Asset Prices and Asset Correlations in Illiquid Markets

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Abstract

We build a new asset pricing framework to study the effects of aggregate illiquidity on asset prices, volatilities and correlations. In our framework the Black-Scholes economy is obtained as the limiting case of perfectly liquid markets. The model is consistent with empirical studies on the effects of illiquidity on asset returns, volatilities and correlations. We present the model, study its qualitative properties and estimate stocks' sensitivities to aggregate liquidity (βs) using nine years data for 24 randomly sampled stocks traded on the NYSE. These sensitivity parameters (βs) determine the effect that aggregate illiquidity has on expected returns, volatilities, correlations, CAPM-betas and Sharpe ratios. We find clear capitalization and sector patterns for liquidity βs.

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1 Introduction

In recent years, considerable effort has been directed towards understanding the impact of illiquidity on asset pricing. There are two main streams of research: the first one is mainly devoted to the empirical assessment of the effects of illiquidity, while the second is mainly theoretical and tries to embody illiquidity into asset pricing theory. The literature also distinguishes between *aggregate* and *stock-specific* liquidity. The former is market-wide, fluctuates over time, and has different impacts on different stocks. The latter is a stock property relating to the fact that, for example, for some small capitalization stocks, bid/ask quotes might be unreliable due to unfrequent trading activity. The empirical literature is vast (see, among others, Amihud (2002), Amihud and Mendelson (1986), Chordia, Roll and Subrahmanyam (2000, 2001), Lo, Petrov and Wierbicki (2003), and Pastor and Stambaugh (2002)) and equally concerned with the two notions of liquidity, while the theoretical literature seems to be more devoted to studying the stock-specific liquidity either by explicitly modeling the price impact of trades of different size (for example, Cetin, Jarrow and Protter (2002), Papanicolaou and Sircar (1998), and Schoenbucher and Wilmott (2000)) or by introducing transaction costs in asset pricing models (for example, Lo, Mamaysky and Wang (2004), Vayanos (2004), and Acharya and Pedersen (2004)).

Since Mandelbrot's (1963) and Fama's (1965) work , it is a stylized fact that volatilities and correlations change over time. Aggregate illiquidity seems to be a major cause of excess comovement across assets. Practitioners know by experience that asset correlations and volatilities jump in illiquid markets. However, so far, models have failed to capture the dynamics of correlation arising from illiquidity to the point that:

> "*At some firms reliance on historical estimates of correlation and volatility are treated with skepticism, because of the simple fact that these historical estimates fail miserably in times of market stress*" (Bhansali and Wise, 2001).

There is a long list of stylized facts associated to illiquidity. Among others, Amihud and Mendelson (1987) and Amihud (2002) analyze the effects of stock-specific and aggregate illiquidity, respectively, on excess returns. They both find that excess returns increase with illiquidity. Pastor and Stambaugh (2002) provide evidence that aggregate liquidity tends to be low when market volatility is high. By measuring the Sharpe ratio of a sample portfolio before and after liquidity shocks, Lo, Petrov and Wierbicki (2003) show that the illiquidity-return property extends to Sharpe ratios.

The contribution of this paper is twofold.

First, to the best of our knowledge, this is the first theoretical model that studies the effects of aggregate liquidity/illiquidity on asset returns volatility and correlations. A recent paper by Longstaff (2004) introduces aggregate illiquidity in a continuous-time exchange economy with two assets and heterogenous agents. Stock specific and aggregate liquidity are also considered in

Acharya and Pedersen (2004). However, these papers do not focus on volatilities and correlations. Therefore, our results are complementary to those of Acharya and Pedersen (2004) and Longstaff (2004).

Second, our asset-pricing framework allows us to model the effects of aggregate illiquidity on returns, volatilities and correlations, while reproducing all the stylized facts mentioned above. Each stock has a different sensitivity to aggregate illiquidity (liquidity β). We embed the Black-Scholes economy in our model in two ways: first, if a stock's liquidity β is zero, that stock follows the usual lognormal price process. Second, in the fictitious case of a "perfectly liquid" economy, all securities follow lognormal processes. The stock's β quantifies the impact of aggregate illiquidity on returns, volatilities and correlations. Stocks with a higher β will have expected returns, volatilities, correlations, CAPM-betas and Sharpe ratios which fluctuate more with aggregate illiquidity. Moreover, the model can be easily estimated using standard GMM techniques. We analyze 24 randomly sampled stocks traded on the NYSE for the period January 1995 - September 2003. We study two stocks for every combination of market capitalization and sector. We consider large, medium and small capitalization and Consumer Discretionary, Industrials, Utilities and IT sectors. We find the following clear patterns: Consumer Discretionary, Industrials and Utilities exhibit a β that decreases in capitalization. That is, small caps stocks, as predictable, are more sensitive to market-wide liquidity. IT stocks exhibit the opposite pattern: larger stocks have higher βs. In terms of magnitude, IT has by far the highest liquidity βs, and Utilities the lowest.

The paper is related to the Arbitrage Pricing Theory literature. In fact, our theoretical model can be viewed as a three-factor model where the factors are, respectively, an idiosyncratic, a market and an aggregate liquidity factor. The paper is closely related to Pastor and Stambaugh (2002), who empirically investigate whether market liquidity is an important state variable for asset pricing, and to Acharya and Pedersen (2004) who developed a theoretical asset pricing model with liquidity risk. In our model, volatility, covariances and correlations depend on the liquidity factor. There is a growing literature on these issues - see for example, Ang, Hodrick, Xing and Zhang (2004).

In section 2 we present the model. For every stock, we derive the market clearing price by inverting a market-clearing condition (Proposition 1). In section 3 we study the properties of the derived price processes and show how the model fits the stylized facts mentioned above. In section 4 we describe moment conditions and the estimation technique, and in section 5 we construct a liquidity measure needed for the estimation. The data used for the empirical application are presented in section 6, and section 7 contains the estimation results. Section 8 contains brief conclusions. The proof of proposition 1 is in the appendix.

2 The Model

In this section we present the model. We introduce a reference probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$, with $0 \leq t \leq T$ for some fixed time T. We assume that the usual conditions on the coefficients of the stochastic differential equations below hold. We use S_i to denote the price of stock i. In the model, S_i is obtained through a market-clearing condition: $S_i(t)$ is the one and only price that makes the demand for security i clear supply at time t. The demand for stock i at time t depends on three factors:

1. The stock-specific information at time t (idiosyncratic factor).

2. The systematic factor (e.g. information about the market at time t).

3. The level of aggregate liquidity at time t.

More precisely, we make the following assumptions:

Assumption 1 *(Information Process)*
The information process for stock i follows

$$dI_i = \mu_i I_i dt + \eta_i I_i dW_i,$$

with

$$\mathbb{E}(dW_i dW_j) = \rho_{ij} dt.$$

Assumption 2 *(Liquidity Discount Factor)*
L_i is the "liquidity discount factor" for stock i and is defined by

$$L_i(t) = \exp(-\beta_i X(t)),$$

with

$$dX(t) = v(t)dt + v(t)dW_0.$$

v is a market liquidity measure defined below and β_i is a non-negative constant. We assume $dW_0 \perp dW_i$ for all i.

