^{Skew-T}(\eta_1),\ldots,F_n^{Skew-T}(\eta_n);\alpha_t) & (3.34)
\end{aligned}
$$

where α is the Clayton dependence parameter, which can be made time varying.
Similar approaches are proposed in Patton (2004b), Rockinger and Jondeau (2005) and Granger, Patton, and Terasvirta (2005). However, these papers focus on bivariate applications only, and no Value at Risk measurement is made.

4 The Empirical analysis

4.1 Models Specification and Cases Examined

The goal of this work is to evaluate what are the main determinants when doing VaR forecasts for a portfolio of assets. Based on our previous analysis, we consider seven elements:

1. The choice of the *marginals distribution*: we compare the standard Normal and the standardized Skew-T.

2. The specification of the *conditional moments of the marginals*, ranging from the mean till the kurtosis. As for the mean we compare a simple constant mean specification and an Auto-Regressive model of order one. As for the variance, a simple constant variance specification and a Threshold-GARCH(1,1) model to take the leverage effect into account, see Glosten, Jaganathan, and Runkle (1993). When working with the Skew-T distribution, we can specify a dynamic model for the conditional skewness parameter and/or the conditional degrees of freedom, as well. We propose here a specification similar to Hansen(1994) and Rockinger and Joundeau (2005).

   Therefore, a general AR(1)-Threshold GARCH(1,1) model with dynamic skewness and kurtosis, for the continuously compounded returns $y_t = 100[\log(P_t) - \log(P_{t-1})]$, is given by:

$$
\begin{aligned}
y_t &= \mu + \phi_1 \; y_{t-1} + \varepsilon_t && (4.1)\\
\varepsilon_t &= \eta_t \sqrt{h_t}, \;\; \eta_t \overset{i.i.d.}{\sim} \text{Skew} - \text{T}(0, 1, \nu_t, \lambda_t) && (4.2)\\
h_t &= \omega + \alpha \varepsilon_{t-1}^2 + \gamma \varepsilon_{t-1}^2 D_{t-1} + \beta h_{t-1} && (4.3)\\
\lambda_t &= \Lambda \left(\zeta + \delta \cdot \varepsilon_{t-1} \right) && (4.4)\\
\nu_t &= \Gamma \left(\theta + \tau \cdot \varepsilon_{t-1} \right) && (4.5)
\end{aligned}
$$

   where $D_{t-1} = 1$ if $\varepsilon_{t-1} < 0$, and 0 otherwise, $\Lambda(\cdot)$ is a modified logistic transformation designed to keep the conditional skewness parameter λ_t in (-1, 1) at all times, while $\Gamma(\cdot)$ is a logistic transformation designed to keep the conditional degrees of freedom in (2, 30) at all times (see Hansen 1994). We avoid an autoregressive specification, in so far as it may lead to spuriously significant parameters (see Joundeau and Rockinger 2003, for a proof). Moreover, we tried different specifications, but with similar results and increased computational time. This is why we resort to this simple modeling.

3. The choice of the *type of copula*: we compare the Normal copula, the T-copula, the Grouped-T and the Clayton copula. As for the Grouped-T, we classify the assets in 7 groups according to their credit rating: 1) AAA; 2) AA (AA+,AA,AA−); 3) A (A+,A,A−); 4) BBB (BBB+,BBB,BBB−); 5) BB (BB+,BB,BB−); 6) B (B+,B,B−); 7) Not rated.

4. The specification of the *conditional copula parameters*: we consider a Normal copula both with a constant correlation matrix R and a dynamic R_t, where in the latter case we use a DCC(1,1) model as in (3.9-3.10). A T-copula with constant correlation matrix R and degrees of freedom ν, as well as with a dynamic R_t and constant ν[6]. As

[6] We discarded a dynamic specification for ν since the numerical maximization of the log-likelihood failed to converge most of the time or the dynamic coefficients were not significant.

for the Grouped-T, we consider the constant case only, since a dynamic specification has not been developed yet, and is a topic of current research. Finally, we consider for the Clayton Copula a dynamic specification, where the parameter α_t follows something akin to a ARMA(1,1) process:

$$\alpha_t = e^{(\kappa+\rho\cdot\alpha_{t-1}+\xi\cdot\varepsilon_{t-1})} \tag{4.6}$$

where e is the exponential function, used to keep α in $(0, +\infty)$ at all times.

Besides we consider the effects of these additional elements, too:

5. The *degree of assets' riskiness*;
6. The *portfolio dimensionality*;
7. The *time sample used for VaR backtesting.*

In order to evaluate how important the cited *first four* elements are, we generate 1-day out-of-sample VaR forecasts by using the following conditional multivariate distributions:

i. *Constant Normal copula + Normal marginals* with Constant mean and Constant Variance;

ii. *Constant Normal copula + Normal marginals* with Constant mean and a T-GARCH(1,1) for the Variance;

iii. *Constant Normal copula + Normal marginals* with an AR(1) specification for the mean and a T-GARCH(1,1) for the Variance;

iv. *DCC(1,1) Normal copula + Normal marginals* with an AR(1) specification for the mean and a T-GARCH(1,1) for the Variance;

v. *Constant Normal copula + Skew-T marginals* with an AR(1) specification for the mean and a T-GARCH(1,1) for the Variance, Constant Skewness parameter, and Constant Degrees of Freedom;

vi. *Constant Normal copula + Skew-T marginals* with an AR(1) specification for the mean and a T-GARCH(1,1) for the Variance, Dynamic Skewness parameter, and Dynamic Degrees of Freedom;

vii. *DCC(1,1) Normal copula + Skew-T marginals* with an AR(1) specification for the mean and a T-GARCH(1,1) for the Variance, Dynamic Skewness parameter, and Dynamic Degrees of Freedom;

viii. *Constant T-copula + Skew-T marginals* with an AR(1) specification for the mean and a T-GARCH(1,1) for the Variance, Constant Skewness parameter, and Constant Degrees of Freedom;

ix. *Constant T-copula + Skew-T marginals* with an AR(1) specification for the mean and a T-GARCH(1,1) for the Variance, Dynamic Skewness parameter, and Dynamic Degrees of Freedom;

x. *DCC(1,1) T-copula + Skew-T marginals* with an AR(1) specification for the mean and a T-GARCH(1,1) for the Variance, Constant Skewness parameter, and Constant Degrees of Freedom;

xi. *DCC(1,1) T-copula + Skew-T marginals* with an AR(1) specification for the mean and a T-GARCH(1,1) for the Variance, Dynamic Skewness parameter, and Dynamic Degrees of Freedom;

xii. *Constant Grouped T copula + Skew-T marginals* with an AR(1) specification for the mean and a T-GARCH(1,1) for the Variance, Constant Skewness parameter, and Constant Degrees of Freedom[7];

xiii. *Dynamic Clayton copula + Skew-T marginals* with an AR(1) specification for the mean and a T-GARCH(1,1) for the Variance, Constant Skewness parameter, and Constant Degrees of Freedom;

In order to evaluate the *fifth element*, i.e. the portfolio riskiness, we apply the previous multivariate models to two different portfolios:

I. A risky portfolio composed of two assets rated BB by Standard and Poor's during the time sample when we computed the out-of-sample VaR forecasts;

II. A less risky portfolio composed of two assets rated AAA by Standard and Poor's during the time sample when we computed the out-of-sample VaR forecasts;

In order to evaluate the *sixth* element, i.e. the portfolio dimensionality, we examined also two large portfolios:

a. A portfolio composed of thirty Dow-Jones stocks;

b. A portfolio composed of one hundred Nasdaq stocks[8]

Finally, in order to evaluate the *seventh* element, i.e. the time sample used for VaR backtesting, we consider for the forth portfolio only, these two different cases:

Sample A: A time sample of 1000 observations ranging between 01/12/1999 and 21/11/2003[9];

Sample B: A time sample of 1000 observations ranging between 13/12/2001 and 01/12/2005, that is after the burst of the high-tech bubble;

We expect a multivariate normal model to be sufficient for the second sample, while this should not be the case for the first one, when financial markets were in turmoil.

