

# Validating and Calibrating Agent-Based Models: A Case Study.

(Preliminary Version)

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## Abstract

In this paper we deal with some validation and calibration experiments on the CATS model proposed in Gallegati et al. (2003a, 2004b).

The CATS model has been extensively used to replicate a large number of scaling type stylized facts with a remarkable degree of precision. For such purposes, the simulation of the model has been performed entering ad hoc parameter values and using the same initial set up for all the agents involved in the experiments.

Nowadays alternative robust and reliable validation techniques for determining whether the simulation model is an acceptable representation of the real system are available. Moreover many distributional and goodness-of-fit tests have been developed while several graphical tools have been proposed to give the researcher a quick comprehension of actual and simulated data.

This paper discusses some validation experiments performed with the CATS model. In particular starting from a sample of Italian firms included in the CEBI database, we perform several ex-post validation experiments over the simulation period 1982-2000. In the experiments, the model parameters have been estimated using actual data and the initial set up consists of a sample of agents in 1982. The CATS model is then simulated over the period 1982-2000. Using alternative validation techniques, the simulations' results are ex-post validated respect to the actual data. The results are promising in that they show the good capabilities of the CATS model in reproducing the observed reality.

Finally we have performed a first calibration experiment via indirect inference, in order to ameliorate our estimates. Even in this case, the results are interesting.

*JEL classification:* C15, C16, D31, E37, L11, L16, O31.

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## 1. Introduction

Mainstream economics adopts the classical mechanical approach of 19th century physics, based upon the reductionist principle, according to which one can understand the aggregate, simply analyzing its single elements. The microfoundation of macroeconomics in the (New) Classical tradition is based on the hope that the aggregate behaviour is the magnification of the single agent's behaviour on a larger scale. The application of the reductionist framework implies that the so-called overlapping principle holds true, i.e. the dynamics of a (linear) model can be decomposed into its constituent parts through the representative agent (RA) framework.

The microeconomic foundations of general equilibrium models must be based, according to mainstream economics, on an optimizing RA, fully rational and omniscient. Unfortunately, "there are no assumptions on [...] isolated individuals which will give us the properties of aggregate behavior which we need to obtain uniqueness and stability. Thus we are reduced to making assumptions at the aggregate level which cannot be justified by the usual individualistic assumptions. This problem is usually avoided in the macroeconomic literature by assuming that the economy behaves like an individual. Such an assumption cannot be justified in the context of the standard economic model and the way to solve the problem may involve rethinking the very basis on which this model is founded." (Hildenbrand and Kirman, 1988, p. 239).

The quantum revolution of the last century radically changed the perspective in contemporary physics. According to the holistic approach, the aggregate is different from the sum of its components because of the interaction of particles. In the social sciences, a step in this direction is taken by the agent-based modeling (ABM) strategy.

Agent-based models, which are increasingly applied in economics (Tesfatsion, 2002; Axelrod, 1997), have been developed to study the interaction of many heterogeneous agents. In a sense they are based on new microfoundations, according to a bottom-up approach. They follow a holistic methodology as opposed to the reductionist approach of the mainstream economics. One builds a model starting from simple behavioral rules at the single agent level. Through interactions some aggregate statistical regularities emerge so that they can not be inferred from the individual level. This emergent behaviour often feeds back to individual agents making their rules change (they may evolve in an adaptive way). According to this approach, macroeconomics is not a set of equations that occurs by summation and averaging of the individual decisions, but it is a SOC (Self-Organized Critical) phenomenon that rises from the micro-level.

As already mentioned, ABM and simulations have been extensively used in many scientific fields, including economics, in the last decade (Axelrod, 1997; Ax-

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tell, 2000). However, in recent years only, researchers have started considering the issue of validation: that is whether a model and its results may be considered correct. As Sargent (1998) puts it: "This concern is addressed through model verification and validation. Model validation is usually defined to mean substantiation that a computerized model within its domain of applicability possesses a satisfactory range of accuracy consistent with the intended application of the model". This is not at all a secondary problem, in fact, only a correct model can be considered a suitable model.

In this paper we deal with some validation experiments of the CATS model proposed in Gallegati et al. (2003a, 2004b).

The CATS model has been extensively used (see, for example, Gallegati et al., 2003b, 2004a, 2005; Delli Gatti, 2004) to replicate a large number of stylized facts with a remarkable degree of precision and, for these purposes, the simulation of the model has been performed entering ad hoc parameters' values and using the same initial set up for all the agents involved in the experiments. It must be recalled that the above mentioned analyses have been performed following Kaldor's suggestion: "construct a hypothesis that could account for these stylized facts, without necessarily committing himself on the historical accuracy" (Kaldor, 1965, page 178).

In this paper our intentions are more ambitious: using an initial set up of actual data (a sample of Italian firms in 1982) we aim to verify if the CATS model, simulated over a period for which actual data are fully available (the interval 1982-2000), is an acceptable representation of the real system. In other words we intend to perform an ex-post validation of the model.

Alternative distributional and goodness-of-fit tests, discussed in Prabhakar et al. (2003) and Kleiber and Kotz (2003), are performed and some graphical tools (Embrechts, 1997) are proposed in order to give the reader a quick comprehension of actual and simulated data.

In the validation exercise, over the simulation period 1982-2000, we use a sample of 18304 Italian firms included in the CEBI database. The model parameters have been estimated using actual data and the initial set up consists of the sample data of the year 1982. The CATS model is then simulated over the period 1982-2000 and the simulations' results are ex-post validated with respect to actual data.

The model reproduces, in a short (medium) term horizon, a good percentage of the output actual data. The two samples (simulated and observed data) belong to the same distribution with a confidence interval of 95%. Moreover the model also reproduces the firms' growth dynamics at a micro level, while less satisfying is the simulation for the behaviour of the very small and very large firms.

We have then performed a first simultaneous calibration-validation of the model via indirect inference, following the techniques presented in Gourieroux et al.

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(1993; 1996). This procedure allows us to ameliorate our estimates. For these reasons we believe indirect inference can be a very powerful tool for validating agent-based models.

The paper is organized as follows: Section 2 presents the state of the art for the validation of agent-based models; Section 3 introduces the model we have studied and validated; Section 4 describes the database we used and the empirical evidence we aim to investigate; Section 5 shows the proceeding of the validation procedure; Sections 6-7 deal with Indirect Inference for calibrating ACE models; while Section 8 concludes.

## 2. Empirical Validation of Agent-based Models (ABM)

As Leigh Tesfatsion points out in her important website on Agent-based Computational Economics<sup>1</sup>, the validation of ACE models is becoming one of the major points in the agenda of those researchers, who work according to the agent-based approach.

In the literature, looking at the main methodological aspects, there are three different ways of validating computational models:

1. descriptive output validation, matching computationally generated output against already available actual data. This kind of validation procedure is probably the most intuitive one and it represents a fundamental step towards a good model's calibration;
2. predictive output validation, matching computationally generated data against yet-to-be-acquired system data. Obviously, the main problem concerning with this procedure is essentially due to the delay between the simulation results and the final comparison with actual data. This may cause some difficulties when trying to study long time phenomena. Anyway, since prediction should be the real aim of every model<sup>2</sup>, predictive output validation must be considered an essential tool for an exhaustive analysis of a model meant to reproduce reality.
3. input validation, ensuring that the fundamental structural, behavioral and institutional conditions incorporated in the model reproduce the main aspects of the actual system. This is what we can call *ex ante* validation: the researcher, in fact, tries to introduce the correct parameters in the model before running it. The information about parameters can be obtained analyzing actual data, thanks to the common empirical analysis. Input validation is obviously a necessary step one has to take before calibrating the model.