Assumption 3 *(Supply and Demand Functions and Market Clearing)*
The supply for stock i is fixed for simplicity and denoted by $\overline{S_i}$. The demand function has the form

$$D^i = \Psi^i \left(\frac{(I_i)^{\gamma_i}}{L_i S_i} \right), \ \Psi^i \text{ any smooth and increasing function,} \tag{1}$$

where $\gamma_i = \frac{k_i}{\eta_i} > 0$ *for some* $k_i > 0$*,* S_i *is the clearing price and* I_i *and* L_i *are defined above.*

The market clearing price S_i *is defined implicitly by the market-clearing condition*

$$\Psi^i \left(\frac{(I_i)^{\gamma_i}}{L_i S_i} \right) = \overline{S_i}. \tag{2}$$

Assumption 1 means that the stock's information process can be decomposed into an idiosyncratic factor and a market factor. For example, we can write

$$\eta_i dW_i = \eta_i \left(\alpha_i dW_i^{\perp} + \sqrt{1-\alpha_i^2} dW_{MKT} \right),$$

where $dW_i^{\perp}$ is orthogonal to $dW_j^{\perp}$ for $i \neq j$ and $dW_i^{\perp}$ is orthogonal to dW_{MKT} for each i, for some α_i. We are therefore assuming that the information for two stocks i and j, and hence the clearing prices of stocks i and j, can be correlated through the market factor. This assumption is very important for the model, since it will allow us to decompose correlation into *market correlation* and *illiquidity correlation*. The idiosyncratic factor can be easily linked to the stock specific dividend flow. Longstaff (2004) considers a two asset economy and each asset produces a stream of dividends which is assumed to follow a geometric Brownian motion. This is very similar to our assumption 1.

The effect of aggregate illiquidity is clear from Assumption 2 together with equation (2): the more $v(t)$ moves away from zero, the greater the perturbation in the market clearing equation (2). Ito's formula applied to L_i yields

$$dL_i(t) = L_i(t) \left(\left(-\beta_i v(t) + \frac{1}{2}\beta_i^2 v(t)^2 \right) dt - \beta_i v(t) dW_0 \right), \tag{3}$$

and another application of Ito's formula gives the solution to (3) as (for simplicity we put $L_i(0) = 1$)

$$L_i(t) = \exp\left\{ -\beta_i \left(\int_0^t v(s)ds + \int_0^t v(s)dW_0 \right) \right\}. \tag{4}$$

Assumption 3 allows us to invert the market clearing condition and solve for the price. Note that the "rational condition" (similarly, Papanicolaou and Sircar (1998) or Schoenbucher and Wilmott (2000)) that demand decreases in price holds (dropping indices for clarity):

$$D_S = -\frac{I^{\gamma}}{S^2 L}\Psi' < 0,$$

and that second order derivatives exist and are continuous. Hence, by restricting our analysis to the family of functions in (1), we can invert the equilibrium equation (2) and recover the equation satisfied by S. The family (1) has been previously used by Papanicolaou and Sircar

(1998), and has the property of yielding the Black-Scholes economy when markets are liquid (see Proposition 1 below). In our setting it has another nice property, namely, the clearing price does not contain Ψ or any derivatives of Ψ (see the proof of proposition 1 in the appendix). That is, *the market clearing price is independent of the specification of* Ψ.

The invertibility of Ψ allows us to solve for S, giving

$$S = \frac{1}{L}\left(\frac{I^{\gamma}}{\Psi^{-1}(\overline{S})}\right). \tag{5}$$

When $v(t) \equiv 0$ and $L \equiv 1$, (5) reduces to

$$S_{BS} = \left(\frac{I^{\gamma}}{\Psi^{-1}(\overline{S})}\right). \tag{6}$$

We will show that under assumptions 1, 2 and 3 the S_{BS} in (6) is a geometric Brownian motion, hence the Black-Scholes world is embedded in our model as the price corresponding to the $v(t) \equiv 0$ case. Furthermore, comparing (5) and (6), we see that

$$S = \frac{1}{L(t)} S_{BS}$$

and substitution using (4) gives

$$S = \exp\left\{\beta\left(\int_0^t v(s)ds + \int_0^t v(s)dW_0\right)\right\} S_{BS}. \tag{7}$$

We compute the following conditional expectation:

$$\begin{aligned}
\mathbb{E}\left(S|S_{BS}\right) &= \mathbb{E}\left\{\exp\left\{\beta\left(\int_0^t v(s)ds + \int_0^t v(s)dW_0\right)\right\} S_{BS}|S_{BS}\right\} \\
&= S_{BS}\exp\left\{\beta\int_0^t v(s)ds\right\}\mathbb{E}\left\{\exp\beta\left(\int_0^t v(s)dW_0\right)\right\} \\
&= S_{BS}\exp\left\{\int_0^t \beta v(s) + \frac{1}{2}\beta^2 v(s)^2 ds\right\}.
\end{aligned}$$

For $[\beta \times v(s)]$ small (positive or negative), the first term in the integral will dominate the second term, and the expression $\exp\left\{\int_0^t \beta v(s) + \frac{1}{2}\beta^2 v(s)^2 ds\right\}$ can be interpreted as a convenience yield. For $v(t)$ positive (negative), we can have simultaneous shortages (gluts) in the n markets and the exponential term multiplying S_{BS} in (7) can be interpreted as a positive (negative) convenience yield similarly to Jarrow (2001) or Jarrow and Turnbull (1997).

Jarrow and Turbull's argument is as follows: Suppose we have two identical bonds, with the only difference that one has a liquidity problem and the other has not, and it is not possible to construct synthetically the illiquid bond. We denote by $B_l(t,T)$ the bond with the liquidity problem and with $B(t,T)$ the liquid bond. Since it is not possible to construct synthetically the illiquid bond, arbitrage arguments can not be used to force the equality between the prices of the two bonds, and one of the following inequalities must hold:

$$B_l(t,T) \geq B(t,T) \text{ in a shortage;} \tag{8}$$

$$B_l(t,T) \leq B(t,T) \text{ in a glut.} \tag{9}$$

Therefore a function $g(t,T)$ exists such that $B_l(t,T) = \exp(-g(t,T))B(t,T)$. In a shortage (glut), $g(t,T) < 0$ ($g(t,T) > 0$) and the exponential is interpreted as a positive (negative) convenience yield obtained for holding the bond. Similarly, in our case,

$$\mathbb{E}\left(S|S_{BS}\right) \geq S_{BS} \text{ when } v\left(t\right) \text{ is small and positive;} \tag{10}$$

$$\mathbb{E}\left(S|S_{BS}\right) \leq S_{BS} \text{ when } v\left(t\right) \text{ is small and negative.} \tag{11}$$

i.e., the stock will on average trade below (above) its Black-Scholes price. We therefore interpret the value of $v(t)$ as the level of market liquidity at time t. Shortages correspond to $v(t) > 0$, while gluts correspond to $v(t) < 0$. We will prove in Proposition 1 below that the case $v\left(t\right) \equiv 0$ yields the Black-Scholes economy, and in the next section we will show how, as $v\left(t\right)$ increases in absolute value, the market clearing price processes will exhibit many of the stylized facts about the effects of illiquidity on asset prices.

