A new framework for firm value using copulas

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Abstract

In this paper we present some contingent claim analysis' models for the firm value. We focus on two different approaches: the structural (Merton) approach and a new one that treats the asset value as a claim on the firm's securities. The non-observability of the assets' value in structural models can be overcome using the bivariate contingent claim analysis and copula theory. First we consider the case of the complete markets followed by the general case of incomplete markets. In the latter we provide the lower and upper bound of the firm's value, using no-arbitrage arguments.

Keywords: Firm value, No Arbitrage, Structural models, Bivariate option, Copula, Incomplete Markets

JEL classification: G12, G30, G32.

1 Structural models

Structural models base the evaluation of firm related securities on the structural firm variables, i.e. the firm's Assets and Debt values. Those models date from the early seventies. Both the classic papers by Black and Scholes [5] and by Merton [17] point out that the liabilities of a corporate firm may be priced as plain vanilla options. Needless to say, the straightforward use of the Black-Scholes valuation formulas requires some basic assumptions on the behavior of Assets, no-arbitrage opportunities and continuous hedging.

Definition 1 *The* value of a firm *is the value of its Assets.*

A very common assumption in structural models (e.g., see [18], [19], [15], [12]) is:

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Assumption 2 *The value A_t of the firm follows a geometric Brownian motion*

$$\mathrm{d}A_t = \mu A_t \,\mathrm{d}t + \sigma A_t \,\mathrm{d}W_t, \tag{1.1}$$

where W_t is a Wiener process and the drift and volatility coefficients μ and σ do not depend on the capital structure of the firm

$$q_t = \frac{B_t}{E_t}, \tag{1.2}$$

i.e. on how the Assets' value A_t is split into Equity value E_t and Bonds value B_t:

$$A_t = B_t + E_t. \tag{1.3}$$

The independence of (μ, σ) on q_t simply translates the Miller-Modigliani theorem (see [21], [22]). Merton [16] obtains equation (1.1) starting from a general intertemporal equilibrium model of the assets market and assumes *" the known net flows of the firm* [... such as] *dividends, coupon payments, sinking fund payments, and principal repayments* [... and] *the future issuance of new securities, e.g., equity or debt, where the timing, terms, and proceeds are known for certain"* have no value (e.g., see Merton [17], [18], [19], and Mason and Merton [15]).

A crucial problem affects the structural approach: the Assets' value A_t is a non-observable variable (e.g., see Leland [14], Trigeorgis [32], Apabhai et al. [2], and Elizade [7]).

For the purpose of our analysis, we give an outline only of the early structural models: the Merton and the Black & Cox. A review of the literature on structural models is beyond our scope, therefore for a detailed analysis we refer to Ammann [1], Elizade [7], and Bieleki et al. [3].

1.1 The Merton model

In the Merton [17] model, debts are assumed to be zero coupon bonds with a total face value of D at a future expiration date $T > t$. Debts are risky, as their final value is contingent on A_T. Moreover, it is also assumed that the firm does not issue new debts of equivalent or senior rank in period $[t, T]$ and does not make payments for dividends or share repurchase. In this case we get the final values of equity and debt:

$$E_T = \max(A_T - D, 0), \tag{1.4}$$
$$B_T = \min(D, A_T) = D + \min(A_T - D, 0). \tag{1.5}$$

The Equity value can be considered the value $c_t(A_t, D)$ of a European call written on the Assets' value, with expiration T and strike price D. In a similar way, the bond-holders' position is equivalent to the one of who have bought the firm outright from the stockholders and have written to them an option to buy back the firm at T for D (similar remarks may be found, e.g., in [31] and [11, sec. 6.14]). Given the instantaneous risk-free interest rate r, the put-call parity relation allows to summarize all this as follows:

$$\underset{\text{Firm's value}}{A_t} = \underbrace{\underset{\substack{\text{Bonds value}\\ \text{as if risk-free}}}{De^{-r(T-t)}} - \underset{\substack{\text{value of the put written}\\ \text{by bondholders}}}{p_t(A_t, D)}}_{\text{value of risky bonds}} + \underset{\substack{\text{Equity value}\\ \text{(call hold by stockholders)}}}{c_t(A_t, D)}.$$