Since the empirical validation of agent-based models is still a brand new topic,

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at the moment there are only a limited number of contributions in the literature dealing with it, as summarized below.

In their paper, Axtell et al. (1996) develop the basic concepts and methods of an alignment process for agent-based models. In particular they show how alignment can be used to test if two different computational models can be considered similar in terms of behaviour and output.

In Carley (1996), there's a first stress on model validation issues, even if the attention of the author is still focusing on computational modeling in general.

A very interesting experiment can be found in the paper by Gilli and Winker (2003), in which the authors present an agent-based exchange market model and introduce a global optimization algorithm for calibrating the model's parameters via simulation.

In Castillo et al. (2003), the authors describe several statistical techniques researchers can use in order to select and validate their models.

In Troitzsch (2004), there is a comprehensive list of all the issues concerning the validation of simulation models to describe and predict real world phenomena.

In Fagiolo et al. (2005), one can finally find a very interesting discussion about the ways agent-based modelers have tried to face the empirical validation of their models. The authors briefly review the standard approaches to model validation employed by mainstream economists and then point out the main differences dealing with ABM validation. The paper concludes with some suggestions regarding the methodological aspects of validation.

Finally, without any presumption of being complete and exhaustive, we cannot forget the mainly theoretical and methodological contributions by Sargent (1998), Klevmarken (1998), Epstein (1999), Axelrod (2003), and Judd (2005).

### **3. The CATS model**

The model we are presenting was first introduced in Gallegati et al. (2003a) to study financial fragility and power laws. Here, it's modified to better reproduce actual data, according to the input validation principle we have mentioned above.

Following the ACE philosophy, it is a simple model, since it makes use of straightforward and clear-cut assumptions. Simplicity is one of the main qualities of agent-based models, which are considered good models only if able to reproduce and explain empirical evidences, without being too complicated or making too many assumptions. In other words, the simpler is the model, the easier is reading and interpreting the results.

Consider a sequential economy<sup>3</sup>, with time running  $t = 1, 2, \dots$ , populated by firms and banks. Two markets are opened in each period: the market for a homogeneous produced good and the market for credit. As in the levered aggregate supply class of models first developed in Greenwald and Stiglitz (1990, 1993), our

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model is fully supply-determined, in the sense that firms can sell all the output they optimally decide to produce. Due to informational imperfections in the equity market, firms can raise funds only from the credit market, apart from retained profits from previous periods. This assumption seems to reflect the examined reality in Italy, since new equity issues were rarely a financial option for Italian firms in the observed period. Moreover, the full distribution of dividends to shareholders was expensive, due to the fiscal system. In a perfect environment, without taxes, corporations would not have preferences among these financial options, as shown by the Modigliani-Miller theorem.

Hence, in our setting, the demand for credit is related to investment expenditure and it is fully satisfied at the fixed banks' interest rates: i.e. total credit supply always equals the demand for it. This hypothesis helps us in identifying a suitable proxy of the individual interest rates (namely, the average interest rate), since we have no reliable information related to them.

Let us then briefly describe the main features of the model in the remaining part of the section.

At any time  $t = 1, \dots, 6$ , the economy consists of  $N_t$  firms, each belonging to two different sets (small firms and large ones)<sup>4</sup>, depending on their size, and facing different levels of risk (price shocks). This assumption is different from the original one (Gallegati et al., 2003a) with a single risk level.

Every firm  $i \in N_t$  produces the output  $Y$  according to a linear production function, in which capital ( $K_{it}$ ) is the only input<sup>5</sup>:

$$Y_{it} = \phi_{it}K_{it}. \quad (1)$$

For each firm  $i$  the productivity  $\phi_{it}$  in  $t = 1$  corresponds to its actual productivity (estimated on the CEBI data in 1982) and it evolves according to the following formula:

$$\phi_{it} = \phi_{it-1} + \varrho_{it}\sqrt{\phi_{it-1}}, \quad \text{where } \varrho_{it} = \frac{M}{2}, \quad (2)$$

with  $M \sim U(0, 2)$ , if the firm is small, and to

$$\phi_{it} = \phi_{i1}, \quad (3)$$

if large. All this reproduces the evidence from our database<sup>6</sup>.

Each firm's demand for goods is affected by an *iid* idiosyncratic real shock. Since arbitrage opportunities are imperfect, the individual selling price is the random outcome of a market process around the average market price  $P_t$  of the output, according to the law  $P_{it} = u_{it}P_t$ , where  $E(u_{it}) = \mu$  and  $\sigma_{u_{it}}^2 < +\infty$ . Actual data suggest to split the price generator process into two different processes, depending on firms' size. For the sake of simplicity we assume that  $u_{it}$  follows two

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different uniform distributions: small firms get a high average price and a stronger volatility, while big firms face more concentrated prices with a lower mean. This assumption has a justification in the analysis of actual data: small firms, in fact, show a stronger volatility in their revenues and profits.

Summarizing, if  $U_1$  is the distribution of  $u_{it}$  if  $i$  is small and  $U_2$  if  $i$  is large, we have that  $\mu^{U_1} > \mu^{U_2}$  and  $\sigma_{U_1}^2 > \sigma_{U_2}^2$ .

Since, by assumption, credit is the only external source of finance for firms, the firm can finance its capital expenditure by recurring to net worth ( $A_{it}$ ) or bank loans ( $L_{it}$ ), i.e.  $K_{it} = A_{it} + L_{it}$ . At the exogenous real interest rate  $\bar{r}$ , at each time  $t$  debt commitments for every firm are equal to  $\bar{r}L_{it}$ . Since, for the sake of simplicity, there are no dividends distributed to shareholders, financing costs equal debt commitments. Therefore, profit/loss ( $\pi_{it}$ ) in real terms is:

$$\pi_{it} = u_{it}Y_{it} - \bar{r}L_{it} \quad (4)$$

In our model a firm goes bankrupt if its net worth becomes negative, that is to say  $A_{it} < 0$ . The law of motion of  $A_{it}$  is, for hypothesis,

$$A_{it} = A_{it-1} + \pi_{it}. \quad (5)$$

As in Greenwald and Stiglitz (1993), we assume that the probability of bankruptcy ( $\text{Pr}^b$ ) is directly incorporated into the firm's profit/loss function: bankruptcy is costly and increasing with the firm's size. In particular we have chosen a quadratic cost function:

$$C^b = cY_{it}^2 \quad c > 0 \quad (6)$$

Finally, each firm, by maximizing its objective function, determines its optimal capital stock  $K_{it}^*$ :

$$\max_{K_{it}} \Gamma_{it} = E(\pi_{it}) - E(C^b). \quad (7)$$

and the demand for credit.

#### 4. The Database and the Empirical Evidence

All our validation experiments, together with the subsequent empirical analysis, are based on firm-level observations from the CEBI database, for the period 1982-2000. CEBI, formerly developed by Bank of Italy, is now maintained by Centrale dei Bilanci Srl.

Thanks to several queries on the database, we have collected a sample of 18304 Italian non-financial firms, all satisfying the following: (i) no missing data in each year; (ii) reliable data for capital, employees and costs. For each firm and year, we have data on equities, along term debts and loans, short term debts, total capital,

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gearing ratio, solvency ratio, debt ratio, number of employees, cost of employees and revenues.