In the literature on illiquidity, it is customary to distinguish between the *paper value* and the *liquidation value* of a security/portfolio. The distinction arises from the sparsity of price quotes in illiquid markets, so that the liquidation value might come as a sudden surprise the moment a long position has to be liquidated. In our economy, there always exists a price that clears the whole market supply. We are therefore modeling the security's liquidation price, the price at which we have enough buyers to clear the market.

Expression (7) can also be phrased in the *Arbitrage Pricing Theory with Illiquidity* framework of Cetin, Jarrow and Protter (2002), where it would be the trade-independent, stochastic case.

In appendix A1, we prove the following.

Proposition 1 *Under Assumptions 1, 2 and 3, the market clearing price S_i solves*

$$\begin{aligned} dS_i &= \left(\widetilde{\mu}_i + \beta_i v(t) + \frac{1}{2}\beta_i^2 v(t)^2\right) S_i dt \\ &\quad + \beta_i v(t) S_i dW_0 + k_i S_i dW_i, \end{aligned} \tag{12}$$

where for notational convenience we have put $\widetilde{\mu}_i = \mu_i\gamma_i + \frac{1}{2}\eta_i^2\gamma_i(\gamma_i - 1)$*. For the case* $v \equiv 0$*, we have*

$$dS_i = \widetilde{\mu}_i S_i dt + k_i S_i dW_i.$$

Furthermore, when $v \equiv 0$*,* $Corr(\frac{dS_i}{S_i}, \frac{dS_j}{S_j}) = \rho_{ij}$.[1]

In our economy when markets are perfectly liquid, the equilibrium asset price is the familiar Black-Scholes price. In an interesting paper He and Leland (1993) show that the Black-Scholes price is the equilibrium price of an economy with a representative agent with von Neumann-Morgenstern preferences. We may interpret the liquidity discount factor as a perturbation to the Black-Scholes economy. When there is no perturbation our results recover the Black-Scholes price, which, following He and Leland (1993), is an equilibrium price in a wider sense. On the other hand, when markets are illiquid, the asset price moves away form the Black-Scholes price.

In the next section, we investigate the market clearing price process when $v(t) \neq 0$.

3 Properties of the Derived Price Processes

In this section we show how $v(t) \neq 0$ can significantly affect expected returns, variances and correlations of our derived price processes and compare these effects to the literature on illiquidity and in particular to some stylized facts found in the empirical literature.

Expected Returns. From equation (12) it is easy to see that expected returns are equal to

$$\mathbb{E}\left(\frac{dS_i}{S_i}\right) = \left(\widetilde{\mu}_i + \beta_i v(t) + \frac{1}{2}\beta_i^2 v(t)^2\right) dt.$$

For small $v(t)$, the extra term $\beta_i v(t) + \frac{1}{2}\beta_i^2 v^2(t)$ will increase or decrease the expected return for stock i depending on whether $v(t)$ is positive or negative (see the convenience yield interpretation given in the previous section), the magnitude of the effect being determined by the value of β_i, the stock's sensitivity to aggregate illiquidity (liquidity beta). In contrast, for large (both positive *and* negative) values of $v(t)$, the quadratic term will dominate the linear one and expected returns will be increased. Stocks' expected returns are therefore always increased by sharp drops in market-wide liquidity.

Previous research on the return-illiquidity relation has focused on quoted bid/ask spreads as a proxy for illiquidity. Among others, Amihud and Mendelson (1986) and Amihud (2002) find evidence that asset returns include a significant premium for illiquidity. Figure 1 shows how our model includes a premium for illiquidity in expected returns.

[1]To test the coherence of the model we checked whether the variance-covariance matrix of this economy is positive semi-definite. The proof is in the appendix.

Variances. The variance for the return of security i is given by

$$Var\left(\frac{dS_i}{S_i}\right) = \left(k_i^2 + \beta_i^2 v(t)^2\right) dt.$$

The positivity of the second term on the right implies that the variance increases with $v(t) \neq 0$. A similar result is obtained, for example, by Papanicolaou and Sircar (1998) and Schoenbucher and Wilmott (2000), who study the effects of the presence in the market of a large trader who uses dynamic replication strategies. They show that the large trader's activity perturbs market equilibrium prices causing an increase in the volatility of the underlying asset. Figure 2 shows how illiquidity may increase volatilities of asset returns dramatically.

Correlations. It is easy to show that the covariance between the returns of two stocks i and j is given by

$$Cov\left(\frac{dS_i}{S_i}, \frac{dS_j}{S_j}\right) = \left(k_i k_j \rho_{ij} + \beta_i \beta_j v(t)^2\right) dt$$

while the correlation between returns is

$$Corr\left(\frac{dS_i}{S_i}, \frac{dS_j}{S_j}\right) = \frac{\left(k_i k_j \rho_{ij} + \beta_i \beta_j v(t)^2\right)}{\left\{\left(k_i^2 + \beta_i^2 v^2(t)\right)\left(k_j^2 + \beta_j^2 v^2(t)\right)\right\}^{1/2}}.$$

This expression is always increasing in $v(t)^2$, and the magnitude is again determined by the value of the liquidity beta. The higher the values of the liquidity-betas, the more correlations will increase with $v(t)^2$. It is well known in the financial industry that historical estimates of correlation become unreliable in times of high market illiquidity, when it is perceived that "true"correlations suddenly reach very high values (close to one) (see, Bhansali and Wise (2001)). Our model is able to reproduce this stylized fact. Its correlation structure changes with the levels of aggregate liquidity and can, in times of market stress, produce a substantial increase in the value of correlation for stocks which have high liquidity betas. Figures 3 and 4 display two possible correlation structures as functions of $v(t)$. The effects are dampened for high values of ks and low values of βs (Figure 3). k is the volatility of stock returns in the Black-Scholes economy - perfect liquidity $v(t) = 0$. Therefore, high values of k imply that stock returns are always very volatile. The impact of illiquidity shocks is, in this case, less strong. On the other hand, for high values of βs and low values of ks (Figure 4), the effects of aggregate illiquidity to correlations are remarkable. As a consequence, the CAPM-beta of the stocks in our economy will also fluctuate with aggregate liquidity levels.

Sharpe Ratios. The Sharpe ratio for the price process is given by

$$SR(a_i) = \frac{\left(\widetilde{\mu}_i + \beta_i v(t) + \frac{1}{2}\beta_i^2 v(t)^2\right) - r}{\sqrt{k_i^2 + \beta_i^2 v(t)^2}},$$

r being the riskless short rate in our economy. Figure 5 shows how Sharpe ratios will tend to increase with $v(t)$ away from zero. Lo, Petrov and Wierbicki (2003), using different liquidity measures, study the change that occurs in the Sharpe ratio of a portfolio when filtering out of the portfolio at various dates the securities that fall below a given "liquidity threshold" (specified in terms of the different measures). They find that the Sharpe ratio of the portfolio is always decreased by filtering out the more "sensitive" stocks, suggesting that illiquidity increases the stock's Sharpe ratio.