This way, the total value of the risky bonds is the difference between their present value $De^{-r(T-t)}$, computed as they were non-risky, and the value $p_t(A_t, D)$ of a European put with the same underlying, expiration and exercise price of the call held by the stockholders. The Black-Scholes values of those vanilla options are

$$\begin{cases} c_t(A_t, D) = A_t \mathrm{N}(d_1) - De^{-r(T-t)} \mathrm{N}(d_2), \\ p_t(A_t, D) = De^{-r(T-t)} \mathrm{N}(-d_2) - A_t \mathrm{N}(-d_1), \end{cases} \tag{1.6}$$

with

$$d_{1,2} = \frac{\ln \dfrac{A_t}{D} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{1.7}$$

$\mathrm{N}(\cdot)$ being the standard normal distribution function.

Structural models are used to evaluate corporate bonds, accounting for the credit risk linked to the default probability of the firms. As remarked above, the main problem is we cannot observe the Assets'value A_t and hence neither its volatility σ. In order to evaluate A_t and σ, different methods have been proposed. Following Ronn & Verma [26] and Schellhorn & Spellman [29], we present the construction of a system of equations that can be solved in A_t and σ, starting from observable variables. From the Itô's lemma the volatility function of $E_t(A_t, D)$ is $\sigma A_t \frac{\partial E_t}{\partial A}$. If we suppose that the equity's volatility $\sigma_E(t, E_t)$ has the form $\sigma_E(t, E_t) = \sigma_E E_t$, with $\sigma_E > 0$, we can write the system

$$\begin{cases} \hat{E}_t = c(A_t, D) \\ \hat{\sigma}_E \hat{E}_t = \sigma A_t \frac{\partial E_t}{\partial A} \end{cases} \tag{1.8}$$

in which all quantities but A_t and σ can be measured for a firm with traded equity. The hat indicates the observed market values[1]. It is also common to use ρD, with $\rho \in [0, 1]$, as the final asset value that triggers the defaults. From Black & Scholes formula we can compute the partial derivative $\frac{\partial E_t}{\partial A} = \mathrm{N}(d_1)$, with d_1 as in (1.7). The system (1.8) has no explicit solution, but it can be easily solved numerically.

Merton's model has been extended in several ways. Some extensions deal with different types of securities (e.g. coupon bonds, callable bonds, mortgages, convertible bonds, variable rate bonds). Other extensions treat the valuation of claims with different maturities, seniority or special properties. In the next section we present the Black & Cox extension of the Merton model.

1.2 The Black & Cox model

The Merton model constrains the default to only happening at the final date T. In their model, Black & Cox [4] introduce the bondholders' right to bankrupt the firm at any time $t \leq T$: *"if the value of the firm falls to a specified level, which may change over time, then the bondholders are entitled to force the firm into bankruptcy and obtain the ownership of the assets."*, see [4, p. 355]. This feature makes the Equity and Debt barrier American derivatives written on the Assets' value. Black & Cox assume the same asset

[1] The volatility value $\hat{\sigma}_E$ can be estimated from the time series of stock prices or can be obtained from the value of derivatives written on the stocks.

value dynamics as the Merton model. Moreover, the bankruptcy level is $Ce^{-\gamma(T-t)}$ and the dividend payout is aA_t. In this case the debt evaluation formula is

$$B^{BC}(t, A_t) = De^{-r(T-t)}\left[\mathrm{N}(z_1) - y^{2\theta-2}\mathrm{N}(z_2)\right] + A_t e^{-a(T-t)}\Big[\mathrm{N}(z_3) + y^{2\theta}\mathrm{N}(z_4) + y^{\theta+\zeta}e^{a(T-t)}\mathrm{N}(z_5) + y^{\theta-\zeta}e^{a(T-t)}\mathrm{N}(z_6) - y^{\theta-\eta}\mathrm{N}(z_7) - y^{\theta-\eta}\mathrm{N}(z_8)\Big], \quad (1.9)$$

where the quantities defined in table 1 are functions of the following known parameters and values:

- r, risk-free interest rate,
- a, dividend payout rate,
- γ, rate of evolution of the bankruptcy level,
- σ, assets' volatility,
- A_t, Assets' value,
- T, maturity.