Recent explorations (Gallegati et al., 2005) in industrial dynamics have detected two important empirical regularities, which are so widespread across countries and persistent over time to be characterized as universal laws:

1. The distribution of firms' size is right skewed and can be described by a Zipf or power law probability density function (Gallegati et al., 2003b; Gaffeo et al., 2003; Axtell, 2000; Ramsden, Kiss-Haypal, 2000; Okuyama et al., 1999; Quandt, 1966a-b; Simon, 1955);
2. Firms' growth rates are Laplace distributed, belonging to the Subbotin's Family (Stanley et al, 1996; Bottazzi, Secchi, 2003);

Gallegati et al.(2004b) have shown analytically that 1-2 (and other regularities we don't deal with) determine several industrial, financial and business cycle facts (see those papers for a review of the empirical literature.) A model should therefore be able to replicate the empirical evidence 1-2, and our validation exercise is focused on it.

The following section will present the validation exercise, i.e. if the above presented CATS model successfully deals with evidences 1-2.

## 5. Simulation and Results

Our validation exercise is run with a sample of 18304 firms over the period 1982-2000,

The validation procedure we have used is quite new for agent-based models, but it is based on some well-known results of extreme value theory, mainly as far as the analytical tests are concerned (see, for example, Bianchi et al., 2005, or Embrechts, 1997)<sup>7</sup>. Appendix A contains a detailed description of this procedure.

In  $t = 1$ , each firm is initialized with its actual data from 1982: net worth, loans, productivity and so on. The market interest rate is exogenous and, each year, it is equal to the average historical interest rate, as in the Bank of Italy yearly Survey.

In each period actual data from the CEBI database are compared with the simulated data produced by the model. In particular our analysis can be divided into two different approaches: a pointwise analysis, meant to evaluate the evolution of the single firm, in order to study the predictive power of the model; and a distributional analysis, whose aim is to look for regularities.

Our experiments can be considered a first ex-post validation of the CATS model, that is to say a first step, necessary to develop all the subsequent analysis. In Cirillo (2006), the interested reader can find a very complete analysis of all



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the data we have used, together with a detailed description of all the empirical evidences we will deal with.

Let us consider the total capital dynamics. Accepting a maximum deviation of  $\pm 15\%$  between observed and simulated data in 2000 (that is a very low composite yearly deviation rate), we succeed in reproducing 15192 firms over 18304 (83%). As Figure 1 shows, the tails of the firms' size distribution is not adequately fitted. Similar results can be found in the previous years (in 1986, for example, the percentage of fitted firms is 76%, in 1990 it's 80% and 79% in 1996) and analyzing the pooled distributions (79,5%).<sup>8</sup>

In order to verify the real goodness of these results, that's verifying if they are due to the goodness of the model rather than to the characteristics of data (few years, few firms with respect to the universe and so on), we have performed an empirical analysis of actual data. In Figure 2 one can observe the comparison, by the means of a Zipf's plot, between actual total capital in 1982 and 2000. The evidence is quite clear: there is a substantial difference between the two amounts of data. Accepting the usual 15% deviation, only 8% of the firms (essentially the smallest ones) can be considered as fitting the data. In fact, even if both distributions belong to the Paretian case<sup>9</sup>, their parameters are different. Several analytical tools, such as the Kolmogorov-Smirnov's statistics and the Kruskal-Wallis' test, confirm all this. Another informative picture is Figure 3, that summarizes the evolution of total capital over time: all the distributions are rather different.

Even if we calculate the average growth rate of firms from 1982 to 2000 and then we multiply the initial data in 1982 for this coefficient, in 2000 we succeed in fitting "only" 44% of the firms (see figure 4). Since our model can fit 83% of them, it must be considered as better performing.

Figure 1 also shows that both observed and simulated capital distributions are particularly skewed, with fat right tails (decreasing linear relationship in the plot). This reproduces a widely accepted result (Zipf, 1932), according to which firms' size is power law distributed<sup>10</sup> (Axtell, 2000; Gaffeo et al., 2003; Gabaix, 2004).

We have performed many graphical and analytical tests to check if our two samples (observed and simulated data) may be considered belonging to the same distribution.

A first Quantile-Quantile plot (Figure 5) supports the idea of a unique distribution for both samples, since there is a clear linear relationship between the observed and simulated data.

Another graphical test, the Box Plot (Figures 6 and 7), shows that the two distributions have many similarities. For example, the median (red line) is almost the same and it's not centered in the box, indicating two skewed distributions. Moreover, both distributions present a great number of outliers (red plus) in the right tails, underling the possible presence of fat tails.

The same results are supported by the Generalized Kolmogorov-Smirnov Test<sup>11</sup>

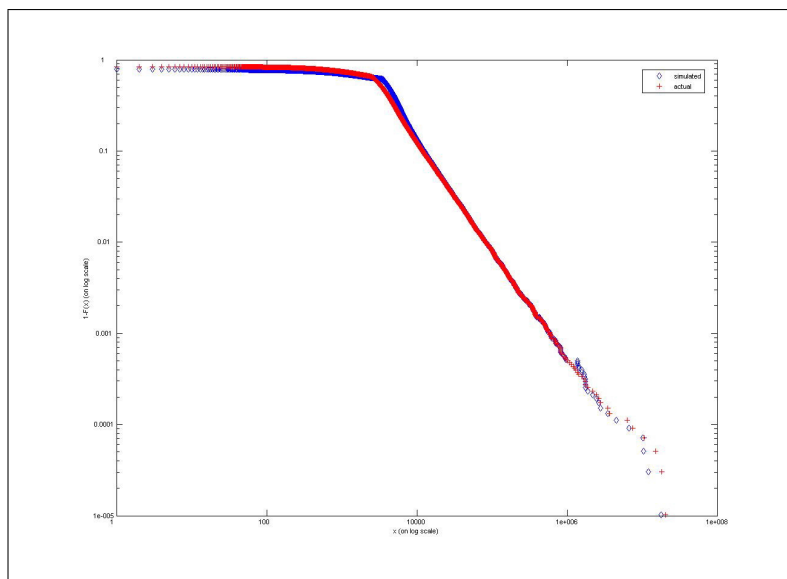


Figure 1: Zipf's Plot of the total capital distributions.

(Prabhakar et al., 2003) with a confidence interval of 95%. Therefore, it's possible to say that our two samples belong to the same distribution or, to be more exact, to the same mixture.

In particular, excluding the right Paretian tails, that we have separately analyzed (we trimmed them after a threshold study, see Cirillo (2006)), we found out that our data follow a Weibull distribution. Figure ?? shows a Weibull Plot<sup>12</sup> of the observed capital distribution (again, after excluding the largest firms).

As far as the right tails of the two distributions, the Mean Excess Function versus Threshold Plot (MEPLOT) clearly shows a Paretian behaviour. An upward sloping mean excess function, as in Figure 8, indicates a heavy tail in the sample distribution. That is why, thanks to the semiparametric Hill's method<sup>13</sup>, we have decided to estimate the shape parameters of the two samples, in order to see if data have a similar behaviour in the right tails<sup>14</sup>.