In this section we showed how our model is able to reproduce all the empirical findings of the effects of aggregate illiquidity on asset returns, volatilities, correlations and Sharpe ratios. In what follows, we focus on estimating the parameters' model.

4 The Econometric Approach

Although the model is developed in a continuous-time setup, the econometric approach is based on a discrete-time specification. This is not a novelty in the finance literature: Brennan and Schwartz (1982), Dietrich-Campbell and Schwartz (1986) and Chan (1992), among others, used this approach - see also Gourieroux, Monfort and Renault (1993).

Starting from the equilibrium price process in Proposition 1 and applying Ito's Lemma to derive the rate of return of the price process, the discrete-time econometric model can easily be written as follows

$$\begin{aligned} r_{i,t} &= \left(\widetilde{\mu}_i - \frac{1}{2}k_i\right) + \beta_i v_t + u_{i,t} && (13) \\ \mathbb{E}\left(u_{i,t}\right) &= 0 && (14) \\ \mathbb{E}\left(u_{i,t}^2\right) &= k_i^2 + \beta_i^2 v_t^2 && (15) \\ \mathbb{E}\left(u_{i,t}u_{j,t}\right) &= k_i k_j \rho_{ij} + \beta_i \beta_j v_t^2 && \forall i \neq j \end{aligned}$$

where $r_{i,t} = \ln\left(S_{i,t}/S_{i,t-1}\right)$ and $i = 1, 2, ...N$.

It is important to note that the processes in (13)-(15) are only an approximation of the continuous-time time processes implied by Proposition 1.[2] In fact, by measuring rates of returns of the price process over discrete-time intervals, entails integrals on the right-hand side of equation (12) and, therefore, in equations (13)-(15). We are approximating the integrals with

[2]See Longstaff (1989).

the area of the rectangle defined by the upper bound value of the process. However, it is possible to show that the approximation error is of second-order importance over short intervals.

The estimation procedure adopted is GMM - see Chan (1992) and Gourieroux, Monfort and Renault (1993). The GMM technique is very flexible and does not require any distributional assumption about the return process. As long as the distribution of asset returns is stationary and ergodic with finite relevant moments, the model's parameters can be easily estimated. Following the standard GMM approach, we define

$$f_t(\theta) = \begin{pmatrix} u_{i,t} \\ u_{i,t}\nu_t \\ u_{i,t}^2 - \left(k_i^2 + \beta_i^2\nu_t^2\right) \\ u_{i,t}u_{j,t} - \left(k_i k_j \rho_{ij} + \beta_i\beta_j\nu_t^2\right) \end{pmatrix}$$

where $\theta = \left\{\widetilde{\mu}_i, k_i, \beta_i, \rho_{ij}\right\}$, $i = 1, 2, ...N$. Equations (13)-(15) imply $\mathbb{E}\left(f_t(\theta)\right) = 0$. The GMM technique consists of replacing the population moments with the sample moments $g(\theta) = T^{-1}\sum_{t=1}^{T} f_t(\theta)$. The parameters' vector can be estimated by minimizing the quadratic form

$$J(\theta) = g(\theta)' W(\theta) g(\theta)$$

where $W(\theta)$ is a positive-definite symmetric matrix. The total number of parameters in θ is equal to $3N + N(N-1)/2$, where N is the number of assets considered in the estimation procedure. This implies that we have exact identification.

To account for autocorrelation, cross-correlation and heteroskedasticity, we computed robust standard errors using the Newey-West procedure.

5 Measuring Market Illiquidity

In our model expected returns, volatilities and correlations, as well as Sharpe ratios and CAPM-βs, are functions of v_t. To estimate the model we need a proxy for v_t.[3] While liquidity is still an elusive concept, there is, nonetheless, consensus on the following definition which is simple and serves our purposes.

Definition 2 *Liquidity is the ability to trade quickly any amount at the market price with no additional cost.*

An illiquid market, therefore, is characterized by the possibility of a sudden rise/drop of a security's price with modest trading volume. In the literature, the most common stock-specific

[3] An alternative approach would be to estimate v_t from a dynamic panel model.

liquidity measures used are bid-ask spreads, volume and turnover.[4] Higher bid-ask spreads correspond to higher transaction costs, while it is believed that low volume and turnover make it easier for a single trade to move the equilibrium price. The main problem with these measures is that they are one-dimensional and can be inconsistent, meaning that a stock could be considered illiquid in terms of bid-ask spreads but liquid in terms of volume and *vice versa*. We propose a simple measure that can capture a sudden rise/drop of a security's price with modest trading volume:

$$\widetilde{v}_t = \frac{r_{i,t}}{V_{i,t}} \tag{16}$$

where $r_{i,t}$ and $V_{i,t}$ denote the time t return and trading volume for security i, respectively.[5] Note that this measure inherits the sign of the time t return, but normalizes the return by the trading activity. We therefore expect the measure to oscillate around zero and to have positive (negative) peaks when a substantial positive (negative) price movement is accompanied by modest volume.

When moving from stock-specific to aggregate illiquidity measures, it is customary to define an aggregate liquidity measure as an average of the specific liquidity measures of each stock in the sample.[6] However, this makes the aggregate measure highly dependent on the composition and size of the sample of stocks. By using the measure (16) directly for the S&P500 index return and volume, we avoid the aggregation problem.

To understand how this measure of liquidity works, we produce a simple example in Table 1.

	Daily Returns	Daily Volume (Billions)	$\widetilde{v}_t$
Liquid Market	0.07%	0.0135	0.052
Illiquid Market	-3.6%	0.005	-7.20

Table 1: Measuring Illiquidity: An Example.

The data for the liquid market are the average return and volume of the S&P500 index for the period January 2003 - September 2003.

The value of $\widetilde{v}_t$ is, in this case, close to zero, indicating a situation in which the market is fairly liquid and there is not any unbalances between demand and supply. Since illiquidity is mostly associated to sudden price drops accompanied by tiny volume, the Illiquid Market in Table 1 represents a situation in which the index falls by 3.6% and volume is at the minimum in the analyzed sample. In this case, $\widetilde{v}_t$ is large and negative as a result of a large negative movement in the price set off by small trading activity.

[4] See, among others, Amihud & Mendelson (1986), Lo, Petrov & Wierbicki (2003).
[5] Amihud (2002) used a similar measure given by the ratio of absolute returns and trading volume.
[6] See, for example, Amihud (2002), and Chordia, Roll & Subrahmanyam (2001).

6 Data

We analyze daily data over the period 04 January 1995 - 15 September 2003. The data are taken from the Yahoo! Finance website. We collected daily data for the S&P500 index, both price and volume, in order to construct our liquidity measure, $\widetilde{v}_t$. For the stock data we selected companies in terms of sector and market capitalization. In particular, we consider four sectors - Consumer Discretionary, Industrials, IT and Utilities - and three market capitalization - large, medium and small caps. For each sector we analyze six stocks, two for each market capitalization, therefore our sample is composed by a total of 24 companies. The sectors analyzed represent a wide spectrum of stock market performances and, for each sector and capitalization, we randomly selected companies that were continuously traded on the NYSE during the period analyzed. We are aware that we are considering a small sample that may not be sufficient in order to identify systematic patterns. However, our results already show capitalization and sector patterns for only 24 stocks. We leave a systematic empirical analysis of the model for future research.