Descriptive graphs (level of portfolio value, daily returns, density of the daily returns vs. normal and QQ-plots against the normal distribution) for each portfolio are given in Figures 1-4. We suppose to invest 1$ in every asset for sake of simplicity.
The density graphs and the QQ-plot against the normal distribution show that all returns distributions exhibit asymmetric fat tails. Moreover, the returns graphs exhibit the usual volatility clustering observed in financial markets.

[7]The Grouped-T copula coincides with the T-copula for bivariate portfolios.
[8]The complete list of stocks for all four portfolios is reported in Appendix B.
[9]This sample was used for all four portfolios.

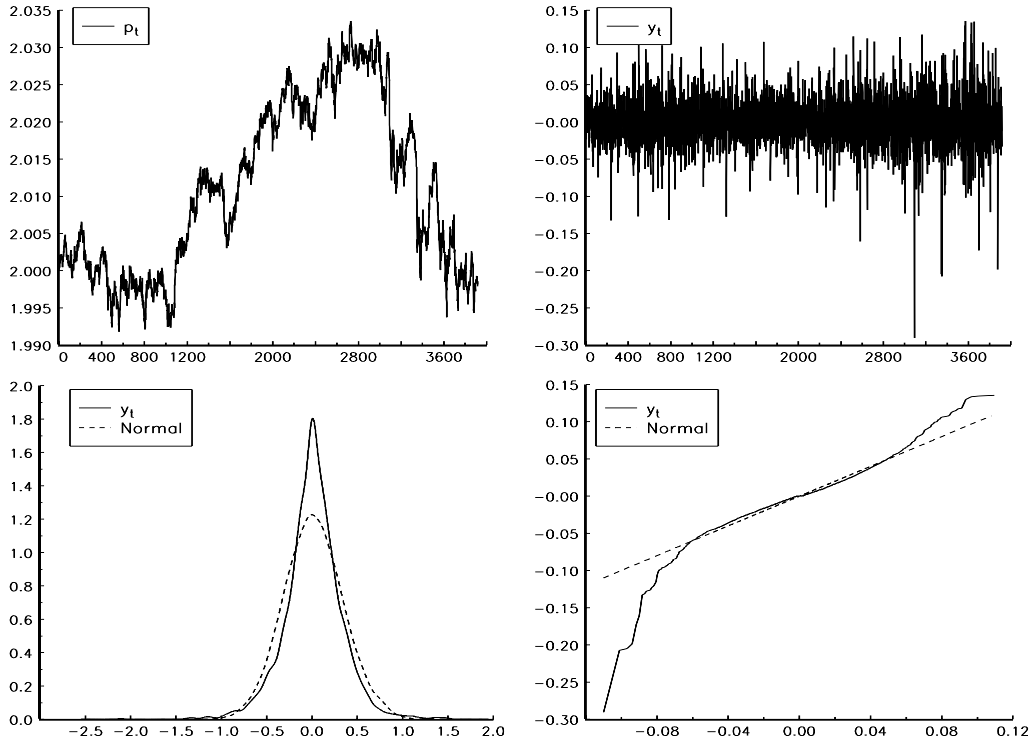

Figure 1: Portfolio value in level, daily returns, daily returns density (versus normal) and QQ-plot against the normal distribution (18/11/1988-20/11/2003). Portfolio composed of two assets rated BB.

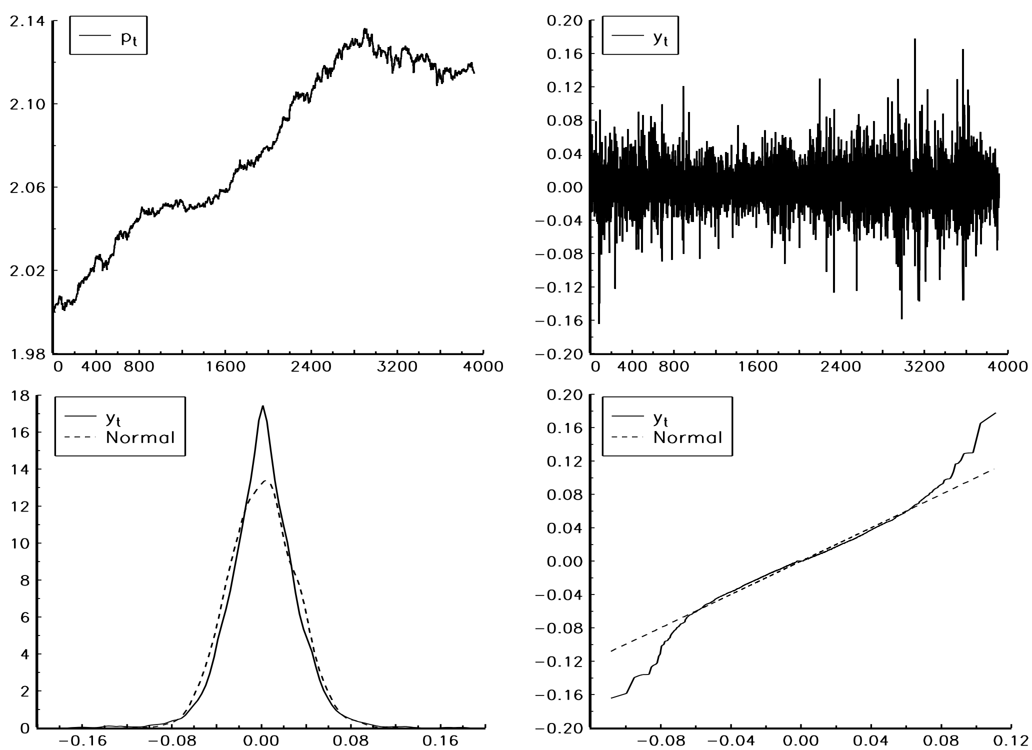

Figure 2: Portfolio value in level, daily returns, daily returns density (versus normal) and QQ-plot against the normal distribution (18/11/1988-20/11/2003). Portfolio composed of two assets rated AAA.

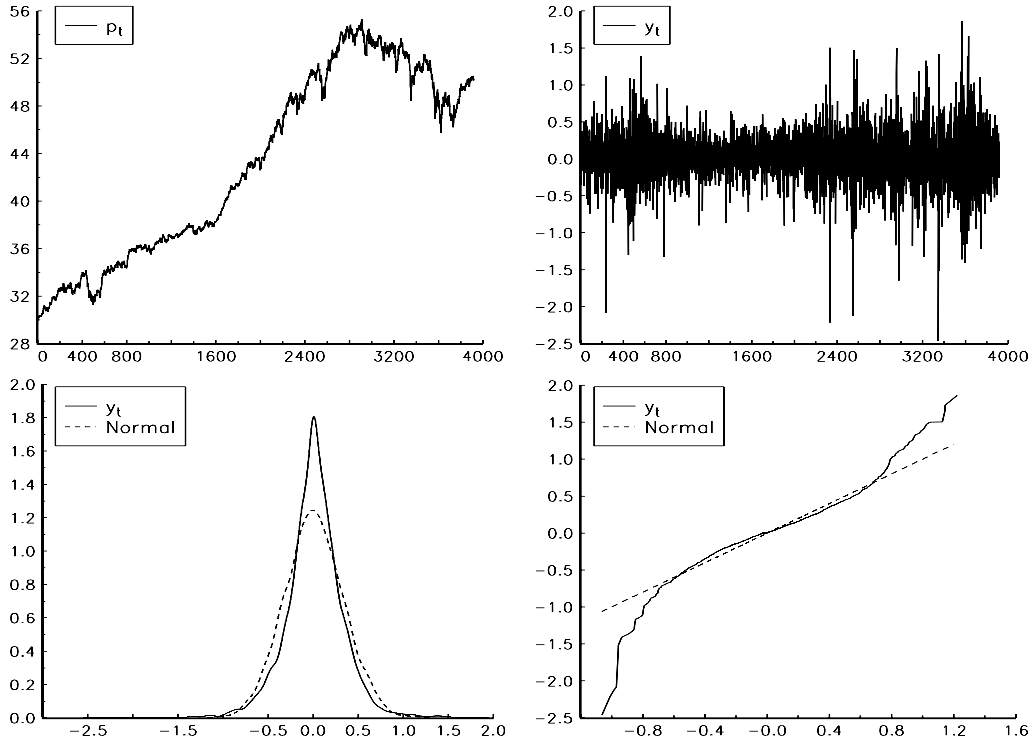

Figure 3: Portfolio value in level, daily returns, daily returns density (versus normal) and QQ-plot against the normal distribution (18/11/1988-20/11/2003). Portfolio composed of thirty Dow-Jones stocks.