Figure 9 reports the Hill's estimates of the shape parameter for the simulated capital, while Figure 10 refers to observed data. In the first case  $\alpha = 1.63$ , while in the second one  $\alpha = 1.66$ . Hence, the two parameters are very similar (Figure 11) and belong to the Paretian field ( $0.8 < \alpha < 2$ )<sup>15</sup>, but we cannot claim that the two tails behave in the same way. Simulated capital, in fact, shows a slightly heavier tail (since its alpha is lower<sup>16</sup>), demonstrating that we slightly overestimated the

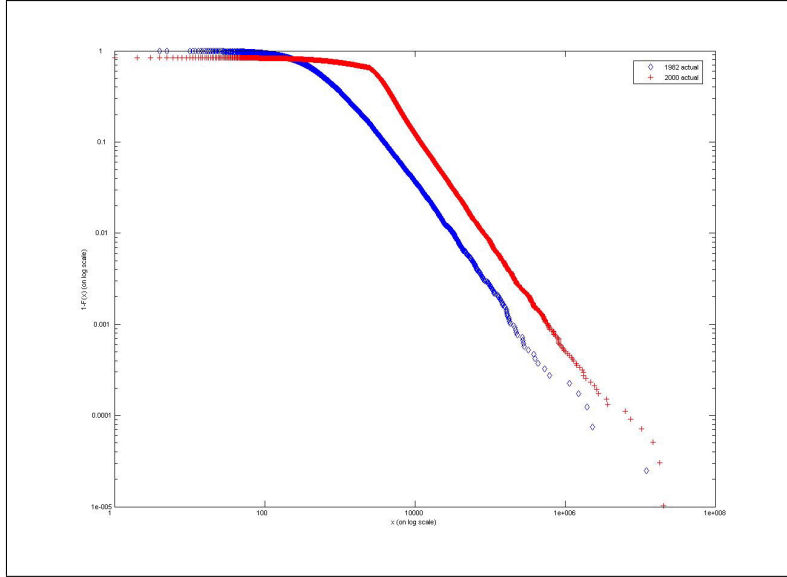


Figure 2: Comparison between actual total capital in 1982 and 2000.

observed values.

As far as net worth is concerned, accepting a maximum deviation of  $\pm 15\%$  between actual and simulated data in 2000, we succeed in reproducing 14461 firms over 18304 (79%)<sup>17</sup>. This number is lower than that of total capital, indicating some more problems of fitting<sup>18</sup>.

Other positive results, see Figure 12, are the skewness of the two distributions and the presence of a clear Paretian behaviour in both actual and simulated net worth. Hill's estimates of the shape parameters both show heavy right tails: actual data present  $\alpha = 1.57$ , while the simulation produces  $\alpha = 1.52$ .

As far as the possibility of a unique distribution for the two samples, the two-sided generalized Kolmogorov-Smirnov test rejects such a null hypothesis. On the contrary the one-sided right version of the test<sup>19</sup> is not rejected, indicating that we get a better fitting of medium and big firms, but we fail in forecasting the smallest one (see in Figure 12), as in Bianchi et al. (2005).

The results we get about loans are very similar to those of the total capital: we succeed in fitting 14827 firms out of 18304 (81%).

Moreover, similarly for total capital and net worth, both graphical and analytical tests support the idea of a unique distribution for both actual and simulated debt data.

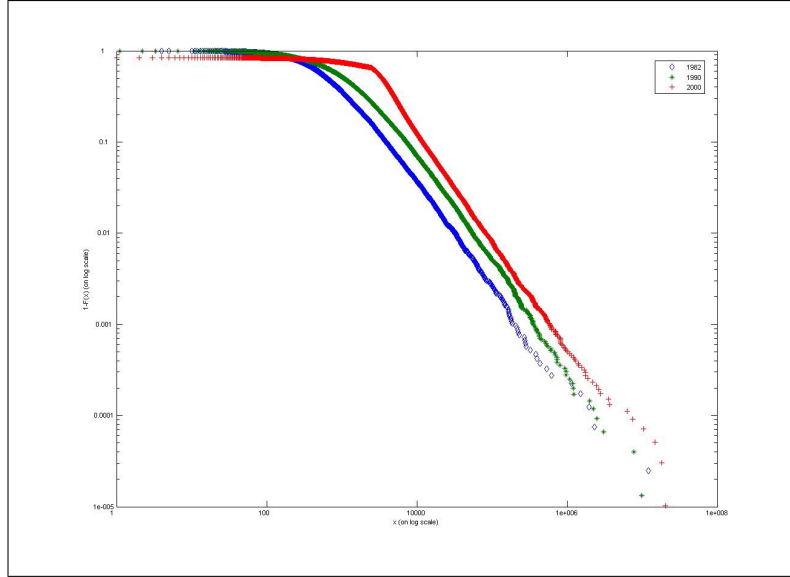


Figure 3: Evolution of total capital over time.

As in Fujiwara (2003), the distribution of loans is also power law. The Hill's estimates of the shape parameters of the Paretian right tails are  $\alpha = 1.69$  for the actual data and  $\alpha = 1.61$  for the simulated ones, demonstrating an overestimate of biggest firms.

Finally, analyzing the ratio between net worth and debt we find out that, apart from some exceptions<sup>20</sup>, it is almost constant for each firm over time. In other words, if firm  $i$  has a ratio of  $x\%$  in 1982, it shows a very similar ratio in 2000,

As far as firms' growth rates are concerned, several studies (Axtell, 2000; Bottazzi and Secchi, 2002; Hall, 1987) find a tent-shape behaviour. In particular, the Laplace and Lévy distributions seem to provide the best fitting (Bottazzi and Secchi, 2003; Gabaix, 2004).

We have investigated if the empirical distributions of growth rates (in terms of capital) belong to the well-known Subbotin's family (Subbotin, 1923), which represents a generalization of several particular cases, such as Laplace and Gaussian distributions. The functional form of Subbotin's family is:

$$f(x, a, b) = \frac{1}{2ab^{\frac{1}{b}}\Gamma(1 + \frac{1}{b})} e^{-\frac{1}{b}|\frac{x-\mu}{a}|^b}, \quad (8)$$

where  $\mu$  is the mean,  $b$  and  $a$  two different shape parameters and  $\Gamma$  is the standard

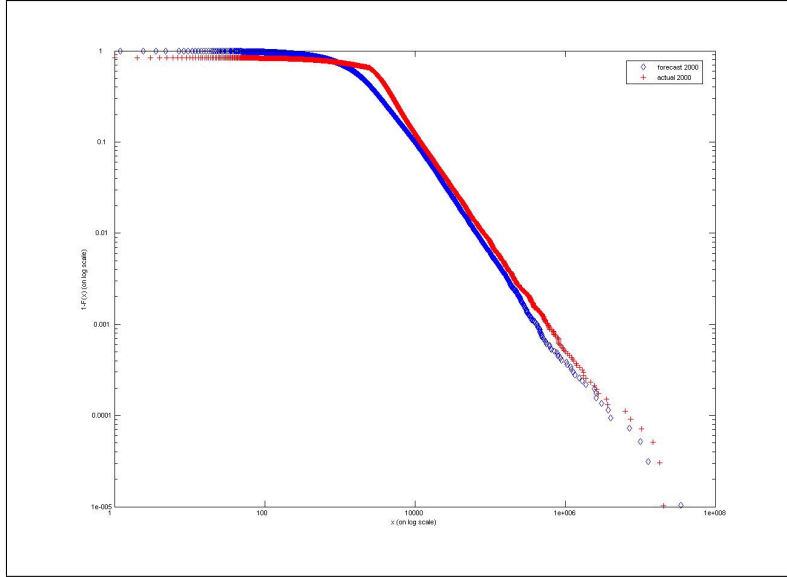


Figure 4: Comparison between actual total capital in 2000 and naive forecast.

Gamma. If  $b \rightarrow 1$  the Subbotin distribution becomes a Laplace, a Gaussian for  $b \rightarrow 2$ .

Using the maximum likelihood method<sup>21</sup>, we have estimated the three Subbotin's parameters on our data. Table 1 contains the results.