Illiquidity Measure $\widetilde{v}_t$. Figure 6 graphs the illiquidity measure $\widetilde{v}_t$. We first computed the S&P500 daily return and then divided by the daily volume.[7] We deleted the day before and after Christmas, New Year, 4th of July and Thanksgiving. It is evident that this measure is very volatile and shows clusters. Recall that downward (upward) spikes represent modest trading volume with large negative (positive) price movements. Since illiquidity is mostly associated to the risk of sudden price drops, we concentrate our attention to downward peaks only. Figure 6 shows that this measure is able to capture the market illiquidity generated by the Asian crisis (October 1997), the Russian Default/LTCM crisis (August 1998), the technology crisis (April 2000), the war in Iraq (March 2003). Note that 9/11 is not detected as a liquidity crisis, because of the significant trading activity that took place when market reopened a week later. Table 2, first column, reports summary statistics for $\widetilde{v}_t$. The mean is small and positive indicating that over the sample period analyzed, there has been, on average, excess demand. This implies, that over the sample period analyzed, stock market prices increased. Skweness and kurtosis indicate that $\widetilde{v}_t$ is not normally distributed and the ADF test[8] provides evidence in favor of the $I(0)$ alternative.

While our measure of illiquidity is able to detect many of the major liquidity crises of the past 9 years (see Figure 6), it is questionable whether every single movement in the measure should be seen as a relevant shift in aggregate liquidity. We believe that this measure performs

[7] One might argue that our liquidity measure is simply capturing returns dynamics. A graphical analysis of $\widetilde{v}_t$ and of the return process shows that this is not the case. To the best of our knowledge, we are the first to propose this liquidity measure. However, Pastor and Stambaugh (2002) use a a similar measure. In particular, they multiply the return sign by the dollar volume. Amihud (2002) measures illiquidity by the ratio between absolute returns and volume.

[8] Results not reported.

well on its peaks but contains noise. In Brunetti and Caldarera (2004) we use wavelet denoising techniques to separate the noise from the signal.

Stock Returns We exclude from our sample days without transaction for every stock. The inclusion of those days could cause a distortion due to artificial serial correlation, a problem that could particularly affect small cap stocks.

Table 2 reports summary statistics for the 24 asset returns analyzed. We group them according to sector. It is interesting to note that for the Consumer, Industrials and Utilities sectors the standard deviation of daily returns increases as market capitalization decreases. This is not the case for IT, where standard deviations are generally high when compared to other sectors. We also point out that extreme values for daily returns (max/min) are sharper as capitalization decreases. As for the standard deviation, IT does not have this pattern in capitalization. The distributions of returns exhibit excess kurtosis and negative skewness.

7 Results

For simplicity, we estimated the model two stocks at a time. For each combination of sector and capitalization, we have a different set of estimates. We list below the results grouped by sector.

Consumer Discretionary βs increase as capitalization decreases. The average β for Consumer Discretionary is the second highest after IT at about 0.52. The correlations between the information processes (ρ_{ij}) are the smallest in the sample, and never significantly different from zero.

Industrials βs increase as capitalization decreases. The average β for Industrials is the third highest after Consumer Discretionary at about 0.41. The correlations between the information processes (ρ_{ij}) increase from zero to 0.11 as capitalization decreases.

IT βs decrease as capitalization decreases. The average β for IT is the highest at about 0.8. The correlations between the information processes (ρ_{ij}) increase from zero to 0.18 as capitalization decreases. IT exhibits a strong capitalization effect for correlations, since βs decrease while ks increase. As we move down in capitalization, more and more correlation is explained by the constant part.

Utilities βs increase as capitalization decreases. The average β for Utilities is the lowest at about 0.21. The correlations between the information processes (ρ_{ij}) decrease from 0.3849 to

0.144 as capitalization decreases. Utilities also exhibit a strong capitalization effect for correlations, since βs increase while ks decrease. As we move down in capitalization, more and more correlation is explained by the non-constant part.

8 Conclusions

We have presented a new asset pricing framework to study the effects of aggregate illiquidity on asset price volatilities and correlations. We have shown how the model is a natural extension of the Black-Scholes world and incorporates the stylized facts from the empirical literature. We have estimated stocks' liquidity βs for a sample of 24 stocks. Although the size of the sample does not allow us to draw definitive conclusions, there is evidence of capitalization patterns. Verification of these patterns constitutes an interesting topic for future research.

On the one hand, a limitation of our approach is that the equilibrium price does not derive from the optimization behavior of agents. This is an interesting area for future research. On the other hand, however, an advantage of our framework is that it is conveniently tractable and the model can be easily estimated using standard econometrics.

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9 Appendix: Proofs

9.1 Proposition 1

For demand functions of type (1), we can apply the implicit function theorem to equations (1), to conclude that there exist unique functions Z_i such that $S_i = Z_i(L_i, I_i)$, with $Z_i \in C^1$, $i = 1, 2$. Since the calculations are identical, we will only deal with the case the case $i = 1$,and drop the index. If Z is C^2, we can use Ito's formula to get

$$dS = \frac{\partial Z}{\partial L} dL + \frac{\partial Z}{\partial I} dI + \frac{1}{2} \frac{\partial^2 Z}{\partial L^2} (dL)^2 + \frac{1}{2} \frac{\partial^2 Z}{\partial I^2} (dI)^2 \tag{A1}$$

where the first and second derivatives are

$$\frac{\partial Z}{\partial L} = -\frac{D_L}{D_S} \text{ and } \frac{\partial Z}{\partial I} = -\frac{D_I}{D_S},$$

$$\frac{\partial^2 Z}{\partial L^2} = -\frac{D_{LL}}{D_S} + 2\frac{D_{LS}D_L}{D_S^2} - \frac{D_{SS}D_L^2}{D_S^3}$$

and

$$\frac{\partial^2 Z}{\partial I^2} = -\frac{D_{II}}{D_S} + 2\frac{D_{IS}D_I}{D_S^2} - \frac{D_{SS}D_I^2}{D_S^3}.$$

Note that the terms

$$\frac{\partial^2 Z}{\partial L \partial I} = \frac{\partial^2 Z}{\partial I \partial L}$$

are irrelevant because of the assumed orthogonality (see Assumption 2).

Computing the relevant derivatives, we obtain the expressions

$$\begin{aligned} Z_L &= -\frac{S}{L}, \\ Z_I &= \frac{\gamma S}{I}, \\ Z_{LL} &= \frac{2S}{L^2}, \end{aligned}$$

and

$$Z_{II} = \frac{\gamma(\gamma-1)S}{I^2}.$$

Substituting in (A1) and using Assumptions 1 and 2 for the dynamics of L and I, we obtain

$$\begin{aligned} dS_i &= \left(\mu_i\gamma_i + \frac{1}{2}\eta_i^2\gamma_i(\gamma_i-1) + \beta_i v(t) + \frac{1}{2}\beta_i^2 v(t)^2\right) S_i dt \\ &\quad +\beta_i v(t) S_i dW_0 + k_i S_i dW_i. \end{aligned}$$

Note that since the price process does not depend on Ψ, each security could be characterized by a different Ψ, as long as Ψ satisfies (1) in Assumption 3.