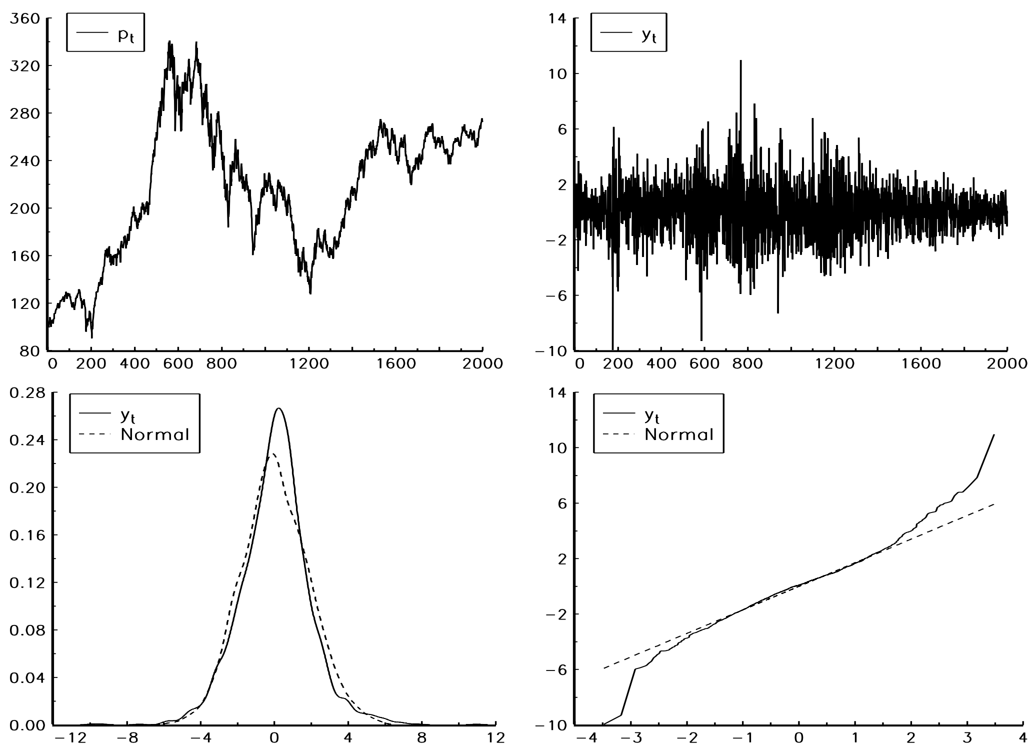

Figure 4: Portfolio value in level, daily returns, daily returns density (versus normal) and QQ-plot against the normal distribution (18/12/1997-30/11/2005). Portfolio composed of one hundred Nasdaq stocks.

4.2 VaR Estimation

We generate portfolio VaR forecasts at the 1 % - 5 % probability levels, that is VaR levels for long positions, and at the 95 % - 99 % probability levels, that is for short positions, too. The predicted one-step-ahead VaR forecasts are then compared with the observed portfolio losses and both results are recorded for later assessment.
A general algorithm for estimating the 1%, 5%, 95%, and 99% VaR over a one-day holding period for a portfolio P of n assets with invested positions equal to $M_i, i = 1, \ldots, n$, is the following:

1. Simulate j = 100,000 scenarios for each asset log-returns, $\{y_{1,t}, \ldots, y_{n,t}\}$, over the time horizon $[t-1, t]$, using a general multivariate distribution like in (3.14), by using this procedure:

   (a) First, *generate a n random variate* $(u_{1,t}, \ldots, u_{n,t})$ *from the copula* $\hat{C}_t$ forecasted at time t, which can be Normal, Student's T, Clayton, or Grouped-T. For a detailed review of copula simulation see Cherubini, Vecchiato, and Luciano (2004), while for the Grouped-T see Daul, Giorgi, Lindskog, and McNeil (2003).

   (b) Second, get a vector $n \times 1$ $\mathbf{Q_t}$ of *standardized asset log-returns* $\eta_{i,t}$ by using the inverse functions of the forecasted marginals at time t, which can be *Normal*, or *Skew-T*:

   $$\mathbf{Q_t} = (\eta_{1,t}, \ldots, \eta_{n,t}) = \left(F_1^{-1}(u_{1,t}; \hat{\alpha}_1), \ldots, F_n^{-1}(u_{n,t}; \hat{\alpha}_n)\right)$$

   (c) Third, *rescale the standardized assets log-returns* by using the forecasted means and variances, estimated with AR-GARCH models:

   $$\{y_{1,t}, \ldots, y_{n,t}\} = \left(\hat{\mu}_{1,t} + \eta_{1,t} \cdot \sqrt{\hat{h}_{1,t}}, \quad \ldots, \quad \hat{\mu}_{nt} + \eta_{n,t} \cdot \sqrt{\hat{h}_{n,t}}\right) \tag{4.7}$$

   (d) Finally, repeat this procedure for $j = 100,000$ times.

2. By using these 100,000 scenarios, the portfolio P is being revaluated at time t, that is:

   $$P_t^j = M_{1,t-1} \cdot \exp(y_{1,t}) + \ldots + M_{n,t-1} \cdot \exp(y_{n,t}), \quad j = 1...100,000 \tag{4.8}$$

3. Portfolio *Losses* in each scenario j are then computed[10]:

   $$\text{Loss}_j = P_{t-1} - P_t^j, \quad j = 1, \ldots, 100,000 \tag{4.9}$$

4. The calculus of the 1%, 5%, 95%, 99% VaR is now straightforward:

   a) We order the 100,000 Loss_j in increasing order ;

   b) 1% VaR is the 99000^{th} ordered scenario;

   c) 5% VaR is the 95000^{th} ordered scenario.

   b) 95% VaR is the 5000^{th} ordered scenario (i.e. the 5% VaR for short positions);

   c) 99% VaR is the 1000^{th} ordered scenario (i.e. the 1% VaR for short positions).

[10]Possible profits are considered as negative losses.

Finally, we compared the performance of the competing models over 1000 observations using the previously discussed back-testing techniques:

- Kupiec's unconditional coverage test;
- Christoffersen's conditional coverage test;
- Hansen's SPA test.
- Loss functions to evaluate VaR forecasts accuracy.

4.3 VaR Results

Tables 1-4 reports the real VaR exceedances N/T, the p-values p_{UC} of Kupiec's Unconditional Coverage test, and the p-values p_{CC} of Christoffersen's Conditional Coverage test, for the VaR forecasts at 1% - 5% - 95% - 99% probability levels, relative to the four considered portfolios:

1. A bivariate portfolio composed of two assets rated BB by S&P;
2. A bivariate portfolio composed of two assets rated AAA by S&P;
3. A portfolio composed of thirty Dow-Jones stocks;
4. A portfolio composed of one hundred Nasdaq stocks;

over a time sample of 1000 observations ranging between 01/12/1999 and 21/11/2003. Table 5 reports the same tests for the portfolio composed of one hundred Nasdaq stocks, but over a time sample of 1000 observations ranging between 13/12/2001 and 01/12/2005, instead.
Tables 6-10 in the appendix reports the two loss functions discussed in Section 2.2. Finally, tables 11-15 provides Hansen's SPA test consistent p-value as well as a lower (upper) bound for the true p-value. A low p-value (less that .05.10) is informative that the benchmark model is inferior to one or more of the competing models[11].