$y = \frac{Ce^{-\gamma(T-t)}}{A_t}$, $\delta = \left(r - a - \gamma - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(r-\gamma)$, $z_1 = \frac{\ln\frac{A_t}{D} + \left(r-a-\frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}$,	$\theta = \frac{r-a-\gamma+\frac{1}{2}\sigma^2}{\sigma^2}$, $\zeta = \frac{\sqrt{\delta}}{\sigma^2}$, $\eta = \frac{\sqrt{\delta-2\sigma^2 a}}{\sigma^2}$, $z_2 = z_1 + \frac{2\ln y}{\sqrt{\sigma^2(T-t)}}$,
$z_3 = -\sqrt{\sigma^2(T-t)} - z_1$,	$z_4 = z_2 + \sqrt{\sigma^2(T-t)}$,
$z_5 = \frac{\ln y + \zeta\sigma^2(T-t)}{\sqrt{\sigma^2(T-t)}}$,	$z_6 = z_5 - 2\zeta\sqrt{\sigma^2(T-t)}$,
$z_7 = \frac{\ln y + \eta\sigma^2(T-t)}{\sqrt{\sigma^2(T-t)}}$,	$z_8 = z_7 - 2\eta\sqrt{\sigma^2(T-t)}$.

Table 1:

From formulas (1.3) and (1.9) the equity value is $E_t^{BC} = A_t - B_t^{BC}$.

The Black & Cox model improves on the classical Merton model and better fits the actual defaults behavior. The Black & Cox model (like the Merton Model), however, is limited by the non-observability and volatility of the Assets' value. As a solution to this problem, we can adapt the Ronn & Verma approach to the Black & Cox framework. The following system can be obtained:

$$\begin{cases} E_t = A_t - B^{BC}(t, A_t) \\ \sigma_E E_t = \sigma A_t \left(1 - \frac{\partial B^{BC}}{\partial A_t}\right) \end{cases} \quad (1.10)$$

Even though these equations are more complex than (1.8), it is possible to write them down in a more explicit form and to solve them numerically.

2 Pricing the firm's value as a bivariate contingent claim using copulas

The basic assumption in structural approach is that the Assets' value is exogenous, so it can be treated as the underlying in an option pricing framework. This means that in structural models the Assets' value does not depend on the dynamics of the firm related securities, and therefore the Equity has a residual value.

Some recent facts have shown that this approach can not fit actual situations . For example, consider the Fiat-GM negotiation that took place in 2000 and 2004. In this case, the value of Fiat's plants and buildings did depend on how they were funded. See [25] and [28] for details. Therefore, the value of physical assets can not be exogenous.

In this section, we propose a model that substantially changes the point of view with regard to structural models. Owing to the financialization of the economy, it is worth treating the market value of the Assets as a claim on the traded securities: stocks and bonds. We thus model the dynamics of stocks and bonds and then endogenously evaluate the Assets. We remark that this approach overcomes the problem of the non-observability of the Assets' value (and its volatility).

2.1 A bivariate contingent claim

In order to solve the observability issue, we propose treating the Assets as a bivariate contingent claim written on the traded securities of the firm, i.e. stocks and bonds. Unlike the structural approach, we now obtain the Assets value from the processes of traded, i.e. observable, securities. This way the underlyings are observable, while the contingent claim is not. Obviously this approach is suited for quoted firms only, but even structural models resort to quoted firms to solve systems (1.8) and (1.10).

A contingent claim can be written in the general form as

$$G\left(g\left(S_1(T), S_2(T)\right); T\right),$$

where $G(\cdot)$ is a univariate pay-off function which identifies the derivative contract, $g(\cdot)$ is a bivariate function which describes how the 2 underlying securities determine the final cash-flows, S_i denotes the price of the i^{th} underlying security and T is the contract maturity.