9.2 Variance-Covariance Matrix Positive Semi-definite

We prove the statement for the economy with two stocks.

Denoting by $Cov_{i,j}$ the 2×2 variance-covariance matrix for the returns of stocks i and j, and using the expressions in section 3, we see that

$$Cov_{i,j} = \begin{pmatrix} k_i^2 + {\beta_i}^2 v^2(t) & k_i k_j \rho_{ij} + \beta_i\beta_j v(t)^2 \\ k_i k_j \rho_{ij} + \beta_i\beta_j v(t)^2 & k_j^2 + {\beta_j}^2 v(t)^2 \end{pmatrix}.$$

Since the diagonal elements are sums of squares, we only have to check that $\det Cov_{i,j} \geq 0$ to prove positive semidefiniteness. We have

$$\det Cov_{i,j} = \left(k_i^2 + {\beta_i}^2 v^2(t)\right)\left(k_j^2 + {\beta_j}^2 v^2(t)\right) - \left(k_i k_j \rho_{ij} + \beta_i \beta_j v(t)^2\right)^2. \tag{17}$$

From $-1 \leq \rho_{ij} \leq 1$ (since ρ_{ij} is the correlation in the Black-Scholes economy) and the condition $k_i > 0$, $\forall i$ (see Assumption 3), we have $\left(k_i k_j \rho_{ij} + \beta_i \beta_j v(t)^2\right) \leq \left(k_i k_j + \beta_i \beta_j v(t)^2\right)$, hence

$$-\left(k_i k_j \rho_{ij} + \beta_i \beta_j v(t)^2\right)^2 \geq -\left(k_i k_j + \beta_i \beta_j v(t)^2\right)^2$$

and using this inequality in (16) gives:

$$\begin{aligned}
\det Cov_{i,j} &\geq \left(k_i^2 + {\beta_i}^2 v^2(t)\right)\left(k_j^2 + {\beta_j}^2 v(t)^2\right) - \left(k_i k_j + \beta_i \beta_j v(t)^2\right)^2 \\
&= \left(k_i \beta_j - k_j \beta_i\right)^2 v(t)^2 \geq 0.
\end{aligned}$$

	v_t	Consumers Disc.					
		Large caps		Medium Caps		Small Caps	
		JCP	MHP	MBG	RDA	FLE	KDE
$Mean\,(\%)$	0.0847	-0.318	0.0822	0.0271	-0.0608	-0.0512	0.1830
$St.Dev.\,(\%)$	1.4798	2.6406	1.7376	2.8771	2.2315	3.3783	6.7116
$Min\,(\%)$	-10.2529	-12.9685	-11.9421	-22.9697	-18.0283	-32.7949	-27.4436
$Max\,(\%)$	6.3057	16.2094	13.3779	20.8207	16.0803	24.2668	31.1436
$Ske.$	-0.2849	0.4404	0.2821	-0.1982	0.1586	-0.7185	0.2353
$Kurt.$	5.8176	6.8924	7.4634	11.0638	9.8073	17.2214	5.2672
		Industrials					
		Large Caps		Medium Caps		Small Caps	
		CBE	LMT	SPW	VCI	JLG	ROP
$Mean\,(\%)$		0.0210	0.0482	0.0724	0.0354	0.0515	0.0477
$St.Dev.\,(\%)$		2.0873	2.070	2.5449	2.1825	3.4785	2.4796
$Min\,(\%)$		-21.4690	-14.7863	-22.1084	-29.5723	-40.2789	-24.5315
$Max\,(\%)$		24.2594	13.7242	13.4784	10.9038	28.3461	15.2155
$Ske.$		-0.0915	-0.1064	-0.4528	-1.3255	-0.8278	-0.4029
$Kurt.$		21.3939	8.7932	9.3369	22.8074	19.9375	9.8475
		IT					
		Large Caps		Medium Caps		Small Caps	
		AMD	CA	ARW	SY	CDT	TNL
$Mean\,(\%)$		-0.0455	0.0517	-0.0261	-0.0604	0.0172	0.0882
$St.Dev.\,(\%)$		4.6033	3.6373	3.0688	4.0809	3.8255	3.6020
$Min\,(\%)$		-39.1595	-36.6794	-19.8451	-53.1395	-18.9906	-46.6486
$Max\,(\%)$		23.2193	19.8252	41.8360	23.1877	25.2343	22.3915
$Ske.$		-0.3275	-0.4564	1.3988	-0.5653	0.2609	-0.6860
$Kurt.$		9.1798	11.1375	24.3066	20.0439	5.6281	20.7479
		Utilities					
		Large Caps		Medium Caps		Small Caps	
		D	PCG	EAS	POM	CNL	SWX
$Mean\,(\%)$		0.0754	0.0451	0.0770	0.013	0.0708	0.0478
$St.Dev.\,(\%)$		1.4288	2.3166	1.5125	1.4785	1.6657	1.6208
$Min\,(\%)$		-13.6716	-14.0273	-7.6685	-7.7542	-14.4082	-9.9835
$Max\,(\%)$		8.3728	19.5586	13.8534	9.2885	11.4033	9.3160
$Ske.$		-0.9684	0.2854	0.6188	0.3585	-0.6332	0.2006
$Kurt.$		13.7734	12.5733	9.1211	6.9332	12.4071	6.3867

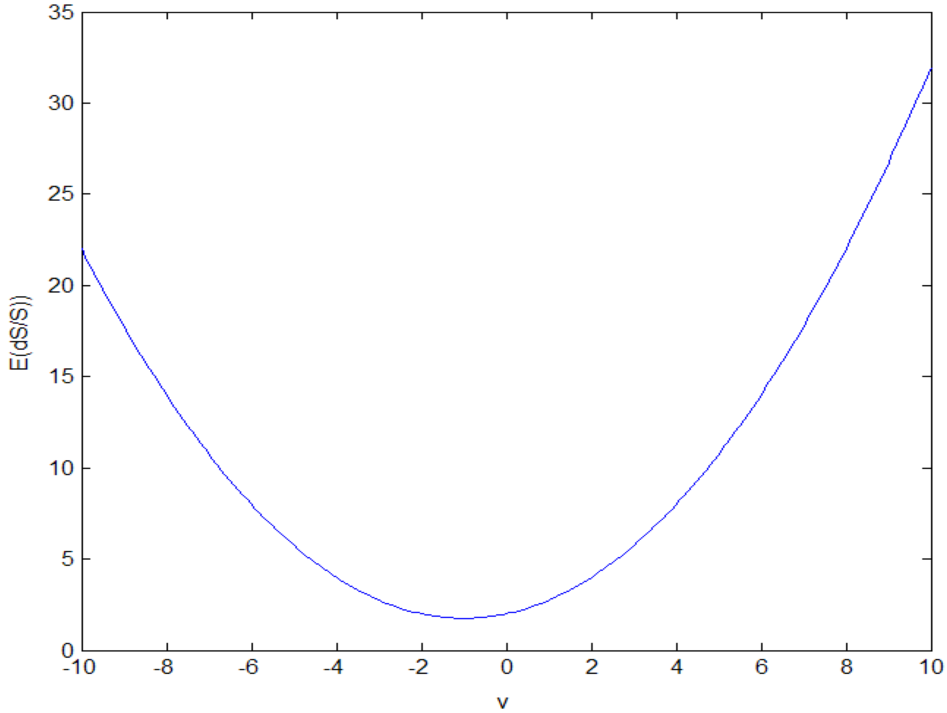
Table 2: Summary Statistics.