At a first glance, observed and simulated growth rates show several similarities:

1. The two means are very close to zero;
2. Since  $b$  is very near to 1, both distributions are in the field of attraction of the Laplacian case<sup>22</sup>. Figure 13 supports this evidence since it's tent-shaped;

	<b>observed data</b>	<b>Simulated data</b>
$\mu$	-0.0022 (0.0010)	0.0038 (0.0013)
$a$	0.0611 (0.0244)	0.0653 (0.0238)
$b$	1.0421 (0.3215)	1.0217 (0.4235)
$-\loglik$	2.1321	3.0119

Table 1: Estimated Subbotin's Parameters (standard errors in brackets)

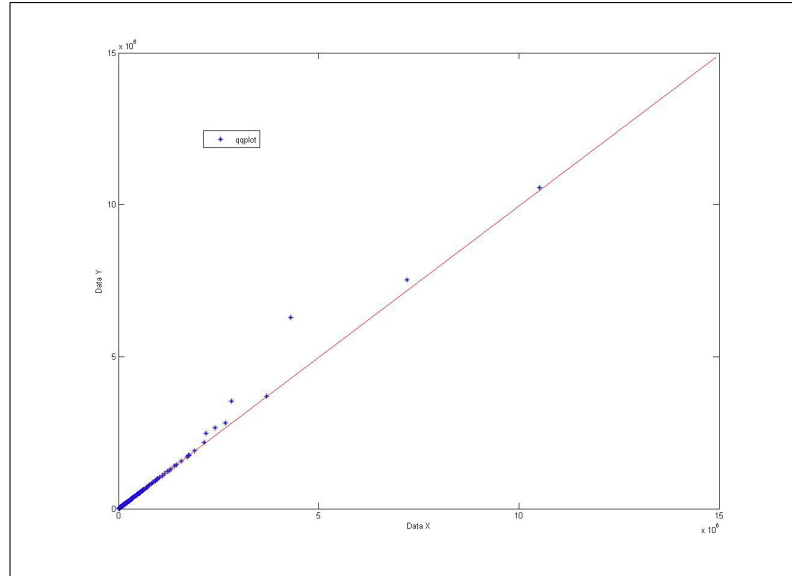


Figure 5: Q-Q Plot of the two Capital distributions.

3. The values of  $a$ , the Laplacian shape parameter, are not very different in both cases, even if simulated data show slightly fatter tails ( $0.065 > 0.061$ ), see Figure 13.

Overall, the CATS model is able to mimic firms' growth dynamic, once again with some discrepancies as far as the tails are concerned.

## 6. Indirect inference: introduction

The aim of this section is to briefly introduce Indirect Inference (Gourieroux et al., 1993, 1996) as estimation and calibration method.

Indirect inference can be understood as a generalization of simulated GMM. Quite often in economics, since the actual model has a complicated structure (in our situation the real industrial model), direct inference is intractable for analytical reasons<sup>23</sup>. If the structural (true) model is easily simulated for any fixed parameter value in the parameter space, indirect inference is probably one of the best procedures to use. According to indirect inference, estimation of the structural parameters consists of two steps. The first one involves the use of an easy-to-compute auxiliary model in order to achieve consistent and asymptotically normal

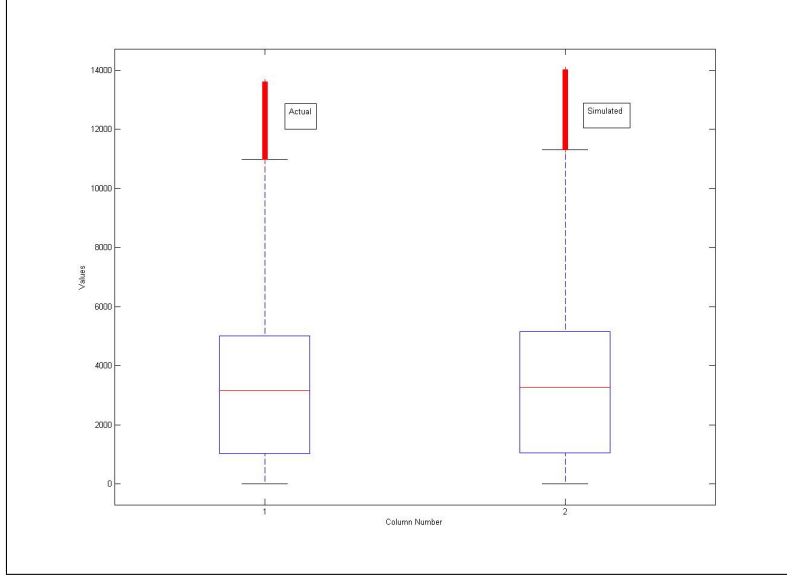


Figure 6: Box Plot of simulated and actual Capital after trimming.

estimates for some auxiliary parameters (pseudo true value). In the second step, simulations are used to correct the discrepancy of the auxiliary parameters from the structural ones.

The idea of indirect inference is as follows. Given a parameter  $\phi$ , let

$$\tilde{y}^h(\phi) = \{\tilde{y}_0^h, \tilde{y}_1^h, \dots, \tilde{y}_T^h\} \quad (9)$$

be data simulated from the true model where  $h=1, \dots, H$  with  $H$  being the number of simulated paths<sup>24</sup>. Now, let's match various functions of the simulated data with those of actual data in order to get estimates of the parameters.

Suppose  $Q_T$  is the objective function of a certain estimation method applied to an auxiliary model indexed by the parameter  $\phi$ . Define the related estimator based on the observed data by

$$\tilde{\theta}_T = \arg \min_{\phi \in \Theta} Q_T(y). \quad (10)$$

The corresponding estimator based on the  $h$ -th simulated path is defined by

$$\tilde{\theta}_T^h(\phi) = \arg \min_{\phi \in \Theta} Q_T(\tilde{y}^h(\phi)), \quad (11)$$

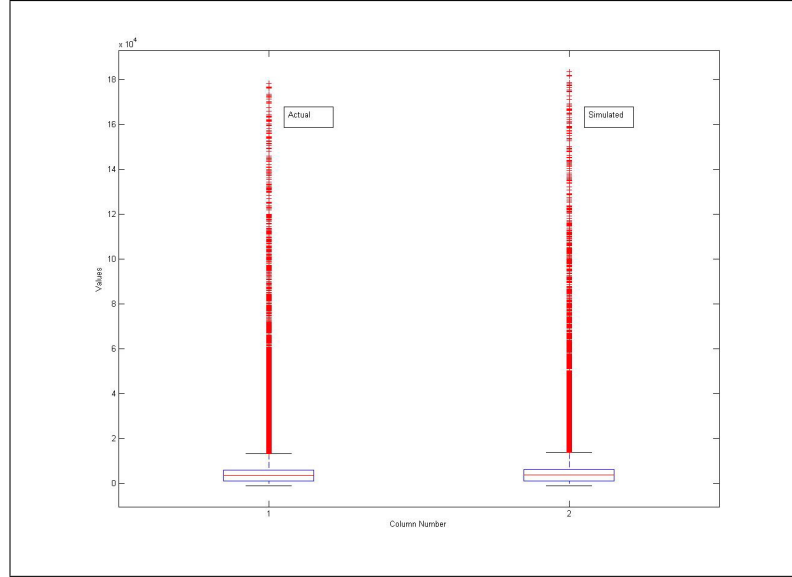


Figure 7: Boxplot of simulated and actual Capital (no trim).

where  $\Theta$  is a compact set.

The indirect inference estimator is set as

$$\phi_{T,H}^{II} = \arg \min_{\phi \in \Phi} \left\| \hat{\theta}_T - \frac{1}{H} \sum_{h=1}^H \tilde{\theta}_T^h(\phi) \right\|, \quad (12)$$

with  $\Phi$  compact set. If  $H$  tends to infinity, the indirect inference estimation becomes

$$\phi_{T,H}^{II} = \arg \min_{\phi \in \Phi} \left\| \hat{\theta}_T - E(\tilde{\theta}_T^h(\phi)) \right\|. \quad (13)$$

Let's define  $b_T(\phi) = E(\tilde{\theta}_T^h(\phi))$  as the *binding function*. When the number of parameters in the auxiliary model is the same as that in the true model and the function  $b$  is invertible, the indirect inference estimator is equal to

$$\phi_T^{II} = b_T^{-1}(\phi). \quad (14)$$

As far as all the properties of Indirect Inference (unbiasedness, efficiency and so on), we refer to the original works of Gouriéroux et al. (1993,1996).