Models	**Long position**						**Short position**					
	1%			**5%**			**1%**			**5%**		
	N/T	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}
Model i.	4.20%	0.000	0.000	8.90%	0.000	0.000	3.50%	0.000	0.000	7.90%	0.000	0.000
Model ii.	3.00%	0.000	0.000	6.80%	0.013	0.026	1.60%	0.079	0.166	5.10%	0.885	0.911
Model iii.	3.00%	0.000	0.000	6.80%	0.013	0.026	1.60%	0.079	0.166	5.10%	0.885	0.911
Model iv.	2.80%	0.000	0.000	6.70%	0.019	0.033	1.50%	0.139	0.266	4.90%	0.884	0.952
Model v.	2.50%	0.000	0.000	8.10%	0.000	0.000	1.00%	1.000	0.904	5.50%	0.475	0.419
Model vi.	2.70%	0.000	0.000	7.90%	0.000	0.000	0.90%	0.746	0.875	5.60%	0.393	0.406
Model vii.	2.40%	0.000	0.001	7.70%	0.000	0.001	1.00%	1.000	0.904	5.30%	0.666	0.708
Model viii.	2.50%	0.000	0.000	8.10%	0.000	0.000	0.90%	0.746	0.875	5.50%	0.475	0.419
Model ix.	2.60%	0.000	0.000	8.00%	0.000	0.000	0.90%	0.746	0.875	5.60%	0.393	0.406
Model x.	2.30%	0.000	0.002	7.50%	0.001	0.003	0.90%	0.746	0.875	5.30%	0.666	0.708
Model xi.	2.40%	0.000	0.001	7.50%	0.001	0.002	1.00%	1.000	0.904	5.20%	0.773	0.707
Model xiii.	2.40%	0.000	0.001	7.90%	0.000	0.000	1.40%	0.231	0.400	6.60%	0.027	0.062

Table 1: Kupiec's and Christoffersen's tests for the portfolio rated **BB** by S&P.

[11]We do not report the parameter estimates for the AR, T-GARCH, or DCC models estimated as these are not of direct interest. The complete set of results is available from the authors upon request.

Models	Long position						Short position					
	1%			5%			1%			5%		
	N/T	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}
Model i.	3.30%	0.000	0.000	8.30%	0.000	0.000	2.70%	0.000	0.000	7.20%	0.003	0.005
Model ii.	1.70%	0.043	0.096	5.50%	0.475	0.775	1.90%	0.011	0.027	5.00%	1.000	0.959
Model iii.	1.80%	0.022	0.053	5.50%	0.475	0.775	1.80%	0.022	0.047	4.90%	0.884	0.928
Model iv.	1.70%	0.043	0.096	5.40%	0.566	0.847	1.70%	0.043	0.077	4.70%	0.660	0.490
Model v.	1.50%	0.139	0.266	6.30%	0.069	0.166	1.40%	0.231	0.216	5.20%	0.773	0.948
Model vi.	1.60%	0.079	0.166	6.60%	0.027	0.084	1.20%	0.538	0.283	5.10%	0.885	0.966
Model vii.	1.50%	0.139	0.266	6.20%	0.093	0.243	1.30%	0.362	0.259	5.00%	1.000	0.673
Model viii.	1.50%	0.139	0.266	6.60%	0.027	0.065	1.40%	0.231	0.216	5.10%	0.885	0.966
Model ix.	1.50%	0.139	0.266	6.60%	0.027	0.084	1.30%	0.362	0.259	5.20%	0.773	0.948
Model x.	1.50%	0.139	0.266	6.20%	0.093	0.217	1.30%	0.362	0.259	5.00%	1.000	0.673
Model xi.	1.40%	0.231	0.400	6.20%	0.093	0.243	1.20%	0.538	0.283	4.90%	0.884	0.622
Model xiii.	1.40%	0.231	0.400	6.40%	0.051	0.149	1.50%	0.139	0.165	6.00%	0.159	0.364

Table 2: Kupiec's and Christoffersen's tests for the portfolio rated **AAA** by S&P.

Models	Long position						Short position					
	1%			5%			1%			5%		
	N/T	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}
Model i.	4.10%	0.000	0.000	9.70%	0.000	0.000	3.40%	0.000	0.000	9.30%	0.000	0.000
Model ii.	2.40%	0.000	0.001	6.30%	0.069	0.192	1.50%	0.139	0.165	5.80%	0.257	0.499
Model iii.	2.40%	0.000	0.001	6.20%	0.093	0.243	1.50%	0.139	0.165	5.80%	0.257	0.210
Model iv.	2.20%	0.001	0.004	5.80%	0.257	0.514	1.20%	0.538	0.706	5.00%	1.000	0.959
Model v.	2.10%	0.002	0.007	6.70%	0.019	0.061	1.10%	0.754	0.833	5.30%	0.666	0.202
Model vi.	2.20%	0.001	0.004	6.80%	0.013	0.044	1.20%	0.538	0.283	5.30%	0.666	0.908
Model vii.	1.80%	0.022	0.046	6.10%	0.122	0.300	1.00%	1.000	0.895	4.50%	0.461	0.621
Model viii.	1.90%	0.011	0.026	6.70%	0.019	0.061	1.00%	1.000	0.895	5.30%	0.666	0.776
Model ix.	1.90%	0.011	0.026	6.80%	0.013	0.044	1.00%	1.000	0.895	5.80%	0.257	0.365
Model x.	1.80%	0.022	0.046	6.20%	0.093	0.243	0.90%	0.746	0.867	4.90%	0.884	0.928
Model xi.	1.90%	0.011	0.026	6.30%	0.069	0.192	1.00%	1.000	0.895	5.10%	0.885	0.966
Model xii.	2.40%	0.000	0.001	7.10%	0.004	0.014	1.30%	0.362	0.549	5.50%	0.475	0.774
Model xiii.	2.70%	0.000	0.000	9.00%	0.000	0.000	7.00%	0.000	0.000	13.70%	0.000	0.000

Table 3: Kupiec's and Christoffersen's tests for the portfolio composed of **thirty Dow Jones stocks**

Models	Long position						Short position					
	1%			5%			1%			5%		
	N/T	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}
Model i.	3.20%	0.000	0.000	7.30%	0.002	0.002	1.00%	1.000	0.913	4.60%	0.557	0.735
Model ii.	2.50%	0.000	0.000	7.80%	0.000	0.000	0.50%	0.079	0.209	4.20%	0.233	0.416
Model iii.	2.50%	0.000	0.000	8.00%	0.000	0.000	0.50%	0.079	0.209	4.10%	0.178	0.355
Model iv.	2.60%	0.000	0.000	8.10%	0.000	0.000	0.40%	0.030	0.094	4.10%	0.178	0.073
Model v.	2.30%	0.000	0.001	8.10%	0.000	0.000	0.40%	0.030	0.094	4.10%	0.178	0.355
Model vi.	2.00%	0.005	0.014	7.60%	0.000	0.001	0.30%	0.009	0.033	3.20%	0.005	0.007
Model vii.	1.90%	0.011	0.028	8.00%	0.000	0.000	0.20%	0.002	0.008	3.60%	0.033	0.102
Model viii.	2.00%	0.005	0.014	8.10%	0.000	0.000	0.40%	0.030	0.094	3.80%	0.070	0.045
Model ix.	1.80%	0.022	0.054	7.80%	0.000	0.000	0.30%	0.009	0.033	3.30%	0.009	0.011
Model x.	1.90%	0.011	0.028	7.90%	0.000	0.000	0.20%	0.002	0.008	3.50%	0.022	0.073
Model xi.	1.90%	0.011	0.028	7.60%	0.000	0.001	0.20%	0.002	0.008	3.40%	0.014	0.050
Model xii.	2.10%	0.002	0.006	7.90%	0.000	0.000	0.20%	0.002	0.008	3.10%	0.003	0.005
Model xiii.	11.20%	0.000	0.000	22.30%	0.000	0.000	14.10%	0.000	0.000	21.30%	0.000	0.000

Table 4: Kupiec's and Christoffersen's tests for the portfolio composed of **one hundred Nasdaq stocks**, using for VaR backtesting the time sample 01/12/1999-21/11/2003