Consumers Disc.	Large caps		Medium Caps		Small Caps	
	JCP & MHP		MBG & RDA		FLE & KDE	
	Param.	St.Err.	Param.	St.Err.	Param.	St.Err.
μ_1	2.7480	(0.2302)	3.5142	(0.2875)	4.9242	(0.5920)
μ_2	1.1602	(0.1008)	2.0864	(0.1679)	22.0375	(1.4955)
k_1	2.3775	(0.0912)	2.6577	(0.1083)	3.1643	(0.1872)
k_2	1.4926	(0.0623)	2.0876	(0.0757)	6.6157	(0.2233)
β_1	0.5480	(0.0409)	0.5254	(0.0452)	0.5673	(0.0494)
β_2	0.4244	(0.0241)	0.3758	(0.0365)	0.5382	(0.1333)
ρ	0.0059	(0.0338)	-0.0132	(0.0287)	-0.0044	(0.0186)
Obs.	2090		2090		1932	
Industrials	Large Caps		Medium Caps		Small Caps	
	CBE & LMT		SPW & VCI		JLG & ROP	
	Param.	St.Err.	Param.	St.Err.	Param.	St.Err.
μ_1	1.6082	(0.2257)	2.8403	(0.2359)	5.5388	(0.5790)
μ_2	1.9491	(0.1667)	2.0770	(0.2431)	2.7750	(0.2122)
k_1	1.8333	(0.1202)	2.3687	(0.1006)	3.3250	(0.1776)
k_2	1.9631	(0.0840)	2.0363	(0.1234)	2.3493	(0.0910)
β_1	0.4775	(0.0340)	0.4436	(0.0433)	0.4863	(0.0591)
β_2	0.3129	(0.0291)	0.3742	(0.0336)	0.3780	(0.0422)
ρ	-0.0378	(0.0464)	0.0601	(0.0294)	0.1133	(0.0246)
Obs.	2090		2089		2090	
IT	Large Caps		Medium Caps		Small Caps	
	AMD & CA		ARW & SY		CDT & TNL	
	Param.	St.Err.	Param.	St.Err.	Param.	St.Err.
μ_1	8.0548	(0.6732)	3.3440	(0.5231)	6.4893	(0.4521)
μ_2	4.5177	(0.5276)	7.000	(0.8680)	5.2024	(0.5736)
k_1	4.0470	(0.1642)	2.6210	(0.1923)	3.6116	(0.1192)
k_2	3.0160	(0.1795)	3.7745	(0.2312)	3.2177	(0.1828)
β_1	1.0465	(0.0876)	0.7613	(0.0646)	0.6021	(0.0639)
β_2	0.9702	(0.0525)	0.7393	(0.0758)	0.7731	(0.0689)
ρ	0.0751	(0.0430)	0.0701	(0.0318)	0.1788	(0.0300)
Obs.	2090		2088		2083	
Utilities	Large Caps		Medium Caps		Small Caps	
	D & PCG		EAS & POM		CNL & SWX	
	Param.	St.Err.	Param.	St.Err.	Param.	St.Err.
μ_1	0.9973	(0.1188)	1.1303	(0.0922)	1.2670	(0.1576)
μ_2	2.6227	(0.2914)	1.002	(0.0905)	1.1894	(0.0791)
k_1	1.3699	(0.0898)	1.4622	(0.0546)	1.5618	(0.1050)
k_2	2.2780	(0.1285)	1.4181	(0.0569)	1.5256	(0.0467)
β_1	0.1932	(0.02181)	0.1841	(0.0230)	0.2758	(0.0238)
β_2	0.1996	(0.0311)	0.1989	(0.0203)	0.2607	(0.0272)
ρ	0.3849	(0.0361)	0.3167	(0.0303)	0.1446	(0.0330)
Obs.	2084		2088		2086	

Table 3: Estimation Results. Robust Standard Errors in Parenthesis.