By using this framework, we can express the final value of the firm A_t as

$$A_T = G\left(E_T, B_T; T\right) = \max\left(E_T + B_T, 0\right) I_{[(E_T \geq 0),(0 \leq B_T \leq D)]}, \tag{2.11}$$

where I is the indicator function.

While this bivariate pricing problem is already quite complex in a standard Gaussian world, the evaluation task becomes even more difficult when we consider the well-known evidence of departures from normality, given by skewness, kurtosis, smile and term structure effects of volatility. Moreover, because of the limited liquidity of many financial assets, such as risky bonds, the problem of incomplete markets arises. Given this reality, jointly taking into account non-normality of yields, the yields' dependence structure and markets incompleteness, seems to be a very challenging mission.

A possible way out of this stalemate is to resort to *copula* theory, as first proposed in the seminal work by Rosemberg [27] and generalized by Cherubini et al. [6]. We deal with a complex non-normal joint distribution by separating two issues: (i) we work with non-Gaussian marginal probability distributions and (i) we use a copula to combine these distributions in a bivariate setting. The bivariate pricing kernel can therefore be written as a function of univariate pricing functions. Besides, copula theory can provide us with upper and lower bounds that we can use when we deal with incomplete markets.

2.2 Copula theory: the mathematical background

In what follows, the definition of a copula function and some of its basic properties are given. The reader interested in more detail can refer to Nelsen [23] and Joe [13]. Since in the sequel we want to price the firm's value as a bivariate claim, here we stick to the bivariate copula: nonetheless, most of the results carry over to the general multivariate setting.

A 2-dimensional copula is basically a bivariate cumulative distribution function with uniform distributed margins in $[0, 1]$. If we consider X_1, X_2 to be random variables, we have:

Definition 3 *The copula of* (X_1, X_2)*, where* $X_1 \sim F_1$*,* $X_2 \sim F_2$*, and* F_1*,* F_2 *are continuous, is the joint distribution function of* $U_1 \equiv F_1(X_1)$*,* $U_2 \equiv F_2(X_2)$*.*

The variables U_1, U_2 are the '*probability integral transforms*' of X_1 and X_2, and follow a Uniform$(0, 1)$ distribution, regardless of the original distribution, F_i. Thus a copula is a joint distribution of Uniform$(0, 1)$ random variables.

Proposition 4 *A bivariate copula is a function* C *of two variables* u_1 *and* u_2*, with the following properties:*

1. *The range of* $C(u_1, u_2)$ *is the unit interval* $[0, 1]$*;*
2. $C(u_1, u_2) = 0$ *if any* $u_i = 0$*, for* $i = 1, 2$*;*
3. $C(1, u_i) = u_i$ *, for all* $u_i \in [0, 1]$*;*
4. $C(u_1, u_2)$ *is* 2*-increasing in the sense that for* $V_{C_t}([u_{11}, u_{12}] \times [u_{21}, u_{22}]) \equiv C_t(u_{12}, u_{22}) - C_t(u_{11}, u_{22}) - C_t(u_{12}, u_{21}) + C_t(u_{11}, u_{21}) \geq 0$ *for all* $u_{11}, u_{12}, u_{12}, u_{22} \in [0, 1]$*, such that* $u_{11} \geq u_{12}$ *and* $u_{21} \geq u_{22}$*.*

The first three conditions provide the lower bound on the distribution function and ensures that the marginal distributions are uniform. The condition that every rectangle V_{C_t} is non-negative ensures that the probability of observing a point in the region $[u_1, u_2] \times [v_1, v_2]$ is non-negative. This definition shows that C is a bivariate distribution function with uniformly distributed margins. Copulae have many useful properties, such as uniform continuity and (almost everywhere) existence of all partial derivatives, just to mention a few (see Nelsen [23], Theorem 2.2.4 and Theorem 2.2.7). Now we present the Sklar's theorem (see Sklar [30]), which justifies the role of copulas as dependence functions:

Theorem 5 (Sklar's theorem, 1959) *Let* H *denote a* 2*-dimensional distribution function with margins* F_1 *and* F_2*. Then, there exists a copula* C *such that for all real* (x_1, x_2)

$$H(x_1, x_2) = C(F_1(x_1), F_2(x_2)).$$

If all the margins are continuous, then the copula is unique. Moreover, the converse of the above statement is also true[2].