Models	Long position						Short position					
	1%			5%			1%			5%		
	N/T	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}	**N/T**	p_{UC}	p_{CC}
Model i.	0.70%	0.314	0.573	2.40%	0.000	0.000	0.00%	0.000	0.000	1.00%	0.000	0.000
Model ii.	1.20%	0.538	0.715	4.60%	0.557	0.702	0.10%	0.000	0.001	2.60%	0.000	0.000
Model iii.	1.10%	0.754	0.843	5.00%	1.000	0.949	0.10%	0.000	0.001	2.90%	0.001	0.002
Model iv.	1.20%	0.538	0.715	5.00%	1.000	0.949	0.10%	0.000	0.001	2.70%	0.000	0.001
Model v.	0.80%	0.510	0.755	5.10%	0.885	0.687	0.00%	0.000	0.000	2.60%	0.000	0.000
Model vi.	0.60%	0.170	0.376	4.00%	0.133	0.189	0.00%	0.000	0.000	1.70%	0.000	0.000
Model vii.	0.60%	0.170	0.376	5.20%	0.773	0.707	0.00%	0.000	0.000	2.60%	0.000	0.000
Model viii.	0.60%	0.170	0.376	5.10%	0.885	0.958	0.00%	0.000	0.000	2.70%	0.000	0.001
Model ix.	0.50%	0.079	0.208	4.10%	0.178	0.254	0.00%	0.000	0.000	1.70%	0.000	0.000
Model x.	0.70%	0.314	0.573	4.90%	0.884	0.916	0.00%	0.000	0.000	2.40%	0.000	0.000
Model xi.	0.60%	0.170	0.376	4.80%	0.770	0.861	0.00%	0.000	0.000	2.30%	0.000	0.000
Model xii.	0.60%	0.170	0.376	4.60%	0.557	0.702	0.00%	0.000	0.000	1.80%	0.000	0.000
Model xiii.	8.90%	0.000	0.000	19.10%	0.000	0.000	13.10%	0.000	0.000	19.90%	0.000	0.000

Table 5: Kupiec's and Christoffersen's tests for the portfolio composed of **one hundred Nasdaq stocks** using for VaR backtesting the time sample 13/12/2001-01/12/2005

The previous tables highlight elements of sure interests. If we consider the seven determinants discussed in section 4.1 the major insights can be summarized as follows:

1. Choice of the *marginals distribution*:

   (a) The multivariate normal with no dynamics at all is the worst model for almost all quantiles and portfolios, being either too aggressive or too conservative.

   (b) The Skew-T distribution usually presents the most precise VaR forecasts, according to the tests and Loss functions used. However, the normal distribution with a T-GARCH specification is sufficient to model all the quantiles for a investment grade portfolio, as well as the 1% and 5% quantiles for short positions for all the considered portfolios. This result is confirmed both by Kupiec's and Christoffersen's tests, and Hansen's SPA test.

2. Specification of the *conditional moments of the marginals*

   (a) The AR specification of the mean is not relevant in all cases;

   (b) The GARCH specification for the variance is absolutely fundamental to have good VaR forecasts, whatever the marginal distribution is;

   (c) Allowing for dynamics in the skewness and degrees of freedom parameters of the Skew-T produces more conservative VaR forecasts in almost all cases;

3. Choice of the *type of copula*:

   (a) The T-copula usually produces better results than the Normal copula according to loss functions. However, once Skew-T marginals and a T-GARCH specification are considered, these differences are no more statistically significant at the 5 % level by using the SPA test, and at the 10 % level or higher if a dynamic specification for the Normal copula is used. This evidence differs slightly from what reported in Chen, Fan, and Patton (2004), who developed two goodness-of-fit tests to compare alternative models of dependence and found strong evidence against the normal copula when the number of assets increases, but little evidence against the T copula. Two possible reasons may be that they filtered the raw returns using a simple GARCH(1,1) model without a leverage effect, together with a Normal distribution for the marginals. Our analysis highlighted that Skew-T modelling is a better choice when risky assets and volatile markets are considered. Moreover, Newey and Steigerwald (1997) showed that Quasi-ML estimators can give not consistent estimates under certain conditions, particularly when the true distribution is not symmetric.

(b) The Grouped-T compared slightly worse to the T-copula. This result may be due either to the lack of dynamic modelling or to the wrong choice of the criteria used to separate the assets in different groups, or both. We used the credit rating for its ease of use, but more refined methods like the cluster analysis can be considered. However, we left it as a topic of future research.

(c) The Clayton copula performed quite well with bivariate portfolios in the left tail of the distribution, as expected. When the number of assets increases is no more a viable choice.

4. Specification of the *conditional copula parameters*

   (a) Dynamic specification always provides better results, where the major gains are achieved when moving from a constant normal copula to a dynamic DCC(1,1) Normal copula, rather than when a T-copula is involved. This result together with the previous 3a) highlights that a T-copula is no more distinguishable from a Normal copula when dynamic Skew-T marginals and dynamic dependence is considered.

5. *Degree of assets' riskiness*;

   (a) The T-GARCH specification is sufficient to model most of the leptokurtosis in the data when dealing with investment grade assets. This is not the case for riskier assets and a Skew-T distribution is a better choice.

   (b) However, none of the considered models were able to pass Kupiec's and Christoffersen's tests for the 1% and 5% VaR (long positions), when the bivariate risky BB portfolio was examined.

6. *Portfolio dimensionality*;

   (a) When the number of assets increases, the use of a dynamic Skew-T marginal is required if the 1% and 5% VaR for long positions are of concern. Dynamic normal marginals are sufficient for short positions, instead.

   (b) A dynamic Normal copula is sufficient for long positions quantiles, while a simple constant normal copula can do the job when dealing with short positions.

7. *Time sample used for VaR backtesting.*

   (a) As for long positions, when the first sample 01/12/1999-21/11/2003 with high volatility was considered, the only models able to pass the Kupiec's and Christoffersen's test for the 1% VaR at the 1% level, were the ones with dynamic Skew-T marginals together with dynamic Normal copula or constant/dynamic T-copula. However, none of them passed the tests for the 5% VaR instead. On the other hand, multivariate Normal models proved to be sufficient when the second sample 13/12/2001-01/12/2005 with low volatility was examined.

(b) As for short positions, multivariate normal models proved to be sufficient when the first sample was considered, while almost all models (except the Clayton model) were too conservative for the second sample.

These results seem to point out that a Skew-T distribution with a T-GARCH(1,1) specification, constant skewness and constant degrees of freedom parameters, together with a dynamic DCC(1,1) Normal copula, should be a good compromise for precise VaR estimates when dealing with long positions. A multivariate normal model with dynamic marginals and a constant copula may be a better choice if short positions are of concern.

5 Conclusions

The goal of this paper has been that of comparing different approaches to model multivariate log-returns to find out what are the main determinants when doing VaR forecasts for a portfolio of assets. To achieve this goal, we introduced a general multivariate framework by means of copulas to unify past approaches and propose new extensions.
We found out that univariate skewness and kurtosis modelling play a crucial role when portfolio dimensionality and riskiness increase, and long positions are of concern. However, we showed that once a dynamic Skew-T model is taken into account, a dynamic Normal copula provides VaR estimates that are not statistically different from a constant or dynamic T-copula. Multivariate Normal models proved to be sufficient when short positions were of interest, instead.
Furthermore, we pointed out that none of the models was able to pass the tests for long positions VaR quantiles when risky portfolios were considered, both low and high dimensional ones: therefore, more flexible models are required. A possible venue for future research is to consider Levy processes for both marginal and copula modelling.
A second extension that can be made is to use more sophisticated methods to separate the assets into homogenous groups for a Grouped-T copula. Besides, a possible dynamic specification could be considered, too.
Finally, an alternative to DCC modelling for high-dimensional portfolios could be that of decomposing the joint distribution into many conditional bivariate distributions, and estimating them separately to improve computational tractability.

Note: This article is the result of the joint work of the three authors. However, Section 1 was written by Carta, De Giuli and Fantazzini jointly, Section 2 by DeGiuli and Fantazzini, Section 3 by Fantazzini, Section 4 by Fantazzini and Carta, while Section 5 by Fantazzini.