Figure 1

Expected return as a function of v,
for mu=2, k=.5, beta=.5



Figire 2

Variance as a function of v
k=2, all other parameters are from figure 1

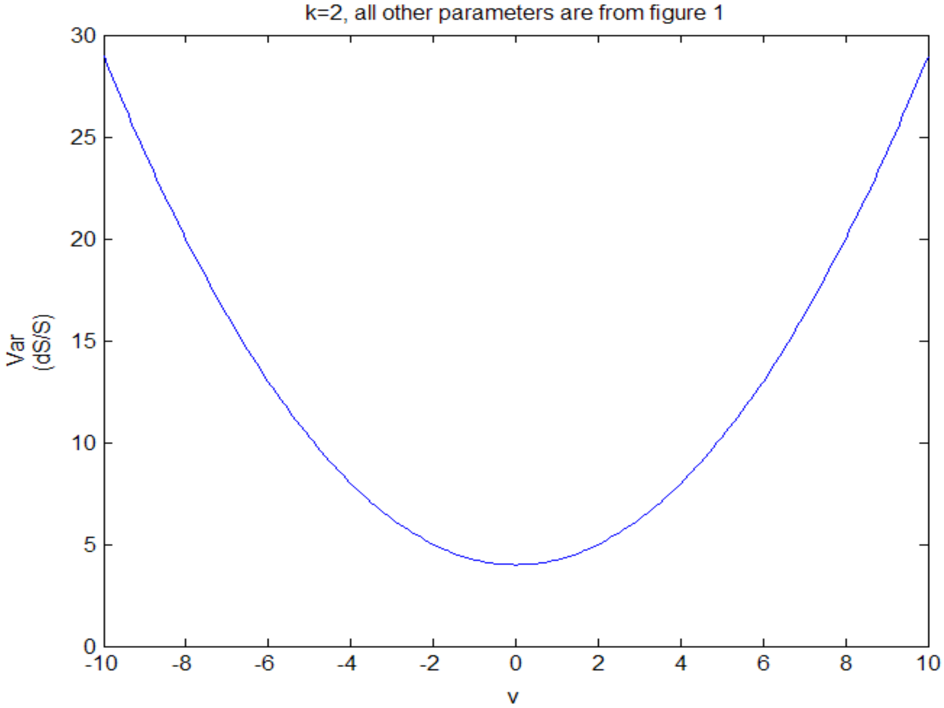


Figure 3

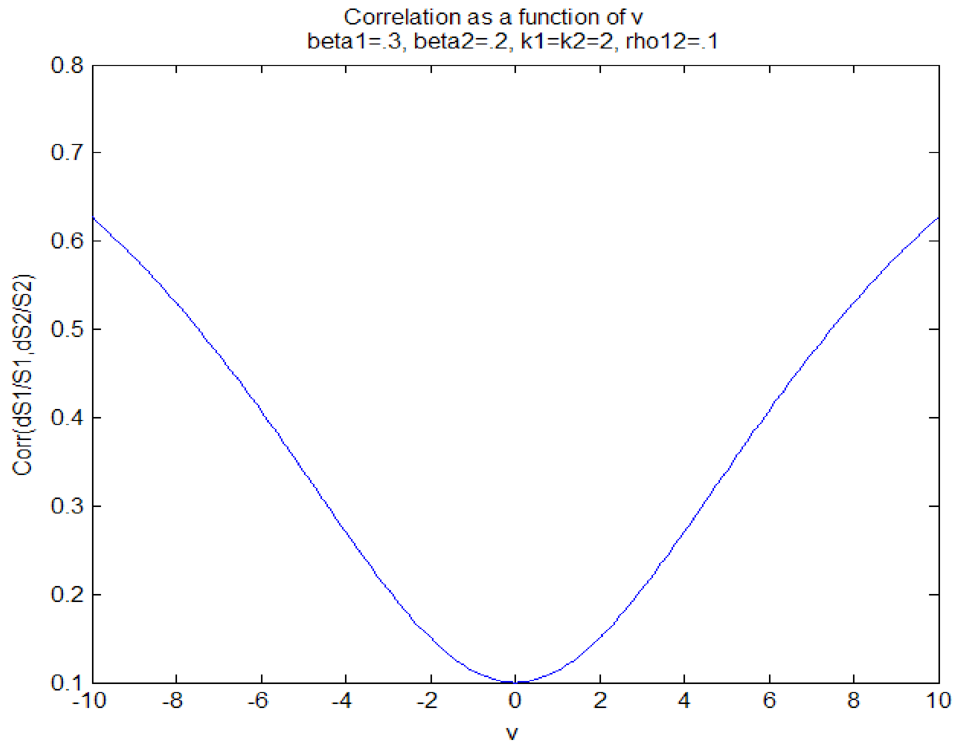


Figure 4

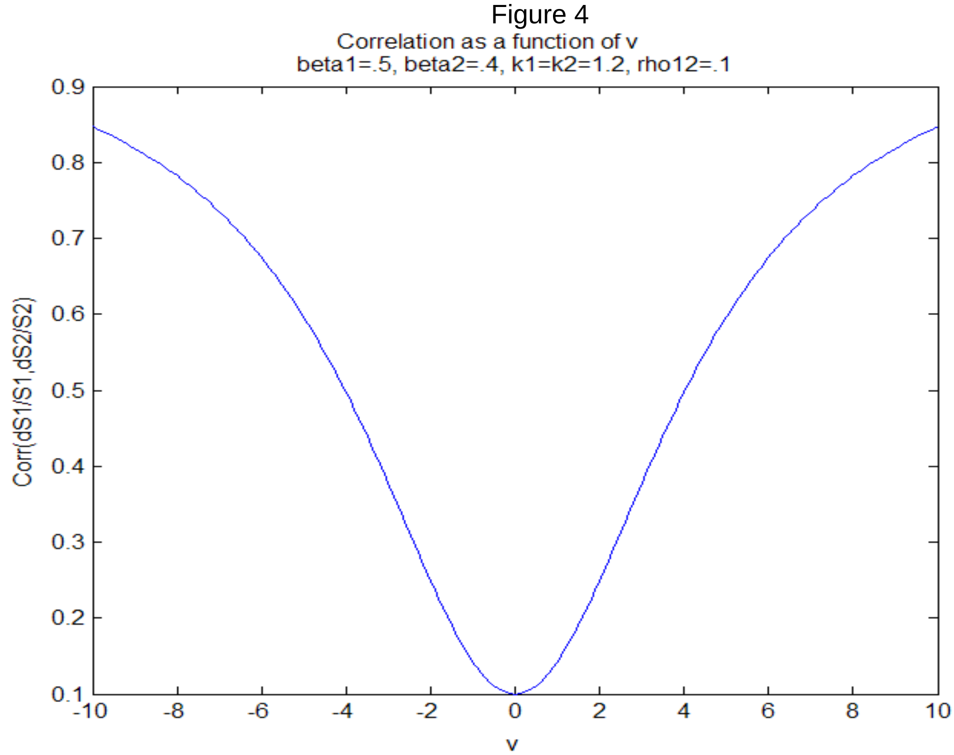


Figure 5

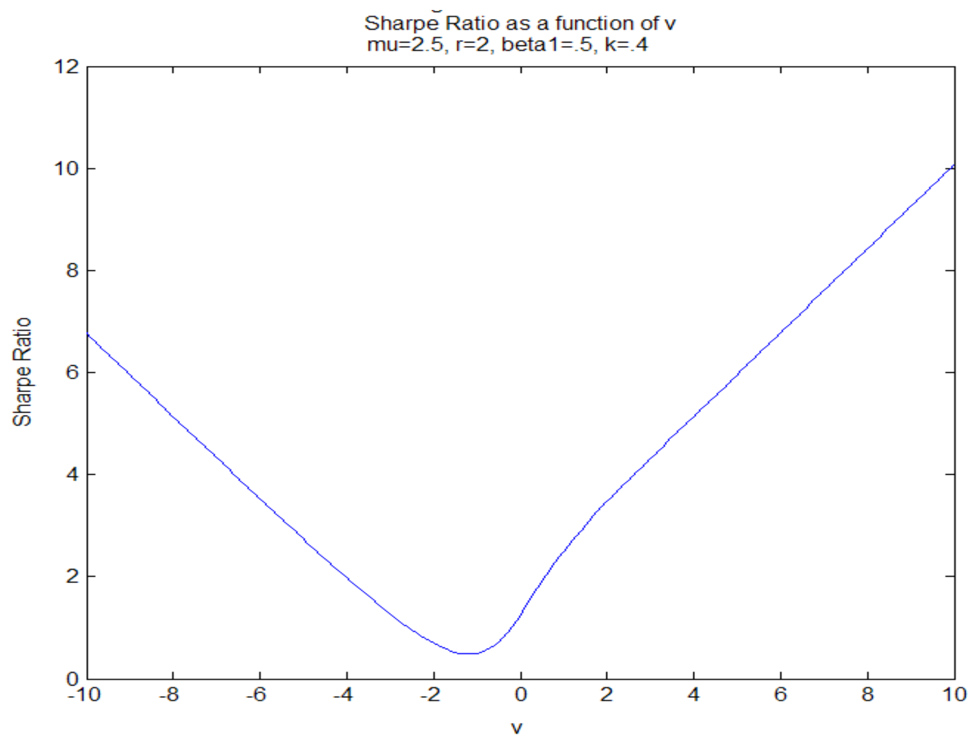


Figure 6
Illiquidity Measure