The latter statement is the most interesting for bivariate density modeling, since it implies that we may link together any $n = 2$ univariate distributions, of any type (not necessarily from the same family), with any copula in order to get a valid bivariate distribution.

A copula is a function that, when applied to univariate marginals, results in a proper bivariate pdf (probability distribution function): since this pdf embodies all the information about the random vector, it contains all the information about the dependence structure of its components. Using copulas in this way splits the distribution of a random vector into individual components (marginals) with a dependence structure between them (the copula) without losing any information.

For what follows in the paper, three specific copulas are of main interest: the *product* copula, the *minimum* and the *maximum* copulas. As for the first, the copula representation of a joint distribution H degenerates into the so-called product copula $C(u_1, u_2) = u_1 \cdot u_2$, if and only if X_1 and X_2 are independent.

As for the others, they derive from the well-known Fréchet-Hoeffding result in probability theory, stating that every bivariate joint distribution function H is constrained between the bounds

$$\max(F_1(x_1) + F_2(x_2) - 1, 0) \leq H(x_1, x_2) \leq \min(F_1(x_1), F_2(x_2)).$$

As a consequence of Sklar's theorem, the Fréchet-Hoeffding bounds exist for copulas too:

$$\max(u_1 + u_2 - 1, 0) \leq C(u_1, u_2) \leq \min(u_1, u_2)$$

which are commonly denoted by $W$and M. In correspondence of the extreme copula bounds, there is perfect positive and negative dependence between the variables, and every variable can be obtained as a deterministic function of the other (see Embrechts et al. [9], [8] for a proof).

2.3 The case of complete markets

When the market is complete, it is common knowledge that a bivariate contingent claim of the type given by (2.11) can be exactly replicated and its price is uniquely determined. Moreover, there is a unique risk-neutral probability distribution $Q(E, B|F_t)$, with density function denoted by $q(E, B|F_t)$, which represents the pricing kernel of the economy.

Remembering that $G(E_T, B_T; T)$ is the bivariate claim pay-off, its pricing function $g(E_t, B_t; t)$ can be expressed as

$$A_t = g(E_t, B_t; t) = P(t, T) \int_0^{\infty} \int_0^{D} G(E_T, B_T; T) q(E_T, B_T|F_t) \, dE_T \, dB_T, \tag{2.12}$$

where D is the bond face value, while $P(t, T)$ is the risk-free discount factor, which, for the sake of simplicity, we assume deterministic or independent of E_T and B_T. However, a

[2] See Nelsen [23] for a proof, Theorem 2.10.9.

more general extension is possible if we change the measure to the forward risk-neutral one, as shown in Cherubini et al. [6]. Besides, the marginal conditional distributions $Q_E(E|F_t)$ and $Q_B(B|F_t)$, with densities $q_E(E|F_t)$ and $q_B(B|F_t)$, can be derived as follows

$$q_E(E|F_t) = \int_0^D q(E_T, B_T|F_t)\ dB_T\ ,$$

$$q_B(B|F_t) = \int_0^\infty q(E_T, B_T|F_t)\ dE_T\ ,$$

so that a univariate contingent claim written on E or B , can be priced as

$$g(E_t;t) = P(t,T)\int_0^\infty G(E_T;T)q_E(E_T|F_t)\ dE_T$$

$$g(B_t;t) = P(t,T)\int_0^D G(B_T;T)q_B(B_T|F_t)\ dB_T$$

Let's now consider a *digital* option, which pays a fixed sum if the price of the underlying asset is higher (or lower) than a strike level K, i.e. a call (put) digital option. We can set this amount to one unit of currency without loss of generality. Given the risk-neutral distribution Q_E or Q_B for assets E_T or B_T, their price are

$$DC_E = \text{Digital Call}(E_T, K_1) = P(t,T)Q_E(E_T > K_1),$$
$$DC_B = \text{Digital Call}(B_T, K_2) = P(t,T)Q_B(B_T > K_2).$$