References

Basle Committee on Banking Supervision (1996): *Supervisory Framework for the Use of Backtesting in Conjunction with the Internal Models Approach to Market Risk Capital Requirements.*

Bauwens, L., and S. Laurent (2004): "A New Class of Multivariate Skew Densities, with Application to GARCH Models," *Journal of Business and Economic Statistics*, forthcoming.

Bauwens, L., S. Laurent, and J. Rombouts (2006): "Multivariate GARCH models: a survey," *Journal of Applied Econometrics*, forthcoming.

Blanco, C., and G. Ihle (1999): "How good is your VaR? Using backtesting to assess system performance," *Financial Engineering News*, August, 1-2.

BOLLERSLEV, T. (1990): "Modelling the Coherence in Short-run Nominal Exchange Rates: A Multivariate Generalized ARCH Model," *The Review of Economics and Statistics*, 72(3), 498–505.

BOUYE', E., V. DURRLEMAN, A. NIKEGHBALI, G. RIBOULET, AND T. RONCALLI (2001): "Copulas: an open field for risk management," Discussion paper, Groupe de Recherche Opérationnelle, Crédit Lyonnais.

CHEN, X., Y. FAN, AND A. PATTON (2004): "Simple Tests for Models of Dependence Between Multiple Financial Time Series with Applications to U.S. Equity Returns and Exchange Rates," Discussion Paper 483, Financial Markets Group, London School of Economics.

CHERUBINI, U., W. VECCHIATO, AND E. LUCIANO (2004): *Copula Methods in Finance*. Wiley.

CHRISTOFFERSEN, P. F. (1998): "Evaluating Interval Forecats," *International Economic Review*, 39, 841–862.

DAUL, S., E. D. GIORGI, F. LINDSKOG, AND A. MCNEIL (2003): "The grouped t-copula with an application to credit risk," *Risk*, 16, 73 – 76.

DEMARTA, S., AND A. J. MCNEIL (2005): "The t Copula and Related Copulas," *International Statistical Review*, 73, 111 – 129.

EMBRECHTS, P., F. LINDSKOG, AND A. J. MCNEIL (2003): "Modelling dependence with copulas and applications to risk management,," in *Handbook of heavy tailed distributions in finance*, ed. by R. ST. Elsevier/North-Holland, Amsterdam.

ENGLE, R. (2002): "Dynamic Conditional Correlation - A Simple Class of Multivariate GARCH Models," *Journal of Business and Economic Statistics*, 20, 339–350.

ENGLE, R. F., AND K. SHEPPARD (2001): "Theoretical and Empirical properties of Dynamic Conditional Correlation Multivariate GARCH," NBER Working Papers 8554, National Bureau of Economic Research, Inc.

GIOT, P., AND S. LAURENT (2003): "Value-at-Risk for Long and Short Positions," *Journal of Applied Econometrics*, 18, 641–664.

GLOSTEN, L. R., R. JAGANATHAN, AND D. RUNKLE (1993): "On the Relation between the Expected Value and the Volatility of the Normal Excess Return on Stocks," *Journal of Finance*, 48, 1779–1801.

GRANGER, C., A. PATTON, AND T. TERASVIRTA (2005): "Common Factors in Conditional Distributions for Bivariate Time Series," *Journal of Econometrics*, forthcoming.

HANSEN, B. E. (1994): "Autoregressive Conditional Density Estimation," *International Economic Review*, 35(3), 705–730.

HANSEN, P. (2005): "A Test for Superior Predictive Ability," *Journal of Business and Economic Statistics*, forthcoming.

HANSEN, P., AND A. LUNDE (2005): "A Forecast Comparison of Volatility Models: Does Anything Beat a GARCH(1,1)?," *Journal of Applied Econometrics*, forthcoming.

JOE, H., AND . J. XU (1996): "The estimation method of inference functions for margins for multivariate models," Discussion Paper 166, Department of Statistics, University of British Columbia.

JOE, U. (1997): *Multivariate Models and Dependence Concepts*. Chapman Hall, London.

JONDEAU, E., AND M. ROCKINGER (2003): "Conditional volatility, skewness, and kurtosis: existence, persistence, and comovements," *Journal of Economic Dynamics and Control*, 27, 1699 – 1737.

KUPIEC, P. (1995): "Techniques for verifying the accuracy of risk measurement models," *Journal of Derivatives*, 2, 173–184.

LOPEZ, J. (1998): "Methods for evaluating value-at-risk estimates," Federal Reserve Bank of New York research paper, n. 9802.

NEWEY, W., AND D. STEIGERWALD (1997): "Asymptotic Bias for Quasi Maximum Likelihood Estimators in Conditional Heteroskedasticity Models," *Econometrica*, 3, 587–599.

PATTON, A. (2004a): "Modelling Asymmetric Exchange Rate Dependence," *International Economic Review*, forthcoming.

——— (2004b): "On the Out-of-Sample Importance of Skewness and Asymmetric Dependence for Asset Allocation," *Journal of Financial Econometrics*, 2(1), 130 – 168.

ROCKINGER, M., AND E. JONDEAU (2005): "The Copula-GARCH Model of Conditional Dependencies: An International Stock-Market Application," *Journal of International Money and Finance*, forthcoming.

Appendix

A Loss functions tables

Models	**Long position**				**Short position**			
	1%		**5%**		**1%**		**5%**	
	Lopez(98)	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**
Model i.	42.04	15.44	89.10	47.88	79.20	47.73	35.13	18.82
Model ii.	30.02	8.46	68.06	30.72	51.16	30.06	16.11	12.25
Model iii.	30.02	8.49	68.06	30.63	51.16	30.09	16.11	12.36
Model iv.	28.02	7.71	67.06	28.85	49.16	28.74	15.11	11.69
Model v.	25.02	7.24	81.07	37.51	55.17	31.39	10.10	9.39
Model vi.	27.02	7.55	79.07	37.64	56.17	32.13	9.10	9.56
Model vii.	24.01	6.81	77.07	35.47	53.17	30.65	10.10	9.29
Model viii.	25.01	6.73	81.08	37.97	55.17	32.00	9.10	9.00
Model ix.	26.02	7.05	80.08	38.44	56.17	32.66	9.10	9.31
Model x.	23.01	5.86	75.07	35.48	53.17	30.27	9.09	8.66
Model xi.	24.01	6.40	75.07	35.65	52.17	31.07	10.09	8.82
Model xiii.	24.01	6.55	79.07	37.62	66.18	36.53	14.11	11.13

Table 6: Loss Functions for the portfolio rated **BB** by S&P.

Models	**Long position**				**Short position**			
	1%		**5%**		**1%**		**5%**	
	Lopez(98)	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**
Model i.	33.05	14.29	83.10	42.38	72.08	33.20	27.04	11.17
Model ii.	17.02	5.31	55.05	21.00	50.05	18.24	19.01	4.74
Model iii.	18.02	5.53	55.06	21.72	49.04	17.95	18.01	4.43
Model iv.	17.02	5.18	54.05	20.05	47.04	17.11	17.01	4.23
Model v.	15.02	4.96	63.06	25.33	52.05	18.15	14.01	2.90
Model vi.	16.02	5.50	66.06	26.97	51.05	18.16	12.01	2.84
Model vii.	15.02	4.54	62.06	23.22	50.04	17.19	13.01	2.66
Model viii.	15.02	4.61	66.06	25.67	51.05	18.54	14.01	2.66
Model ix.	15.02	4.68	66.06	25.52	52.05	18.42	13.01	2.72
Model x.	15.02	4.28	62.06	23.43	50.04	17.46	13.01	2.47
Model xi.	14.02	4.30	62.06	23.45	49.04	17.29	12.01	2.59
Model xiii.	14.02	3.94	64.06	23.96	60.05	20.84	15.01	3.73

Table 7: Loss Functions for the portfolio rated **AAA** by S&P.