For all these reasons, indirect inference can be thought as a simultaneous way of calibrating and validating models



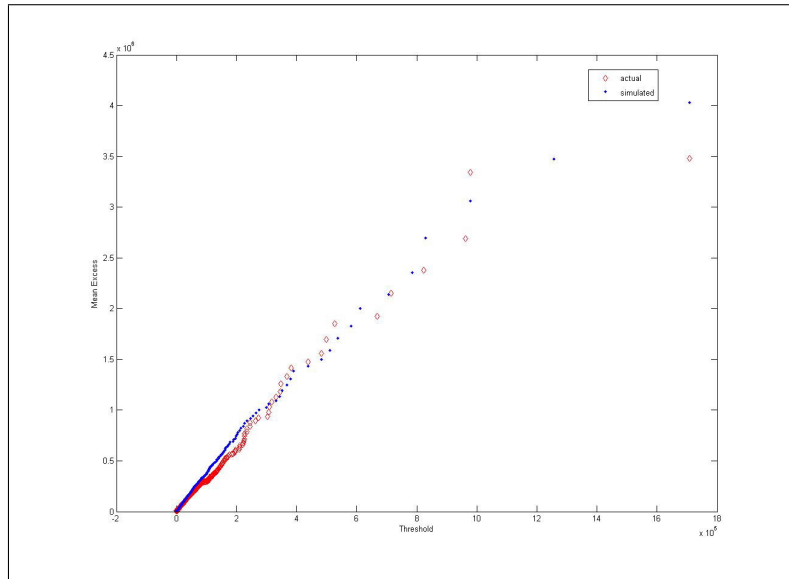


Figure 8: Meplot of observed and simulated total capital.

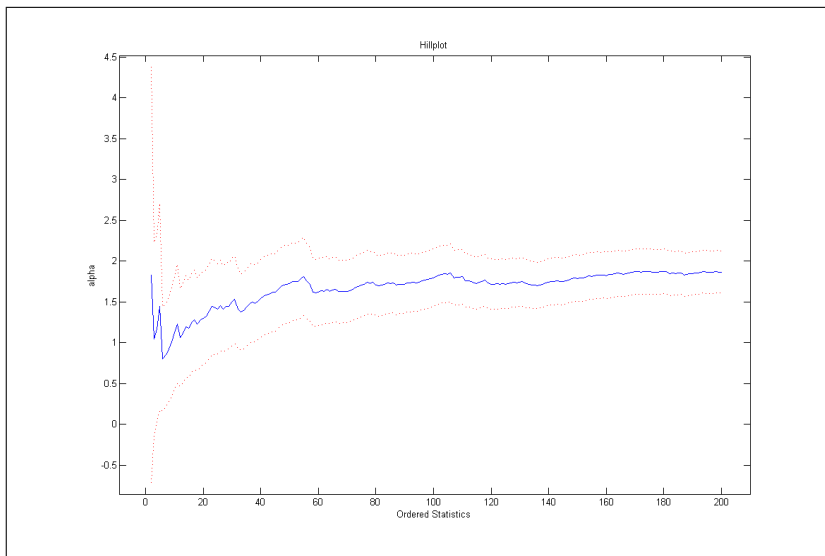


Figure 9: Hill Plot of the simulated Capital

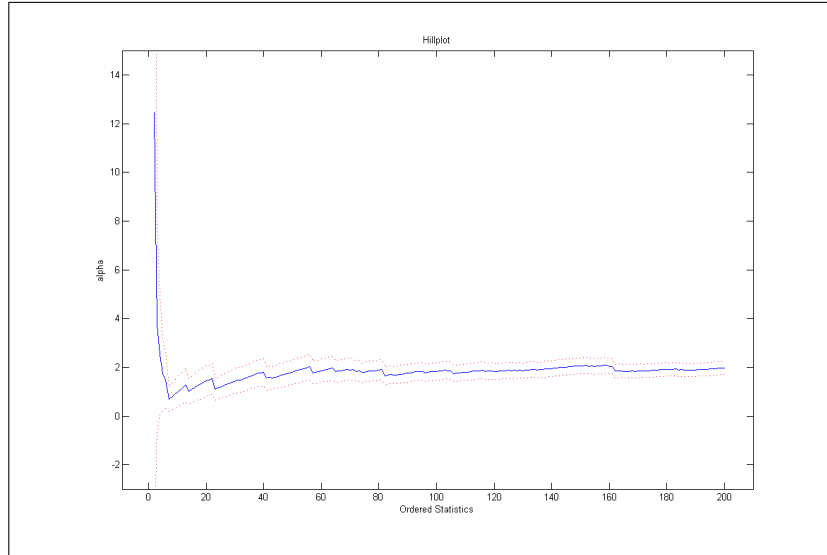


Figure 10: Hill Plot of the actual capital

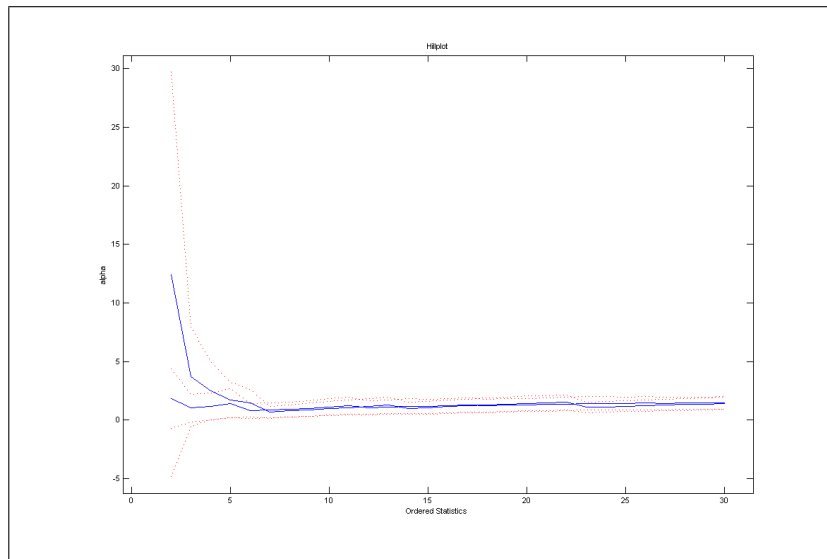


Figure 11: Comparison of the two Hill Plots.

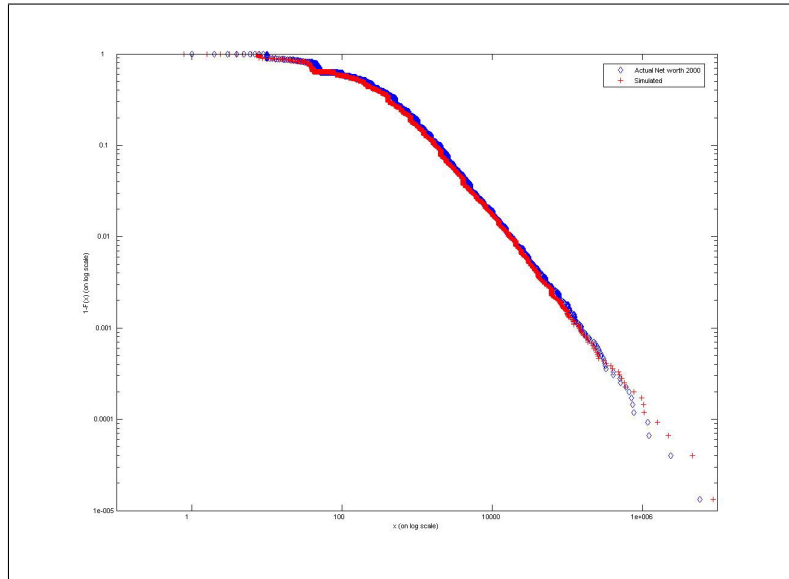


Figure 12: Zipf's Plots of the net worth distributions: actual and simulated data.