Then this result follows:

Definition 6 *In a complete market, the prices of univariate digital options are equal to the discounted values of risk-neutral probability distributions.*

In order to write the bivariate pricing function, the final step is to consider the extension of Sklar's theorem to conditional distribution:

Theorem 7 *For any bivariate conditional distribution function $Q(E,B|F_t)$ with margins $Q_E(E|F_t)$, and $Q_B(B|F_t)$, there exists a copula $C(u_E, v_B)$ such that*

$$Q(E,B|F_t) = C(Q_E(E|F_t), Q_B(B|F_t)).$$

Besides, given two conditional distributions $Q_E(E|F_t)$, $Q_B(B|F_t)$ and a copula function $C(u_E, v_B)$, the function $C(Q_E(E|F_t), Q_B(B|F_{tt}))$ is a bivariate conditional distribution function[3].

This theorem enables us to separate the effects of the marginal pricing kernels and the dependence structure of the underlying assets.

[3]See Patton [24] for a proof.

Now Consider the case of a bivariate digital option paying one unit of currency if both assets E_T and B_T are higher than strike prices K_1 and K_2, respectively, and denote it D_{EB}.

By using the previous Definition 6 and Theorem 7, we can finally write the (discounted) risk-neutral probability distribution $Q(E, B|F_t)$ as a copula function taking the forward values of univariate digital options as arguments:

$$D_{EB}(K_1, K_2) = P(t,T)\ Q(E, B|F_t) = P(t,T)\ C_{EB}\left(\frac{DC_E}{P(t,T)}, \frac{DC_B}{P(t,T)}\right),$$

where C_{EB} is a particular copula function, the survival copula[4].

Coming back to our initial bivariate pricing function $g(E_t, B_t; t)$ given by (2.12), we can write that integral representation using the relationship between the joint density and the copula and marginal densities[5]

$$q(E, B|F_t) = c_{EB}\left(Q_E, Q_B|F_t\right) \cdot q_E\left(Q_E|F_t\right) \cdot q_B\left(Q_B|F_t\right),$$

where c_{EB} is the density associated with the copula function. With this result in hand, the firm's value price $A_t = g(E_t, B_t; t)$ can be expressed as

$$A_t = P(t,T) \int_0^{\infty} \int_0^{D} G(E_T, B_T; T) c_{EB}\left(Q_E, Q_B|F_t\right) q_E\left(Q_E|F_t\right) q_B\left(Q_B|F_t\right) dE_T\ dB_T\,.$$

The above integral can be simplified according to the assumptions made within the analysis. Since here we want to propose a general procedure to use as a building block for a firm's value, we leave possible extensions of this model to future research.

2.4 The case of incomplete markets

Market incompleteness is determined by the fact that options are not traded for a continuum of strike prices, so that the derivative cannot be directly estimated from option prices observed on the market. In incomplete markets, a *perfect hedge* does not exist for each and every contract

From the pricing viewpoint, this new dimension of risk implies the selection of a risk neutral probability from among many possible candidates to compute the price. One could resort to expected utility to give a preference rank for the probabilities in the set, or one could select a range of prices consistent with the no-arbitrage assumption. The replicating strategies corresponding to the bounds of this range are known as *super-replicating portfolios*.

Similarly to what we did in the complete market case, we start with a bivariate digital product. This is because recovering a contingent claim pricing kernel amounts to pricing a digital option that pays one unit, if the two events take place. However, we now drop any reference to Sklar's theorem or any other probability theory argument, and focus our attention on no-arbitrage pricing only.

[4]See Nelsen [23] for more details.