Models	**Long position**				**Short position**			
	1%		**5%**		**1%**		**5%**	
	Lopez(98)	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**
Model i.	49.42	20.69	113.66	57.68	107.28	45.78	41.42	14.86
Model ii.	26.50	6.69	71.58	27.33	66.40	19.96	19.10	5.55
Model iii.	26.54	6.73	70.57	27.21	66.43	20.13	19.13	5.66
Model iv.	24.22	6.11	65.99	24.85	58.29	19.39	16.13	5.41
Model v.	23.41	6.46	76.37	30.50	61.20	19.29	14.59	4.42
Model vi.	24.30	6.35	77.08	29.87	61.40	19.74	15.69	4.62
Model vii.	19.78	5.18	69.07	25.57	53.03	17.91	13.53	4.25
Model viii.	20.91	5.37	76.40	31.12	61.50	20.27	13.30	3.85
Model ix.	20.92	5.39	77.32	30.92	66.55	20.35	13.42	4.04
Model x.	19.49	4.45	70.34	26.72	57.08	18.25	12.20	3.62
Model xi.	20.39	4.21	70.97	25.06	59.47	19.99	13.30	3.81
Model xii.	26.42	6.79	80.45	31.60	63.53	20.54	16.75	4.71
Model xiii.	29.79	7.12	103.19	47.80	153.83	70.57	79.89	27.61

Table 8: Loss Functions for the portfolio composed of **thirty Dow Jones stocks**

Models	Long position				Short position			
	1%		5%		1%		5%	
	Lopez(98)	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**
Model i.	136.91	9.60	359.00	35.99	40.82	2.37	139.22	12.40
Model ii.	81.65	5.26	289.61	28.79	23.43	1.34	106.29	8.99
Model iii.	82.70	5.28	290.96	28.48	24.89	1.48	109.61	9.32
Model iv.	78.86	4.72	285.18	28.03	18.50	1.09	103.48	8.87
Model v.	73.39	4.47	294.75	29.22	14.20	0.86	97.77	8.00
Model vi.	68.78	4.07	283.58	27.55	12.98	0.71	85.83	6.43
Model vii.	67.57	4.34	285.10	27.82	7.11	0.50	81.47	6.87
Model viii.	63.87	3.90	291.15	29.29	12.78	0.75	96.28	8.26
Model ix.	58.27	3.34	284.11	27.55	11.27	0.77	86.87	6.95
Model x.	60.26	3.73	276.24	27.12	8.75	0.59	83.21	7.07
Model xi.	59.61	3.67	270.78	26.66	8.64	0.58	81.75	6.97
Model xii.	67.75	3.96	289.19	27.51	9.33	0.60	82.51	6.73
Model xiii.	444.90	52.64	994.14	184.70	384.82	62.50	697.32	146.10

Table 9: Loss Functions for the portfolio composed of **one hundred Nasdaq stocks** using for VaR backtesting the time sample 01/12/1999-21/11/2003

Models	Long position				Short position			
	1%		5%		1%		5%	
	Lopez(98)	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**	**Lopez(98)**	**Blanco-Ihle(99)**
Model i.	15.52	1.39	70.36	7.92	0.00	0.00	13.99	1.48
Model ii.	23.03	1.67	102.75	13.23	1.01	0.02	33.86	3.78
Model iii.	22.89	1.81	108.71	13.81	1.00	0.02	37.12	4.05
Model iv.	23.68	1.75	108.61	13.79	1.02	0.04	35.55	4.22
Model v.	18.10	1.42	109.38	14.00	0.00	0.00	33.03	3.59
Model vi.	12.95	0.93	89.55	10.94	0.00	0.00	20.77	1.83
Model vii.	16.48	1.42	110.29	13.93	0.00	0.00	33.02	3.60
Model viii.	15.36	1.27	110.58	14.35	0.00	0.00	34.16	3.66
Model ix.	11.37	0.86	90.72	10.97	0.00	0.00	21.10	2.01
Model x.	16.79	1.34	107.44	13.81	0.00	0.00	31.01	3.55
Model xi.	15.65	1.31	105.46	13.52	0.00	0.00	29.82	3.46
Model xii.	14.37	1.15	98.90	11.88	0.00	0.00	22.99	2.45
Model xiii.	207.14	32.25	512.84	139.07	237.49	53.25	424.96	126.16

Table 10: Loss Functions for the portfolio composed of **one hundred Nasdaq stocks** using for VaR backtesting the time sample 13/12/2001-01/12/2005

Benchmark	Long position						Short position					
	1%			5%			1%			5%		
	Lower	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*
Model i.	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Model ii.	0.050	0.050	0.052	0.224	0.224	0.533	0.050	0.052	0.054	0.155	0.236	0.333
Model iii.	0.043	0.045	0.046	0.109	0.223	0.534	0.037	0.037	0.038	0.100	0.218	0.304
Model iv.	0.051	0.051	0.055	0.577	0.920	0.997	0.038	0.038	0.040	0.716	0.989	0.998
Model v.	0.007	0.008	0.008	0.001	0.001	0.001	0.157	0.157	0.406	0.028	0.028	0.032
Model vi.	0.104	0.108	0.117	0.002	0.002	0.002	0.053	0.053	0.070	0.035	0.035	0.039
Model vii.	0.092	0.104	0.113	0.009	0.010	0.011	0.043	0.043	0.055	0.074	0.074	0.098
Model viii.	0.130	0.194	0.313	0.002	0.003	0.003	0.157	0.673	0.753	0.044	0.044	0.047
Model ix.	0.152	0.231	0.266	0.006	0.006	0.006	0.071	0.072	0.124	0.042	0.042	0.044
Model x.	0.602	0.894	0.961	0.016	0.020	0.023	0.818	0.991	0.998	0.062	0.077	0.104
Model xi.	0.467	0.727	0.854	0.020	0.021	0.024	0.194	0.194	0.534	0.106	0.145	0.302
Model xiii.	0.426	0.667	0.721	0.004	0.005	0.005	0.032	0.032	0.036	0.000	0.000	0.000

Table 11: Hansen's SPA test for the portfolio rated **BB** by S&P.

Benchmark	**Long position**						**Short position**					
	1%			**5%**			**1%**			**5%**		
	Lower	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*
Model i.	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000	0.000
Model ii.	0.129	0.129	0.156	0.499	0.615	0.808	0.034	0.034	0.036	0.358	0.430	0.516
Model iii.	0.104	0.104	0.118	0.419	0.419	0.745	0.044	0.044	0.048	0.329	0.727	0.766
Model iv.	0.147	0.147	0.173	0.604	0.767	0.955	0.047	0.047	0.054	0.752	0.986	0.994
Model v.	0.301	0.489	0.546	0.015	0.017	0.019	0.208	0.208	0.233	0.238	0.238	0.279
Model vi.	0.186	0.207	0.250	0.013	0.013	0.013	0.566	0.758	0.869	0.215	0.215	0.282
Model vii.	0.303	0.410	0.484	0.010	0.011	0.012	0.212	0.212	0.326	0.190	0.208	0.248
Model viii.	0.314	0.539	0.590	0.007	0.007	0.007	0.249	0.290	0.396	0.241	0.261	0.319
Model ix.	0.331	0.378	0.463	0.010	0.010	0.010	0.352	0.548	0.634	0.183	0.183	0.234
Model x.	0.343	0.487	0.545	0.011	0.013	0.016	0.358	0.605	0.717	0.131	0.163	0.193
Model xi.	0.374	0.859	0.890	0.018	0.020	0.020	0.685	0.892	0.946	0.251	0.366	0.475
Model xiii.	0.564	0.989	0.990	0.007	0.007	0.009	0.137	0.158	0.186	0.003	0.003	0.003

Table 12: Hansen's SPA test for the portfolio rated **AAA** by S&P.