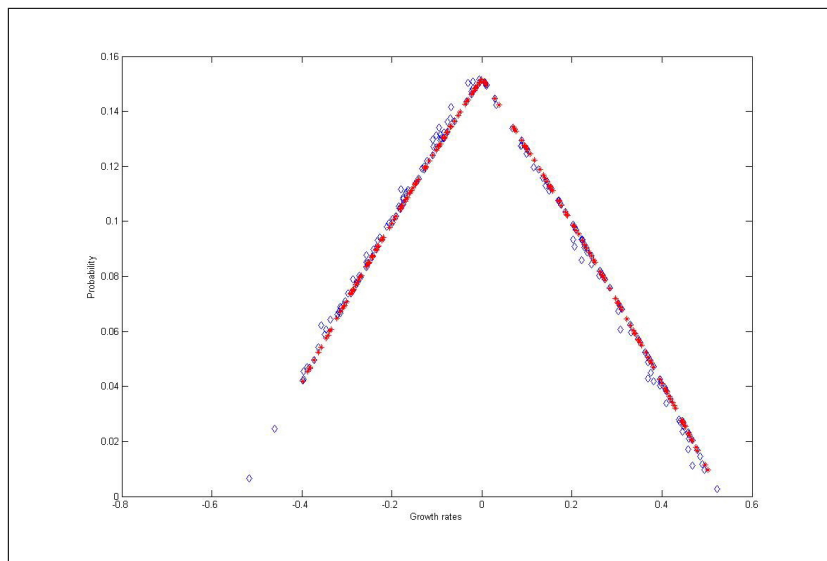


Figure 13: Empirical distributions of actual and simulated growth rates.

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## 7. Indirect Inference: Simulation results

The first step of our calibration and validation procedure consists in defining both the true and the auxiliary model. For our analysis the true model is the Italian Industrial Market, so we assume that the CEBI database comes from a process we don't explicitly know. The CATS models, on the contrary, represents the auxiliary model we are able to simulate.

As we have seen before, the distributions of actual and simulated data are very similar. In particular they show very close moments and scale parameters, while the greater differences are related to the shape parameters  $\alpha^i$ , that's to say the two right tails. Our aim is to calibrate the shape parameters, in order to improve our capability of reproducing empirical data. If we reduce the distance between the shape parameters of actual and simulated data, we can reach our goals. Figure 14 gives a graphical representation of our method.

Unfortunately, we cannot directly calibrate the shape parameters. Anyway, we can make them change, acting on other parameters of the auxiliary model, as required by indirect inference. Since our model is very sensitive to the price generator processes, we have decided to calibrate the supports of the two generators. Starting from the values we have determined with a simple grid method (see Appendix A), we let them vary until the distance between the two right tails is minimized. The procedure we have used is the one by Gouriéroux et al. (1996), which makes use of a quadratic loss function. In order to have perfect identification, besides shape parameters, we calibrate also the scale ones.

The results we achieve are quite promising, even if preliminary. In particular, after calibration via indirect inference, in 2000 we succeed in reproducing the total capital of 16108 firms over 18304, so we ameliorate our estimates<sup>25</sup> passing from 83% to 88%. Similar results are available for the other years (80% in 1986, 84 in 1990 and 81% in 1996) and the pooled distribution. Figure 15 shows the new comparison between actual and simulated total capital in 2000 after calibration via indirect inference.

So, modifying the two supports of the price generators, indirect inference allow us to reduce the distance between the shape parameters of the two distributions:  $\alpha^{simulation}$  passes from 1,63 to 1,646.

## 8. Conclusions and future research

Even if the results we have presented are preliminary they shows that, in the interval 1982-2000, the simple CATS model, firstly introduced by Gallegati et al. (2003a) and slightly modified for these experiments (see section 3), has good capabilities in replicating empirical evidence, with few exceptions.

More reliable results could be obtained improving the specification of the

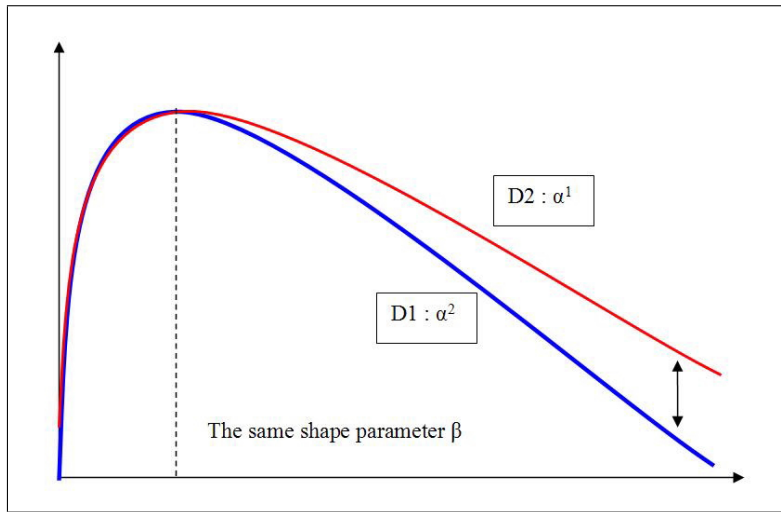


Figure 14: Graphical explanation of our calibration method.

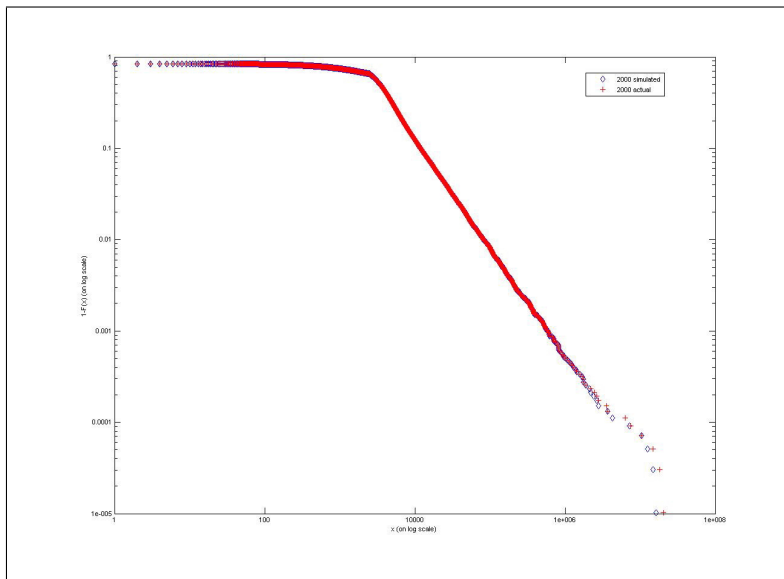


Figure 15: Comparison between actual and simulated total capital in 2000 after calibration via indirect inference.

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model, and then defining new parameters to be calibrated.