[5]Just take the first derivatives w.r.t. E and B. For more details see also Cherubini et al. [6] Joe [13] and Nelsen [23].

Here, we want to show that we can use univariate digital options to hedge the bivariate one: since we focus on bivariate pricing, we assume that we may replicate and price the two univariate digital options with exercise date T, underlying markets E and B, and strike prices K_1 and K_2, respectively.

We now break the sample space into the four relevant regions, as shown in Table 2, in order to facilitate the proofs of static arbitrage relationship. We present the payoffs of the different assets and the relative prices observed in the market in Table 3.

	State H	*State L*
State H	$E \geq K_1,\, B \geq K_2$	$E \geq K_1,\, B < K_2$
State L	$E < K_1,\, B \geq K_2$	$E < K_1,\, B < K_2$

Table 2: Sample space bivariate digital option

	Price	*HH*	*HL*	*LH*	*LL*
Digital Option asset E	DC_E	1	1	0	0
Digital Option asset B	DC_B	1	0	1	0
Risk Free Asset	$P(t,T)$	1	1	1	1
Bivariate digital option	?	1	0	0	0

Table 3: Prices and payoffs for different assets

The problem is to use no-arbitrage arguments to recover the price of the bivariate digital option. In order to find the super-replication strategies leading to pricing bounds for the bivariate digital option, one way is to compare its pay-off with the one portfolios of the single digital options and the risk-free asset:

Theorem 8 *The no-arbitrage price $D_{EB}(K_1,K_2)$ of a bivariate digital option is bounded by the following inequality:*

$$\max(DC_E + DC_B - P(t,T)) \leq D_{EB}(K_1,K_2) \leq \min(DC_E, DC_B).$$

Proof We can immediately check that if $\max\left[DC_E + DC_B - P(t,T), 0\right] > D_{EB}\left(K_1, K_2\right)$ there is an arbitrage opportunity, since I can invest in the bivariate digital option and the risk-free asset, and fund the investment with univariate digital options. By the same token, if $D_{EB}(K_1,K_2) > \min(DC_E, DC_B)$ it will be possible to exploit arbitrage profits by issuing the bivariate digital option and investing the proceedings in the cheaper of the univariate digital options. ■

If we use the forward prices an interesting result emerges:

$$\max\left(\frac{DC_E}{P(t,T)} + \frac{DC_B}{P(t,T)} - 1, 0\right) \leq \frac{D_{EB}(K_1,K_2)}{P(t,T)} \leq \min\left(\frac{DC_E}{P(t,T)}, \frac{DC_B}{P(t,T)}\right). \tag{2.13}$$

The two bounds that constrain the range of the forward price of the bivariate digital option are the Fréchet-Hoeffding bounds. We remark that these bounds simply emerged from no-arbitrage considerations.

The forward price of this bivariate digital option represents the pricing kernel for a bivariate contingent claim. The last step is to show that this kernel is indeed a bivariate copula:

Proposition 9 *The bivariate pricing kernel* $\frac{D_{EB}(K_1,K_2)}{P(t,T)}$ *is a copula of the type*

$$C_{EB}\left(\frac{DC_E}{P(t,T)}, \frac{DC_B}{P(t,T)}\right),$$

since it must fulfil the following requirements to rule out arbitrage opportunities:

1. *It is defined in* $I^2 = [0,1] \times [0,1]$ *and takes values in* $I[0,1]$*;*
2. *For every* v *and* z *of* I^2, $C_{EB}(v,0) = 0 = C_{EB}(0,z)$, $C_{EB}(v,1) = v$, $C_{EB}(1,z) = z$*;*
3. *For every rectangle* $[v_1, v_2] \times [z_1, z_2]$ *in* I^2*, with* $v_1 \leq v_2$ *and* $z_1 \leq z_2$*,*

$$C_{EB}(v_2, z_2) - C_{EB}(v_2, z_1) - C_{EB}(v_1, z_2) + C_{EB}(v_1, z_1) \geq 0.$$