Benchmark	**Long position**						**Short position**					
	1%			**5%**			**1%**			**5%**		
	Lower	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*
Model i.	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Model ii.	0.025	0.025	0.026	0.104	0.116	0.160	0.026	0.028	0.031	0.005	0.005	0.005
Model iii.	0.037	0.038	0.039	0.126	0.130	0.293	0.027	0.027	0.028	0.009	0.009	0.009
Model iv.	0.046	0.048	0.055	0.672	0.964	0.998	0.054	0.063	0.074	0.044	0.087	0.121
Model v.	0.046	0.050	0.060	0.004	0.004	0.004	0.099	0.099	0.172	0.063	0.075	0.105
Model vi.	0.052	0.052	0.059	0.006	0.007	0.007	0.086	0.094	0.126	0.053	0.063	0.082
Model vii.	0.020	0.020	0.026	0.071	0.168	0.284	0.043	0.043	0.050	0.810	0.810	1.000
Model viii.	0.185	0.289	0.384	0.007	0.008	0.008	0.216	0.451	0.587	0.040	0.046	0.055
Model ix.	0.180	0.271	0.364	0.006	0.006	0.007	0.180	0.251	0.317	0.008	0.008	0.008
Model x.	0.725	0.952	0.998	0.046	0.052	0.085	0.669	0.943	0.995	0.032	0.078	0.118
Model xi.	0.258	0.266	0.632	0.062	0.073	0.098	0.186	0.438	0.633	0.025	0.033	0.040
Model xii.	0.026	0.028	0.028	0.004	0.004	0.004	0.059	0.059	0.072	0.016	0.017	0.018
Model xiii.	0.004	0.004	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 13: Hansen's SPA test for the portfolio composed of **thirty Dow Jones stocks**

Benchmark	**Long position**						**Short position**					
	1%			**5%**			**1%**			**5%**		
	Lower	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*
Model i.	0.002	0.002	0.002	0.004	0.004	0.004	0.075	0.075	0.107	0.007	0.007	0.007
Model ii.	0.004	0.004	0.004	0.016	0.018	0.020	0.096	0.115	0.143	0.006	0.006	0.006
Model iii.	0.009	0.009	0.009	0.014	0.015	0.015	0.091	0.093	0.116	0.007	0.007	0.007
Model iv.	0.024	0.026	0.027	0.044	0.065	0.081	0.099	0.099	0.128	0.001	0.001	0.001
Model v.	0.028	0.030	0.032	0.007	0.007	0.007	0.109	0.115	0.129	0.000	0.000	0.000
Model vi.	0.060	0.066	0.101	0.089	0.174	0.229	0.207	0.375	0.417	0.386	0.446	0.772
Model vii.	0.079	0.085	0.112	0.011	0.015	0.015	0.536	0.840	0.901	0.646	0.748	0.839
Model viii.	0.114	0.162	0.219	0.001	0.001	0.001	0.104	0.115	0.130	0.001	0.001	0.001
Model ix.	0.503	0.749	0.953	0.101	0.159	0.203	0.240	0.492	0.522	0.315	0.315	0.520
Model x.	0.023	0.029	0.032	0.004	0.005	0.007	0.223	0.482	0.529	0.211	0.244	0.356
Model xi.	0.377	0.671	0.779	0.649	0.974	0.998	0.313	0.793	0.808	0.778	0.916	0.958
Model xii.	0.060	0.066	0.095	0.063	0.077	0.087	0.252	0.760	0.793	0.660	0.889	0.972
Model xiii.	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 14: Hansen's SPA test for the portfolio composed of **one hundred Nasdaq stocks** using for VaR backtesting the time sample 01/12/1999-21/11/2003

Benchmark	Long position						Short position					
	1%			5%			1%			5%		
	Lower	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*	*Lower*	*Cons.*	*Upper*
Model i.	0.179	0.235	0.320	0.508	0.508	1.000	0.357	0.727	0.944	0.552	0.971	0.998
Model ii.	0.022	0.022	0.022	0.002	0.002	0.002	0.074	0.074	0.231	0.001	0.001	0.001
Model iii.	0.021	0.021	0.023	0.000	0.000	0.000	0.082	0.082	0.252	0.001	0.001	0.001
Model iv.	0.039	0.039	0.039	0.000	0.000	0.000	0.082	0.082	0.235	0.003	0.003	0.003
Model v.	0.077	0.077	0.084	0.000	0.000	0.000	0.375	0.726	0.943	0.003	0.003	0.003
Model vi.	0.096	0.242	0.310	0.005	0.006	0.014	0.376	0.743	0.944	0.071	0.078	0.164
Model vii.	0.106	0.108	0.156	0.001	0.001	0.001	0.360	0.707	0.928	0.003	0.003	0.003
Model viii.	0.086	0.093	0.144	0.000	0.000	0.000	0.356	0.721	0.934	0.004	0.004	0.004
Model ix.	0.795	0.997	0.997	0.017	0.017	0.023	0.359	0.707	0.939	0.078	0.078	0.174
Model x.	0.102	0.102	0.121	0.000	0.000	0.000	0.394	0.746	0.950	0.005	0.005	0.005
Model xi.	0.123	0.123	0.170	0.000	0.000	0.000	0.370	0.730	0.927	0.004	0.004	0.004
Model xii.	0.078	0.118	0.164	0.003	0.003	0.003	0.329	0.729	0.921	0.065	0.065	0.092
Model xiii.	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 15: Hansen's SPA test for the portfolio composed of **one hundred Nasdaq stocks** using for VaR backtesting the time sample 13/12/2001-01/12/2005

B List of Analyzed Stocks

- *Bivariate portfolio rated BB*: Eastman Kodak - General Motors.

- *Bivariate portfolio rated AAA*: Exxon Mobil - Microsoft.

- *Multivariate portfolio composed of 30 Dow Jones stocks*:

3M	At&.T	Alcoa & Altria Gp.	American Express	
Boeing	Caterpillar	Citigroup	Coca Cola	Du Pont
Eastman Kodak	Exxon Mobil	General Electric	General Motors	Hewlett - Packard
Home Depot	Honeywell Intl.	Intel	Intl.Bus.Mach.	Intl.Paper
Jp Morgan Chase	Johnson & Johnson	Mcdonalds	Merck	Microsoft
Procter & Gamble	Sbc Communications	United Technologies	Wal Mart Stores	Walt Disney

- *Multivariate portfolio composed of 100 Nasdaq stocks*:

Apple Computer	Adobe Systems Inc	Autodesk Inc	Altera Cp	Applied Materials
Amgen	Amazon.Com Inc	Amer. Power	Apollo Gp Inc	Ace Comm Corp
Bed Bath & Beyond	Bea Systems Inc	Biogen Idec Inc	Biomet Inc	Actel Cp
Cdw Corp	Acxiom	Celgene	Chiron	Check Point Software
C.H. Robinson	Comcast Cp	Comverse Tech Inc	Costco Wholesale	Cisco Sys Inc
Cintas	Authentidate Hldg	Citrix Systems	Dell Inc	Echostar Commun
Dollar Tree Store	Ade Corp	Lm Ericsson Adr	Electronic Arts	Express Scripts
Expeditors Intl	Fastenal Co	Fiserv Inc	Flextronics Intl	Genzyme
Gilead Sciences	Bell Microproduct	Iac-Interactive	Intel Cp	Intuit Inc
Inter Tel Inc	Wegener	Jds Uniphase	Captaris Inc	Kl A-Tencor
Lamar Advertis	Adc Telecommunicat	Linear Technology	Lincare Hldgs Inc	Lam Research
Adtran Inc	Microchip Tech	Applied Innovation	Medimmune Inc	Mercury Interact
Millennium Pharm	Molex Inc	Cobra Electronics	Microsoft	Maxim Integrated
Network Appliance	Audiovox	Novellus Systems	Oracle Corp	Paychex Inc
Paccar Inc	Patterson Companies	Pet Smart Inc	Pixar	Qualcomm Inc
Qlogic	Colt Telecom	Ross Stores Inc	Sanmina-Sci Corp	Starbucks
Siebel Systems	Amer Cap Strategie	D&E Communication	Sigma Aldrich	Assoc Banc Cp
Alabama Natl Bncp	Alfa	Staples Inc	Smurfit-Stone Cont	Sun Microsys Inc
Amcore Financial	Teva Pharm Inds	Tellabs Inc	Ameritrade Hldg	Whole Foods
Eci Telecom Ltd	Xilinx Inc	Amer Natl Ins	Dentsply Intl Inc	Yahoo Inc