In future validation experiments, we intend to modify the model specification, endogenising the banking sector (see Vagliasindi et al., 2005) and the price generator process and including a labor market module. Moreover, we hope to match our database with other dataset in order to increase the available information, which is a fundamental aspects for correctly calibrating simulation models.

## Notes

<sup>1</sup><http://www.econ.iastate.edu/tesfatsi/empvalid.htm>

<sup>2</sup>Validation is not the end of the study process. Indeed, it must be considered an intermediate step, necessary to ameliorate the model in order to make predictions.

<sup>3</sup>In a sequential economy (Hahn, 1982) spot markets open at given dates, while future markets do not operate.

<sup>4</sup>According to the Italian Fiscal Law, to which we referred in writing this paper, a firm is considered: “*small*”, if it has less than 50 employees; “*medium*” if it has between 51 and 250 employees; “*large*” if it has more than 250 employees. In our sample, the percentage of firms is:  $\approx 56\%$  small,  $\approx 31\%$  medium,  $\approx 13\%$  large. In 1996 the smallest firm shows 2 employees, while the largest one 7308.

<sup>5</sup>Capital stock never depreciates.

<sup>6</sup>In fact, as statistically tested, small firms show an increasing productivity, while the large ones present an almost constant one.

<sup>7</sup>But also in Sargent et al. (2000), Sargent (1998), Kleijen (1998), Gumbel (1958).

<sup>8</sup>As in Ijiri et al. (1997), the use of pooled distribution is possible since the single distributions show similar slopes.

In this paper, almost all the figures refer to year 2000.

<sup>9</sup>This is quite obvious: in only six years one cannot expect a distribution to change its shape and family. Finally this is what we can find in most of all the empirical studies about firms.

<sup>10</sup>A power law behaviour in firms’ size is essentially due to the random iid micro-multiplicative shocks (Solomon, 1995; Gabaix, 2004) and the presence of the

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(bankruptcy) lower bound we have modelled. As Lutz et al. (1995) show a system with power laws tails distributions have divergent first and second moments, so the law of large numbers does not hold and the system is not ergodic. All this has disruptive consequences for mainstream economics (Davidson, 1986).

<sup>11</sup>The Generalised or *Two Sample* Kolmogorov-Smirnov test is a variation of the classical Kolmogorov-Smirnov test.

Given  $N$  data points  $Y_1, Y_2, \dots, Y_n$  the empirical distribution function (ECDF) is defined as

$$F_N = \frac{n(i)}{N}, \quad (15)$$

where  $n(i)$  represents the number of points less than  $Y_i$ . As one can see, this step function increases by  $\frac{1}{N}$  for each data point.

The Kolmogorov-Smirnov test is based on the maximum distance between the ECDF and the theoretical cumulative distribution one wants to test ( $F^T$ ):

$$D = \max_{1 \leq i \leq N} \left| F^T(Y_i) - \frac{i}{N} \right|. \quad (16)$$

On the contrary, the two sample K-S test, instead of comparing an empirical distribution function to the theoretical one, compares two different ECDF, that is

$$D = |F_1(i) - F_2(i)|, \quad (17)$$

where  $F_i$  is the empirical distribution for sample  $i$ .

The generalised K-S statistic can be defined as:

$H_0 : F_1 = F_2 \rightarrow$  the two samples come from the same distribution

$H_1 : F_1 \neq F_2 \rightarrow$  the two samples come from different distributions

To decide the results of the test, the values of  $D$  are compared to the critical values obtained from Kolmogorov and Smirnov's table.

<sup>12</sup>That's a quantile-quantile plot with a theoretical Weibull distribution.

<sup>13</sup>The well-known Hill's Estimator  $\xi$ , together with the Pickard's one, is the most used way to determine the shape parameter  $\alpha = \frac{1}{\xi}$  of a distribution belonging to the GEV family.

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In particular

$$\xi = \frac{1}{k-1} \sum_{i=1}^{k-1} \ln x_{i,N} - \ln x_{k,N} \quad \text{for } k \geq 2, \quad (18)$$

where  $k$  is the upper order statistics and  $N$  the sample size.

<sup>14</sup>Similar results can be obtained with standard MLE.

<sup>15</sup>Once again the results concerning the pooled distributions are very similar. The reason can be found in the words of Ijiri et al. (1977): “We conclude that when two or more Pareto distributions are pooled together, the resulting distribution is Pareto if and only if all the distributions have similar slopes [...]. This result is important in dealing with the aggregation of empirical firm size distributions.”

<sup>16</sup>As clearly showed in Kleiber and Kotz (2003), the Pareto density has a polynomial right tail that varies at infinity with index  $(-\alpha - 1)$ , implying that the right tail is heavier as  $\alpha$  is smaller.

<sup>17</sup>80% in 1986, 78% in 1996.

<sup>18</sup>In particular this may depend on the hypothesis of the model that: 1) firms cannot raise funds on the equity market, 2) profits are entirely retained in the firm and 3) as suggested by the referee, that all firms face the very same interest rate. However, these simplifying hypothesis, typical of CATS model, do not seem to affect too much the robustness of our validation results.

<sup>19</sup> $H_0 : F_1^+(x) = F_2^+(x)$ .

$H_1 : F_1^+(x) > F_2^+(x)$ .

<sup>20</sup>While validating our model, we have experienced several experiments on interest rates, finding out an interesting thing.

Those firms showing a decreasing net worth/debt ratio are the same that obviously go bankrupt if the interest rates rise. All this is interesting since the decreasing ratio is almost completely due to a monotonically deteriorating equity ratio (Beaver, 1966; Gallegati et al., 2005; Bianchi et al., 2005).

<sup>21</sup>The results are very similar, using the method of moments.

<sup>22</sup>Some authors prefer a truncated Lévy distribution. The *querelle* is open. See Kleiber and Kotz (2003).



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<sup>23</sup>It is useful when the moments and the likelihood function of the true model are difficult to deal with, but the true model is amenable to data simulation.

<sup>24</sup>The number of simulated observations must be the same as the number of actual observations for the purpose of the bias calibration.

<sup>25</sup>We succeed in better reproducing smaller firms.

## A Validation Procedure: some notes

The aim of this appendix is to describe the procedure we have used to validate the CATS model.

All the codes and the programs have been written in Fortran90©, while all the graphics have been developed with Matlab7©.

As far as the simulation of the CATS model is concerned, it can be useful to underline the following aspects:

1. In  $t = 1$  (1982), when the simulation starts, every firm is initialized with its actual data from the database. These data are: net worth, loans and productivity. The current version of the model has a recursive structure so that parameters  $\phi_{it}$  have been consistently estimated using, firm by firm, ordinary least squares. Then productivity evolves according to the laws of motions presented in 2 and 3;
2. The parameter  $M$  in 2 follows a uniform distribution, whose support  $(0, 2)$  has been ad hoc calibrated, thanks to several replications;
3. The interest rate is equal to the average historical one;
4. The two different uniform distributions we have used to model the idiosyncratic shocks on prices show support  $(0.6, 2.6)$  for small firms and support  $(0.6, 1.4)$  for the large ones. This supports have been inductively calibrated with a grid method, considering the results of several alternative replications, in order to get the best fitting values;
5. Every year the following data are stored in order to be compared with actual data: net worth, loans, total capital, productivity, growth rates, paid interests, total output, aggregate output.

Our analysis of data can be divided into two different approaches: a pointwise analysis, meant to evaluate the evolution of the single firm, in order to study the predictive power of the model; and a distributional analysis, whose aim is to look for general regularities.

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In Embrechts (1997), one can find a quite complete list of all the tests a researcher should perform in analyzing data, while Kleijnen (1998) deals with the theoretical implications of validation.

Further information on this procedure is present in Bianchi et al. (2005).

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