Proof Since the prices of the digital options cannot be higher than the risk-free asset B, the forward prices of both the univariate and bivariate digital are bounded in the unit interval, and the first condition follows. The second condition follows from the no-arbitrage inequality 2.13 by substituting the values 0 and 1 for $v = DC_E/P(t,T)$ or $z = DC_B/P(t,T)$. The final condition can be shown by taking two different strike prices $K_{11} > K_{12}$ for the asset E, and $K_{21} > K_{22}$ for asset B. Denote with $v_1 = DC_E(K_{11})$ the forward price of the first digital option corresponding to the strike K_{11}, with $v_2 DC_E(K_{12})$ that of the first digital option for the strike K_{12} and use an analogous notation for the second asset B. Then, the third condition above can be rewritten as

$$D_{EB}(K_{12}, K_{22}) - D_{EB}(K_{12}, K_{21}) - D_{EB}(K_{11}, K_{22}) + C_{EB}(K_{11}, K_{21}) \geq 0.$$

This implies that a spread position in bivariate options paying one unit if the two underlying assets end in the region $[K_{12}, K_{11}] \times [K_{22}, K_{21}]$ cannot have a negative value. ■

As a result, the arbitrage free pricing kernel of a bivariate contingent claim is a copula taking the univaraite pricing kernels as arguments, and super-replicating portfolios given by the Fréchet-Hoeffding bounds.

$$\max\left(\frac{DC_E}{P(t,T)} + \frac{DC_B}{P(t,T)} - 1, 0\right) \leq C_{EB}\left(\frac{DC_E}{P(t,T)}, \frac{DC_B}{P(t,T)}\right) \leq \min\left(\frac{DC_E}{P(t,T)}, \frac{DC_B}{P(t,T)}\right).$$

With the previous results , we can now state this proposition:

Proposition 10 *The super-replicating strategies for the firm's value* A_t *in an incomplete market for bivariate contingent claims, are given by the following inequality:*

$$\begin{aligned} P(t,T)\int_0^\infty\int_0^D G(E_T, B_T; T)\ \left[\max\left(q_E\left(\frac{DC_E}{P(t,T)}|F_t\right) + q_B\left(\frac{DC_B}{P(t,T)}|F_t\right) - 1, 0\right)\right]\ dE_T\ dB_T \leq \\ \leq A_t = g(E_t, B_t; t) \leq \\ \leq P(t,T)\int_0^\infty\int_0^D G(E_T, B_T; T)\ \left[\min\left(q_E\left(\frac{DC_E}{P(t,T)}|F_t\right), q_B\left(\frac{DC_B}{P(t,T)}|F_t\right)\right)\right]\ dE_T\ dB_T\,, \end{aligned} \tag{2.14}$$

where $q_E(\cdot)$ *and* $q_B(\cdot)$ *are the densities of univariate pricing kernels* $\frac{DC_E}{P(t,T)}$ *and* $\frac{DC_B}{P(t,T)}$.

Proof Just take the densities of the previous bounds (2.14) and substitute them in the pricing equation of the firm's value (2.12). ■

3 Conclusions

In the first part of this paper we outline the early structural models, underlying that, although it is possible to use them to compute the market Assets' value, they do have certain limitations. Primarily, the underlying process is not observable, so it may be necessary to use an approach, such as Ronn & Verma like models, to evaluate the Assets' value implied by stock and bond market prices. Moreover, as remarked by Eom et al. [10], the empirical performances of structural models is quite disappointing. Overall, the increasing financialization of the economy is undermining the basic structural assumption: the exogeneity of Assets' dynamics. As proposed in the second part of our paper, the copula approach seems to be the most suitable way of overcoming the limits of structural models. We therefore treat the Assets'value as a bivariate contingent claim with two observable underlyings: the stock and bond market prices. In this way, we can evaluate the firm value according to its capital structure. The copula approach also has another advantage, we can use it in the case of incomplete markets as well, finding the no-arbitrage bounds of the market Assets' value. Therefore, as a building block of a new framework to compute Assets values, able to model current capital structures taking into account more realistic assumptions than standard structural models, we propose the copula approach.

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